The Many Faces of Modal Logic
Day 5: Compositionality

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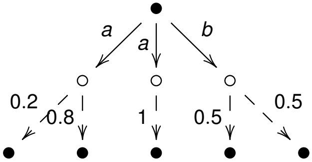
The Plan for Today

Expanding the scope

- Fusion / modularity
- Global assumptions
- Nominals
- Fixpoints
- First-order logic

Example: Modal Logic of Segala Systems

Functor $TX = \mathcal{P}(A \times \mathcal{D}X)$:

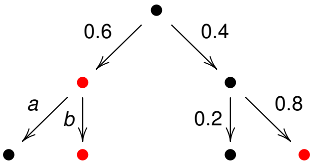


- Sorts u ('uncertain'), n ('non-deterministic')
- Operators $\Box_a : u \to n$, $L_p : n \to u$

$$\mathcal{L}_n \ni \phi ::= \top \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \Box_a \psi$$
$$\mathcal{L}_u \ni \psi ::= \top \mid \psi_1 \wedge \psi_2 \mid \neg\psi \mid L_p \phi.$$

Example: Modal Logic of Alternating Systems

Functor $TX = \mathcal{P}(A \times X) + \mathcal{D}X$:



- Sorts u ('uncertain'), n ('non-deterministic'), c ('choice')
- Operators

$$L_p : c \to u \qquad \Box_a : c \to n \qquad + : u, n \to c$$

Example: Conditional Logic *CK*

Functor

$$CX = (2^X \to \mathcal{P}(X))$$

so

$$C = G \circ \langle id, \mathcal{P} \rangle \quad \text{where} \quad G(X, Y) = 2^X \to Y.$$

$\to$ Embed into two-sorted logic:

- Sorts c ('conditional'), n ('non-deterministic')
- Operators $\Rightarrow : c, n \to c$ $\quad \Box : c \to n$

Then

$$\alpha \Rightarrow \beta \rightsquigarrow \alpha \Rightarrow \Box\beta.$$

Logical Features

An *n*-ary feature $(\Lambda, \mathcal{R})$ consists of

- A set Λ of modal operators with profiles

$$L : i_1, \ldots, i_k \to * \qquad (1 \leq i_j \leq n)$$

- A set $\mathcal{R}$ of one-step rules

$$\frac{\phi_1; \ldots; \phi_n}{\psi} \qquad \begin{array}{l}(\text{rank } 0)\\(\text{rank } 1)\end{array}$$

Logical Features: Examples

Non-Determinism: unary feature N, $\Box : 1 \to *$

Uncertainty: unary feature U, operators $L_p : 1 \to *$, $p \in [0,1] \cap \mathbb{Q}$.

Choice: binary feature S, operator $+ : 1,2 \to *$.

Fusion: binary feature P, operators $[\pi_i] : i \to *$, $i = 1,2$.

Conditionality: binary feature C, operator $\Rightarrow : 1,2 \to *$.

Logical Features: Semantics

Structure for n-ary feature $\mathsf{F} = (\Lambda, \mathcal{R})$:

- functor $[\![\mathsf{F}]\!] : \mathbf{Set}^n \to \mathbf{Set}$
- for $L : i_1, \ldots, i_k \to *$, predicate lifting

$$[\![L]\!] : (\mathcal{Q} \circ P_{i_1}^{op}) \times \cdots \times (\mathcal{Q} \circ P_{i_k}^{op}) \to \mathcal{Q} \circ [\![\mathsf{F}]\!]^{op}$$

($\mathcal{Q}$ contravariant powerset, P_i projection)

such that $\mathcal{R}$ is (one-step) sound.

Semantics: Examples

Choice: $[\![\mathsf{S}]\!](X,Y) = X + Y$

$$[\![+]\!]_{X,Y}(A,B) = A + B \subseteq X + Y.$$

Fusion: $[\![\mathsf{P}]\!](X,Y) = X \times Y$;

$$[\![\pi_1]\!]_{X,Y} A = \{(x,y) \mid x \in A\}.$$

Conditionality: $[\![\mathsf{C}]\!](X,Y) = \mathcal{Q}X \to Y$;

$$[\![\Rightarrow]\!]_{X,Y}(A,B) = \{f : \mathcal{Q}X \to Y \mid f(A) \in B\}.$$

Feature Combination: Gluings

- Feature expressions t over set $\mathcal{S}$ of sorts:

$$t ::= a \mid \mathsf{F}(t_1, \ldots, t_n) \qquad a \in \mathcal{S}, \mathsf{F}\ n\text{-ary feature.}$$

- Gluing: family $\mathsf{G} = (t_a)_{a \in \mathcal{S}}$ of feature expressions.

- G flat iff all t_a have the form $\mathsf{F}(a_1, \ldots, a_n)$.

The Modal Logic of a Gluing

- types: proper subexpressions of the t_a
- obvious operator/rule profiles modulo $t_a \sim a$
- feature expression s induces composite functor $[\![s]\!] : \mathbf{Set}^{\mathcal{S}} \to \mathbf{Set}$, hence $[\![\mathsf{G}]\!] : \mathbf{Set}^{\mathcal{S}} \to \mathbf{Set}^{\mathcal{S}}$
- Semantics over $[\![\mathsf{G}]\!]$-coalgebra $(X, \xi) = ((X_a)_{a \in \mathcal{S}}, (\xi_a)_{a \in \mathcal{S}})$: $[\![\phi : s]\!] \subseteq [\![s]\!]X$,

$$\begin{aligned} [\![L(\phi_1, \dots, \phi_n) : \mathsf{F}(s_1, \dots, s_n)]\!] &= [\![L]\!]([\![\phi_1]\!], \dots, [\![\phi_n]\!]) \\ [\![L(\phi_1, \dots, \phi_n) : a]\!] &= \xi_a^{-1} \; [\![L]\!]([\![\phi_1]\!], \dots, [\![\phi_n]\!]) \end{aligned}$$

Gluings: Examples

Segala Systems: $(n \to \mathsf{N}(\mathsf{U}(n)))$

Alternating Systems: $(c \to \mathsf{S}(\mathsf{U}(c), \mathsf{N}(c)))$

Conditional Logic ***CK*****:** $(c \to \mathsf{C}(c, \mathsf{K}(c)))$

Alternatively: flat gluings, e.g.

Segala Systems: $(u \to \mathsf{U}(n), n \to \mathsf{N}(u))$

Generally: G has flattening $\mathsf{G}^\flat$ inducing the same syntax:
Turn all types into sorts (above: n and $u := \mathsf{U}(n)$).

Reduction to Flat Gluings

Theorem: ϕ satisfiable in G iff ϕ satisfiable in $G^\flat$.
Proof of 'if' by Example:

$$\begin{array}{ccc}
(X, Y) & \xrightarrow{(id,\xi_u)} & (X, [\![N]\!] Y) \\
\downarrow{\scriptstyle (\xi_n,\xi_u)} & & \downarrow{\scriptstyle (([\![N]\!]\xi_u)\xi_n, id)} \\
([\![N]\!] Y, [\![U]\!] X) & \xrightarrow[([\![N]\!]\xi_n, id)]{} & ([\![N]\!][\![U]\!] X, [\![N]\!] Y)
\end{array}$$

Coalgebra morphisms preserve satisfaction.

Everything is Compositional

- Reduce to flat gluings
- Coalgebraic meta-theorems generalize straightforwardly to flat gluings
  - Completeness
  - Complexity
  - …

Description Logics

- Notational variant of modal logic

$$\text{Recall: } \exists R = \Diamond_R \qquad \forall R = \Box_R$$

- Reasoning with terminologies and assertions:
  - TBoxes: global assumptions imposed to hold everywhere in a model

$$\exists \text{studies}.\,\text{coalgebraicProblem} \sqsubseteq \exists \text{hasCondition}.\,\text{headache}$$

  - ABoxes: knowledge about individuals

$$\text{studies}(\text{Alice}, \text{streamProblem1}) \qquad \text{coalgebraicProblem}(\text{streamProblem1})$$

Global Consequence

C ϕ-model if $\forall x \in C.\, x \models \phi$

ψ ϕ-satisfiable if there exist a ϕ-model C and $x \in C$ such that $x \models \psi$.

Without blocking, tableaux fail to terminate : for $\phi = \Diamond p$, $\psi = p$:

$$\cfrac{\cfrac{p, \Diamond p}{p, \Diamond p}}{\dots}$$

Branches can be exponentially long before blocking
(implement binary counter via ϕ)

Global consequence in K is EXPTIME-complete

Models for TBoxes

as in previous completeness proof

- Closed set $\Sigma \ni \phi, \psi$
- States:

$$S = \{\Gamma \subseteq \Sigma \mid \Gamma \text{ MCS}, \phi \in \Gamma\}$$

- Find coherent structure on S by one-step completeness

Type Elimination

Take $S_0 = \{\Gamma \subseteq \Sigma \mid \Gamma \text{ Hintikka}\}$, and put

$$FS = \{\Gamma \in S \mid \underbrace{\exists t \in TS.\, t \models \heartsuit \overbrace{\hat{\rho}}^{=\{\Delta \in S \mid \rho \in \Delta\}} \iff \heartsuit \rho \in \Gamma}_{\text{Coherence}}\}$$

→ solve a one-step satisfiability (OSS) problem:

$$(\theta, \chi) : (\text{rank } 0, \text{rank } 1)$$

$$(X, \tau, t) \models (\theta, \chi) \iff X, \tau \models \theta \wedge t \models_\tau \chi.$$

For OSS, need upper bound

- polynomial in θ and exponential in χ
- that is, exponential in χ. So:

Theorem OSS EXPTIME in $\chi \implies$ TBox reasoning in EXPTIME

Example: Presburger Constraints

(Following Demri/Lugiez)

$$\sum a_i \#(\psi_i) \geq b$$

- OSS: Solve system of integer linear inequalities.
- ILP (Papadimitriou 1981): Binary size of solutions
  - polynomial in number of inequalities, times
  - logarithmic in number of variables
- Hence, here: polynomial in rank-1 part.
- NPSPACE (in χ) Algorithm:
  - Guess multiplicities for $s \in S$ successively
  - Keep track of the $\#\psi_i$ only.
- NPSPACE $\subseteq$ EXPTIME:
  - Depth-first search in exponentially large state space.

Hybrid Logic

- Nominals = designators $i, j, \ldots$ for individual states
- Hybrid valuations π:

$$|\pi(i)| = 1 \qquad c, \pi \models i \iff c \in \pi(i) \quad (\iff \pi(i) = \{c\})$$

- Satisfaction operators $@_i$ (jump to i)

$$c, \pi \models @_i \phi \iff \pi(i), \pi \models \phi$$

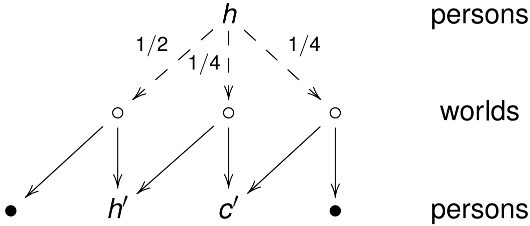
E.g.

$$\models L_p i \wedge L_q j \rightarrow L_q i \vee L_1(\neg(i \wedge j))$$

in probabilistic hybrid logic

The Uncertain Progeny of Henry VIII

h = Henry VIII
h' = Henry Carey
c' = Catherine Carey



Combined TBox and ABox e.g.

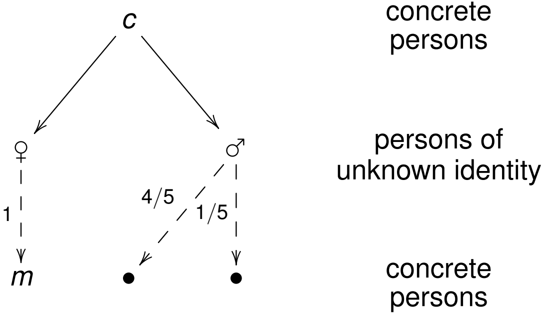
$$\text{king} \to L_{4/5} \exists \text{child}.\, \text{illegitimate}$$

$$@_h L_{3/4} \exists \text{child}.\, h'$$

The Uncertain Progeny of Catherine of Aragon

c = Catherine of Aragon

m = Queen Mary



$$\text{queen} \to \forall \text{child}.\, \neg L_1 \text{illegitimate}$$

$$@_c \exists \text{child}.\, L_1 \text{queen}$$

Axiomatizing Coalgebraic Hybrid Logic

- $@$ commutes with propositional connectives, e.g. $\neg @_i \phi \leftrightarrow @_i \neg \phi$
- $i \sim j :\equiv @_i j$ is an equivalence, e.g. $@_i j \wedge @_j k \to @_i k$
- Agreement: $@_i @_j \phi \leftrightarrow @_j \phi$
- $@$-generalization: $\phi / @_i \phi$
- $@$-introduction: $i \wedge \phi \to @_i \phi$
- make-or-break: $@_i \phi \to (\heartsuit \psi \leftrightarrow \heartsuit(@_i \phi \wedge \psi))$
- And, of course, the one-step rules.

Relativized Congruence

$$\frac{\phi \to (\psi \leftrightarrow \chi)}{\phi \to (\heartsuit\psi \leftrightarrow \heartsuit\chi)} \quad (\phi \text{ @-formula})$$

is derivable: Under ϕ,

$$\begin{aligned} \heartsuit\psi &\leftrightarrow \heartsuit(\psi \wedge \phi) && \text{(MOB)} \\ &\leftrightarrow \heartsuit(\chi \wedge \phi) && \text{(assumption)} \\ &\leftrightarrow \heartsuit\chi && \text{(MOB)} \end{aligned}$$

Fixing an ABox

- Σ = closure of ϕ under subformulas, negation and @
  - normalize to stay finite
- Fix maximally consistent $K \subseteq @\Sigma = \{\psi \in \Sigma \mid \psi \text{ @-formula}\}$
- Put $S_K = \{\Gamma \subseteq \Sigma \mid \Gamma \text{ MCS}, K \subseteq \Gamma\}$
- Put $\pi(a) = \{\Gamma \in S_K \mid a \in \Gamma\}$
  - Modality-free axioms imply that π is hybrid

The Relativized Existence Lemma

There exists a coherent coalgebra on S_K

Proof:

- Top-level decomposition $\phi = \phi_0 \sigma$
- Non-existence is $S_K, \tau \models \neg\phi_0$ where

$$\tau(a) = \{\Gamma \in S_K \mid \sigma(a) \in \Gamma\}$$

- $\neg\phi_0$ one-step derivable from propositional χ valid over S_K, τ
- For these, $\vdash \bigwedge K \to \chi\sigma$
- Then $\vdash \bigwedge K \to \neg\phi_0\sigma$ by relativized congruence, contradiction.

Further Results on Coalgebraic Hybrid Logic

- EXPTIME with TBox by Type Elimination
- PSPACE via forest models
- In some cases: strong completeness via named models
  - with pure axioms (e.g. $\Diamond\Diamond i \to \Diamond i$)
  - with $\downarrow$:

    $$c, \pi \models \downarrow i.\phi \iff c, \pi[i \mapsto c] \models \phi$$

    Axiom:

    $$\text{(DA)} \qquad @_i((\downarrow j.\phi) \leftrightarrow \phi[i/j]).$$