The Many Faces of Modal Logic
Day 3: Algebraic Semantics

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Detour Through Algebraic Semantics

Goal. Coherence Conditions for Completeness, i.e. $\text{Log}(T) \subseteq \text{Log}(\mathcal{R})$, or: 'enough' rules to generate all semantically valid formulae.

Cheap Trick. Use *algebraic semantics* (first)

- logical connectives $\wedge, \vee, \Box, \dots$ are like term-constructors $+, *, \dots$ in algebra
- obey algebraic rules, e.g. $a \wedge b = b \wedge a$
- algebraic semantics has cheap completeness theorem.

Duality. Use *algebraic completeness* to establish *coalgebraic* (or frame) completeness.

Algebraic Semantics

Given: modal similarity type Λ.

Modal Algebras = tuples $A = (A, [\![\cdot]\!])$ where

- A Boolean algebra
- $[\![\heartsuit]\!] : A^n \to A$ for $\heartsuit \in \Lambda$ n-ary.

Algebraic Interpretation over Λ-algebra A, valuation $\theta : \mathcal{V} \to A$

$$[\![p]\!]\theta = \theta(p) \qquad [\![\heartsuit(\phi_1, \dots, \phi_n)]\!]\theta = [\![\heartsuit]\!]([\![\phi_1]\!]\theta, \dots, [\![\phi_n]\!]\theta)$$

and propositional connectives via Boolean algebra structure.

For $\phi \in \mathcal{F}(\mathcal{V})$ write $A, \theta \models \phi$ if $[\![\phi]\!]\theta = \top$.

Coalgebras Induce Algebras

Given: Λ-structure T and $(C, \gamma) \in \mathrm{Coalg}(T)$.

Induced Λ-algebra $(\mathcal{P}(C), [\![\cdot]\!])$ where

$$[\![\heartsuit]\!](A_1, \ldots, A_n) = \gamma^{-1} \circ [\![\heartsuit]\!]_C(A_1, \ldots, A_n)$$

Alignment Lemma. Let $(C, \gamma) \in \mathrm{Coalg}(T)$, $\theta : \mathcal{V} \to \mathcal{P}(C)$. Then

$$C, c, \theta \models \phi \iff c \in [\![\phi]\!]\theta$$

where $(\mathcal{P}(C), [\![\cdot]\!])$ is the induced Λ-algebra.

Slogan. Every T-coalgebra is a Λ-algebra, in a way that preserves logical validity. *How about the other way around?*

Algebraic Completeness

Logic of a class of Algebras. For $\mathcal{A}$ class of Λ-algebras,

$$\mathrm{Log}(\mathcal{A}) = \{\phi \in \mathcal{F}(\Lambda) \mid [\![\phi]\!]\theta = \top \text{ for all } A \in \mathcal{A}, \theta : \mathcal{V} \to A\}$$

Soundness of $\mathcal{R}$ with respect to $\mathcal{A}$: $\mathrm{Log}(\mathcal{R}) \subseteq \mathrm{Log}(\mathcal{A})$

Completeness of $\mathcal{R}$ with respect to $\mathcal{A}$: $\mathrm{Log}(\mathcal{A}) \subseteq \mathrm{Log}(\mathcal{R})$

Valid Rules. ϕ/ψ (not necessarily rank-1) *valid* over Λ-algebra $(A, [\![\cdot]\!])$ if

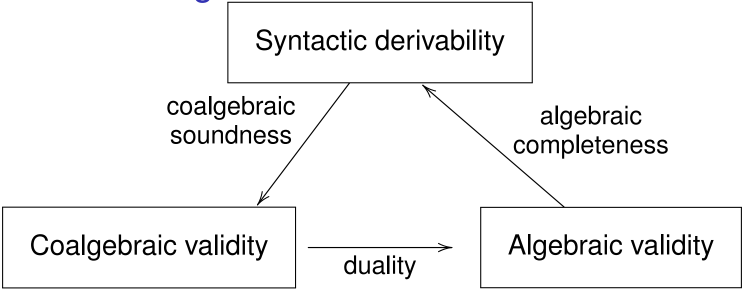
$$[\![\psi]\!]\theta = \top \text{ whenever } [\![\phi]\!]\theta = \top$$

for all $\theta : \mathcal{V} \to A$.

Algebras determined by a set of rules.

$$\mathrm{Alg}(\mathcal{R}) = \{A\ \Lambda\text{-algebra} \mid \text{ all } \phi/\psi \in \mathcal{R} \text{ valid over } A\}$$

Algebraic vs Coalgebraic Semantics



Coalgebraic Soundness.

- follows from one-step soundness (already done)

Algebraic Completeness.

- is easy: Lindenbaum Construction (our next step)

Duality.

- show contrapositive: model construction (later today)

Lindenbaum Says: Algebraic Completeness is Easy

Given. Set $\mathcal{R}$ of Λ-Rules determining class $\mathcal{A} = \mathsf{Alg}(\mathcal{R})$ of algebras.

Lindenbaum Algebra. Let $\phi \sim \psi \iff \phi \leftrightarrow \psi \in \mathsf{Log}(\mathcal{R})$ and

$$A = (\mathcal{F}(\Lambda)/\sim, [\![\cdot]\!]) \text{ with } [\![\heartsuit]\!]([\phi]_\sim) = [\heartsuit\phi]_\sim$$

Then A is a well-defined Λ-algebra.

Trivial Lemma. $\mathcal{R} \vdash \phi \iff [\![\phi]\!]\theta = \top$ where $\theta(p) = [p]$.

Algebraic Completeness. $\mathsf{Log}(\mathcal{A}) \subseteq \mathsf{Log}(\mathcal{R})$.

Proof. The Lindenbaum algebra A lies in $\mathcal{A}$.

Aside: From Axioms to Rules

Easy: e.g.

$$(K) \qquad \Box(a \to b) \to \Box a \to \Box b$$

is already a rule $\top / \psi$.

Normalize to $\psi \in \mathrm{Prop}(\Lambda(V))$:

$$\frac{c \leftrightarrow (a \to b)}{\Box c \to \Box a \to \Box b}.$$

Transform to CNF / Clause:

$$\frac{c \wedge a \to b \qquad c \vee a \qquad b \to c}{\Box c \wedge \Box a \to \Box b}.$$

Aside: From Rules to Axioms

Boolean unification: Given ϕ/ψ rank 1, $\kappa \models \phi$ put

$$\sigma(a) = \begin{cases} a \wedge \phi, & \text{if } \kappa(a) = \bot; \\ \phi \to a & \text{otherwise.} \end{cases}$$

Then

$$\models \phi \to (a \leftrightarrow \sigma(a)) \qquad \models \phi\sigma$$

(2nd claim: case distinction over whether $\tau \models \phi$ for valuation τ) so

$$\psi\sigma \quad \text{replaces} \quad \frac{\phi}{\psi}$$

(given the congruence rule!)

From Rules to Axioms: Example

Monotonicity rule

$$\frac{a \to b}{\Box a \to \Box b}$$

$\kappa(a)$	$\kappa(b)$	$\sigma(a)$	$\sigma(b)$	$\psi\sigma$
$\top$	$\top$	a	$a \vee b$	$\Box a \to \Box(a \vee b)$
$\bot$	$\bot$	$a \wedge b$	b	$\Box(a \wedge b) \to \Box b$
$\bot$	$\top$	$a \wedge b$	$a \vee b$	$\Box(a \wedge b) \to \Box(a \vee b)$

The Hard Part: Duality and Model Constructions

Goal. If ϕ is valid in $\mathsf{Alg}(\mathcal{R})$ then ϕ is valid in $\mathsf{Coalg}(T)$ (subject to coherence $\mathcal{R} \leftrightarrow T$).

Dually:

- if ϕ is satisfiable in some algebra
- then ϕ is satisfiable in some *finite* algebra (*filtration*)
- then ϕ is satisfiable in some T-coalgebra (*model construction*)

First Question. Given Λ-algebra A, what is the carrier C of a model?

Interlude: Stone Duality

First Goal. From a Boolean algebra A construct a set of “points” $\mathrm{Uf}(A)$ such that $A \subseteq \mathcal{P}(\mathrm{Uf}(A))$ subalgebra

Second Goal. equip $\mathrm{Uf}(A)$ with a T-structure $\gamma : \mathrm{Uf}(A) \to T\mathrm{Uf}(A)$

Heuristics.
Suppose that we have already constructed $\mathrm{Uf}(A)$ such that $A \subseteq \mathcal{P}(\mathrm{Uf}(A))$ is a sub-algebra.

- every $u \in \mathrm{Uf}(A)$ determines a subset $\{a \in A \mid u \in a\} \subseteq A$
  – the set of propositions true at u
- these sets are “saturated” in a way that we will make precise

Ultrafilters

Let A be a Boolean algebra.

Partial Order on A

$$a \leq b \iff a \wedge b = a$$

Filters are subsets $F \subseteq A$ that are

- up-closed: $a \in F$ and $a \leq b$ implies $b \in F$
- meet-closed: $a, b \in F$ implies $a \wedge b \in F$

Ultrafilters are filters $F \subseteq A$ that are

- proper, i.e. $\bot \notin F$; and
- $a \vee b \in F$ implies $a \in F$ or $b \in F$.
- Equivalently: for each a, exactly one of $a, \neg a$ is in F
- Equivalently: F is a maximal proper filter

Handy Things About Ultrafilters

Ultrafilters exist. Let A be a Boolean algebra, $F \subseteq A$ such that

$$a_1 \wedge \cdots \wedge a_n \neq \bot$$

for all (finitely many) $a_1, \ldots, a_n \in F$.
Then there exists an ultrafilter $u \subseteq A$ with $F \subseteq u$.

Proof. Extend F to a (proper) filter, use Zorn's lemma (!).

Ultrafilters Determine Truth. Let A be a Boolean algebra and $a \in A$.
Then $a = \top$ iff $a \in u$ for all $u \in \mathrm{Uf}(A)$.

Proof. If not, $\neg a \neq \bot$ extends to an ultrafilter u with $a \notin u$.

From Boolean Algebras to Powerset Algebras

Let A be a Boolean algebra and $\mathrm{Uf}(A)$ the set of ultrafilters on A. Define

$$\begin{aligned} j : A &\to \mathcal{P}(\mathrm{Uf}(A)) \\ a &\mapsto \hat{a} = \{u \in \mathrm{Uf}(A) \mid a \in u\}. \end{aligned}$$

This is clearly a Boolean algebra morphism.

Stone's Theorem. j is injective

From Boolean Algebras to Powerset Algebras

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Stone's Theorem. j is injective
(and hence makes A a subalgebra of $\mathcal{P}(\mathsf{Uf}(A))$)

Stone Duality in the Finite

...is much more harmless:

- *Atoms* in a BA are minimal elements $\neq \bot$.
- A finite, $u \in \mathrm{Uf}(A)$: $\bigwedge u$ atom, $u = \{b \in A \mid b \geq \bigwedge u\}$
- So $\mathrm{Uf}(A) \cong$ atoms in A
- $j : A \cong \mathcal{P}(\mathrm{Uf}(A))$, i.e. j is also surjective:
  - **Proof:** $\{a_1, \ldots, a_n\} = j(a_1 \vee \cdots \vee a_n)$.

Roadmap for Completeness

Goal.

ϕ coalgebraically *valid* implies ϕ *derivable*.

Contrapositive.

If ϕ is *not derivable*, then $\neg\phi$ is coalgebraically *satisfiable*.

Algebraic Completeness.

If ϕ is *not derivable*, then $\neg\phi$ is *algebraically* satisfiable.

Need to show.

Algebraic satisfiability implies *coalgebraic* satisfiability.

Coherent Structures

Goal. Given finite Λ-algebra A, construct $\gamma : \mathrm{Uf}(A) \to T\mathrm{Uf}(A)$ with

$$\mathrm{Uf}(A), u \models \phi \iff [\![\phi]\!]_A \in u$$

viewing $A \cong \mathcal{P}(\mathrm{Uf}(A))$ as a powerset algebra.

Definition. $\gamma : \mathrm{Uf}(A) \to T\mathrm{Uf}(A)$ *coherent* if

$$[\![\heartsuit]\!]_A a \in u \iff \gamma(u) \in [\![\heartsuit]\!]_{\mathrm{Uf}(A)} \hat{a}$$

where for $a \in A$ we put $\hat{a} = \{u \in \mathrm{Uf}(A) \mid a \in u\}$.

The Truth Lemma

Truth Lemma. Let $\gamma : \mathsf{Uf}(A) \to T\mathsf{Uf}(A)$ be coherent. Then

$$\mathsf{Uf}(A), u \models \phi \iff [\![\phi]\!]_A \in u \iff u \in \widehat{[\![\phi]\!]_A}$$

(i.e. $[\![\phi]\!]_{\mathsf{Uf}(A)} = \widehat{[\![\phi]\!]_A}$)

Proof. Induction on formulae using coherence for modal operators:

$$\mathsf{Uf}(A), u \models \heartsuit\phi \iff \gamma(u) \in [\![\heartsuit]\!]_{\mathsf{Uf}(A)}([\![\phi]\!]_{\mathsf{Uf}(A)}) \stackrel{\mathsf{IH}}{=} [\![\heartsuit]\!]_{\mathsf{Uf}(A)}\widehat{[\![\phi]\!]_A}$$

$$\stackrel{\text{coherence}}{\iff} \underbrace{[\![\heartsuit]\!]_A [\![\phi]\!]_A}_{= [\![\heartsuit\phi]\!]_A} \in u$$

Do Coherent Structures Exist?

Approach. Let ϕ be satisfiable in $\mathsf{Alg}(\mathcal{R})$

- i.e. $[\![\phi]\!]_A \neq \bot$ for some Λ-algebra A
- construct coherent structure $\gamma : \mathsf{Uf}(A) \to T\mathsf{Uf}(A)$
- then there is $u \in \mathsf{Uf}(A)$ so that $\mathsf{Uf}(A), u \models \phi$
- this shows that algebraic satisfiability implies coalgebraic satisfiability.

Next Step. Coherent structures exist on finite $\mathsf{Uf}(A)$.

Recall. $\mathcal{R}$ is one-step sound if $\mathsf{Log}_1(\mathcal{R}) \subseteq \mathsf{Log}_1(T)$.

One-Step Completeness. $\mathcal{R}$ is one-step complete with respect to T if $\mathsf{Log}_1(T) \subseteq \mathsf{Log}_1(\mathcal{R})$.

One-Step Completeness: Intuition

Idea. $\mathcal{R}$ is one-step complete if $\mathcal{R}$ is strong enough to derive all one-step validities $\phi \in \text{Prop}(\Lambda(\text{Prop}(\mathcal{V}))$.

Equivalent Characterisation. $\mathcal{R}$ is one-step complete, if:

- for all sets X and all valuations $\theta : \mathcal{V} \to \mathcal{P}(X)$
- for all $\rho \in \text{Prop}(\Lambda(\mathcal{V}))$ with $[\![\rho]\!]\theta = TX$

we have that ρ is derivable

- from all $\psi\sigma$ where $\phi/\psi \in \mathcal{R}$ and $[\![\phi\sigma]\!]\theta = \top$
- using only propositional reasoning.

One-Step Completeness: Examples

Example. Take the modal logic K and the set of rules comprising

$$\frac{a_1, \ldots, a_n \to a_0}{\Box a_1 \wedge \cdots \wedge \Box a_n \to \Box a_0}$$

for each $n \geq 0$ (clearly derivable in K). If

$$TX, \sigma \models \bigwedge_i \Box p_i \to \bigvee_j \Box q_j$$

then

$$\bigcap_i \sigma(p_i) \in \bigcap_i [\![\Box]\!]_X(\sigma(p_i)) \subseteq \bigcup_j [\![\Box]\!]_X(\sigma(q_j))$$

– i.e. there is j such that

$$\bigcap_i \sigma(p_i) \subseteq \sigma(q_j)$$

which we use as rule premiss in a one-step deduction.

More Examples

The rule sets seen previously (graded / probabilistic / coalition / conditional logic) are one-step complete.

(Not always as easily.)

Coherent Structures on Finite Algebras

Existence Lemma. Let $A \in \mathrm{Alg}(\mathcal{R})$ *finite*, $\mathcal{R}$ one-step complete for T. Then there is a coherent structure $\gamma : \mathrm{Uf}(A) \to T\mathrm{Uf}(A)$.

Proof. For $u \in \mathrm{Uf}(A)$ we just need to pick $\gamma(u)$ from the set

$$\bigcap_{[\![\heartsuit]\!] a \in u} [\![\heartsuit]\!]_{\mathrm{Uf}(A)} \hat{a} \quad \cap \bigcap_{[\![\heartsuit]\!] a \notin u} (T\mathrm{Uf}(A) - [\![\heartsuit]\!]_{\mathrm{Uf}(A)} \hat{a}).$$

If this set were empty, the (finite!) clause

$$\chi = \bigvee_{[\![\heartsuit]\!] a \in u} \neg \heartsuit p_a \vee \bigvee_{[\![\heartsuit]\!] a \notin u} \heartsuit p_a$$

would be valid over TX under $\hat{\theta}(p_a) = \hat{a}$.

Existence Lemma (cont'd)

One-step completeness: $\chi = \bigvee_{[\![\heartsuit]\!] a \in u} \neg \heartsuit p_a \vee \bigvee_{[\![\heartsuit]\!] a \notin u} \heartsuit p_a$ valid under $\hat{\theta}$, hence propositionally derivable from

$$\psi\sigma \qquad (\phi/\psi \in \mathcal{R}, \qquad \underbrace{[\![\phi\sigma]\!]\hat{\theta} = \top = X}_{\iff \theta(\phi\sigma) = \top \text{ in } A \text{ where } \theta(p_a) = a} \qquad)$$

Copy this derivation to show $\theta(\chi) = \top$ in A, hence $\theta(\neg\chi) = \bot$
but by construction $\theta(\neg\chi) \in u$, contradiction to u proper.

Filtrations, or: chopping off the infinite

Last Step. If $[\![\phi]\!]\theta \neq \bot$ in some Λ-algebra A, then A can be chosen finite.

Filtrations. Let A be a Λ-algebra, $B \subseteq A$ a finite Boolean sub-algebra, and $u \subseteq E(u) \in \mathrm{Uf}(A)$ for all $u \in \mathrm{Uf}(B)$. Define $[\![\heartsuit]\!]_B : B \to B$ by

$$[\![\heartsuit]\!]_B b = \bigvee \{ \bigwedge u \mid u \in \mathrm{Uf}(B), [\![\heartsuit]\!]_A b \in E(u) \}$$

Then $(B, [\![\cdot]\!])$ is a *filtration* of A. We have

$$[\![\phi]\!]_B \theta = [\![\phi]\!]_A \theta$$

whenever $[\![\rho]\!]_A \theta \in B$ for all subformulae ρ of ϕ.

Proof. Induction on formulae, and using properties of ultrafilters.

Filtrations Preserve Rules

Non-Iterative Rules are of the form ϕ/ψ where $\mathrm{rk}(\phi) = 0$ and $\mathrm{rk}(\psi) \leq 1$ (and $\mathrm{rk}(\rho)$ is the nesting depth of modal operators). (Generalizes rank-1)

Filtrations preserve non-iterative rules. (cf. Lewis 1974) Let A be a Λ-algebra, $B \subseteq A$ a filtration and ϕ/ψ a non-iterative rule. If ϕ/ψ is valid on A, then ϕ/ψ is valid on B.

Proof. We may assume that ψ is a clause over literals $\heartsuit p$ and variables $p \in \mathcal{V}$. If $B, \theta \models \phi$, then $A, \theta \models \phi$ whence $A, \theta \models \psi$. For $u \in \mathrm{Uf}(B)$, at least one disjunct l of ψ lies in $E(u)$

- $l = \pm p$: $\theta(p) \in u \iff \theta(p) \in E(u)$, since $\theta(p) \in B$.
- $l = \pm\heartsuit p$: $[\![\heartsuit]\!]_B \theta(p) \in u \iff \bigwedge u \leq [\![\heartsuit]\!]_B \theta(p) \iff [\![\heartsuit]\!]_A \theta(p) \in E(u)$

Putting Things Together

Let $\mathcal{R}$ be one-step sound and complete with respect to T.

Main Theorem. The following are equivalent for $\phi \in \mathcal{F}(\Lambda)$

1. $\phi \in \mathsf{Log}(\mathcal{R})$
2. $\phi \in \mathsf{Log}(T)$
3. $[\![\phi]\!]\theta = \top$ in all finite $A \in \mathsf{Alg}(\mathcal{R})$
4. $[\![\phi]\!]\theta = \top$ in all $A \in \mathsf{Alg}(\mathcal{R})$

Proof. Using coalgebraic soundness, finite model construction, filtration, and Lindenbaum algebra.

Dissecting Things Further: the FMP

Observation. Turning finite algebras into models gives *finite* models.

Small Model Property. If $\phi \in \mathcal{F}(\Lambda)$ is satisfiable, then ϕ is satisfiable on a frame (C, γ) with $|C| \leq 2^{|\phi|}$

Proof. If ϕ is satisfiable, then ϕ is satisfiable in Lindenbaum algebra, hence in the filtration on the Boolean subalgebra B generated by the subformulae of ϕ. By Duality, ϕ satisfiable in $\mathrm{Uf}(B)$, which has the claimed size (atoms can be written as finite conjunctions of subformulas of ϕ).

Dissecting Even Further: Non-Iterative Logics

Preservation Lemma. Let A be a finite Λ-algebra, and ϕ/ψ a non-iterative rule valid on A. Then

$$\mathsf{Uf}(A), u, \theta \models \psi \text{ whenever } \mathsf{Uf}(A), u, \theta \models \phi$$

for all $u \in \mathsf{Uf}(A)$ where $\mathsf{Uf}(A) = (\mathsf{Uf}(A), \gamma)$ is the coherent structure on $\mathsf{Uf}(A)$.

Proof. Extending the truth lemma we have

$$\mathsf{Uf}(A), u, \hat{\theta} \models \phi \iff u \in \theta(\phi)$$

for all valuations $\theta : \mathcal{V} \to A$. The claim follows as every valuation $\mathcal{V} \to \mathcal{P}(\mathsf{Uf}(A))$ arises as $\hat{\theta}$ for some $\theta : \mathcal{V} \to A$ as A is *finite*, hence $\mathcal{P}(\mathsf{Uf}(A)) \cong A$..

Non-Iterative Completeness

The *model class* of a set $\mathcal{R}_1$ of non-iterative rules

$$\mathrm{Frm}(\mathcal{R}_1) = \{C \in \mathrm{Coalg}(T) \mid C, \sigma \models \psi \text{ whenever } C, \sigma \models \phi \;\; (\sigma : \mathcal{V} \to \mathcal{P}(C))\}$$

is the set of frames that validate all rules in $\mathcal{R}_1$.

Completeness for restricted Frame Classes. Let $\mathcal{R}_0$ be one-step sound and complete, and $\mathcal{R}_1$ be non-iterative. Then

$$\mathrm{Log}(\mathcal{R}_0 \cup \mathcal{R}_1) = \mathrm{Log}(\mathrm{Frm}(\mathcal{R}_1))$$

that is, $\mathcal{R}_0 \cup \mathcal{R}_1$ is sound and complete with respect to the class of frames that validate $\mathcal{R}_1$.

Final Question for Today

Q. We get completeness from one-step completeness. But do one-step complete rule sets even exist?

Proposition. The set of all one-step sound rank-1 rules is one-step complete.

Proof. Let $[\![\psi]\!]\theta = TX$ for $\theta : \mathcal{V}_0 \to \mathcal{P}(X)$ and finite $\mathcal{V}_0 \subseteq \mathcal{V}$. Put $\phi = \bigwedge\{\chi \in \mathsf{Prop}(\mathcal{V}_0) \mid [\![\chi]\!]\theta = \top\}$. Then ϕ/ψ is one-step sound.

Summary for Today. Coalgebraic Logics can always be axiomatised by rank-1 rules / axioms. Tomorrow, we'll do this (more) efficiently!