Dualities in Algebraic Logic

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13 February 2018
LAC 2018, Melbourne

Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
- Final remarks

Algebraic Logic

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- abstract algebraic logic:
  study Logic using methods from (universal) algebra

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  i.e. there are contravariant functors linking C and D

A Fundamental Duality

verbal visual

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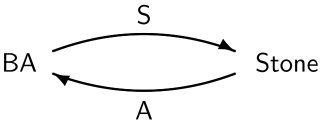
Stone duality:



Variants of Stone duality

- Heyting algebra vs Esakia spaces
- compact regular frames vs compact Hausdorff spaces
- distributive lattices vs Priestley spaces
- modal algebras vs topological Kripke structures
- cylindric algebras vs . . .
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Contravariance In all these examples both categories are concrete!

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Aim:

- introduce TKS
- develop duality between MA and TKS

Modal Algebras

- $\mathbb{A} = (A, \vee, -, \bot, \Diamond)$ is a modal algebra if
  - $(A, \vee, -, \bot)$ is a Boolean algebra
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- Modal algebras are (the simplest) Boolean Algebras with Operators

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- KS is the category of Kripke structures with bounded morphisms

Stone spaces

- A (topological) space is a pair (S, τ) where τ is a topology on S
- A Stone space is a space (S, τ) where τ is
  - compact,
  - Hausdorff
  - zero-dimensional (i.e. it has a basis of clopen sets)
- Stone is the category of Stone spaces and continuous functions

Stone duality

From Stone spaces to Boolean algebras: $(\cdot)^*$

Objects Given (S, τ) take $(S, \tau)^* := (Clp(\tau), \cup, \sim_S, \varnothing)$

Arrows Given $f : (S', \tau') \to (S, \tau)$ define $f^* : Clp(\tau) \to Clp(\tau')$

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The functors $(\cdot)^*$ and $(\cdot)_*$ witness the dual equivalence of BA and Stone.

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This is a natural duality evolving around the schizophrenic object 2

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- Open Problem characterize the ultrafilter structures modulo isomorphism

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- TKS is the category with
  - objects: topological Kripke structures
  - arrows: continuous bounded morphism

Topological modal duality

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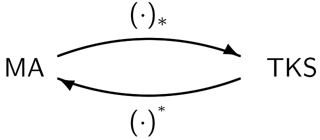

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Folklore Subdirect irreducibility is related to rootedness

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Roots

Auxiliary definitions

- $R^\omega := \bigcup_{n>0} R^n$,
  - where $R^0 := Id_S$ and $R^{n+1} := R \circ R^n$
- $R(s) := \{t \in S \mid Rst\}$
- $r \in S$ is a root of $\mathbb{S}$ if $S = R^\omega(r)$
- $\mathbb{S}$ is rooted if its collection $W_\mathbb{S}$ of roots is non-empty

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Proposition (Rautenberg)
$\mathbb{A}$ is s.i. iff $\mathbb{A}_*$ has a largest nontrivial, closed hereditary subset.

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Observations

Proposition For any modal algebra $\mathbb{A}$:
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(2) $Q^\omega \subseteq Q^\star$
(3) $Q^\star(u)$ is hereditary for any ultrafilter u
(4) $Q^\star(u)$ is closed for any ultrafilter u
(5) $Q^\star(u) = \overline{Q^\omega(u)}$ for any ultrafilter u
(6) $\langle Q^\star \rangle$ maps opens to opens
(7) If Q is transitive then $Q = Q^\omega = Q^\star$

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Suggestion Develop the modal theory of $Q^\star$

Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
- Final remarks

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- $\mathsf{V}(\mathbb{X}) := \langle K(\mathbb{X}), \upsilon_\tau \rangle$ is the Vietoris space of $\mathbb{X}$.

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Different presentation:

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Fact The Vietoris construction preserves various properties, including:

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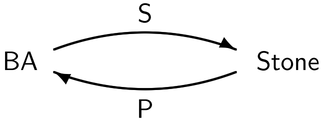
Observation Stone duality and the Vietoris functor:



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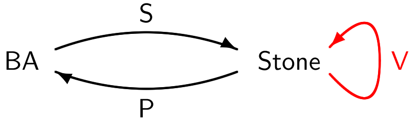

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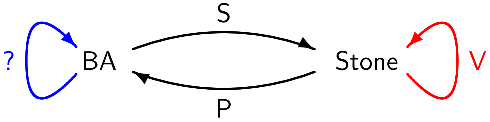

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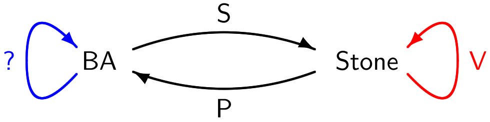

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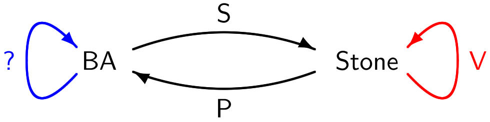

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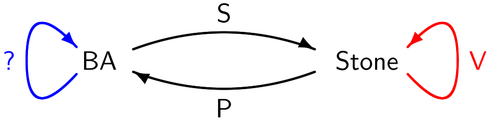

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Theorem
Topological Kripke frames are Vietoris coalgebras over Stone

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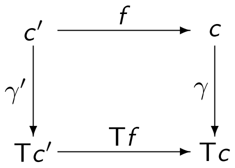

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{\scriptstyle \gamma'} \big\downarrow & & \big\downarrow {\scriptstyle \gamma} \\
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\end{array}
$$

Examples:

- Kripke structures are P-coalgebras over Set
- deterministics finite automata are coalgebras over Set

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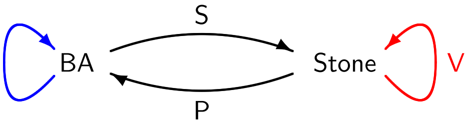

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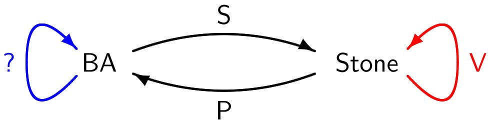
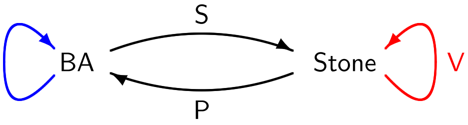


Modal Logic Dualizes the Vietoris Functor



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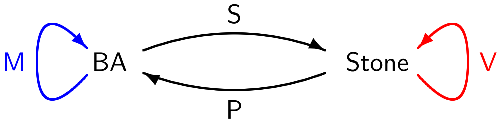

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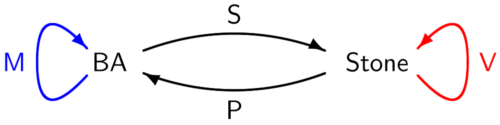

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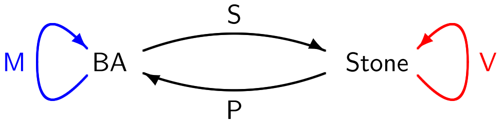

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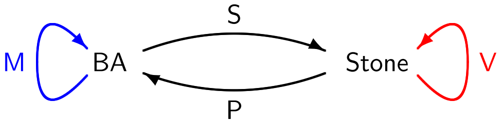

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The topological modal duality is an algebra|coalgebra duality

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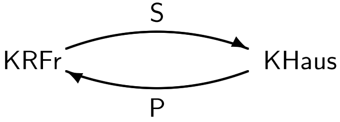

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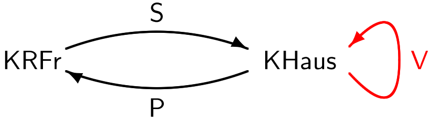

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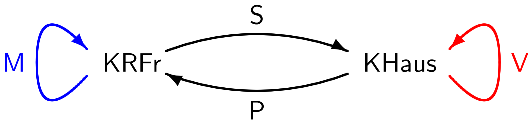
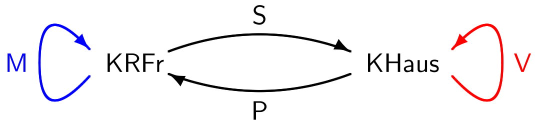


Variation: Pointfree Topology

Frames/Locales provide pointfree versions of topologies.



Geometric modal logic dualizes/axiomatizes the Vietoris functor
(Johnstone)

Vietoris pointfree (Johnstone Functor)

Given a frame $\mathbb{L}$, define $L_\Box := \{\Box a \mid a \in L\}$ and $L_\Diamond := \{\Diamond a \mid a \in L\}$.

$$
\mathsf{M}\mathbb{L} := \mathsf{Fr}\langle L_\Box \uplus L_\Diamond \mid \quad
\begin{array}{ll}
\Box(\bigwedge A) = \bigwedge_{a\in A} \Box a & (A \in \mathsf{P}_\omega L) \\
\Diamond(\bigvee A) = \bigvee_{a\in A} \Diamond a & (A \in \mathsf{P}_\omega L) \\
\\
\Box a \wedge \Diamond b \leq \Diamond(a \wedge b) & \\
\Box(a \vee b) \leq \Box a \vee \Diamond b & \\
\\
\Box(\bigsqcup A) = \bigsqcup_{a\in A} \Box a & (A \in \mathsf{P}L \text{ directed}) \\
\Diamond(\bigsqcup A) = \bigsqcup_{a\in A} \Diamond a & (A \in \mathsf{P}L \text{ directed}) \\
\\
\rangle
\end{array}
$$

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- ...and formulate the functor M accordingly, in terms of ∇

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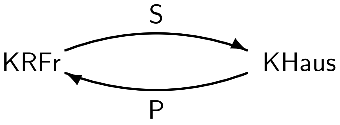

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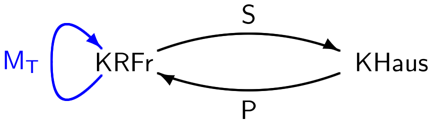

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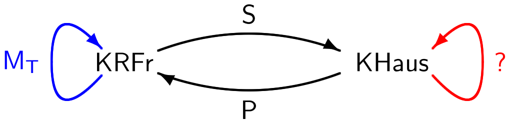

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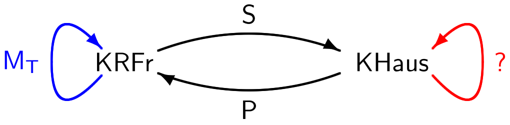

Describe the dual of M_T for an arbitrary set functor T!

Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
- Final remarks

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