Cut elimination and Semi-completeness

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HO, *A unified algebraic approach to cut elimination via semi-completeness*, in: Philosophical Logic: Current Trends in Asia, Springer, pp.19 - 43, 2017.

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Cut elimination

Cut elimination is one of the most important syntactic properties in sequent systems.

$$\frac{\Gamma \Rightarrow \Lambda, \alpha \quad \alpha, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Lambda, \Pi} \ (\mathrm{cut})$$

A standard way of showing cut elimination is proof-theoretic. It consists of combinatorial analysis of proof structures, with a constructive procedure for eliminating each application of cut rule, using double induction.

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We show that an idea introduced by S. Maehara in [Mae] will provide a uniform framework for understanding semantical proofs of both types.

- S. Maehara (1991): *Lattice-valued representation of the cut elimination theorem*, [Mae] Tsukuba J. of Math. 15.

We assume that each sequent is an expression of the form $\Gamma \Rightarrow \Delta$, where both Γ and Δ are *multisets* of formulas. As examples, we consider the following two sequent systems.

- **GS4** for modal logic **S4**, which is obtained from **LK** by adding the following two rules for $\Box$;

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow) \qquad \frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Rightarrow \Box 1)$$

  Here, $\Box\Gamma$ denotes the sequence of formulas $\Box\alpha_1, \ldots, \Box\alpha_m$ when Γ is $\alpha_1, \ldots, \alpha_m$.

- the multiple-succedent sequent system **LJ**$'$ (known also as **G3im**) for intuitionistic logic, obtained from **LK** by restricting rules $(\Rightarrow\to)$ and $(\Rightarrow \neg)$ of **LJ**$'$ to the following form;

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to) \qquad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}\ (\Rightarrow \neg)$$

◦ How semantical proofs go

Let $\mathbf{S}^-$ be the system obtained from a sequent system $\mathbf{S}$ by deleting cut rule. A standard semantical proof of cut elimination for $\mathbf{S}$ is to provide a proof of the *completeness* of the cut-free system $\mathbf{S}^-$ with respect to a class of *Kripke frames* or of *algebras* for $\mathbf{S}$, i.e. it is to show that (1) implies (3).

1. $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not provable in $\mathbf{S}^-$,

3. $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not valid in an $\mathbf{S}$ algebra (or an $\mathbf{S}$ frame).

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Maehara's idea is to introduce the following semantical condition (2) between (1) and (3).

1. $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not provable in $\mathbf{S}^-$,
2. $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not true under a quasi-valuation,
3. $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not valid in an $\mathbf{S}$ algebra (or an $\mathbf{S}$ frame).

Modal algebra

An algebra $\mathbf{A} = \langle A, \cap, \cup, ', 1, \Box \rangle$ is a modal algebra, if $\langle A, \cap, \cup, 1, ' \rangle$ is a Boolean algebra and $\Box$ is a unary operator on A satisfying $\Box 1 = 1$, and $\Box(a \cap b) = \Box a \cap \Box b$ for all $a, b \in A$.

◦ Quasi-valuations

A pair (k, K) of mappings k and K from the set Ω of all modal formulas to A is a quasi-valuation on $\mathbf{A}$, if it satisfies the following conditions;

- $k(\alpha) \leq K(\alpha)$,
- $k(\alpha \wedge \beta) \leq k(\alpha) \cap k(\beta)$ and $K(\alpha) \cap K(\beta) \leq K(\alpha \wedge \beta)$,
- $k(\alpha \vee \beta) \leq k(\alpha) \cup k(\beta)$ and $K(\alpha) \cup K(\beta) \leq K(\alpha \vee \beta)$,
- $k(\neg\alpha) \leq K(\alpha)'$ and $k(\alpha)' \leq K(\neg\alpha)$,
- $k(\Box\alpha) \leq \Box k(\alpha)$ and $\Box K(\alpha) \leq K(\Box\alpha)$.

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- $k(\neg\alpha) \leq K(\alpha)'$ and $k(\alpha)' \leq K(\neg\alpha)$,
- $k(\Box\alpha) \leq \Box k(\alpha)$ and $\Box K(\alpha) \leq K(\Box\alpha)$.

When $k(\alpha) = K(\alpha)$ for every α, the mapping K is no other than a usual valuation.

Quasi-valuations can be defined also on other algebras, e.g. Heyting algebras and residuated lattices in general.

For example, a pair of mappings k and K from Z to a Heyting algebra A is a quasi-valuation on a Heyting algebra $\mathbf{A}$ if it satisfies the following conditions.

- $k(\alpha) \leq K(\alpha)$ for $\alpha \in Z$,
- $k(\alpha \wedge \beta) \leq k(\alpha) \cap k(\beta)$ and $K(\alpha) \cap K(\beta) \leq K(\alpha \wedge \beta)$ for $\alpha \wedge \beta \in Z$,
- $k(\alpha \vee \beta) \leq k(\alpha) \cup k(\beta)$ and $K(\alpha) \cup K(\beta) \leq K(\alpha \vee \beta)$ for $\alpha \vee \beta \in Z$,
- $k(0) = 0_A$,
- $k(\alpha \to \beta) \leq K(\alpha) \to k(\beta)$ and $k(\alpha) \to K(\beta) \leq K(\alpha \to \beta)$ for $\alpha \to \beta \in Z$.

Lemma (quasi-valuation lemma)

Suppose that f is a valuation and (k, K) is a quasi-valuation on $\mathbf{A}$, respectively, such that $k(p) \leq f(p) \leq K(p)$ for every propositional variable p. Then, $k(\alpha) \leq f(\alpha) \leq K(\alpha)$ for every formula α.

Thus, k and K can be regarded as a *lower* and an *upper approximation* of a valuation f, respectively.

◦ Maehara's Lemma

Lemma (Maehara's Lemma)

For all formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, *if*

$$g(\alpha_1) \cap \ldots \cap g(\alpha_m) \leq g(\beta_1) \cup \ldots \cup g(\beta_n)$$

holds for every valuation g *on a modal algebra* $\mathbf{A}$, *then*

$$(*) \quad k(\alpha_1) \cap \ldots \cap k(\alpha_m) \leq K(\beta_1) \cup \ldots \cup K(\beta_n)$$

holds for every quasi-valuation (k, K) *on* $\mathbf{A}$.

Proof. For a given (k, K) on $\mathbf{A}$, take any valuation g on $\mathbf{A}$ satisfying $k(p) \leq g(p) \leq K(p)$ for any variable p. By quasi-valuation lemma, $k(\gamma) \leq g(\gamma) \leq K(\gamma)$ for every formula γ. From our assumption,

$$g(\alpha_1) \cap \ldots \cap g(\alpha_m) \leq g(\beta_1) \cup \ldots \cup g(\beta_n).$$

Therefore,

$$\begin{aligned} k(\alpha_1) \cap \ldots \cap k(\alpha_m) &\leq g(\alpha_1) \cap \ldots \cap g(\alpha_m) \\ &\leq g(\beta_1) \cup \ldots \cup g(\beta_n) \leq K(\beta_1) \cup \ldots \cup K(\beta_n). \end{aligned}$$

To put it another way,

Corollary

Let **S** *be a sequent system for a modal logic* **M**. *Then, (a) implies always (b).*

(a) $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ *is provable in* **S**,

(b) $k(\alpha_1) \cap \ldots \cap k(\alpha_m) \leq K(\beta_1) \cap \ldots \cap K(\beta_m)$
holds for every quasi-valuation (k, K) *on any* **M**-*algebra* **A**.

Thus, (2) always implies (3) in the following, if we assume the completeness of **S**.

1. $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not provable in $\mathbf{S}^-$,
2. $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not true under a quasi-valuation,
3. $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not valid in an **M**-algebra (or an **M**-frame).

Hence, cut elimination holds for **S** when (1) implies (2).

Definition (Semi-completeness)

A sequent system **T** is semi-complete w.r.t. a class $\mathcal{C}$ of **M**-algebras, when for all formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$,

- if the inequality

$$(*) \quad k(\alpha_1) \cap \ldots \cap k(\alpha_m) \leq K(\beta_1) \cup \ldots \cup K(\beta_n)$$

  holds for each **M**-algebra $\mathbf{A} \in \mathcal{C}$ and each quasi-valuation (k, K) on **A**,

- then the sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is provable in **T**.

Hence, if $\mathbf{S}^-$ is semi-complete then cut elimination holds for **S**. In fact, cut elimination for **S** implies semi-completeness of $\mathbf{S}^-$.

Existing algebraic proofs

Here are some references to papers on algebraic proofs of cut elimination except [Mae].

- M. Okada and K. Terui (1999) — for linear logic,
- F. Belardinelli, P. Jipsen and HO (2004) for substructural and modal logics: *Algebraic aspects of cut elimination* [BJO], Studia Logica 77,
- A. Ciabattoni, N. Galatos and K. Terui (2012) — algebraic proofs and MacNeille completions.

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1. It consists of *embedding* a given *Gentzen structure* (for $\mathbf{GS4}^-$) into an **S4** algebra (quasi-embedding). The process is regarded as a generalization of Dedekind-MacNeille completions.
2. In fact, this quasi-embedding lemma is a special case of our quasi-valuation lemma.

Semantical proofs using Kripke frames

Here are some references to semantical proofs of cut elimination.

- M. Fitting (1973) — for modal and intuitionistic logics,
- O. Lahav and A. Avron (2014) — introducing a "unified semantic framework"
- HO (2015) — an early attempt to the present topic: *Semantical approach to cut elimination and subformula property in modal logic*, in: Structural Analysis of Non-Classical Logics,

We will first explain an example of a semantical proof of cut elimination using Kripke frames for a sequent system **GS4** (due to M. Takano). Then, we show how the proof can be incorporated into semi-completeness arguments.

◦ Canonical models

The proof goes similarly to a standard proof of *Kripke completeness* of **S4** using canonical models. Recall that $\mathbf{GS4}^-$ denotes the system **GS4** without the cut rule.

- A pair (Σ, Θ) of subsets Σ and Θ of the set Ω of modal formulas is ($\mathbf{GS4}^-$) consistent (in Ω) if any sequent of the form $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not provable in $\mathbf{GS4}^-$ for $\alpha_1, \ldots, \alpha_m \in \Sigma$ and $\beta_1, \ldots, \beta_n \in \Theta$.
- A pair (Σ, Θ) of subsets Σ and Θ of Ω is ($\mathbf{GS4}^-$) saturated in Ω, if it is maximally consistent in Ω, i.e. it is consistent and moreover for any $\gamma \in \Omega \backslash (\Sigma \cup \Theta)$, neither $(\Sigma \cup \{\gamma\}, \Theta)$ nor $(\Sigma, \Theta \cup \{\gamma\})$ is consistent.

Due to lack of cut rule in $\mathbf{GS4}^-$, we cannot expect the following.

- If (Σ, Θ) is consistent, then either $(\Sigma \cup \{\gamma\}, \Theta)$ or $(\Sigma, \Theta \cup \{\gamma\})$ is consistent for any formula γ in Ω.

Thus, the union $\Sigma \cup \Theta$ is not always equal to Ω for a saturated pair (Σ, Θ).

But still we can show the following by using Zorn's lemma, as the set of all consistent pairs is inductive.

Lemma (saturation)

For every consistent pair (Σ, Θ) *there exists a saturated pair* (Σ^*, Θ^*) *such that* $\Sigma \subseteq \Sigma^*$ *and* $\Theta \subseteq \Theta^*$.

Define a Kripke model $\langle W, R, V\rangle$ as follows.

- W is the set of all saturated pairs (Σ, Θ) in Ω,
- For every $(\Sigma, \Theta), (\Lambda, \Pi) \in W$, the relation $(\Sigma, \Theta)R(\Lambda, \Pi)$ holds iff $\Sigma_{\Box} \subseteq \Lambda_{\Box}$, where $\Gamma_{\Box} = \{\beta; \Box\beta \in \Gamma\}$,
- The valuation V is defined by $V(p) = \{(\Sigma, \Theta) \in W; p \in \Sigma\}$, for every propositional variable p.

We have that

1. the structure $\langle W, R \rangle$ is a Kripke frame for **S4**,
2. for each formula $\alpha \in \Omega$ and each $(\Sigma, \Theta) \in W$,
   - if $\alpha \in \Sigma$ then $(\Sigma, \Theta) \models \alpha$,
   - if $\alpha \in \Theta$ then $(\Sigma, \Theta) \not\models \alpha$.

(cf. semi-valuations in Schütte (1960))

The above (2) can be shown inductively by the following downward saturation of each saturated pair (Σ, Θ).

◦ Downward saturation

I. The case where α is of the form $\beta \wedge \gamma$.

- if $\beta \wedge \gamma \in \Sigma$ then both β and γ are in Σ,
- if $\beta \wedge \gamma \in \Theta$ then either β or γ is in Θ.

II. The case where α is of the form $\beta \vee \gamma$.

- if $\beta \vee \gamma \in \Sigma$ then either β or γ are in Σ,
- if $\beta \vee \gamma \in \Theta$ then both β and γ are in Θ.

III. The case where α is of the form $\neg\beta$.

- if $\neg\beta \in \Sigma$ then β is in Θ,
- if $\neg\beta \in \Theta$ then β is in Σ.

IV. The case where α is of the form $\Box\beta$.

- if $\Box\beta \in \Sigma$ then $\beta \in \Lambda$ for each (Λ, Π) such that $(\Sigma, \Theta)R(\Lambda, \Pi)$,
- if $\Box\beta \in \Theta$ then $\beta \in \Pi$ for some (Λ, Π) such that $(\Sigma, \Theta)R(\Lambda, \Pi)$.

◦ Cut elimination in model theoretic way

Theorem (Cut elimination)

If a sequent $\Gamma \Rightarrow \Delta$ is not provable in $\mathbf{GS4}^-$*, then* $\Gamma \Rightarrow \Delta$ *is not valid in a Kripke frame for* **S4***.*

Proof. If $\Gamma \Rightarrow \Delta$ is not provable in $\mathbf{GS4}^-$ then there exists a saturated pair (Σ, Θ) such that $\Gamma \subseteq \Sigma$ and $\Delta \subseteq \Theta$. Then in our Kripke model $\langle W, R, V \rangle$ for **S4**, we have that $(\Sigma, \Theta) \models \alpha$ for all $\alpha \in \Gamma$ and $(\Sigma, \Theta) \not\models \beta$ for all $\beta \in \Delta$. Therefore, $\Gamma \Rightarrow \Delta$ is not valid in $\langle W, R \rangle$.

Theorem (Cut elimination)

If a sequent $\Gamma \Rightarrow \Delta$ is not provable in **GS4**$^{-}$, *then $\Gamma \Rightarrow \Delta$ is not valid in a Kripke frame for* **S4**.

Proof. If $\Gamma \Rightarrow \Delta$ is not provable in **GS4**$^{-}$ then there exists a saturated pair (Σ, Θ) such that $\Gamma \subseteq \Sigma$ and $\Delta \subseteq \Theta$. Then in our Kripke model $\langle W, R, V \rangle$ for **S4**, we have that $(\Sigma, \Theta) \models \alpha$ for all $\alpha \in \Gamma$ and $(\Sigma, \Theta) \not\models \beta$ for all $\beta \in \Delta$. Therefore, $\Gamma \Rightarrow \Delta$ is not valid in $\langle W, R \rangle$.

But, why doesn't this argument work for **GS5**$^{-}$?

◦ Semi-completeness of GS4^{-}

We will transform our semantical proof of cut elimination of **GS4** mentioned above into a proof of semi-completeness. Recall that

- W is the set of all saturated pairs,
- the relation $(\Sigma, \Theta)R(\Lambda, \Pi)$ holds iff $\Sigma_\Box \subseteq \Lambda_\Box$, where $\Gamma_\Box = \{\beta; \Box\beta \in \Gamma\}$,

Now, the power set $\wp(W)$ with $\Box_R$ forms a modal algebra $\mathbf{A}^*$, which is in fact an **S4**-algebra, where $\Box_R S$ for S $(\subseteq W)$ is defined by the set

$$\{(\Sigma, \Theta) : \textit{for each } (\Lambda, \Pi), \textit{ if } (\Sigma, \Theta)R(\Lambda, \Pi) \textit{ then } (\Lambda, \Pi) \in S\}.$$

Define (k, K) on $\mathbf{A}^*$ by

- $k(\alpha) = \{(\Sigma, \Theta) : \alpha \in \Sigma\}$,
- $K(\alpha) = \{(\Sigma, \Theta) : \alpha \notin \Theta\}$.

By downward saturation of each $(\Sigma, \Theta) \in W$, (k, K) is shown to be a quasi-valuation.

Lemma (Semi-completeness of **GS4**$^-$)

Assume that $k(\alpha_1) \cap \ldots \cap k(\alpha_m) \subseteq K(\beta_1) \cup \ldots \cup K(\beta_n)$ *holds in* $\mathbf{A}^*$. *Then the sequent* $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ *is provable in* **GS4**$^-$.

Proof. From our assumption, for any $(\Sigma, \Theta) \in W$, if $\alpha_i \in \Sigma$ for all i then (Σ, Θ) belongs to $k(\alpha_1) \cap \ldots \cap k(\alpha_m)$ and hence to $K(\beta_1) \cup \ldots \cup K(\beta_n)$. Thus, $\beta_j \notin \Theta$ for some j. Now suppose that the above sequent is not provable in **GS4**$^-$. Then, there exists $(\Sigma^*, \Theta^*) \in W$ such that $\alpha_i \in \Sigma^*$ for all i and also $\beta_j \in \Theta^*$ for all j. But this contradicts our assumption.

Similar arguments work for some other modal logics and also for a multiple-succedent system **LJ**$'$ for intuitionistic logic.

Semi-completeness in systems for predicate logics

Arguments about semi-completeness work well also for sequent systems for modal and substructural predicate logics. In these cases, algebraic structures of the form $\langle \mathbf{A}, D \rangle$ with a *complete* algebra $\mathbf{A}$ and a nonempty set D for individual domain are taken. Quasi-valuations on such an algebraic structure must satisfy the following;

- $k(\forall x\alpha) \subseteq \bigcap\{k(\alpha[d/x]) : d \in D\}$ and $\bigcap\{K(\alpha[d/x]) : d \in D\} \subseteq K(\forall x\alpha)$ for $\forall x\alpha \in Z$,
- $k(\exists x\alpha) \subseteq \bigcup\{k(\alpha[d/x]) : d \in D\}$ and $\bigcup\{K(\alpha[d/x]) : d \in D\} \subseteq K(\exists x\alpha)$ for $\exists x\alpha \in Z$.

We can extend "model-theoretic proofs" of cut elimination to proofs for sequent systems for modal predicate logics, including the predicate extension **GQS4** of **GS4**, and also for intuitionistic predicate logic **QLJ**′.

So far so good. But if we want to transform this proof into semi-completeness, we will face some technical difficulties since it is necessary to construct algebraic structures corresponding to Kripke frames with varying domains. Then, how?

◦ Expanded algebraic structures

To overcome this problem, we introduce expanded algebraic structures. The triple $\langle \mathbf{A}, D, \phi \rangle$ is an expanded algebraic structure for modal (intuitionistic) predicate logic if

- $\mathbf{A}$ is a complete modal (Heyting, resp.) algebra,
- D is a nonempty set,
- ϕ is a mapping from D to A satisfying that $\bigcup \{\phi(d) : d \in D\} = 1_{\mathbf{A}}$, (and moreover $\phi(d) \leq \Box\phi(d)$ for each $d \in D$ for a modal structure.)

Valuations over expanded algebraic structures are defined similarly to those over usual algebraic structures, except

- $f(\forall x \alpha) = \bigcap \{ \phi(d) \to f(\alpha[d/x]) : d \in D\}$,
- $f(\exists x \alpha) = \bigcup \{ \phi(d) \wedge f(\alpha[d/x]) : d \in D\}$.

◦ Semi-completeness w.r.t. expanded structures

Lemma (Completeness of **GQS4** w.r.t. expanded structures)

A sequent is provable in **GQS4** *iff it is valid in every expanded algebraic structure for modal predicate logic* **QS4***.*

Theorem (Semi-completeness w.r.t. expanded structures)

The sequent system $\mathbf{GQS4}^{-}$ *is semi-complete w.r.t. expanded algebraic structures for intuitionistic predicate logic. Similarly for* $\mathbf{QLJ}'^{-}$*.*

◦ Semi-completeness w.r.t. expanded structures

Lemma (Completeness of GQS4 w.r.t. expanded structures)

A sequent is provable in **GQS4** *iff it is valid in every expanded algebraic structure for modal predicate logic* **QS4**.

Theorem (Semi-completeness w.r.t. expanded structures)

The sequent system $\mathbf{GQS4}^{-}$ *is semi-complete w.r.t. expanded algebraic structures for intuitionistic predicate logic. Similarly for* $\mathbf{QLJ}'^{-}$.

Later I found that as for $\mathbf{QLJ}'^{-}$, our proof is a simplified version of the proof given in Part 3 "Algebraic Models" of

- A.G. Dragalin, *Mathematical Intuitionism: Introduction to Proof Theory*, AMS (1988).

★ The framework due to Maehara can cover many of existing standard semantical proofs of cut elimination, whether they are algebraic ones or model-theoretic ones.

★ Algebras constructed in their proofs can be regarded as a generalization of either MacNeille completions or complex algebras.