The algebraic method for Constraint Satisfaction Problems

LAC 2018

Institute of Mathematics for Industry, Kyushu University and La Trobe University

Marcel Jackson

Constraints and satisfaction

Constraint

A tuple of variables, and a target relation on some domain

Constraint satisfaction problem

Given some constraints, can they be satisfied?

3SAT

Conjunction of clauses:

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \wedge \neg x_2 \wedge \neg x_4) \wedge \ldots$$

Can the instance be satisfied?

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Can the instance be satisfied?

- ▷ The quintessential NP-complete classic

a classic catch by John Dyson 1981

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Can the instance be satisfied?

As a CSP

Each clause is a constraint:

- Clause $(\neg x_1 \vee x_2 \vee \neg x_3)$ means (x_1, x_2, x_3) must lie in

$$\{000, 001, 010, 011, 100, \sout{101}, 110, 111\}$$

Not-all-equal 3SAT

Conjunction of clauses:

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \wedge \neg x_2 \wedge \neg x_4) \wedge \ldots$$

Can the instance be satisfied with each clause containing a true literal and a false literal?

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Can the instance be satisfied with each clause containing a true literal and a false literal?

- ▷ Another well-known NP-complete classic.

A Warrick Capper classic

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Solvability of linear equations

A system of equations over $\mathbb{Z}_2$:

$$
\begin{array}{ccccccccc}
x_1 & + & x_2 & & & +x_4 & = & 1 \\
 & & x_2 & + & x_3 & +x_4 & = & 1 \\
x_1 & & & + & x_3 & +x_4 & = & 0 \\
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\end{array}
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- Easily solved in polynomial time using Gaussian elimination. It has its own complexity class $\oplus\mathtt{L}$ (“parity $\mathtt{L}$”)

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As a CSP

Each equation is a constraint:

- Equation $x_2 + x_3 + x_4 = 1$ means (x_2, x_3, x_4) is constrained to be in

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Directed graph unreachability

A directed graph, a "start" vertex s and a "finish" vertex t. Is there *no* directed path from s to t?

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- Easily solved in polynomial time (and nondeterministic logspace).
- A fundamental computational problem in computational complexity

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Each edge is a constraint:

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AND:

- s is constrained to be 1 while t is constrained to be 0

Schaefer's Theorem

So far, all these problems have domain $0, 1$.

Schaefer's Theorem (1979)

A Boolean satisfiability problem is either solvable in $\mathtt{P}$ or $\mathtt{NP}$-complete

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Given a graph $G = (V, E)$, can we colour the vertices V by $\{0, 1, 2\}$ so that adjacent vertices have different colours?

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- Yet another classic $\mathtt{NP}$-complete problem

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The target domain and relations are fixed: “template”

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Database example

Conjunctive database queries (the database is the template, the query the instance)

Ladner's Theorem versus CSPs

Ladner's Theorem

If $\mathtt{P} \neq \mathtt{NP}$ then there are problems in $\mathtt{NP} \setminus \mathtt{P}$ that are not $\mathtt{NP}$-complete.

. . . but in practice there seem to be few natural problems that appear to have this intermediate status

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(very roughly)

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A fixed template CSP is either solvable in $\mathtt{P}$ or is $\mathtt{NP}$-complete.

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▷ *they give a structural characterisation of hardness for an enormous class of natural problems of interest...*

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- as well as how this approach and result can be used to achieve other complexity-theoretic classifications

The CSP Dichotomy Theorem

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- to be explained in due course

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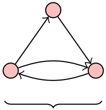
- the hard part

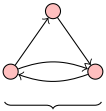
it's really hard

Polymorphism



template $\mathbb{D}$

Polymorphism



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Automorphism

Automorphism: $f : \mathbb{D} \to \mathbb{D}$

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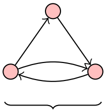
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- the set of automorphisms form a group action on D

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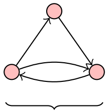

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Polymorphism

Polymorphism: $f : \mathbb{D}^n \to \mathbb{D}$

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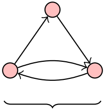

template $\mathbb{D}$

Polymorphism

Polymorphism: $f : \mathbb{D}^n \to \mathbb{D}$

- the set of all polymorphisms forms an exotic algebra on D

Cyclic

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n-ary cyclic polymorphism

$$\forall x_1 \ldots \forall x_n \qquad c(x_1, x_2, \ldots, x_n) = c(x_2, \ldots, x_n, x_1)$$

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A finite algebra has a cyclic term if and only if its variety contains no algebras with essentially trivial term operations (projections)

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- iff (roughly) the template relations can logically define the 3SAT ternary relations by way of primitive positive formulæ

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- iff it has cyclic terms of all prime arities greater than the size of the algebra

Example: Directed graph unreachability

$$\mathbb{R} = \langle \{0, 1\}; \leq, \textcolor{red}{\blacksquare}, \textcolor{cyan}{\blacksquare} \rangle$$

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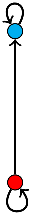

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Cyclic polymorphism

The operations of meet $\wedge$ and join $\vee$ on $0, 1$ are cyclic polymorphisms.

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if...

$$\begin{array}{ccc} a & & b \\ \rotatebox{90}{$\leq$} & \text{and} & \rotatebox{90}{$\leq$} \\ a' & & b' \end{array}$$

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Example: Directed graph unreachability

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The operations of meet $\wedge$ and join $\vee$ on $0, 1$ are cyclic polymorphisms.

and obviously $x_1 \wedge x_2 = x_2 \wedge x_1$ (for all $x_1, x_2 \in \{0, 1\}$)

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Hell and Nešetřil (1990)

A finite graph has tractable CSP if it has a loop or is bipartite and is NP-complete otherwise

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Modern proof by cyclic polymorphism

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$$c(\quad u_1, \quad u_2, \quad \ldots \quad u_{p-1}, \quad u_p)$$

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Many other results follow from polymorphism analysis

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- approximation of CSPs (complexity of solving asymptotically most constraints)
- **Universal Horn class membership...**

CSPs as homomorphism problems

Fixed template $\mathbb{A}$

CSP($\mathbb{A}$) is just the homomorphism problem for homomorphisms into $\mathbb{A}$

$$\overbrace{\mathbb{B}}^{\text{instance}} \xrightarrow{?} \overbrace{\mathbb{A}}^{\text{template}}$$

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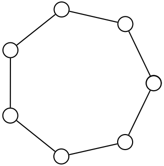
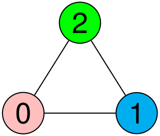


a graph



$\mathbb{K}_3$

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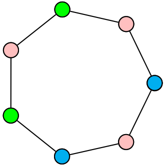

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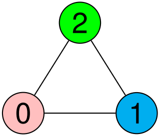

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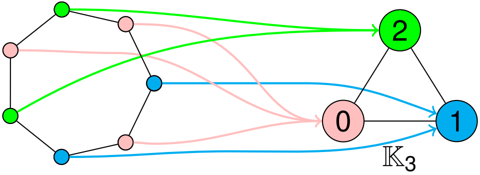

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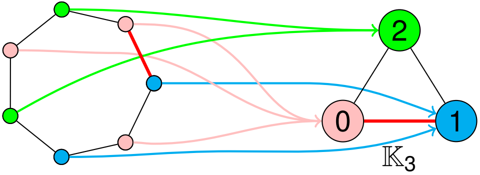

a graph

(3-colouring is an edge-preserving function into $\mathbb{K}_3$)

Universal Horn

A non-constraint is *implied* if every solution maps it inside the target relation.

CSP to universal Horn

- CSP($\mathbb{A}$): is there a homomorphism from $\mathbb{B}$ into $\mathbb{A}$?
- UHorn($\mathbb{A}$): are there no implied constraints?

Universal Horn

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$$\mathbb{R} = \langle \{0, 1\}; \leq, \blacksquare, \blacksquare \rangle$$

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CSP($\mathbb{R}$) is $\mathtt{NL}$-complete, but UHorn($\mathbb{R}$) is first order definable

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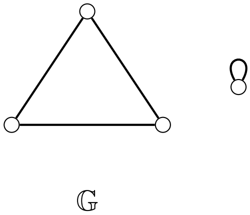

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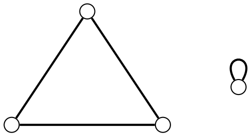
Example

CSP($\mathbb{G}$) is first order definable, but UHorn($\mathbb{G}$) is $\mathtt{NP}$-complete



$\mathbb{G}$

Main Result

Barto, Jackson, Ham (2017)

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(roughly)

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2. make multiple calls to this to solve membership in UHorn($\mathbb{A}$)

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- (ii) means No for UHorn($\mathbb{A}$). Argue (nontrivially) how (i) implies YES for UHorn($\mathbb{A}$).

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