Catalan families in categories

Richard Garner

Macquarie University

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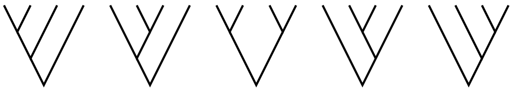
The *Catalan numbers* are given by $C_n = \binom{2n}{n}/(n+1)$:

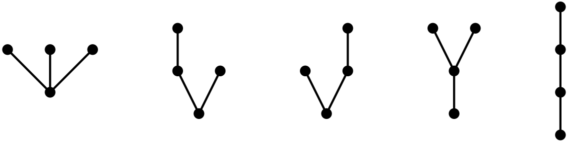
$$1, 1, 2, 5, 14, 42, \ldots$$

and are the solution to many counting problems. . .

There are C_n plane binary trees with $n+1$ leaves:



There are C_n plane trees with $n+1$ vertices:



There are C_n Dyck words of length $2n$:

UDUDUD UUDDUD UDUUDD UUDUDD UUUDDD

A *Catalan family* is a family of sets $(A_n)_{n\in\mathbb{N}}$ with $|A_n| = C_n$.

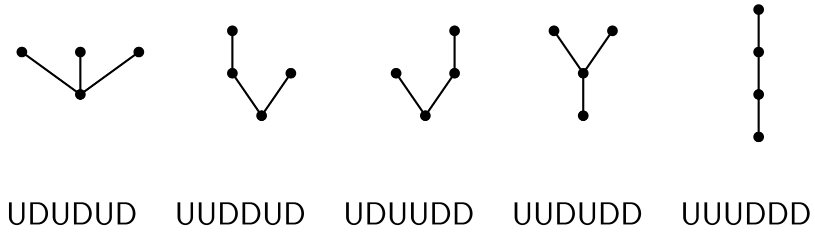
Stanley's "Enumerative combinatorics" and its addendum lists 207 such families, which he invites the reader to show are indeed Catalan families

> *"by exhibiting a simple, elegant bijection $\phi\colon A \to B$ for each pair of families"*

making a total of $21,321$ bijections in all.

The point, of course, is that a Catalan family bears more structure than its mere sequence of cardinalities.

Easy: plane trees versus Dyck words.



Harder: binary trees versus Dyck words.

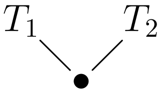
The aim of this talk is to explain this latter bijection in terms of logic, algebra and category theory.

(joint work with Geoff Edington-Cheater).

The Catalan family of binary trees $(B_n)_{n\in\mathbb{N}}$ is endowed with the following structure:

1. There's a constant $e \in B_0$ given by the trivial binary tree $\bullet$;
2. There's an operation $(-) \star (-) \colon B_n \times B_m \to B_{n+m+1}$ with

$$T_1, T_2 \quad \mapsto$$



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And $(B_n)_{n\in\mathbb{N}}$ is *initial* among families with such structure.

The Catalan family of Dyck words $(D_n)_{n\in\mathbb{N}}$ is endowed with the following structure:

1. There's a constant $e \in D_0$—the empty Dyck word;
2. There's an associative operation $(-)\cdot(-)\colon D_n \times D_m \to D_{n+m}$ with unit e, given by concatenation of Dyck words;
3. There's an operation $s\colon D_n \to D_{n+1}$ given by $W \mapsto \mathrm{U}W\mathrm{D}$.

And $(D_n)_{n\in\mathbb{N}}$ is *initial* among families with such structure.

In fact, a further simplification is possible. If we take the disjoint unions of these families, $B = \Sigma_n B_n$ and $D = \Sigma_n D_n$, then we still have universal characterisations:

- The set B of binary trees underlies the initial *pointed magma*. A pointed magma is a set equipped with a constant $e\colon 1 \to X$ and a binary operation $\star\colon X \times X \to X$.

- The set D of Dyck words underlies the initial *motor*. A motor is a set X equipped with unital semigroup structure $(e, \cdot)$ and a unary operation $s\colon X \to X$.

In both cases, the $\mathbb{N}$-grading can be recovered from the initiality.

- We make $\mathbb{N}$ into a pointed magma by taking $e = 0$ and $n \star m = n + m + 1$. Because B is an initial pointed magma, we induce a map $B \to \mathbb{N}$. Its fibres are the Catalan family of binary trees.
- We make $\mathbb{N}$ into a motor by taking $e = 0$, $n \cdot m = n + m$ and $s(n) = n + 1$. Because D is an initial motor, we induce a map $D \to \mathbb{N}$. Its fibres are the Catalan family of Dyck words.

To summarise:

- A *pointed magma* $(X, e, \star)$ is a set endowed with a constant and a binary operation;
- A *motor* $(X, e, \cdot, s)$ is a set endowed with a unital semigroup structure and a (non-homomorphic) unary operation.

The Catalan question to be answered: why do the initial pointed magma and the initial motor coincide?

Write **PtMag** and **Motor** for the categories of pointed magmas and of motors. There is a forgetful functor

$$U\colon \mathbf{Motor} \to \mathbf{PtMag}$$

sending $(X, e, \cdot, s)$ to $(X, e, \star)$, where

$$x \star y = x \cdot s(y)\ .$$

We can exploit this functor in various ways to show that the initial motor and initial pointed magma coincide.

One possibility is to show that the initial pointed magma lifts along U to an initial motor. This is interesting, but much more fun is the converse.

Proposition

$U\colon \mathbf{Motor} \to \mathbf{PtMag}$ *preserves the initial object.*

The initial motor is the motor $D = (D, e, \cdot, s)$ of Dyck words. To show that UD is an initial pointed magma, we must exhibit a unique homomorphism $UD \to M$ into any other pointed magma.

The only reasonable way to construct this homomorphism is by exploiting initiality of D in **Motor**. So we need to build a motor out of the pointed magma M.

Elements of the initial motor D are Dyck words:

$$W = \mathrm{UDUUDDUUDUDD}$$

which we view as *instructions* for a stack machine for a function of type $M \to M$, as follows:

(1) Push the input element $m \in M$ onto the stack.

(2) If W is exhausted, output the top of the stack and stop.

(3) Otherwise, read the next unconsumed element of W.
   - If it is U, push an e onto the stack;
   - If it is D, pop x_1, x_2 off the stack, and push on $x_1 \star x_2$.

(4) Go to (2).

UDUUDDUUDUDD

m

UDUUDDUUDUDD

m

e

UDUUDDUUDUDD

$m \star e$

UDUUDDUUDUDD

$m \star e$

e

UDUUDDUUDUDD

$m \star e$

e

e

UDUUDDUUDUDD

$m \star e$

$e \star e$

UDUUDDUUDUDD

$(m \star e) \star (e \star e)$

UDUUDDUUDUDD

$(m \star e) \star (e \star e)$

e

UDUUDDDUUDUDD

$(m \star e) \star (e \star e)$

e

e

UDUUDDUUDUDD

$(m \star e) \star (e \star e)$

$e \star e$

UDUUDDUUDUDD

$(m \star e) \star (e \star e)$

$e \star e$

e

UDUUDDUUDUDD

$(m \star e) \star (e \star e)$

$(e \star e) \star e$

UDUUDDUUDUDD

$$((m \star e) \star (e \star e)) \star ((e \star e) \star e)$$

So $W = \mathrm{UDUUDDUUDUDD}$ encodes the function

$$\varphi_W(m) = ((m \star e) \star (e \star e)) \star ((e \star e) \star e)\ .$$

Doing similarly for all $W \in D$, we get a function

$$\varphi_{(-)} \colon D \to M^M\ .$$

Having constructed $\varphi_{(-)}$ in a hands-on way, we can now reconstruct it much more efficiently . . .

Indeed, we have that:

- $\varphi_\epsilon(m) = m$;
- $\varphi_{W_1 W_2}(m) = \varphi_{W_2}(\varphi_{W_1}(m))$;
- $\varphi_{\mathrm{U}W\mathrm{D}}(m) = m \star \varphi_W(e)$;

so we can construct $\varphi_{(-)}$ by recursion over D.

Explicitly, to each pointed magma M, we associate the motor $\tilde{M} = (M^M, 1_M, \cdot, s)$ where $\cdot$ and s are defined by

$$(\varphi \cdot \psi)(m) = \psi(\varphi(m)) \qquad \text{and} \qquad s(\varphi)(m) = m \star \varphi(e) \ .$$

Because D is the initial motor, we induce a unique motor homomorphism $D \to \tilde{M}$, which must equal $\varphi_{(-)}$ since it satisfies the same defining clauses.

Proposition

$U \colon \mathbf{Motor} \to \mathbf{PtMag}$ *preserves the initial object.*

Proof.

Let D be the initial motor of Dyck words. We show that UD is an initial pointed magma.

For any $M \in \mathbf{PtMag}$, we construct the motor $\tilde{M}$ as above, and the unique homomorphism $\varphi_{(-)} \colon D \to \tilde{M}$. Applying U gives a map of pointed magmas

$$U\varphi_{(-)} \colon UD \to U\tilde{M} \ .$$

We now check that evaluation at e is a map of pointed magmas $\mathrm{ev}_e \colon U\tilde{M} \to M$, and so obtain a composite:

$$u_M \colon UD \xrightarrow{\varphi_{(-)}} U\tilde{M} \xrightarrow{\mathrm{ev}_e} M \qquad \text{in } \mathbf{PtMag}.$$

It remains to show unicity: thus, if $f\colon UD \to M$ is another map of pointed magmas, then $f = u_M$.

To do so, we prove by induction on W that the following commutes for all $W \in D$:

$$\begin{array}{ccc} D & \xrightarrow{(-)\cdot W} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\varphi_W} & M \end{array}$$

Evaluating at $e \in D$ gives $f(W) = \varphi_W(e) = u_M(W)$ for each W, whence $f = u_M$. $\square$

We can immediately generalise this proof!

Consider a stack machine for sets endowed with:

- A constant e;
- A family of operations $(f_i)_{i \in I}$ with $\operatorname{arity}(f_i) = a_i + 1$.

This receives instructions composed of commands:

- U: "push e"; and
- D_i: "pop $a_i + 1$ elements, apply f_i, and push".

(I.e., RPN for the constant e and operators $(f_i)_{i \in I}$.)

This language is the free unital semigroup M equipped with operations $(s_i)_{i \in I}$ of respective arities a_i. The semigroup operation is still concatenation, while

$$\begin{aligned} s_i \colon M^{a_i} &\to M \\ (W_1, \ldots, W_{a_i}) &\mapsto \mathrm{U}W_1\mathrm{U}W_2 \ldots \mathrm{U}W_{a_i}\mathrm{D}_m \ . \end{aligned}$$

The preceding proof now generalises as follows. We say:

- An $\vec{a}$-*motor* is a unital semigroup endowed with a family of functions $(s_i)_{i \in I}$ of respective arities a_i.
- A *pointed* $\vec{a}$-*magma* is a set endowed with a constant e and a family of functions $(f_i)_{i \in I}$ of respectively arities $a_i + 1$.

There is a forgetful functor $U \colon \vec{a}\text{-}\mathbf{Motor} \to \mathbf{Pt}\text{-}\vec{a}\text{-}\mathbf{Mag}$ sending $(X, e, \cdot, (s_i)_{i \in I})$ to $(X, e, (f_i)_{i \in I})$ where

$$f_i(x_0, \ldots, x_{a_i}) = x_0 \cdot s_i(x_1, \ldots, x_{a_i}) \ .$$

Proposition

$U \colon \vec{a}\text{-}\mathbf{Motor} \to \mathbf{Pt}\text{-}\vec{a}\text{-}\mathbf{Mag}$ *preserves the initial object.*

A special case of this: let each $a_i = 0$.

So each operation f_i have arity 1, and each corresponding s_i has arity 0. Then our result identifies:

- The initial unital semigroup endowed with I constants, i.e., the free unital semigroup I^* on I; and
- The initial set endowed with a constant and I unary operations: i.e., the initial set X with a constant nil and a function $\mathsf{cons}\colon I \times X \to X$.

Even more special case: take $I = 1$. Then we identify the free monoid on one generator with the free set with a constant 0 and a unary operation s.

Further generalisation: let $F\colon \mathbf{Set} \to \mathbf{Set}$ be an endofunctor.

- An *F-motor* is a unital semigroup X endowed with a function $s\colon FX \to X$.
- A *pointed F-magma* is a set X endowed with a constant e and a function $\star\colon X \times FX \to X$.

There is a forgetful functor $U\colon F\text{-}\mathbf{Motor} \to \mathbf{Pt}\text{-}F\text{-}\mathbf{Mag}$ sending $(X, e, \cdot, s)$ to $(X, e, \star)$ where $\star$ is the composite

$$X \times FX \xrightarrow{\mathrm{id} \times s} X \times X \xrightarrow{\cdot} X \ .$$

Proposition

$U\colon F\text{-}\mathbf{Motor} \to \mathbf{Pt}\text{-}F\text{-}\mathbf{Mag}$ preserves the initial object.

Further[2] generalisation: let $\mathcal{C}$ be a cartesian closed category and $F\colon \mathcal{C} \to \mathcal{C}$ an endofunctor.

- An *F-motor* is a unital semigroup X in $\mathcal{C}$ endowed with a morphism $s\colon FX \to X$.
- A *pointed F-magma* is an object $X \in \mathcal{C}$ endowed with a constant $e\colon 1 \to X$ and a morphism $\star\colon X \times FX \to X$.

There is still a forgetful functor $U\colon F\text{-}\mathbf{Motor} \to \mathbf{Pt}\text{-}F\text{-}\mathbf{Mag}$.

Proposition

$U\colon F\text{-}\mathbf{Motor} \to \mathbf{Pt}\text{-}F\text{-}\mathbf{Mag}$ *preserves the initial object.*

Further^3 generalisation: let $\mathcal{C}$ be a right-closed monoidal category and $F\colon \mathcal{C} \to \mathcal{C}$ an endofunctor.

- An F-*motor* is a unital semigroup X in $\mathcal{C}$ endowed with a morphism $s\colon FX \to X$.
- A *pointed* F-*magma* is an object $X \in \mathcal{C}$ endowed with a constant $e\colon I \to X$ and a morphism $\star\colon X \otimes FX \to X$.

There is still a forgetful functor $U\colon F\text{-}\mathbf{Motor} \to \mathbf{Pt}\text{-}F\text{-}\mathbf{Mag}$.

Proposition

$U\colon F\text{-}\mathbf{Motor} \to \mathbf{Pt}\text{-}F\text{-}\mathbf{Mag}$ *preserves the initial object.*

Questions:

- Are there interesting new instantiations of these more general results?
- What does all of this have to do with the Fiore–Plotkin–Turi theory of variable binding?