ON THE CELLULAR DECOMPOSITION AND THE LUSTERNIK-SCHNIRELMANN CATEGORY OF $Spin(7)$

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Abstract. We give a cellular decomposition of the compact connected Lie group $Spin(7)$. We also determine the L-S categories of $Spin(7)$ and $Spin(8)$.

1. Introduction

In this paper, we assume that a space has the homotopy type of a CW-complex. The Lusternik-Schnirelmann category $\operatorname{cat} X$ of a space X is the least integer n such that X is the union of $(n+1)$ open subsets, each of which is contractible in X. G. Whitehead [15] showed that $\operatorname{cat} X \leq n$ if and only if the diagonal map $\Delta_{n+1} : X \to \prod^{n+1} X$ is homotopic to some composition map

$$X \longrightarrow T^{n+1}(X) \longrightarrow \prod^{n+1} X,$$

where $T^{n+1}(X)$ is the fat wedge and $T^{n+1}(X) \to \prod^{n+1} X$ is the inclusion map.

The weak Lusternik-Schnirelmann category $w\mathrm{cat}\, X$ is the least integer n such that the reduced diagonal map $\bar{\Delta}_{n+1} : X \to \wedge^{n+1} X$ is trivial. Then it is easy to see that $w\mathrm{cat}\, X \leq \operatorname{cat} X$, since $\wedge^{n+1} X = \prod^{n+1} X / T^{n+1}(X)$.

The strong Lusternik-Schnirelmann category $\operatorname{Cat} X$ is the least integer n such that there exist a space X' which is homotopy equivalent to X and is covered by $(n+1)$ open subsets contractible in themselves. $\operatorname{Cat} X$ is closely related with $\operatorname{cat} X$, and Ganea and Takens [14] showed that

$$\operatorname{cat} X \leq \operatorname{Cat} X \leq \operatorname{cat} X + 1.$$

Ganea [3] showed that $\operatorname{Cat} X$ is equal to the invariant which is the least integer n such that there is a cofibre sequence

$$A_i \longrightarrow X_{i-1} \longrightarrow X_i$$

where X_0 is a point and X_n is homotopy equivalent to X.

The Lusternik-Schnirelmann category for some Lie groups are determined, such as $\operatorname{cat}(U(n)) = n$ and $\operatorname{cat}(SU(n)) = n-1$ by Singhof [11], $\operatorname{cat}(Sp(2)) = 3$ by Schweitzer [10], $\operatorname{cat}(Sp(3)) = 5$ by Fernández-Suárez, Gómez-Tato, Strom and Tanré [2], and Iwase and Mimura [6], $\operatorname{cat}(SO(2)) = 1$, $\operatorname{cat}(SO(3)) = 3$, $\operatorname{cat}(SO(4)) =$

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4, $\mathrm{cat}(SO(5)) = 8$ by James and Singhof [7]. Some general argument about the Lusternik-Schnirel-mann category implies that $\mathrm{cat}(G_2) = 4$ (see for example [6]).

As is well-known, we have the following isomorphisms:

$$Spin(3) \cong S^3, \quad Spin(4) \cong S^3 \times S^3, \quad Spin(5) \cong Sp(2), \quad Spin(6) \cong SU(4).$$

Thus $Spin(7)$ is the first non-trivial case in determining the cellular decomposition and the Lusternik-Schnirelmann category as well; it is our purpose in this paper.

Theorem 1.1. *We have* $w\mathrm{cat}(Spin(7)) = \mathrm{cat}(Spin(7)) = \mathrm{Cat}(Spin(7)) = 5$.

Since $Spin(8)$ is homeomorphic to $Spin(7) \times S^7$, we obtain the following corollary.

Corollary 1.2. *We have* $w\mathrm{cat}(Spin(8)) = \mathrm{cat}(Spin(8)) = \mathrm{Cat}(Spin(8)) = 6$.

The paper is organized as follows. In Section 2 we give a cellular decomposition of $Spin(7)$ such that $Spin(7)$ contains a subgroup $SU(4)$, which turns out to be useful for determining the Lusternik-Schnirelmann category of $Spin(7)$. In Section 3 we give a cone-decomposition of $SU(4)$, which gives rise to the Lusternik-Schnirelmann category of $Spin(7)$ in Section 4.

2. The cellular decomposition of $Spin(7)$

In this section, we use the notation in [9]. Let $\mathfrak{C}$ be the Cayley algebra. $SO(8)$ acts on $\mathfrak{C}$ naturally since $\mathfrak{C} \cong \mathbb{R}^8$ as $\mathbb{R}$-module. We regard $SO(7)$ as the subgroup of $SO(8)$ fixing e_0, the unit of $\mathfrak{C}$. As is well known, the exceptional Lie group G_2 is defined by

$$G_2 = \{g \in SO(7) \mid g(x)g(y) = g(xy), x, y \in \mathfrak{C}\} = \mathrm{Aut}(\mathfrak{C}).$$

According to [19], the group $Spin(7)$ is the set of the elements $\tilde{g} \in SO(8)$ such that $g(x)\tilde{g}(y) = \tilde{g}(xy)$ for any $x, y \in \mathfrak{C}$, where $g \in SO(7)$ is uniquely determined by $\tilde{g}$:

$$Spin(7) = \{\tilde{g} \in SO(8) \mid g(x)\tilde{g}(y) = \tilde{g}(xy), g \in SO(7), x, y \in \mathfrak{C}\}.$$

It is easy to see that G_2 is the subgroup of $Spin(7)$. Observe that the algebra generated by e_1 in $\mathfrak{C}$ is isomorphic to $\mathbb{C}$. $SU(4)$ acts on $\mathfrak{C}$ naturally, since as $\mathbb{C}$-module $\mathfrak{C} \cong \mathbb{C}^4$ whose basis is $\{e_0, e_2, e_4, e_6\}$. We regard $SU(3)$ as the subgroup of $SU(4)$ fixing e_0 and also as the subgroup of G_2 fixing e_1.

Let D^i be the i-dimensional disc. We define four maps:

$$\begin{aligned}
A &: D^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\} \longrightarrow SO(8),\\
B &: D^2 = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\} \longrightarrow SO(8),\\
C &: D^1 = \{z_1 \in \mathbb{R} \mid z_1^2 \leq 1\} \longrightarrow SO(8),\\
D &: D^2 = \{(w_1, w_2) \in \mathbb{R}^2 \mid w_1^2 + w_2^2 \leq 1\} \longrightarrow SO(8),
\end{aligned}$$

as follows:

$$A(x_1,x_2,x_3)=\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1 &&&& \\
&&&& 1-2X^2 & -2x_1X & -2x_2X & -2x_3X \\
&&&& 2x_1X & 1-2X^2 & 2x_3X & -2x_2X \\
&&&& 2x_2X & -2x_3X & 1-2X^2 & 2x_1X \\
&&&& 2x_3X & 2x_2X & -2x_1X & 1-2X^2
\end{pmatrix},$$

$$B(y_1,y_2)=\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& y_1 & -y_2 & -Y & 0 && \\
&& y_2 & y_1 & 0 & -Y && \\
&& Y & 0 & y_1 & y_2 && \\
&& 0 & Y & -y_2 & y_1 && \\
&&&&&& 1 & \\
&&&&&&& 1
\end{pmatrix},$$

$$C(z_1)=\begin{pmatrix}
1 &&&&&&& \\
& z_1 & 0 & -Z &&&& \\
& 0 & 1 & 0 &&&& \\
& Z & 0 & z_1 &&&& \\
&&&& 1 &&& \\
&&&&& z_1 & 0 & -Z \\
&&&&& 0 & 1 & 0 \\
&&&&& Z & 0 & z_1
\end{pmatrix},$$

$$D(w_1,w_2)=\begin{pmatrix}
w_1 & -w_2 & -W &&&&& \\
w_2 & w_1 && -W &&&& \\
W && w_1 & w_2 &&&& \\
& W & -w_2 & w_1 &&&& \\
&&&& 1 &&& \\
&&&&& 1 && \\
&&&&&& 1 & \\
&&&&&&& 1
\end{pmatrix},$$

where we put for simplicity

$$\begin{array}{ll} X=\sqrt{1-x_1^2-x_2^2-x_3^2}, & Y=\sqrt{1-y_1^2-y_2^2}, \\ Z=\sqrt{1-z_1^2}, & W=\sqrt{1-w_1^2-w_2^2}. \end{array}$$

Lemma 2.1. *The elements $A(x_1,x_2,x_3)$, $B(y_1,y_2)$, $C(z_1)$ and $D(w_1,w_2)$ belong to $Spin(7)$.*

Proof. Apparently the elements $A(x_1, x_2, x_3)$, $B(y_1, y_2)$ and $C(z_1)$ belong to G_2. In the proof, we denote $D(w_1, w_2)$ simply by D. Let D' be the matrix

$$\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& w_1 & -w_2 & W & \\ &&&& w_2 & w_1 && -W \\ &&&& -W && w_1 & -w_2 \\ &&&&& W & w_2 & w_1 \end{pmatrix}.$$

Then we can show by a tedious calculation that $D'xDy = D(xy)$ for any $x, y \in \mathfrak{C}$, which gives us the result. $\square$

Let φ_3, φ_5, φ_6 and φ_7 be maps

$$\begin{aligned} &\varphi_3 : D^3 \longrightarrow Spin(7), \\ &\varphi_5 : D^3 \times D^2 \longrightarrow Spin(7), \\ &\varphi_6 : D^3 \times D^2 \times D^1 \longrightarrow Spin(7), \\ &\varphi_7 : D^3 \times D^2 \times D^2 \longrightarrow Spin(7) \end{aligned}$$

respectively defined by the equalities

$$\begin{aligned} &\varphi_3(\mathbf{x}) = A(\mathbf{x}), \\ &\varphi_5(\mathbf{x}, \mathbf{y}) = B(\mathbf{y})A(\mathbf{x})B(\mathbf{y})^{-1}, \\ &\varphi_6(\mathbf{x}, \mathbf{y}, \mathbf{z}) = C(\mathbf{z})B(\mathbf{y})A(\mathbf{x})B(\mathbf{y})^{-1}C(\mathbf{z})^{-1}, \\ &\varphi_7(\mathbf{x}, \mathbf{y}, \mathbf{w}) = D(\mathbf{w})B(\mathbf{y})A(\mathbf{x})B(\mathbf{y})^{-1}D(\mathbf{w})^{-1}, \end{aligned}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2)$, $\mathbf{z} = (z_1)$ and $\mathbf{w} = (w_1, w_2)$. We define sixteen cells e^j for $j = 0, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 21$ respectively as follows:

$$\begin{array}{lllll} e^0 = \{1\}, & e^3 = \mathrm{Im}\ \varphi_3, & e^5 = \mathrm{Im}\ \varphi_5, & e^6 = \mathrm{Im}\ \varphi_6, & e^7 = \mathrm{Im}\ \varphi_7, \\ e^8 = e^5e^3, & e^9 = e^6e^3, & e^{10} = e^7e^3, & e^{11} = e^6e^5, & e^{12} = e^7e^5, \\ e^{13} = e^6e^7, & e^{14} = e^6e^5e^3, & e^{15} = e^7e^5e^3, & e^{16} = e^6e^7e^3, & e^{18} = e^6e^7e^5, \\ e^{21} = e^6e^7e^5e^3. &&&& \end{array}$$

Let S^7 be the unit sphere of $\mathfrak{C}$. Then we have a principal bundle over it:

$$SU(3) \longrightarrow SU(4) \xrightarrow{p_0} S^7,$$

where $p_0(g) = ge_0$.

Lemma 2.2. *Let $V^7 = D^3 \times D^2 \times D^2$. Then the composite map $p_0\varphi_7 : (V^7, \partial V^7) \to (S^7, e_0)$ is a relative homeomorphism.*

Proof. We express the map $(p_0\varphi_7)|_{V^7\setminus\partial V^7}$ as follows:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = D(\mathbf{w})B(\mathbf{y})A(\mathbf{x})B(\mathbf{y})^{-1}D(\mathbf{w})^{-1}e_0 = \begin{pmatrix} 1-2X^2Y^2W^2 \\ 2x_1XY^2W^2 \\ 2(w_1X - x_1w_2)XY^2W \\ -2(w_2X + x_1w_1)XY^2W \\ 2(-y_1X + x_1y_2)XYW \\ 2(y_2X + x_1y_1)XYW \\ 2x_2XYW \\ 2x_3XYW \end{pmatrix}$$

and hence we have

$$\begin{pmatrix} 1-a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = 2XYW \begin{pmatrix} XYW \\ x_1YW \\ (w_1X - x_1w_2)Y \\ -(w_2X + x_1w_1)Y \\ -y_1X + x_1y_2 \\ y_2X + x_1y_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Since $X > 0$, $Y > 0$, $W > 0$ and $1 - a_0 > 0$, an easy calculation as for the first component in the above equation gives the following equation:

$$XYW = \frac{\sqrt{1-a_2}}{\sqrt{2}}, \tag{2.1}$$

from which we easily obtain

$$x_2 = \frac{a_6}{\sqrt{2(1-a_2)}}, \quad x_3 = \frac{a_7}{\sqrt{2(1-a_2)}}. \tag{2.2}$$

Further we obtain three more equalities from the above equalities:

$$\begin{aligned} &(1-a_0)^2 + a_1^2 = 4X^2Y^4W^4(x_1^2 + X^2), \\ &a_2^2 + a_3^2 = 4X^2Y^4W^2(w_1^2 + w_2^2)(x_1^2 + X^2) = 4X^2Y^4W^2(1 - W^2)(x_1^2 + X^2), \\ &a_4^2 + a_5^2 = 4X^2Y^2W^2(y_1^2 + y_2^2)(x_1^2 + X^2) = 4X^2Y^2W^2(1 - Y^2)(x_1^2 + X^2). \end{aligned}$$

Using these three equalities, we obtain

$$Y^2 = \frac{(1-a_0)^2 + a_1^2 + a_2^2 + a_3^2}{(1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2}, \tag{2.3}$$

$$W^2 = \frac{(1-a_0)^2 + a_1^2}{(1-a_0)^2 + a_1^2 + a_2^2 + a_3^2}. \tag{2.4}$$

It follows from (2.1), (2.3) and (2.4) that

$$X^2 = \frac{(1-a_0)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}{2((1-a_0)^2 + a_1^2)}. \tag{2.5}$$

It follows also from (2.2) and (2.5) that

$$x_1^2 = \frac{a_1^2((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}{2(1-a_0)((1-a_0)^2 + a_1^2)}. \tag{2.6}$$

Since Y, W, X are positive, (2.3), (2.4), (2.5) imply respectively

$$Y = \frac{\sqrt{(1-a_0)^2 + a_1^2 + a_2^2 + a_3^2}}{\sqrt{(1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2}}, \tag{2.7}$$

$$W = \frac{\sqrt{(1-a_0)^2 + a_1^2}}{\sqrt{(1-a_0)^2 + a_1^2 + a_2^2 + a_3^2}}, \tag{2.8}$$

$$X = \frac{\sqrt{(1-a_0)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}}{\sqrt{2((1-a_0)^2 + a_1^2)}}. \tag{2.9}$$

Since the signs of x_1 and a_1 are the same, (2.6) implies that

$$x_1 = \frac{a_1\sqrt{(1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2}}{\sqrt{2(1-a_0)((1-a_0)^2 + a_1^2)}}. \tag{2.10}$$

Now we determine y_1; we have

$$-a_4X + a_5x_1 = 2XYW(x_1^2 + X^2)y_2.$$

Substituting the equations (2.1), (2.9) and (2.10) in the above equation, we obtain

$$y_1 = \frac{a_1a_5 - (1-a_0)a_4}{\sqrt{((1-a_0)^2 + a_1^2)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}}. \tag{2.11}$$

We determine y_2; we have

$$a_4x_1 + a_5X = 2XYW(x_1^2 + X^2)y_2.$$

Substituting the equations (2.1), (2.9) and (2.10) in the above equation, we obtain

$$y_2 = \frac{a_1a_4 + (1-a_0)a_5}{\sqrt{((1-a_0)^2 + a_1^2)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}}. \tag{2.12}$$

We determine w_1; we have

$$a_2X - a_3x_1 = 2XY^2W(x_1^2 + X^2)w_1.$$

Substituting the equations (2.1), (2.7), (2.9) and (2.10) in the above equation, we obtain

$$w_1 = \frac{(1-a_0)a_2 - a_1a_3}{\sqrt{((1-a_0)^2 + a_1^2)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2)}}. \tag{2.13}$$

Finally we determine w_2; we have

$$-a_2x_1 - a_3X = 2XY^2W(x_1^2 + X^2)w_2.$$

Substituting the equations (2.1), (2.7), (2.9) and (2.10) in the above equation, we obtain

$$w_2 = \frac{-a_1a_2 - (1-a_0)a_3}{\sqrt{((1-a_0)^2 + a_1^2)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2)}}. \tag{2.14}$$

Thus we have expressed $x_1, x_2, x_3, y_1, y_2, w_1, w_2$ in terms of $a_0, \cdots, a_7$, that is, the inverse map has been constructed, which completes the proof. □

In a similar way to that of Section 3 of [9], we can obtain the following theorem, which is essentially the same as Yokota's decomposition [17].

Proposition 2.3. *$e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$ thus obtained is a cellular decomposition of $SU(4)$.*

Proof. First we show that $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$. We consider the following three cases:

(1) For the case where $i, j \in \{0, 3, 5, 8\}$; both cells e^i and e^j are in $SU(3)$ and $e^0 \cup e^3 \cup e^5 \cup e^8$ is a cellular decomposition of $SU(3)$. Then we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.

(2) For the case where $i \in \{0, 3, 5, 8\}$ and $j \in \{7, 10, 12, 15\}$; we have $p_0(\mathring{e}^i) = \{e_0\}$ and $p_0(\mathring{e}^j) = S^7 \backslash \{e_0\}$. Then we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$.

(3) For the case where $i, j \in \{7, 10, 12, 15\}$; suppose that $A \in \mathring{e}^i \cap \mathring{e}^j$. Since $\mathring{e}^i = \mathring{e}^7 \mathring{e}^{i-7}$ and $\mathring{e}^j = \mathring{e}^7 \mathring{e}^{j-7}$, we can put $A = A_1 A_2 = A_1' A_2'$ where $A_1, A_1' \in \mathring{e}^7$, $A_2 \in \mathring{e}^{i-7}$ and $A_2' \in \mathring{e}^{j-7}$. We have $A_1 = A_1'$, since $p_0(A_1) = p_0(A_1 A_2) = p_0(A_1' A_2') = p_0(A_1')$ and $p_0|_{\mathring{e}^7}$ is monic. Then we have $A_2 = A_2'$ and the first case shows that $i - 7 = j - 7$, that is, $i = j$. Thus $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. In the proof of Proposition 3.2 [9], it is proved that the boundaries $\dot{e}^3$, $\dot{e}^5$ and $\dot{e}^8$ are included in the lower dimensional cells. Observe that the boundary $\dot{e}^7$ is the union of the following three sets:

$$\{DBAB^{-1}D^{-1} \mid A \in A(\dot{D}^3), B \in B(D^2), D \in D(D^2)\},$$
$$\{DBAB^{-1}D^{-1} \mid A \in A(D^3), B \in B(\dot{D}^2), D \in D(D^2)\},$$
$$\{DBAB^{-1}D^{-1} \mid A \in A(D^3), B \in B(D^2), D \in D(\dot{D}^2)\}.$$

The first set contains only the identity element, since A is the identity element. It is easy to see that the second set is contained in e^3 and that the third set is contained in e^5. We have $\dot{e}^{10} = e^7 \dot{e}^3 \cup \dot{e}^7 e^3 \subset e^7 e^0 \cup e^5 e^3 = e^7 \cup e^8$. We also have $\dot{e}^{12} = \dot{e}^7 e^5 \cup e^7 \dot{e}^5 \subset e^5 e^5 \cup e^7 e^3 = e^8 \cup e^{10}$, and $\dot{e}^{15} = \dot{e}^7 e^5 e^3 \cup e^7 \dot{e}^5 e^3 \cup e^7 e^5 \dot{e}^3 \subset e^5 e^5 e^3 \cup e^7 e^3 e^3 \cup e^7 e^5 = e^8 \cup e^{10} \cup e^{12}$.

Finally, we will show that the inclusion map $e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15} \to SU(4)$ is epic. Let $g \in SU(4)$. If $p_0(g) = e_0$, then g is contained in $SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$. Suppose that $p_0(g) \neq e_0$. There is an element $h \in e^7$ such that $p_0(h) = p_0(g)$. Thus we have $h^{-1} g \in SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$, since $p_0(h^{-1} g) = e_0$. Therefore we have $g \in h(e^0 \cup e^3 \cup e^5 \cup e^8) \subset e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$. □

Remark 2.4. (1) We regard $SO(6)$ as the subgroup of $SO(7)$ fixing e_1. Let $\pi : Spin(6) \to SO(6)$ be the double covering. Then, according to the Proof of Lemma 2.1, $\pi(SU(4)) \subset SO(6)$ so that $\pi|_{SU(4)} : SU(4) \to SO(6)$ is the double covering.

(2) For $1 \leq n \leq 3$, the subcomplex $e^0 \cup e^3 \cup \cdots \cup e^{2n+1}$ is homeomorphic to $\Sigma\mathbb{C}P^n$, which consists of the elements

$$A \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2i\theta} \end{pmatrix} A^{-1} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{-2i\theta} \end{pmatrix}$$

for any elements A in $SU(n+1)$. Moreover, according to Proposition 2.6 of Chapter IV of [13], we have $e^{2i+1}e^{2j+1} \subset e^{2j+1}e^{2i+1}$ for $i < j$; in fact we have $e^{2i+1}e^{2j+1} = e^{2j+1}e^{2i+1}$ (see [19]).

Let S^6 be the unit sphere of $\mathbb{R}^7$ whose basis $\{e_i \mid 1 \leq i \leq 7\}$. We consider the following diagram

$$\begin{tikzcd}
SU(3) \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & S^6 \arrow[d, equal] \\
SU(4) \arrow[r] \arrow[d] & Spin(7) \arrow[r, "p"] \arrow[d, "\pi"'] & S^6 \arrow[d, equal] \\
SO(6) \arrow[r] & SO(7) \arrow[r] & S^6
\end{tikzcd}$$

where the horizontal lines are principal fibre bundles and $p(g) = \pi(g)e_1$.

Lemma 4.1 of [9] implies the following lemma immediately.

Lemma 2.5. *Put $V^6 = D^3 \times D^2 \times D^1$. Then the composite map $p\varphi_6 : (V^6, \partial V^6) \to (S^6, \{e_1\})$ is a relative homeomorphism.*

Now we can state one of our main results.

Theorem 2.6. *The cell complex $e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^7 \cup e^8 \cup e^9 \cup e^{10} \cup e^{11} \cup e^{12} \cup e^{13} \cup e^{14} \cup e^{15} \cup e^{16} \cup e^{18} \cup e^{21}$ gives a cellular decomposition of $Spin(7)$.*

Proof. First we show that $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$. We consider the following three cases:

(1) For the case where $i, j \in \{0, 3, 5, 7, 8, 10, 12, 15\}$; both cells e^i and e^j are in $SU(4)$ and $e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$ is a cellular decomposition of $SU(4)$, whence we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.

(2) For the case where $i \in \{0, 3, 5, 7, 8, 10, 12, 15\}$ and $j \in \{6, 9, 11, 13, 14, 16, 18, 21\}$; we have $p(\mathring{e}^i) = \{e_1\}$ and $p(\mathring{e}^j) = S^6 \backslash \{e_1\}$, whence we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$.

(3) For the case where $i, j \in \{6, 9, 11, 13, 14, 16, 18, 21\}$, suppose that $A \in \mathring{e}^i \cap \mathring{e}^j$. Since $\mathring{e}^i = \mathring{e}^6 \mathring{e}^{i-6}$ and $\mathring{e}^j = \mathring{e}^6 \mathring{e}^{j-6}$, we can put $A = A_1A_2 = A_1'A_2'$, where $A_1, A_1' \in \mathring{e}^6$, $A_2 \in \mathring{e}^{i-6}$ and $A_2' \in \mathring{e}^{j-6}$. We have $A_1 = A_1'$, since $p(A_1) = p(A_1A_2) = p(A_1'A_2') = p(A_1')$ and $p|_{\mathring{e}^6}$ is monic. Then we have $A_2 = A_2'$ and the first case shows that $i - 6 = j - 6$, that is, $i = j$. Thus $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. In Proposition 2.3, it is proved that the boundaries of the cells of $SU(4)$ are included in the lower dimensional cells. In Proof of Theorem 4.2 in [9], we showed that $\dot{e}^6 \subset e^3 \cup e^5$, $\dot{e}^9 \subset e^6 \cup e^8$, $\dot{e}^{11} \subset e^5 \cup e^9$ and $\dot{e}^{14} \subset e^8 \cup e^9 \cup e^{11}$. By using (2) of Remark 2.4, we also obtain

$$\begin{aligned}
\dot{e}^{13} &= e^6\dot{e}^7 \cup \dot{e}^6 e^7 \subset e^{11} \cup e^{12},\\
\dot{e}^{16} &= e^6 e^7 \dot{e}^3 \cup e^6 \dot{e}^7 e^3 \cup \dot{e}^6 e^7 e^3 \subset e^{13} \cup e^{14} \cup e^{15},\\
\dot{e}^{18} &= e^6 e^7 \dot{e}^5 \cup e^6 \dot{e}^7 e^5 \cup \dot{e}^6 e^7 e^5 \subset e^{16} \cup e^{14} \cup e^{15},\\
\dot{e}^{21} &= e^6 e^7 e^5 \dot{e}^3 \cup e^6 e^7 \dot{e}^5 e^3 \cup e^6 \dot{e}^7 e^5 e^3 \cup \dot{e}^6 e^7 e^5 e^3 \subset e^{18} \cup e^{16} \cup e^{14} \cup e^{15}.
\end{aligned}$$

Finally, we will show that the inclusion map $e^0\cup e^3\cup e^5\cup e^6\cup e^7\cup e^8\cup e^9\cup e^{10}\cup e^{11}\cup e^{12}\cup e^{13}\cup e^{14}\cup e^{15}\cup e^{16}\cup e^{18}\cup e^{21} \to Spin(7)$ is epic. Let $g \in Spin(7)$. If $p(g) = e_1$, then g is contained in $SU(4) = e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$. Suppose that $p(g) \neq e_1$. There is an element $h \in e^6$ such that $p(h) = p(g)$. Thus we have $h^{-1}g \in SU(4)$ since $p(h^{-1}g) = e_1$. Therefore we have $g \in h(e^0\cup e^3\cup e^5\cup e^7\cup e^8\cup e^{10}\cup e^{12}\cup e^{15}) \subset e^0\cup e^3\cup e^5\cup e^6\cup e^7\cup e^8\cup e^9\cup e^{10}\cup e^{11}\cup e^{12}\cup e^{13}\cup e^{14}\cup e^{15}\cup e^{16}\cup e^{18}\cup e^{21}$. □

Remark 2.7. Araki [1] also gave a cellular decomposition of $Spin(n)$, but the one we have given here is a cellular decomposition with the minimum number of cells, satisfying the Yokota principle ([17], [18], [19]). As will be seen later, it is effectively used to determine the Lusternik-Schnirelmann category.

It is easy to give a cellular decomposition of $Spin(8)$ using a homeomorphism $Spin(8) \to Spin(7) \times S^7$.

3. The cone-decomposition of $SU(4)$

Obviously there is a filtration $F'_0 = * \subset F'_1 = SU(4)^{(7)} \subset F'_2 = SU(4)^{(12)} \subset F'_3 = SU(4)$. It is well-known that $F'_1 = \Sigma\mathbb{C}P^3 = S^3 \cup e^5 \cup e^7$ and $F'_2 = F'_1 \cup e^8 \cup e^{10} \cup e^{12}$. Thus the integral cohomology $H^n(F'_2;\mathbb{Z})$ is given by

$$H^n(F'_2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\langle 1\rangle & (n=0) \\ \mathbb{Z}\langle y_n\rangle & (n = 3,5,7,8,10,12) \\ 0 & (\text{otherwise}). \end{cases}$$

The action of the squaring operation Sq^2 is given as follows:

$$Sq^2 y_n = \begin{cases} y_{n+2} & \text{for } n = 3, 10, \\ 0 & \text{for } n = 5,7,8,12 \end{cases}$$

where y_n is regarded as an element of the mod 2 cohomology. To give the cone decomposition of $SU(4)$, we use the following homotopy fibration:

$$F \xrightarrow{\Psi} F'_1 \xrightarrow{\iota} F'_2. \tag{3.1}$$

Without loss of generality, we may regard this as a Hurewicz fibration over F'_2.

Firstly we consider the Serre spectral sequence $(E_r^{*,*}, d_r)$ associated with the above fibration, where the generators of $E_2^{*,0}$ for $* \le 7$ are permanent cycles and survive to E_∞-terms. Hence F is 6-connected and the transgression $\tau : H^7(F;\mathbb{Z}) \to H^8(F'_2;\mathbb{Z})$ is an isomorphism to $H^8(F'_2;\mathbb{Z}) \cong \mathbb{Z}\langle y_8\rangle$. Thus $H^7(F;\mathbb{Z}) \cong \mathbb{Z}\langle x_7\rangle$ for some $x_7 \in H^7(F;\mathbb{Z})$. Similarly, the generators in $E_2^{3,7} \cong \mathbb{Z}\langle y_3 \otimes x_7\rangle$ and $E_2^{10,0} \cong H^{10}(F'_2;\mathbb{Z}) \cong \mathbb{Z}\langle y_{10}\rangle$ must lie in the image of differentials d_3 and $d_{10} = \tau : H^9(F;\mathbb{Z}) \to H^{10}(F'_2;\mathbb{Z})$ respectively, and we have that $H^8(F;\mathbb{Z}) = 0$ and $H^9(F;\mathbb{Z}) \cong \mathbb{Z}\langle x_9\rangle \oplus \mathbb{Z}\langle x'_9\rangle$, where the elements x_9 and x'_9 in $H^9(F;\mathbb{Z})$ are corresponding to x_{10} and $y_3 \otimes x_7$ by the transgression τ and d_3 respectively. We remark that the choice of the generator x'_9 is not unique. Continuing this process, we have that $H^{10}(F;\mathbb{Z}) = 0$ and $H^{11}(F;\mathbb{Z}) \cong \mathbb{Z}\langle x_{11}\rangle \oplus \mathbb{Z}\langle x'_{11}\rangle \oplus \mathbb{Z}\langle x''_{11}\rangle \oplus \mathbb{Z}\langle x'''_{11}\rangle$ whose generators are corresponding to x_{12}, $y_3 \otimes x_9$, $y_3 \otimes x'_9$ and $y_5 \otimes x_7$ respectively by the transgression τ and differentials d_3, d_3 and d_5.

Thus the integral cohomology $H^n(F;\mathbb{Z})$ for $0 \le n \le 11$ is given by

$$H^n(F;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\langle 1\rangle & (n=0) \\ \mathbb{Z}\langle x_7\rangle & (n=7) \\ \mathbb{Z}\langle x_9\rangle \oplus \mathbb{Z}\langle x'_9\rangle & (n=9) \\ \mathbb{Z}\langle x_{11}\rangle \oplus \mathbb{Z}\langle x'_{11}\rangle \oplus \mathbb{Z}\langle x''_{11}\rangle \oplus \mathbb{Z}\langle x'''_{11}\rangle & (n=11) \\ 0 & (\text{otherwise}) \end{cases}$$

where x_7, x_9 and x_{11} are transgressive generators in $H^*(F;\mathbb{Z})$. Hence F has, up to homotopy, a cellular decomposition $e^0 \cup e^7 \cup_{\varphi_1} e^9 \cup_{\varphi'_1} e^9_1 \cup_{\varphi_2} e^{11} \cup$ (cells in dimensions ≥ 11), where the cells e^7, e^9 and e^{11} correspond to x_7, x_9 and x_{11} respectively. Then we obtain a subcomplex $A' = e^0 \cup e^7 \cup_{\varphi_1} e^9 \cup_{\varphi'_1} e^9_1 \cup_{\varphi_2} e^{11}$ of F.

Secondly, we determine the attaching maps φ_1 and φ'_1: Let us recall that $\pi_8(S^7) \cong \mathbb{Z}/2\langle \eta_7\rangle$ whose generator η_7 can be detected by Sq^2, the mod 2 Steenrod operation. Since the action of mod 2 Steenrod operation commutes with the cohomology transgression (see [8, Proposition 6.5]), we see that Sq^2x_7 is transgressive, and hence is cx_9 for some $c \in \mathbb{Z}/2$. We know that $\tau x_9 = y_{10} \ne 0$ and $\tau Sq^2 x_7 = Sq^2 \tau x_7 = Sq^2 y_8 = 0$, and hence Sq^2x_7 must be trivial. Thus the attaching maps φ_1 and φ'_1 are both null homotopic and A' is homotopy equivalent to $(S^7 \vee S^9 \vee S^9_1) \cup_{\varphi_2} e^{11}$.

Thirdly we check the composition of projections with the attaching map $\varphi_2 : S^{10} \to S^7 \vee S^9 \vee S^9_1$ to S^9 and S^9_1, which can also be detected by Sq^2. Again by the commutativity of the action of mod 2 Steenrod operation with the transgression, we see that the composition map $\mathrm{pr}_{S^9} \circ \varphi_2 : S^{10} \xrightarrow{\varphi_2} S^7 \vee S^9 \vee S^9_1 \longrightarrow S^9$ represents a generator of $\pi_{10}(S^9) \cong \mathbb{Z}/2\langle \eta_9\rangle$, since $Sq^2 : H^8(F'_2;\mathbb{Z}/2) \to H^{10}(F'_2;\mathbb{Z}/2)$ is non-trivial. If the composition map $\phi_1 = \mathrm{pr}_{S^9_1} \circ \varphi_2 : S^{10} \xrightarrow{\varphi_2} S^7 \vee S^9 \vee S^9_1 \longrightarrow S^9_1$ is non-trivial, we replace φ_2 by the composition of φ_2 and the homotopy equivalence

$\xi : S^7 \vee S^9 \vee S^9_1 \to S^7 \vee S^9 \vee S^9_1$ where $\xi|_{S^7}$ and $\xi|_{S^9_1}$ are the identity maps and $\xi|_{S^9}$ is the unique co-H-structure map $\phi : S^9 \to S^9 \vee S^9_1$; then we obtain that ϕ_1 is trivial, since $2\eta_9 = 0$. Then A' is homotopy equivalent to $((S^7 \vee S^9) \cup_{\varphi_2} e^{11}) \vee S^9_1$. Let A denote the subcomplex $(S^7 \vee S^9) \cup_{\varphi_2} e^{11}$ of A' and $\psi = \Psi|_A : A \to F'_1$.

Lemma 3.1. *F'_2 is homotopy equivalent to $F'_1 \cup_\psi CA$.*

Proof. The image of $H^*(A;\mathbb{Z})$ in $H^*(F;\mathbb{Z})$ under the induced map of the inclusion coincides with the module of transgressive elements with respect to the fibration (3.1) (see [8, Chapter 6]). Thus we may regard that $H^{n-1}(A;\mathbb{Z}) = \delta^{-1}(\iota^*(H^n(F'_2,*;)))\subset H^{n-1}(F;\mathbb{Z})$:

$$\begin{array}{ccccc} H^{n-1}(F;\mathbb{Z}) & \xrightarrow{\delta_F} & H^n(F'_1,F;\mathbb{Z}) & \xleftarrow{\iota_F^*} & H^n(F'_2,*;\mathbb{Z}) \\ \downarrow & & \downarrow & & \| \downarrow \\ H^{n-1}(A;\mathbb{Z}) & \xrightarrow{\delta_A} & H^n(F'_1,A;\mathbb{Z}) & \xleftarrow{\iota_A^*} & H^n(F'_2,*;\mathbb{Z}), \end{array}$$

where ι_F and ι_A are given by ι, and δ_F and δ_A denote the connecting homomorphisms of the long exact sequences for the pairs (F'_1,F) and (F'_1,A), respectively. Thus the image of δ_A is contained in the image of ι_A^* and we also have

$$H^n(F'_1,A;\mathbb{Z}) \cong H^n(F'_1 \cup_\psi CA, CA;\mathbb{Z}) \cong H^n(F'_1 \cup_\psi CA,*;\mathbb{Z}).$$

Since the composition map $A \xrightarrow{\psi} F'_1 \xrightarrow{\iota} F'_2$ is trivial, we can define a map

$$f : F'_1 \cup_\psi CA \longrightarrow F'_2,$$

by $f|_{F'_1} = \iota : F'_1 \to F'_2$ and $f|_{CA} = *$.

In order to prove the lemma, we show that $f^* : H^n(F'_2;\mathbb{Z}) \cong \mathbb{Z} \to H^n(F'_1 \cup_\psi CA;\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism for $n = 3,5,7,8,10,12$. We have a commutative diagram

$$\begin{array}{ccccc} & & H^n(F'_2;\mathbb{Z}) & \xrightarrow{\iota^*} & H^n(F'_1;\mathbb{Z}) \\ & & f^* \downarrow & & \| \downarrow \\ H^n(F'_1 \cup CA, F'_1;\mathbb{Z}) & \xrightarrow{j^*} & H^n(F'_1 \cup CA;\mathbb{Z}) & \xrightarrow{i^*} & H^n(F'_1;\mathbb{Z}), \end{array}$$

where the bottom row is a part of the exact sequence for the pair $(F'_1 \cup CA, F'_1)$. The induced map i^* is an isomorphism for $n \le 7$, since $H^n(F'_1 \cup CA, F'_1;\mathbb{Z}) = 0$ for $n \le 7$ and since ι^* is an isomorphism for $n \le 7$. Then we obtain that f^* is an isomorphism for $n \le 7$. Moreover we can show that $j^* : H^n(F'_1 \cup CA, F'_1;\mathbb{Z}) \to H^n(F'_1 \cup CA;\mathbb{Z})$ is an isomorphism for $n \ge 8$, by considering the exact sequence for the pair $(F'_1 \cup CA, F'_1)$, since we have $H^n(F'_1) = 0$ for $n \ge 8$. To perform the other cases for $n = 8,10,12$, it is sufficient to show that f^* is surjective. In fact, we have a commutative diagram

$$\begin{tikzcd}
H^{n-1}(A;\mathbb{Z}) \arrow[r, "\delta_A"] \arrow[d, "\Sigma"', "\cong"] & H^n(F_1', A;\mathbb{Z}) \arrow[dr, "\cong"] & H^n(F_2', *;\mathbb{Z}) \arrow[l, "\iota_A^*"'] \arrow[d, "f^*"] \\
H^n(\Sigma A, *;\mathbb{Z}) \arrow[r, "\cong"] & H^n(F_1' \cup CA, F_1';\mathbb{Z}) \arrow[u] \arrow[r, "j^*"] & H^n(F_1' \cup CA, *;\mathbb{Z}),
\end{tikzcd}$$

where Σ is the suspension isomorphism. Since j^* is an isomorphism for $n \geq 8$, we obtain that δ_A is an isomorphism for $n \geq 8$. Since the image of δ_A is contained in the image ι_A^*, we see that f^* is surjective for $n \geq 8$, and hence f is a homotopy equivalence. □

Proposition 3.2. *We have* $w\mathrm{cat}(F_i') = \mathrm{cat}(F_i') = \mathrm{Cat}(F_i') = i$.

Proof. The cohomology of F_i' implies that $w\mathrm{cat}(F_i') \geq i$. The cone-decomposition

$$F_1' = \Sigma\mathbb{C}P^3, \quad F_2' \simeq F_1' \cup CA, \quad F_3' = F_2' \cup CS^{14}$$

implies that $\mathrm{Cat}(F_i') \leq i$, which completes the proof. □

4. Proof of Theorem 1.1

We define a filtration $F_0 = * \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset F_5 = Spin(7)$ by

$$\begin{aligned} F_1 &= SU(4)^{(7)}, & F_2 &= SU(4)^{(12)} \cup e^6, \\ F_3 &= SU(4) \cup e^6 \cup e^9 \cup e^{11} \cup e^{13}, & F_4 &= Spin(7)^{(18)}. \end{aligned}$$

We need the following lemma to prove Theorem 4.2.

Lemma 4.1. *We have a homeomorphism of pairs*

$$(CA_1, A_1) \times (CA_2, A_2) = (C(A_1 * A_2), A_1 * A_2).$$

(The proof can be found in p.482-483 of [16].)

Now Theorem 1.1 follows from the following theorem.

Theorem 4.2. *We have* $w\mathrm{cat}(F_i) = \mathrm{cat}(F_i) = \mathrm{Cat}(F_i) = i$.

Proof. The mod 2 cohomology of F_i implies that $w\mathrm{cat}(F_i) \geq i$. Then it is sufficient to show that $\mathrm{Cat}(F_i) \leq i$. Obviously we have a homeomorphism $F_1 = \Sigma\mathbb{C}P^3$. Since the cell e^6 is attached to F_1, we obtain that $F_2 \simeq F_1 \cup C(S^5 \vee A)$ using Lemma 3.1. Since we have $e^9 \cup e^{11} \cup e^{13} = e^6(e^3 \cup e^5 \cup e^7)$, the composition map

$$(CS^5, S^5) \times (C\mathbb{C}P^3, \mathbb{C}P^3) \longrightarrow (CS^5, S^5) \times (\Sigma\mathbb{C}P^3, *) \longrightarrow (F_2 \cup e^9 \cup e^{11} \cup e^{13}, F_2)$$

is a relative homeomorphism. Then we obtain $F_2 \cup e^9 \cup e^{11} \cup e^{13} = F_2 \cup C(S^5 * \mathbb{C}P^3)$ using Lemma 4.1. The cell e^{15} is the highest dimensional cell of $SU(4)$ and is attached to F_2. Then we obtain $F_3 \simeq F_2 \cup C(S^{14} \vee (S^5 * \mathbb{C}P^3))$. Now we consider the following composition map:

$$(C(S^5 * A), S^5 * A) = (CS^5, S^5) \times (CA, A) \longrightarrow (CS^5, S^5) \times (F_2', F_1') \longrightarrow (F_4, F_3).$$

Since we have $e^{14} \cup e^{16} \cup e^{18} = e^6(e^8 \cup e^{10} \cup e^{12})$, the right map is a relative homeomorphism. The left map induces an isomorphism of homologies of pairs so that the map $H_*(F_3 \cup C(S^5 * A), F_3; \mathbb{Z}) \to H_*(F_4, F_3; \mathbb{Z})$ is an isomorphism. Thus we obtain $F_4 \simeq F_3 \cup C(S^5 * A)$. Obviously we have a homeomorphism $F_5 = F_4 \cup CS^{20}$. $\square$

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