A p-complete version of the Ganea Conjecture for co-H-spaces

J. R. Hubbuck and Norio Iwase

Abstract. A finite connected CW complex which is a co-H-space is shown to have the homotopy type of a wedge of a bunch of circles and a simply-connected finite complex after almost p-completion at a prime p.

1. Fundamentals and Results

When Y is a homotopy associative H-space or when X is a (homotopy) associative co-H-space, the set of based homotopy classes $[Z, Y]$ or $[X, Z]$ respectively, is a group natural in the Z argument. If the H-multiplication on Y is not known to be homotopy associative, the induced structure on $[Z, Y]$ is that of an algebraic loop; in particular, left and right inverses exist but they may be distinct. One cannot make as general a statement for $[X, Z]$ when X is a co-H-space. The immediate problem is that whereas the shearing map for an H-space induces isomorphisms of *homotopy* groups, the co-shearing map for a co-H-space induces isomorphisms of *homology* groups. This general situation has been well understood for some decades.

We assume that spaces have the homotopy types of CW-complexes, are based and that maps and homotopies preserve base points. A space X is a co-H-space if there is a comultiplication map $\nu : X \to X \vee X$ satisfying $j \circ \nu \simeq \Delta : X \to X \times X$ where $j : X \vee X \hookrightarrow X \times X$ is the inclusion and Δ the diagonal map. Equivalently, X is a co-H-space if the Lusternik-Schnirelmann category $\mathrm{cat}\, X$ is one.

Statements (1.1) and (1.2) below were shown to be equivalent in [**13**], see also Theorem 0.1 of [**16**].

(1.1) X is a co-H-space and the comultiplication can be chosen so that $[X, Z]$ is an algebraic loop for all Z.

(1.2) The space X has the homotopy type of a wedge of a bunch of circles and a simply-connected co-H-space.

Problem 10 in [**11**] asks "Is any (non-simply-connected) co-H-space of the homotopy type of $S^1 \vee \cdots \vee S^1 \vee Y$ where there may be infinitely many circles and $\pi_1(Y) = 0$?" The positive statement has become known as 'the Ganea conjecture'

1991 *Mathematics Subject Classification.* Primary 55P45.

Key words and phrases. co-H-space, Ganea's conjecture, almost completion, atomic space.

This research was partially supported by Grant-in-Aid for Scientific Research (C)08640125 from The Ministry of Science, Sports and Culture.

for co-H-spaces (see section 6 of [**1**]). The conjecture was resolved thirty years after being raised when the second author constructed in [**16**] infinitely many finite complexes which are co-H-spaces but which do not have the homotopy type described in Problem 10. This leaves open a p-complete version of Ganea's conjecture, and probably more difficult, a p-local version (see Conjecture 1.6 of [**16**]). The rational version was established in [**12**] as a prime decomposition theorem for an 'almost rational' co-H-space. In this note, we address the p-complete problem at a prime p.

Some comments are required on p-completion. The p-completion of a simply connected co-H-space is rarely a co-H-space (unless it is a 'finite torsion space' in the sense of [**8**]) as a wedge of p-complete spaces need not be p-complete; it becomes a co-H-object in a categorical sense, which is adequate for some purposes. More seriously, we are interested in non-simply-connected co-H-spaces and it is shown in [**9**] that when X is a co-H-space,

(1.3) $\pi_1(X)$ is a free group.

We therefore use fibrewise p-completion which we describe after introducing more notation.

Let X be a co-H-space and $B = B\pi_1(X)$, so that B can be chosen as a bunch of circles by (1.3). Let $i : B \to X$ represent the generators of $\pi_1(X)$ associated with the circles and $j : X \to B$ be the classifying map of the universal cover $p : \widetilde{X} \to X$. We may assume that $j{\circ}i \simeq 1_B$, and so B is a homotopy retract of X. Also let $c : X \to C$ be the cofibre of $i : B \to X$, so C is simply connected. One seems tantalizingly close to Ganea's original conjecture as there are homology equivalences $X \to B \vee C$ and $B \vee C \to X$ inducing isomorphisms of fundamental groups.

For each prime p, we consider the fibrewise p-completion of $j : X \to B$, $\widehat{e}^a_p : X \to \widehat{X}^a_p$ which commutes with projections to B. The map $\widehat{e}^a_p$ induces an isomorphism of fundamental groups and acts as standard p-completion on the fibre $\widetilde{X}$ and so $\widetilde{\widehat{X}^a_p} \simeq \widehat{\widetilde{X}}_p$. Following earlier authors, we refer to this fibrewise p-completion for $j : X \to B$ as 'almost p-completion'. A general reference for fibrewise p-completion is [**6**], [**4**] or [**18**]. Also it is shown in [**16**] that a co-H-space X is a co-H-space over B up to homotopy, and so $\widehat{X}^a_p$ becomes a co-H-object over B in the sense of [**17**] and [**8**].

The main result of this note is the following.

THEOREM 1.1. *Let X be a finite, connected, based CW-complex and a co-H-space. After almost p-completion, $\widehat{X}^a_p$ has the homotopy type of $\widehat{(B{\vee}C)}^a_p$ where B is a finite bunch of circles and C is a simply-connected finite complex and a co-H-space.*

Let $Y \to B$ be a fibration with cross-section.

COROLLARY 1.2. *The homotopy set $[\widehat{X}^a_p, \widehat{Y}^a_p]_B$ inherits an algebraic loop structure from C.*

Since C is a simply-connected co-H-space, results of [**21**] imply that $\widehat{C}_p$ can be decomposed, uniquely up to homotopy, as a completed wedge sum of simply-connected p-atomic spaces.

COROLLARY 1.3. *$\widehat{X}^a_p$ has the homotopy type of the almost p-completion of a wedge sum of circles and simply-connected p-atomic spaces.*

The general strategy used to prove Theorem 1.1 first occurs in [**15**] in establishing the Ganea conjecture for complexes of dimension less than 4. The existence of a co-H-multiplication enabled the authors to construct a new splitting $C \to X$ to obtain a homotopy equivalence $B \vee C \to X$. In our case, we adopt techniques for simply-connected p-complete spaces of [**14**] and [**21**] for a similar purpose.

The first author expresses his gratitude to the Kyushu University and the second to the University of Aberdeen for their hospitality.

2. Finite co-H-complexes

We give a different proof of Theorem 3.1 of [**15**] and include a converse statement for completeness.

THEOREM 2.1. *Let X have the homotopy type of a based CW complex. Then X has the homotopy type of a finite complex which is a co-H-space if and only if there are a finite bunch of circles, a connected finite complex D and maps*

$$\rho : B{\vee}\Sigma D \to X \quad \text{and} \quad \sigma : X \to B{\vee}\Sigma D$$

satisfying $\rho\circ\sigma \simeq 1_X$.

PROOF. Let X be a finite complex and a co-H-space. Then by [**10**], X is dominated by $\Sigma\Omega X$, where $\Sigma\Omega X \simeq \Sigma\pi \vee \Sigma(\bigvee_{\tau\in\pi} \Omega_\tau X)$ and $\pi = \pi_1(X)$ is a free group. As X is a finite complex, the rank of π is finite and $B = B\pi$ is a finite bunch of circles and X is dominated by $B \vee \Sigma(\bigvee_{\tau\in\pi} \Omega_\tau X)$. The image of X can be taken as a finite subcomplex of $B \vee \Sigma(\bigvee_{\tau\in\pi} \Omega_\tau X)$ and so there is a finite subcomplex D in $\bigvee_{\tau\in\pi} \Omega_\tau X$ such that $B{\vee}\Sigma D$ dominates X.

Conversely, let X be dominated by $B{\vee}\Sigma W$ where B is a finite bunch of circles indexed by a finite set Λ and W is a connected finite complex. Then $B{\vee}\Sigma W = \Sigma(\Lambda{\vee}W)$, and $\operatorname{cat} X \le \operatorname{cat}(\Sigma(\Lambda{\vee}W)) = 1$. Thus X is a co-H-space. Also X is dominated by the finite complex $B{\vee}\Sigma W$ whose fundamental group is free of finite rank. The finiteness obstruction for X lies in the Whitehead group $Wh(\pi) = K_0(\mathbb{Z}\pi_1(X))/\pm 1$ ([**20**] and [**19**]) which is zero (see [**7**] and [**2**]). Thus X has a homotopy type of a finite complex. This completes the proof. $\square$

Let $P = \sigma\circ\rho : B{\vee}\Sigma D \to B{\vee}\Sigma D$ be the homotopy idempotent given by Theorem 2.1. So P restricted to B can be chosen as the inclusion $\mathrm{in}_B : B \subset B{\vee}\Sigma D$ and P restricted to ΣD lifts to $P_0 : \Sigma D \to \widetilde{B{\vee}\Sigma D}$ where $\widetilde{B{\vee}\Sigma D} \simeq \bigvee_{\tau\in\pi} \tau\cdot\Sigma D$ as ΣD is simply connected. As ΣD is a finite complex, $P_0(\Sigma D)$ is included in a finite subcomplex $\bigvee_{i=1}^t \tau_i\cdot\Sigma D$. So the restriction of P to ΣD equals the composition

$$\Sigma D \xrightarrow{P_0} \bigvee_{i=1}^t \tau_i\cdot\Sigma D \hookrightarrow \widetilde{B{\vee}\Sigma D} \xrightarrow{p} B{\vee}\Sigma D.$$

Therefore we have the commutative diagram

$$\begin{array}{ccc} B{\vee}\Sigma D & \xrightarrow{\quad P \quad} & B{\vee}\Sigma D \\ \Big\downarrow{\scriptstyle 1_B\vee P_0} & & \Big\uparrow{\scriptstyle \{\mathrm{in}_B, p\}} \\ B{\vee}\bigvee_{i=1}^t \tau_i\cdot\Sigma D & \hookrightarrow & B{\vee}\bigvee_{\tau\in\pi} \tau\cdot\Sigma D \end{array} \tag{2.1}$$

which plays a crucial role in the next section.

We need also maps $\sigma' : C \to \Sigma D$ and $\rho' : \Sigma D \to C$ defined by the commutative diagram below in which the columns are cofibrations:

(2.2)
$$\begin{array}{ccccc}
B & = & B & = & B \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & B\vee\Sigma D & \xrightarrow{\rho} & X \\
\downarrow & & \downarrow & & \downarrow \\
C & \xrightarrow{\sigma'} & \Sigma D & \xrightarrow{\rho'} & C
\end{array}$$

chosen so that $\rho'{\circ}\sigma' \simeq 1_C$, as $\rho{\circ}\sigma \simeq 1_X$. Thus the self map $P' = \sigma'{\circ}\rho'$ of ΣD is also a homotopy idempotent. We will investigate the compositions

$$\widehat{X}^a_p \xrightarrow{\widehat{\sigma}^a_p} \widehat{B \vee \Sigma D}^a_p \overset{\phi}{\simeq} \widehat{B \vee \Sigma D}^a_p \xrightarrow{\widehat{1_B \vee \rho'}^a_p} \widehat{B \vee C}^a_p$$

and

$$\widehat{B \vee C}^a_p \xrightarrow{\widehat{1_B \vee \sigma'}^a_p} \widehat{B \vee \Sigma D}^a_p \overset{\phi^{-1}}{\simeq} \widehat{B \vee \Sigma D}^a_p \xrightarrow{\widehat{\rho}^a_p} \widehat{X}^a_p$$

with an appropriate homotopy equivalence ϕ.

3. Almost p-complete co-H-objects

Using the universality of almost p-completion, we have the natural equivalences between homotopy sets.

$$\begin{aligned}
[(\widehat{B\vee\Sigma D})^a_p, (\widehat{B\vee\Sigma D})^a_p]_B &= [B\vee\Sigma D, (\widehat{B\vee\Sigma D})^a_p]_B = [\Sigma D, (\widehat{B\vee\Sigma D})^a_p] \\
&= [\Sigma D,\ (\widetilde{\widehat{B\vee\Sigma D}})^a_p] = [\Sigma D, (\widetilde{\widehat{B\vee\Sigma D}})_p] = [\widehat{\Sigma D}_p, \widehat{\bigvee}_{\tau\in\pi}\tau{\cdot}\widehat{\Sigma D}_p]
\end{aligned}$$

where $\widehat{\bigvee}$ denotes the completed wedge sum. Projecting to its factors $\tau{\cdot}\widehat{\Sigma D}_p$, we have a map

$$\beta : [(\widehat{B\vee\Sigma D})^a_p, (\widehat{B\vee\Sigma D})^a_p]_B = [\widehat{\Sigma D}_p, \widehat{\bigvee}_{\tau\in\pi}\tau{\cdot}\widehat{\Sigma D}_p] \to \textstyle\prod_{\tau\in\pi}\tau{\cdot}[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p]$$

to the product and the image of β contains the sum $\sum_{\tau\in\pi}\tau{\cdot}[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p]$. Indeed β is a continuous homomorphism of topological groups, where the group structure is inherited from the co-H-space ΣD.

We give an alternative description of the closed subgroup which is the image of β. Let $\{g_\tau\}_{\tau\in\pi}$ denote an element of the product $\prod\limits_{\tau\in\pi}\tau{\cdot}[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p] = \prod\limits_{\tau\in\pi}\tau{\cdot}[\Sigma D, \widehat{\Sigma D}_p]$.

Proposition 3.1. *The element $\{g_\tau\}_{\tau\in\pi}$ lies in the image of β if and only if $\{\chi{\circ}g_\tau\}_{\tau\in\pi} \in \sum_{\tau\in\pi}\tau{\cdot}[\Sigma D, K]$ for any map $\chi : \widehat{\Sigma D}_p \to K$ and any space K all of whose homotopy groups are finite p-groups.*

Proof. Let $f : \widehat{\Sigma D}_p \to \widehat{\bigvee}_{\tau\in\pi}\tau{\cdot}\widehat{\Sigma D}_p$ and $\beta(f) = \{f_\tau\}_{\tau\in\pi} \in \prod\limits_{\tau\in\pi}\tau{\cdot}[\Sigma D, \widehat{\Sigma D}_p]$. Since $(\widehat{\bigvee}_{\tau\in\pi}\tau{\cdot}\chi){\circ}f$ lies in $[\widehat{\Sigma D}_p, \widehat{\bigvee}_{\tau\in\pi}\tau{\cdot}K] = [\Sigma D, \bigvee_{\tau\in\pi}\tau{\cdot}K]$, the map $\{\chi{\circ}f_\tau\}_{\tau\in\pi}$ lies in $\sum_{\tau\in\pi}\tau{\cdot}[\Sigma D, K]$ as required. The converse statement holds by naturality and fundamental properties of p-completion. $\square$

Lemma 3.2. $\beta(\widehat{P}_p^a) \in \sum_{\tau\in\pi} \tau\cdot[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p]$.

Proof. The lemma follows from (2.1). □

We now recall results from [**14**] and [**21**]. We define a homomorphism of near-algebras by mapping homotopy classes of self-maps of $\widehat{(B\vee\Sigma D)}_p^a$ over B to the induced endomorphism of $\tilde{H}_*(\widetilde{\widehat{(B\vee\Sigma D)}_p^a};\mathbb{F}_p) \cong \tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p\pi)$ (see [**15**]) the $\mathbb{F}_p$-homology groups of the universal cover

$$\alpha : [\widehat{(B\vee\Sigma D)}_p^a, \widehat{(B\vee\Sigma D)}_p^a]_B \to \mathrm{End}_{\mathbb{F}_p\pi}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p\pi)\}.$$

When B is a point, the same definition gives a homomorphism

$$\alpha_0 : [\widehat{\Sigma D}_p, \widehat{\Sigma D}_p] \to \mathrm{End}_{\mathbb{F}_p}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p)\}.$$

The homomorphisms α and α_0 fit into a commutative diagram

$$\begin{array}{ccc}
[\widehat{(B\vee\Sigma D)}_p^a, \widehat{(B\vee\Sigma D)}_p^a]_B & \xrightarrow{\quad\alpha\quad} & \mathrm{End}_{\mathbb{F}_p\pi}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p\pi)\} \\
\downarrow{\scriptstyle\beta} & & \| \\
\prod_{\tau\in\pi} \tau\cdot[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p] & & \| \\
\uparrow & & \| \\
\sum_{\tau\in\pi} \tau\cdot[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p] & \xrightarrow{\sum_{\tau\in\pi}\tau\cdot\alpha_0} & \sum_{\tau\in\pi} \tau\cdot\mathrm{End}_{\mathbb{F}_p}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p)\}.
\end{array}$$

The topological radical N in the compact, Hausdorff space $[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p]$ is defined by

$$N = \{n \in [\widehat{\Sigma D}_p, \widehat{\Sigma D}_p] \mid h\circ n \text{ is topologically nilpotent for all } h\}.$$

The radical R in the finite ring $\mathrm{End}_{\mathbb{F}_p}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p)\}$ is defined by

$$R = \{r \in \mathrm{End}_{\mathbb{F}_p}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p)\} \mid \text{For any } u, \text{ there is } v \text{ such that } v(1+ur) = 1\}.$$

Then (see section 3 in [**14**]), α_0 induces a homomorphism of rings

$$\alpha_0' : [\widehat{\Sigma D}_p, \widehat{\Sigma D}_p]/N \to \Big\{\mathrm{End}_{\mathbb{F}_p}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p)\}\Big\}/R$$

which is a monomorphism onto its image, which can be identified with $\overset{k}{\underset{i=1}{\oplus}} M(n_i, \mathbb{F}_{q_i})$ for some k, where $\mathbb{F}_q$ is a finite field of characteristic p with q elements.

Lemma 3.3. *There is an isomorphism of rings induced by* α.

$$\alpha' : \sum_{\tau\in\pi} \tau\cdot([\widehat{\Sigma D}_p, \widehat{\Sigma D}_p]/N) \to \overset{k}{\underset{i=1}{\oplus}} M(n_i, \mathbb{F}_{q_i}\pi).$$

Proof. We identify $\mathbb{F}_p\pi\otimes\mathrm{End}_{\mathbb{F}_p}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p)\}$ with $\mathrm{End}_{\mathbb{F}_p\pi}\{\tilde{H}_*(\widehat{\Sigma D}_p;\mathbb{F}_p\pi)\}$ so that the image $\mathbb{F}_p\pi\otimes(\overset{k}{\underset{i=1}{\oplus}} M(n_i,\mathbb{F}_{q_i}))$ of $\alpha' = \sum_{\tau\in\pi}\tau\cdot\alpha_0$ becomes $\overset{k}{\underset{i=1}{\oplus}} M(n_i, \mathbb{F}_{q_i}\pi)$. □

4. The proof of Theorem 1.1

Lemmas 3.2 and 3.3 imply that $\widehat{P}^a_p \in [(\widehat{B\vee\Sigma D})^a_p, (\widehat{B\vee\Sigma D})^a_p]$ is mapped in homology to a direct sum of idempotents $\overset{k}{\underset{i=1}{\oplus}} P_i \in \overset{k}{\underset{i=1}{\oplus}} M(n_i, \mathbb{F}_{q_i}\pi)$. We appeal to work of Bass [**3**]; each P_i defines an $\mathbb{F}_{q_i}\pi$-homomorphism of $(\mathbb{F}_{q_i}\pi)^{n_i}$ and so there exists an $\mathbb{F}_{q_i}\pi$-isomorphism of $(\mathbb{F}_{q_i}\pi)^{n_i}$, A_i say, such that

$$A_i P_i A_i^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & \text{\Large 0} & \\ & & 1 & & \\ & & & 0 & \\ & \text{\Large 0} & & & \ddots & \\ & & & & & 0 \end{bmatrix} \in M(n_i, \mathbb{F}_{q_i}\pi).$$

This matrix also lies in $M(n_i, \mathbb{F}_{q_i})$. Let a ring homomorphism $\epsilon : \mathbb{F}_q\pi \to \mathbb{F}_q$ be defined by $\epsilon(\tau) = 1$ and so $\epsilon(P_i)$ represents $\widehat{P'}^a_p$. We choose $\phi : (\widehat{B\vee\Sigma D})^a_p \to (\widehat{B\vee\Sigma D})^a_p$ lying in $\beta^{-1}(\Sigma_{\tau\in\pi}\tau\cdot[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p])$ and a corresponding $\phi' : \widehat{\Sigma D}_p \to \widehat{\Sigma D}_p$ representing the invertible matrices $\overset{k}{\underset{i=1}{\oplus}} A_i$ in $M(n_i, \mathbb{F}_{q_i}\pi)$ and $\overset{k}{\underset{i=1}{\oplus}} \epsilon(A_i)$ in $M(n_i, \mathbb{F}_{q_i})$ respectively. So ϕ and ϕ' are homotopy equivalences as they induce isomorphisms of homology groups of universal covers by Lemma 3.3. Referring back to (2.2), we have a commutative diagram

$$\begin{array}{ccccccc}
\widehat{X}^a_p & \xrightarrow{\widehat{\sigma}^a_p} & (\widehat{B\vee\Sigma D})^a_p & \xrightarrow{\phi} & (\widehat{B\vee\Sigma D})^a_p & \xrightarrow{\widehat{\rho}^a_p} & \widehat{X}^a_p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widehat{C}_p & \xrightarrow{\widehat{\sigma'}_p} & \widehat{\Sigma D}_p & \xrightarrow{\phi'} & \widehat{\Sigma D}_p & \xrightarrow{\widehat{\rho'}_p} & \widehat{C}_p.
\end{array}$$

The self map $\phi\circ\widehat{P}^a_p\circ\phi^{-1}$ of $(\widehat{B\vee\Sigma D})^a_p$ is a homotopy idempotent represented by the matrix $\overset{k}{\underset{i=1}{\oplus}} A_i P_i A_i^{-1} = \overset{k}{\underset{i=1}{\oplus}} \epsilon(A_i)\epsilon(P_i)\epsilon(A_i)^{-1}$ which also represents $\phi'\circ\widehat{P'}_p\circ\phi'^{-1}$. Therefore this matrix also represents $\widehat{s\circ r}^a_p$ where $s = 1_B\vee(\phi'\circ\widehat{\sigma'}_p) : B\vee\widehat{C}_p \to B\vee\widehat{\Sigma D}_p$ and $r = 1_B\vee(\widehat{\rho'}_p\circ\phi'^{-1}) : B\vee\widehat{\Sigma D}_p \to B\vee\widehat{C}_p$, and so $r\circ s \simeq 1_{B\vee\widehat{C}_p}$. We deduce

$$\beta(\phi\circ\widehat{P}^a_p\circ\phi^{-1}) \simeq \beta((\widehat{s\circ r})^a_p) \quad \mathrm{mod}\ \textstyle\sum_{\tau\in\pi}\tau\cdot N \quad \text{in} \quad \textstyle\sum_{\tau\in\pi}\tau\cdot[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p].$$

Let $f = \widehat{\rho}^a_p\circ\phi^{-1}\circ\widehat{s}^a_p : (\widehat{B\vee C})^a_p \to \widehat{X}^a_p$ and $g = \widehat{r}^a_p\circ\phi\circ\widehat{\sigma}^a_p : \widehat{X}^a_p \to (\widehat{B\vee C})^a_p$. Then

$$g\circ f = (\widehat{r}^a_p\circ\phi\circ\widehat{\sigma}^a_p)\circ(\widehat{\rho}^a_p\circ\phi^{-1}\circ\widehat{s}^a_p) = \widehat{r}^a_p\circ(\phi\circ\widehat{P}^a_p\circ\phi^{-1})\circ\widehat{s}^a_p$$

whose image by β is in $\sum_{\tau\in\pi}\tau\cdot[\widehat{\Sigma D}_p, \widehat{\Sigma D}_p]$ and is homotopic modulo $\sum_{\tau\in\pi}\tau\cdot N$ to that of

$$\widehat{r}^a_p\circ(\widehat{s\circ r})^a_p\circ\widehat{s}^a_p = (\widehat{r\circ s})^a_p\circ(\widehat{r\circ s})^a_p \simeq (\widehat{1_{B\vee\widehat{C}_p}})^a_p\circ(\widehat{1_{B\vee\widehat{C}_p}})^a_p = 1_{(\widehat{B\vee C})^a_p}.$$

Thus the self map $g\circ f$ of $(\widehat{B\vee C})^a_p$ over B induces an isomorphisms of homology groups of the universal cover by Lemma 3.3. Therefore

(4.1) $g \circ f : \widehat{(B \vee C)}_p^a \to \widehat{(B \vee C)}_p^a$ is a homotopy equivalence.

It is routine to check that

(4.2) f and g induce isomorphisms of the $\mathbb{F}_p$-homology groups of universal covers,

(4.3) f and g induce isomorphisms of fundamental groups.

We complete the proof by following [**5**]. Statements (4.1), (4.2) and (4.3) are similar to the conclusion of Lemma 1.6 of [**5**]. One then makes minor changes to the proof of Theorem 1.5 given there to deduce Theorem 1.1.

References

[1] M. Arkowitz, *Co-H-spaces*, Handbook of algebraic topology, 1143-1173, North Holland, Amsterdam, 1995.

[2] H. Bass, *Projective modules over free groups are free*, J. of Algebra **1** (1964), 367–373.

[3] H. Bass, *Algebraic K-theory*, Benjamin, Amsterdam,1968.

[4] M. Bendersky, *A functor which localizes the higher homotopy groups of an arbitrary CW-complex*, Lecture Notes in Math. 418, Springer Verlag, Berlin (1975), 13–21.

[5] I. Berstein and E. Dror, *On the homotopy type of non-simply connected co-H-space*, Ill. Jour. Math. **20** (1976), 528–534.

[6] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer Verlag, Berlin (1972).

[7] P. M. Cohn, *Free ideal rings*, J. of Algebra **1** (1964), 47–69.

[8] M. C. Crabb, J. R. Hubbuck and Kai Xu, *Fields of spaces*, Adams Memorial Symposium on Algebraic Topology, 1 (1990), 241–254, LMS. Lecture Note 175, Cambridge Univ. Press, Cambridge, 1992.

[9] S. Eilenberg and T. Ganea, *On the Lusternik-Schnirelmann category of abstract groups*, Ann. of Math. **65** (1957), 517-518.

[10] T. Ganea, *Cogroups and suspensions*, Invent. Math. **9** (1970), 185–197.

[11] T. Ganea, *Some problems on numerical homotopy invariants*, Lecture Notes in Math. 249, Springer Verlag, Berlin (1971), 13–22.

[12] H. W. Henn, *On almost rational co-H-spaces*, Proc. Amer. Math. Soc. **87**, (1983), 164–168.

[13] P. Hilton, G. Mislin and J. Roitberg, *On co-H-spaces*, Comment. Math. Helv. **53** (1978), 1–14.

[14] J. R. Hubbuck, *Self maps of H-spaces*, Advances in homotopy theory (Cortona, 1988), 105–110, London Math. Soc. Lecture Note Ser., 139, Cambridge Univ. Press, Cambridge, 1989.

[15] N. Iwase, S. Saito and T. Sumi, *Homology of the universal covering space of a co-H-space*, Trans. Amer. Math. Soc. **351** (1999), 4837–4846.

[16] N. Iwase, *Co-H-spaces and the Ganea conjecture*, Topology **40** (2001), 223–234.

[17] I. M. James, *Fibrewise Topology*, Cambridge University Press, Cambridge 1989.

[18] J. P. May, *Fiberwise localization and completion*, Trans. Amer. Math. Soc. **258** (1980), 127–146.

[19] K. Varadarajan, *The Finiteness Obstruction of C.T.C. Wall*, John Wiley & Sons, Inc., New York, 1989.

[20] C. T. C. Wall, *An obstruction to finiteness of CW-complexes*, Bull. Amer. Math. Soc. **70** (1964), 269–270.

[21] K. Xu, *Space of rings*, PhD Thesis at University of Aberdeen (1989).

(J. R. Hubbuck) Department of Mathematical Sciences, University of Aberdeen, Aberdeen AB24 3QY, United Kingdom

E-mail address: `j.hubbuck@maths.abdn.ac.uk`

(N. Iwase) Faculty of Mathematics, Kyushu University, Fukuoka 810-8560, Japan

E-mail address: `iwase@math.kyushu-u.ac.jp`