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ON THE K-RING STRUCTURE OF X-PROJECTIVE n-SPACE

By

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§1. Introduction

Let X be an A_n-space in the sense of Stasheff [10, 11]. We recall that there is a sequence of quasi-fibrations $\{p_k\}$ with fibre X as follows:

$$\begin{array}{ccccccccccc}
X=E^1 & \hookrightarrow & E^2 & \hookrightarrow & E^3 & \hookrightarrow & \cdots & \hookrightarrow & E^n & & \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} & & & & \downarrow{\scriptstyle p_n} & & \\
pt=P^0 & \hookrightarrow & P^1=SX & \hookrightarrow & P^2 & \hookrightarrow & \cdots & \hookrightarrow & P^{n-1} & \hookrightarrow & P^n
\end{array}$$

where 1) *E^k has the same homotopy type as the k-fold join of X,*
2) *E^k is contractible in E^{k+1},*
3) *P^k has the same homotopy type as the mapping cone of p_k.*

The space P^k is called the *X-projective k-space.* The purpose of this paper is to determine the ring structure of the $Z/2$-graded K-ring of the X-projective k-space, $k\geqq 3$. For $k=2$, the structure of $K^*(P^2)$ has been studied, for X satisfying suitable condition, by several authors, e.g. [12]. In §3, we shall construct a spectral sequence of Stasheff type [11], which converges to $K^*(P^n)$, and define an element x in $K^*(X)$ to be A_k-primitive, $k\leqq n$ (Definition 3.3), a property which is proved to be equivalent to the existence of $u\in K^*(P^k)$ satisfying

$$s^*(x)=\iota_2^*\cdots\iota_k^*(u),$$

where s^* is the suspension isomorphism, and ι_i^* is the homomorphism induced from the inclusion ι_i: $P^{i-1}\to P^i$ (Proposition 3.4). In [5], the author gives a certain decomposition of E^k and P^k, $k\leqq n$. Making use of this decomposition and its Mayer-Vietoris exact sequence, we obtain the following Theorems.

THEOREM A. *Let X be* 1*-connected finite A_n-complex, $n\geqq 3$, of rank l.*

Suppose $K^*(X)$ *is generated by* A_k*-primitive elements* $\{x_j\}_{1 \leqq j \leqq l}$, $k \leqq n$, *with* $s^*(x_j) = \iota_2^* \cdots \iota_k^*(u_j)$ *for some* $u_j \in K^*(P^k)$.*) *Then, for any choice of* $\{u_j\}$, *there is an isomorphism of rings over* $K_* = K^{-*}(pt)$:

$$K^*(P^k) \cong K_*^{[k+1]}[u_1, \ldots, u_l] \oplus S_k,$$

where S_k, *as defined in* §3, *is an ideal of* $K^*(P^k)$ *satisfying*

$$\iota_k^*(S_k) = 0,$$

$$S_k \cdot \tilde{K}^*(P^k) = 0,$$

and $K_*^{[k+1]}[u_1, \ldots, u_l]$ *is a truncated polynomial ring over* K_* *of height* $k+1$.

It is an immediate consequence of Theorem B below that, for $k \leqq n-1$, there exist A_k-primitive generators $\{x_j\}$ (see §5). Theorem A generalizes Corollary 2.5 of [4].

THEOREM B. *If* X *is a* 1*-connected finite* A_n*-complex,* $n \geqq 3$, *then the following two conditions are equivalent*:

1) *the spectral sequence* $\{E^{*,*}, d_r\}$ *of Stasheff* [10] *collapses.*
2) $K^*(X)$ *is generated by* A_n*-primitive elements.*

COROLLARY. *Let* X *be a* 1*-connected finite* A_n*-complex,* $n \geqq 3$. *Then* $K^*(X)$ *is an exterior algebra on odd dimensional* A_{n-1}*-primitive elements.*

This paper is organized as follows. Based on the theorem of J. P. Lin and L. Hodgkin, in §2, we present the K-cohomology of X and E^k, $k \leqq n$. In §3, we introduce the spectral sequence of Stasheff type which converges to the K-cohomology of XP^n. Finally in §4 and §5, using results of §2 and §3, we prove the above theorems.

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§ 2. Preliminaries

Let us recall some results which are used to prove Theorem A and Theorem B. If a finite 1-connected CW-complex X is an A_3-space, then J. P. Lin [6, 7, 8] showed that its K-cohomology ring is torsion free.

*) As in Theorem 2.1, $K^*(X)$ becomes an exterior algebra generated by $\{x_j\}$ and deg $x_j = -1$, deg $u_j = 0$.

THEOREM (J. P. Lin). *If X is a 1-connected H-complex of finite type and its Pontrjagin ring with coefficient ring $\boldsymbol{Z}/2$ is associative, then the K-ring of X has no torsion.*

Moreover, making use of Hodgkin's observation in K-rings of Lie groups, we get

THEOREM 2.1 (J. P. Lin, J. R. Hubbuck, L. Hodgkin [8, 4, 3]) *If such X admits an A_n-structure, $n \geqq 3$, then $K^*(X)$ is the exterior algebra on odd dimensional primitives.*

Thus in the remainder of this paper, a space X is always assumed to be a 1-connected finite A_n-complex with n greater than 2. We introduce the following mappings, as defined in [10].

$$\mu_k\colon X \times E^k \longrightarrow E^k,\ 1 \leqq k \leqq n-1,$$

such that

$$E^{k+1} \cong E^k \cup_{\mu_k} X \times C(E^k).$$

By induction, the Mayer-Vietoris exact sequence of $(E^k \cup_{\mu_k} X \times C(E^k);\ E^k \cup_{\mu_k} X \times C_1,\ X \times C_2)$, where $C_1 \cup C_2 = C(E^k)$, $C_1 \cap C_2 = \{*\} \times E^k$, $C_1 \cong [0, 1] \times E^k/[0, 1] \times \{*\}$, where $*$ is the unit of X, $C_2 \cong C(E^k)$, and the homotopy equivalences $E^{k+1} \cong E^k \cup_{\mu_k} X \times C(E^k)$ imply that $K^*(E^k)$ can be described in terms of the elements of $K^*(X)$ (Remark 2.3). The Mayer-Vietoris exact sequence is as follows;

$$\begin{aligned} \cdots \longrightarrow \tilde{K}^*(E^{k+1}) \longrightarrow \tilde{K}^*(X) \oplus \tilde{K}^*(E^k) \xrightarrow{\langle -pr_X^*,\, \mu_k^* \rangle} \\ \tilde{K}^*(X \times E^k) \xrightarrow{\Delta_{k+1}} \tilde{K}^{*+1}(E^{k+1}) \longrightarrow \cdots \end{aligned} \tag{2.1}$$

where pr_X is the canonical projection and Δ_{k+1} is the connecting homomorphism. We recall the inclusions $X \hookrightarrow E^k$ and $E^k \hookrightarrow E^{k+1}$ are all null-homotopic. So Δ_{k+1} is epimorphic. For $x \in \tilde{K}^*(X)$ and $e \in \tilde{K}^*(E^k)$, we define

$$x * e = \Delta_{k+1}(x \times e).$$

Then any element of $\tilde{K}^*(X \times E^k)$ is represented by a linear combination of $x * e$, $x \in \tilde{K}^*(X)$ and $e \in \tilde{K}^*(E^k)$.

LEMMA 2.2. *The homomorphism Δ_{k+1} restricted to $\tilde{K}^*(X) \otimes \tilde{K}^*(E^k)$ is an isomorphism onto $\tilde{K}^*(E^{k+1})$.*

PROOF. This can be obtained by combining the exactness at $\tilde{K}^*(X\times E^k)$ of (2.1) and the fact that the homomorphism $\tilde{K}^*(E^{k+1})\to\tilde{K}^*(X)\oplus\tilde{K}^*(E^k)$ is a zero mapping. Q.E.D.

We define isomorphisms

$$\Delta^{(k+1)}: (\tilde{K}^*(X)\otimes\cdots\otimes\tilde{K}^*(X))^j \longrightarrow \tilde{K}^{j+k}(E^{k+1})$$

by the following formulas,

1) $\Delta^{(1)}$ is the identity mapping,
2) $\Delta^{(k+1)}=\Delta_{k+1}\mu_k(1\otimes\Delta^{(k)})$,

where μ_k: $\tilde{K}^*(X)\otimes\tilde{K}^*(E^k)\to\tilde{K}^*(X\times E^{k+1})$ is the cross product. We denote $\Delta^{(k+1)}(x_1\otimes\cdots\otimes x_{k+1})$ by $x_1*\cdots*x_{k+1}$.

REMARK 2.3. *We may take the module basis* $\{y_{j_1}*\cdots*y_{j_{k+1}}: \{y_t\}$ *is a module basis for* $\tilde{K}^*(X)\}$ *for* $\tilde{K}^*(E^{k+1})$.

According to E. Thomas [12], the element $e\in\tilde{K}^*(E^k)$ is said to be primitive with respect to μ_k iff $\mu_k^*(e)=x\times 1+1\times e'$ for some $x\in\tilde{K}^*(X)$ and $e'\in(E^k)$.

PROPOSITION 2.4. For $x_1,\dots, x_{k-1}\in\tilde{K}^*(X)$, $k\leq n$,

$$(*)\qquad \Delta_k(pr_{E^{k-1}})^*(x_1*\cdots*x_{k-1})$$
$$=\Sigma_{j=1}^{k-1}(-1)^j x_1*\cdots*x_{j-1}*(\Delta_2\bar{M}^*(x_j))*x_{j+1}*\cdots*x_{k-1},$$

where M: $X\times X\to X$ *is the multiplication defining the H-space structure on* X, $\bar{M}^*(x)=M^*(x)-x\times 1-1\times x$ *and* $pr_{E^{k-1}}$: $X\times E^{k-1}\to E^{k-1}$ *is the canonical projection.*

PROOF. Let us consider the following commutative diagram [5];

(2.2)

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & \tilde{K}^*(P^{k-1}) & \longrightarrow & \tilde{K}^*(pt)\oplus\tilde{K}^*(P^{k-2}) & \longrightarrow \\
 & \Big\downarrow{\scriptstyle p_k^*} & & \Big\downarrow{\scriptstyle 0^*\oplus p_{k-1}^*} & \\
\cdots \longrightarrow & \tilde{K}^*(E^k) & \longrightarrow & \tilde{K}^*(X)\oplus\tilde{K}^*(E^{k-1}) & \longrightarrow \\
 & \Big\downarrow{\scriptstyle \mu_k^*} & & \Big\downarrow{\scriptstyle M^*\oplus\mu_{k-1}^*} & \\
\cdots \longrightarrow & \tilde{K}^*(X\times E^k) & \longrightarrow & \tilde{K}^*(X\times X)\oplus\tilde{K}^*(X\times E^{k-1}) & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccccc}
\tilde{K}^*(E^{k-1}) & \xrightarrow{\delta_{k-1}} & \tilde{K}^{*+1}(P^{k-1}) & \longrightarrow \cdots \\
\Big\downarrow{\scriptstyle (pr_{E^{k-1}})^*} & & \Big\downarrow{\scriptstyle p_k^*} & \\
\tilde{K}^*(X\times E^{k-1}) & \xrightarrow{\Delta_k} & \tilde{K}^{*+1}(E^k) & \longrightarrow \cdots \\
\Big\downarrow{\scriptstyle (M\times 1)^*} & & \Big\downarrow{\scriptstyle \mu_k^*} & \\
\tilde{K}^*(X\times X\times E^{k-1}) & \xrightarrow{\Delta'_k} & \tilde{K}^{*+1}(X\times E^k) & \longrightarrow \cdots,
\end{array}
$$

where δ_{k-1}, Δ_k and Δ'_k are the suitable connecting homomorphisms for the Mayer-Vietoris exact sequences. We prove this proposition by induction on k. The formula $(*)$ holds for $k=2$, because we can see

$$
\begin{aligned}
\Delta_2(pr_{E^1})^*(x) &= \Delta_2(1\times x) \\
&= \Delta_2(1\times x)+\Delta_2(-M^*(x)+x\times 1) = -\Delta_2\overline{M}^*(x).
\end{aligned}
$$

Suppose that the formula $(*)$ holds for k. Since $\Delta_{k+1}\mu_k^*=0$ by (2.1), we proceed as follows:

$$
\begin{aligned}
&\Delta_{k+1}(pr_{E^k})^*(x_1*\cdots*x_k) \\
&\quad = \Delta_{k+1}(pr_{E^k})^*(x_1*\cdots*x_k)-\Delta_{k+1}\mu_k^*(x_1*\cdots*x_k) \\
&\quad = \Delta_{k+1}(pr_{E^k})^*(x_1*\cdots*x_k)-\Delta_{k+1}\mu_k^*\Delta'_k(x_1\times(x_2*\cdots*x_k)) \\
&\quad = \Delta_{k+1}(pr_{E^k})^*(x_1*\cdots*x_k)-\Delta_{k+1}\Delta'_k(M\times 1)^*(x_1\times(x_2*\cdots*x_k)) \\
&\quad = \Delta_{k+1}(1\times(x_1*\cdots*x_k))-\Delta_{k+1}\Delta'_k(1\times x_1\times(x_2*\cdots*x_k)) \\
&\qquad\qquad -\Delta_{k+1}(x_1\times\Delta_k(pr_{E^{k-1}})^*(x_2*\cdots*x_k)) \\
&\qquad\qquad -\Delta_{k+1}\Delta'_k(\overline{M}^*(x_1)\times(x_2*\cdots*x_k)).
\end{aligned}
$$

Applying the inductive hypothesis to $\Delta_k(1\times(x_2*\cdots*x_k))=\Delta_k(pr_{E^{k-1}})^*(x_2*\cdots*x_k)$, we have

$$\begin{aligned}\Delta_{k+1}(pr_{E^k})^*(x_1*\cdots*x_k)\\ &= -\Delta_2\bar{M}^*(x_1)*x_2*\cdots*x_k\\ &\quad+\textstyle\sum_{j=2}^{k+1}(-1)^j x_1*\cdots*x_{j-1}*\Delta_2\bar{M}^*(x_j)*x_{j+1}*\cdots*x_k\\ &=\textstyle\sum_{j=1}^{k+1}(-1)^j x_1*\cdots*x_{j-1}*\Delta_2\bar{M}^*(x_j)*x_{j+1}*\cdots*x_k\,. \qquad \text{Q.E.D.}\end{aligned}$$

Since $\Delta_2\bar{M}^*(x)=\sum_{j\geqq 1}u_j*v_j$, where $M^*(x)=\sum_{j\geqq 1}u_j\times v_j+x\times 1+1\times x$, x is primitive if and only if $\Delta_2\bar{M}^*(x)=0$. Hence Lemma 2.2 and Proposition 2.4 imply the following.

PROPOSITION 2.5. *Let $x_1,\dots, x_k$ be non-zero elements of $\tilde{K}^*(X)$. Then the following two statements are equivalent:*

1) *Each x_j is primitive,*
2) *$x_1*\cdots*x_k$ is primitive with respect to μ_k.*

§3. The Stasheff spectral sequence

Suppose that X is an A_n-space. Then there is the following exact couple:

$$(3.1)\qquad \begin{array}{ccc} D^{*,*} & \xrightarrow{\ \alpha\ } & D^{*,*}\\ & {\scriptstyle\gamma}\nwarrow \quad \swarrow{\scriptstyle\beta} & \\ & E^{*,*} & \end{array}$$

where

$$D^{i,j}=\begin{cases}0, & i<0,\\ K^j(pt), & i=0,\\ K^{i+j}(P^i), & 1\leqq i\leqq n,\\ K^{i+j}(P^n), & n+1\leqq i\end{cases}$$

$$E^{i,j}=\begin{cases}0, & i<0 \text{ or } n+1\leqq i,\\ K^j(pt), & i=0,\\ \tilde{K}^{i+j-1}(E^i), & 1\leqq i\leqq n,\end{cases}$$

and α: $D^{i,j}\to D^{i-1,j+1}$ is induced from the inclusion mapping ι_i: $P^{i-1}\to P^i$, $i\leqq n$, and for $i\geqq n+1$, α is the identity. β: $D^{i,j}\to E^{i+1,j-1}$ is induced from the projection mapping $p_{i+1}: E^{i+1} \to P^i$, $i \leqq n-1$, and for $i \geqq n$, β is the zero-mapping. $\gamma: E^{i,j} \to D^{i,j}$ is the connecting homomorphism, $i \leqq n-1$, and for $i \geqq n$, γ is the zero-mapping.

DEFINITION 3.1. We define the submodule $\bar{d}_r$ of $E^{*,*} \oplus E^{*,*}$ and the submodules Ker $\bar{d}_r$, Im $\bar{d}_r$, Dom $\bar{d}_r$ and Ind $\bar{d}_r$ of the module $E^{*,*}$ as follows:

$\bar{d}_r \ni (x, y)$ iff there exists an element $z \in D^{*,*}$ such $y = \beta(z)$, $\alpha^{r-1}(z) = \gamma(x)$,

Ker $\bar{d}_r \in x$ iff $(x, 0) \ni \bar{d}_r$,

Im $\bar{d}_r \ni x$ iff there exists an element $y \in E^{*,*}$ such that $(y, x) \in \bar{d}_r$,

Dom $\bar{d}_r \ni x$ iff there exists an element $y \in E^{*,*}$ such that $(x, y) \in \bar{d}_r$,

Ind $\bar{d}_r \ni x$ iff $(0, x) \in \bar{d}_r$.

Then Ker $\bar{d}_{r-1}$/Im $\bar{d}_{r-1}$ = Dom $\bar{d}_r$/Ind $\bar{d}_r$ is the r-th term $E_r^{*,*}$ of the spectral sequence induced by the above exact couple. Therefore we obtain the following.

PROPOSITION 3.2. *If X is an A_n-space, then there exists a spectral sequence $\{E_r^{*,*}, d_r\}$ which converges to $K^*(P^n)$ such that $d_r = 0$ for $r \geqq n$ and $E_n^{*,*} = E_{n+1}^* = \cdots = E_\infty^{*,*}$.*

DEFINITION 3.3. We say that $x \in \tilde{K}^*(X)$ is A_k-primitive iff $x \in \text{Ker}\, \bar{d}_{k-1}$. x is said to be $(k-1)$-transgressive iff there exists $y \in \tilde{K}^{*+1}(P^{k-1})$ such that $\delta'_k(x) = p_k^*(y)$ where $\delta'_k: \tilde{K}^*(X) \to K^{*+1}(E^k, X)$ is the connecting homomorphism. (cf. E. Thomas [12])

PROPOSITION 3.4. *If X is an A_n-space, the following three statements are equivalent for $x \in \tilde{K}^*(X)$, $k \leqq n$:*

1) *x is A_k-primitive,*
2) *x is $(k-1)$-transgressive,*
3) *$s^*(x) = \iota_2^* \cdots \iota_k^*(y)$ for some $y \in \tilde{K}^{*+1}(P^k)$.*

PROOF. The equivalence of 1) and 3) follows readily from the definition. We are left to prove the equivalence of 2) and 3). Consider the following commutative diagram,

(3.2)

$$\begin{array}{ccccccc}
 & & \tilde{K}^*(X) & = & \tilde{K}^*(X) & = & \tilde{K}^*(X) = \tilde{K}^*(X) \\
 & & \downarrow{\scriptstyle \delta'_k} & & \downarrow & & \cong\downarrow \qquad \quad \Big\downarrow {\scriptstyle s^*}\, \cong \\
K^{*+2}(D^k, E^k) & \xleftarrow{\delta''_k} & K^{*+1}(E^k, X) & \longleftarrow & K^{*+1}(D^k, X) & \longrightarrow & K^{*+1}(CX, X) \\
\uparrow{\scriptstyle \sigma_{k+1}^*} & & \uparrow{\scriptstyle p_k^*} & & \uparrow{\scriptstyle \sigma_{k+1}^*} & & \cong\uparrow \\
K^{*+2}(P^k, P^{k-1}) & \xleftarrow{\delta_k} & \tilde{K}^{*+1}(P^{k-1}) & \xleftarrow{\iota_k^*} & \tilde{K}^{*+1}(P^k) & \xrightarrow{\iota_2^* \cdots \iota_k^*} & \tilde{K}^{*+1}(SX) = \tilde{K}^{*+1}(SX)
\end{array}$$

where D^k is a space, defined in Stasheff [10], such that (D^k, E^k) is homotopically equivalent to $(C(E^k), E^k)$ and for $k \leqq n-1$, D^k is a subspace of E^{k+1}. σ_{k+1} is a mapping from D^k to P^k which is an extension of p_k: $E^k \to P^{k-1}$ for $k \leqq n$ and the restriction of p_{k+1} to D^k for $k \leqq n-1$. The second row with the last term $K^{*+1}(CX, X)$ removed, and the third row with the last term $\tilde{K}^{*+1}(SX)$ removed are exact. Assume that x satisfies 3). Then there exists $y \in K^{*+1}(P^k, pt) = \tilde{K}^{*+1}(P^k)$ such that $s^*(x) = \iota_2^* \cdots \iota_k^*(y)$. From the diagram (3.2), we get

$$\delta_k'(x) = p_k^* \iota_k^*(y),$$

Conversely, assume that x is $(k-1)$-transgressive. Then there exists $y' \in \tilde{K}^{*+1}(P^{k-1})$ such that $\delta_k'(x) = p_k^*(y')$. Since $\sigma_{k+1}^* \tilde{\delta}_k(y') = \delta_k'' p_k^*(y') = \delta_k'' \delta_k'(x) = 0$ and σ_{k+1}^* is an isomorphism, there exists an element $y \in \tilde{K}^{*+1}(P^k)$ such that $y' = \iota_k^*(y)$ and $s^*(x) = \iota_2^* \cdots \iota_k^*(y)$. Q.E.D.

In particular, we remark that, for an element x in $K^*(X)$, where X is an A_∞-space, x is A_∞-primitive iff it is ∞-transgressive.

Next we consider the following commutative diagram:

(3.3)

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \tilde{K}^{*+1}(P^k) & \longrightarrow & \tilde{K}^{*+1}(P^{k-1}) & \longrightarrow & \tilde{K}^{*+1}(E^k) & \longrightarrow & \tilde{K}^{*+2}(P^k) & \longrightarrow \cdots \\
 & & & \uparrow{\scriptstyle \delta_{k-1}} & & \uparrow{\scriptstyle \Delta_k} & & & \\
 & & & \tilde{K}^*(E^{k-1}) & \longrightarrow & \tilde{K}^*(X \times E^{k-1}) & & & \\
 & & & \uparrow{\scriptstyle \Delta^{(k-1)}} & & & & & \\
 & & & (\tilde{K}^*(X) \otimes \cdots \otimes \tilde{K}^*(X))^{*+2-k} & & & & &
\end{array}$$

We denote by P the submodule consisting of all primitive elements of $K^*(X)$ and by D the submodule consisting of all decomposable elements of $K^*(X)$. Then we have $\tilde{K}^*(X) = P \oplus D$ and because the rank of X is l, the rank of P and D are l and $2^l - l - 1$ respectively. We define the following modules.

$$\bar{S}_{k-1} = \sum_{i=1}^{k-1} \tilde{K}^*(X) \otimes \cdots \otimes \tilde{K}^*(X) \otimes \underset{\widehat{i}}{D} \otimes \tilde{K}^*(X) \otimes \cdots \otimes \tilde{K}^*(X),$$

$$\tilde{S}_{k-1} = \Delta^{(k-1)}(\bar{S}_{k-1}) \subseteq \tilde{K}^*(E^{k-1}),$$

$$S_{k-1} = \delta_{k-1}(\tilde{S}_{k-1}) \subseteq \tilde{K}^*(P^{k-1}).$$

Then the rank of $\tilde{S}_{k-1}$ is equal to rank $\bar{S}_{k-1} = (2^l - 1)^{k-1} - l^{k-1}$ and the rank of S_1 is equal to $2^l - l - 1$. We have the following lemma:

LEMMA 3.5. Coker $p_k^*\delta_{k-1}$ *is torsion free and isomorphic to*

$$\frac{\overbrace{P*\cdots*P}^{k}}{P*\cdots*P \cap \operatorname{Im} p_k^*\delta_{k-1}} \oplus \frac{\tilde{S}_k}{\tilde{S}_k \cap \operatorname{Im} p_k^*\delta_{k-1}}$$

PROOF. Take any element $x_{i_1}*\cdots*x_{i_{k-1}}$ from the module basis of $\tilde{K}^*(E^{k-1})$ given in Remark 2.3. As $p_k^*\delta_{k-1}=\delta_k(pr_{E^{k-1}})^*$, the formula (*) of Proposition 2.4 implies that $p_k^*\delta_{k-1}(x_{i_1}*\cdots*x_{i_{k-1}})$ can not be divisible by any integer greater than 1. Hence Coker $p_k^*\delta_{k-1}$ is torsion free. Since $\tilde{K}^*(E^k)\cong P*\cdots*P\oplus\tilde{S}_k$, the lemma follows. Q.E.D.

We mention that the above $\{p_k^*\delta_{k-1}\}$ represent the first differential d_1 of the spectral sequence $\{E_r^{*,*}, d_r\}$. Then Proposition 2.4 tells us that $\{E_1^{k,*}, d_1, k\geqq 0\}$ is the cobar construction of the exterior algebra on odd dimensional primitive elements, $k\leqq n-1$.

PROPOSITION 3.6.

$$d_1(\tilde{S}_{k+1}) \cap \tilde{S}_k = \operatorname{Ker} d_1 \cap \tilde{S}_k, \textit{ for } k\leqq n-1.$$

PROOF. We define a certain differential algebra. Let $(E^{1,*})^j$ be a submodule of $E^{1,*}=K^*(X)$ generated by $\{x_{i_1}\cdots x_{i_j} \mid$ each x_t is the primitive generator$\}$, and let

$$(E^{i,*})^j = \sum_{j_1+\cdots+j_i=j} (E^{1,*})^{j_1}*\cdots*(E^{1,*})^{j_i} \subseteq E^{i,*},$$

then $d_1\colon (E^{i,*})^j \to (E^{i+1,*})^j$ is

$$d_1(x_1*\cdots*x_i)=\begin{cases} \Sigma_{j=1}^{i}(-1)^j x_1*\cdots*(\Delta_2\overline{M}^*(x_j))*\cdots*x_i, & i\leqq n-1,\\ 0, \quad i\geqq n. \end{cases}$$

Let $\bar{E}^{1,*}$ be an exterior algebra of rank l on odd dimensional primitive generators with co-multiplication m. Denote by $\bar{P}$ the submodule of $\bar{E}^{1,*}$ generated by the primitive elements. $\{\bar{E}^{1,j}\}$ are defined to be the submodules of $\bar{E}^{1,*}$ generated by the following set;

$$\{\bar{x}_1\cdots\bar{x}_j \mid \text{each } \bar{x}_t \text{ is primitive}\}.$$

Define

$$\bar{E}^{i,j}=\begin{cases} \boldsymbol{Z}, \ i=0,\\ \sum_{j_1+\cdots+j_i=j} (\bar{E}^{1,j_1})\otimes\cdots\otimes(\bar{E}^{1,j_i}), \ i\geqq 1, \end{cases}$$

equipped with the differential d': $\bar{E}^{i,j} \to \bar{E}^{i+1,j}$ given by

$$d'(\bar{x}_1 \otimes \cdots \otimes \bar{x}_i) = \sum_{j=1}^{i} (-1)^j \bar{x}_1 \otimes \cdots \otimes \bar{x}_{j-1} \otimes (\bar{m}(\bar{x}_j)) \otimes \bar{x}_{j+1} \otimes \cdots \otimes \bar{x}_i,$$

where $\bar{m}(\bar{x}) = m(\bar{x}) - \bar{x} \otimes 1 - 1 \otimes \bar{x}$.

Consider the homomorphism ϕ: $\bar{E}^{i,j} \to (E^{i,*})^j$ given by

$$\phi(\bar{x}_1 \otimes \cdots \otimes \bar{x}_i) = \begin{cases} x_1 * \cdots * x_i, & i \leqq n, \\ 0, & i \geqq n+1. \end{cases}$$

Immediately we see that $d_1 \phi = \phi d'$ and ϕ is isomorphic for $i \leqq n$.

According to J. C. Moore [9] and A. Borel [1],

$$H(\bar{E}^{*,*}) \cong \mathrm{Ext}^{*,*}{}_{\Lambda(\bar{x}'_1, \ldots, \bar{x}'_l)} (\boldsymbol{Z}, \boldsymbol{Z})$$
$$\cong \boldsymbol{Z}[\bar{u}_{1,\ldots,}\bar{u}_l],$$

where $\bar{x}'_j$ is the dual to $\bar{x}_j$, and $\bar{u}_j$ is the image of $\bar{x}_j$ under the transgression. Therefore $\mathrm{Ker}\, d'/\mathrm{Im}\, d' \cong \boldsymbol{Z}[\bar{u}_1, \ldots, \bar{u}_l]$. On the other hand, $d'(\bar{P} \otimes \cdots \otimes \bar{P}) = 0$ and $(\bar{P} \otimes \cdots \otimes \bar{P}) \cap \mathrm{Im}\, d'$ is the module generated by the following set;

$$\{\bar{x}_{i_1} \otimes \cdots \otimes \bar{x}_{i_{j-1}} \otimes (\bar{x}_{i_j} \otimes \bar{x}_{i_{j+1}} - \bar{x}_{i_{j+1}} \otimes \bar{x}_{i_j}) \otimes \bar{x}_{i_{j+2}} \otimes \cdots \otimes \bar{x}_{i_k} : \text{each}$$
$$\bar{x}_t \text{ is a primitive generator}\}.$$

Thus we have the following

$$\Sigma_{k \geqq 0} \frac{\overbrace{\bar{P} \otimes \cdots \otimes \bar{P}}^{k}}{\bar{P} \otimes \cdots \otimes \bar{P} \cap \mathrm{Im}\, d'} \subseteqq \frac{\mathrm{Ker}\, d'}{\mathrm{Im}\, d'} = \boldsymbol{Z}[\bar{u}_1, \ldots, \bar{u}_l].$$

Let us determine the rank of above modules;

$$\mathrm{rank}\, \frac{\bar{P} \otimes \cdots \otimes \bar{P}}{\bar{P} \otimes \cdots \otimes \bar{P} \cap \mathrm{Im}\, d'} = \frac{(l+k-1)!}{(k-1)!l!}$$
$$= \mathrm{rank}\, \boldsymbol{Z}\{n_{i_1} \cdot \cdots \cdot u_{i_k} : 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq l\}$$

Therefore we obtain the following equation:

$$\Sigma_{k \geqq 0} \frac{\bar{P} \otimes \cdots \otimes \bar{P}}{\bar{P} \otimes \cdots \otimes \bar{P} \cap \mathrm{Im}\, d'} = \frac{\mathrm{Ker}\, d'}{\mathrm{Im}\, d'}$$

Since ϕ is an isomorphism for $k \leqq n$, we get

$$\frac{P*\cdots*P}{P*\cdots*P\cap \operatorname{Im} d_1}=\frac{P*\cdots*P\oplus\tilde{S}_k\cap \operatorname{Ker} d_1}{(P*\cdots*P\cap \operatorname{Im} d_1)\oplus(\tilde{S}_k\cap \operatorname{Im} d_1)}$$

and $\tilde{S}_k\cap \operatorname{Ker} d_1=\tilde{S}_k\cap \operatorname{Im} d_1$. Q.E.D.

§4. Proof of Theorem A

Let us recall the following basic theorem.

THEOREM 4.1 (E. Thomas [12]). *Let $x_1,\dots, x_k$ be A_k-primitive elements in $H^{\mathrm{odd}}(X; \boldsymbol{Q})\cong K^{-1}(X)\otimes\boldsymbol{Q}$, and $\{y_i\}$ be elements such that $\iota_2^*\cdots\iota_k^*(y_j)=s^*(x_j)$, $1\leqq j\leqq k$. Then the following formula holds:*

$$\delta_k(x_1*\cdots*x_k)=y_1\cdot\cdots\cdot y_k.$$

PROPOSITION 4.2. *If X satisfies the hypothesis in Theorem A, then the following hold.*

$(A)_k$ $K^*(P^k)\cong K_*^{[k+1]}[u_1,\dots, u_l]\oplus S_k$ *as modules,*

$(B)_k$ $p_k^*|_{S_{k-1}}$ *is monomorphic,*

$(C)_k$ $\delta_k(x_1*\cdots*x_k)=u_1\cdot\cdots\cdot u_k$,

where $\iota_2^*\cdots\iota_k^*(u_i)=s^*(x_i)$, $K^*(X)\cong\Lambda(x_1,\dots, x_l)$.

PROOF. As $K^*(SX)\cong\boldsymbol{Z}\oplus\delta_1(P)\oplus\delta_1(D)$ for $k=1$, we get $(A)_1$, $(C)_1$. The fact "P^0=one point space" implies $(B)_1$. Suppose that $(A)_i$, $(B)_i$ and $(C)_i$ hold for $i\leqq k$. The hypothesis for $K^*(X)$ implies

$$(p_{k+1})^*(K_*^{[k+1]}[u_1,\dots, u_l])=0,$$

$$\operatorname{Im}(p_{k+1})^*=\operatorname{Im}(p_{k+1})^*\delta_k.$$

Proposition 3.6 implies that

$$\operatorname{Ker}\delta_k\cap\tilde{S}_k=p_k{}^*(S_{k-1})\cap\tilde{S}_k=\operatorname{Ker}(p_{k+1})^*\delta_k\cap\tilde{S}_k.$$

Hence $(p_{k+1})^*|_{\delta_k(\tilde{S}_k)}$ is monomorphic. So far we have:

1) $K^*(P^k)=K_*^{[k+1]}[u_1,\dots, u_l]\oplus S_k$,

2) $(p_{k+1})^*(K_*^{[k+1]}[u_1,\dots, u_l])=0$,

3) $(p_{k+1})^*|_{S_k}$ is monomorphic.

Consequently $(B)_{k+1}$ is satisfied. It is clear by Lemma 3.5 that $\operatorname{Coker}(p_{k+1})^*=\operatorname{Coker} d_1$ is torsion free. On the other hand $K^*(P^{k+1})$ has no torsion. Let $x_1,\dots,$ x_l be A_{k+1}-primitive elements in $K^1(X)$, and $\{u_j\}$ be elements such that

$\iota_2^* \cdots \iota_{k+1}^*(u_j)=s^*(x_j)$, $1 \leqq j \leqq l$. Using naturality and injectivity of the Chern character on $K^*(P^k)$, Theorem 4.1 implies $(C)_{k+1}$. The following short exact sequence is valid:

$$(4.1) \qquad 0 \longrightarrow \operatorname{Coker}(p_{k+1})^* \longrightarrow K^*(P^{k+1}) \longrightarrow \operatorname{Ker}(p_{k+1})^* \longrightarrow 0$$

Moreover, by the above observation,

$$\operatorname{Coker}(p_{k+1})^* \cong K_*[u_1, \dots, u_l],$$

$$\operatorname{Ker}(p_{k+1})^* \cong K_*\{u_{i_1} \cdots u_{i_{k+1}} : \iota_2^* \cdots \iota_{k+1}^*(u_j)=s^*(x_j),\ 1 \leqq j \leqq l\} \oplus S_{k+1}.$$

Therefore we obtain an isomorphism $(A)_{k+1}$. Q. E. D.

Proof of Theorem A. Going back to the definition of $\{u_j\}$, it is readily seen that the isomorphism $(A)_k$ of Proposition 4.2 preserves multiplication. This completes the proof of Theorem A. Q. E. D.

§5. Proof of Theorem B

We first assume that the spectral sequence collapses. Proposition 2.5 implies that $\operatorname{Ker}\{d_1 : E_1^{1,1} \cong K^1(X) \to K^0(X) \cong E_1^{2,1}\}$ is the module of primitive elements previously denoted by P. Since $E_1^{1,1} \cong \cdots \cong E_\infty^{1,1}$, we get $d_r(P)=0$ for all r. This means that primitive elements are A_n-primitive. Conversely assume that $d_r(P)=0$ for $r \leqq n-1$. It follows from Theorem A that;

$$\operatorname{Im}((\iota_{k-r})^* \cdots \iota_k^*)=\operatorname{Im}(\iota_{k-r})^* \cong K_*^{[k-r]}[u_1, \dots, u_l],$$

for $k \leqq n$, $r \leqq k-1$, and

$$\operatorname{Ker}((\iota_{k-r})^* \cdots (\iota_{k-1})^*) \cap S_{k-1}=\operatorname{Ker}(\iota_{k-1})^* \cap S_{k-1}.$$

Hence we obtain

$$\begin{aligned} \operatorname{Ker}\bar{d}_r &= (\delta_{k-r})^{-1}((\iota_{k-r+1})^* \cdots (\iota_{k-1})^*(\operatorname{Ker} p_k^*)) \\ &= (\delta_{k-r})^{-1}(\operatorname{Im}(\iota_{k-r+1})^*)=\operatorname{Ker}\bar{d}_1, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}\bar{d}_r &= p_k^*(((\iota_{k-r+1})^* \cdots (\iota_{k-1})^*)^{-1}(\operatorname{Im}\delta_{k-r})) \\ &= p_k^*(\operatorname{Ker}((\iota_{k-r})^* \cdots (\iota_{k-1})^*)) \\ &= p_k^*(\operatorname{Ker}(\iota_{k-1})^*)=\operatorname{Im}\bar{d}_1. \end{aligned}$$

This completes the proof of Theorem B. Q. E. D.

PROOF of COROLLARY. The proof of Proposition 3.6 and Theorem 4.1, allow us to calculate the E_2-term of the Stasheff spectral sequence. We get the following

$$E_2^{*,*} \cong K_*^{[n+1]}[u_1,\dots, u_l] \oplus S_n.$$

Since every u_j is of even total degree, we get

$$d_r(u_j)=0, \quad 2 \leqq r \leqq n-2, \quad 1 \leqq j \leqq l. \qquad \text{Q. E. D.}$$

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