Lemma 7.1 *The following two conditions on R_n are equivalent.*
1) There is a homotopy section of $p : R_n \to B\vee\widetilde{R_n}$.
2) R_n admits a co-H-structure.

Now we show the existence of a homotopy section of $p : B\vee\widetilde{R_n} \to R_n$. We define a map $s_0 : B\vee A_n \to B\vee\widetilde{B\vee A_n} \simeq B\vee \bigvee_{i\in\mathbb{Z}} \tau^i{\cdot}A_n$ as follows:

$$s_0|_B = \mathrm{in}_B : B \to B\vee \bigvee_{i\in\mathbb{Z}} \tau^i{\cdot}A_n,$$
$$s_0|_{A_n} : A_n \overset{\{f,g\}}{\to} A_n\vee A_n \overset{\widetilde{\psi(\tau)}\vee 1_{An}}{\longrightarrow} \tau{\cdot}A_n\vee A_n \overset{(\Psi(\tau^{-1})\circ\mathrm{in}_{\tau\cdot An})\vee 1_{An}}{\longrightarrow} B\vee\tau{\cdot}A_n\vee A_n \subset B\vee \bigvee_{i\in\mathbb{Z}} \tau^i{\cdot}A_n.$$

By (6.1), we have $p_0\circ s_0 \sim 1_B\vee(f+g) \sim 1_B\vee 1_{A_n} = 1_{B\vee A_n}$. Since $n \geq 4$, it follows that $\pi_{n+4}(A_n\vee A_n) \cong \pi_{n+4}(A_n)\oplus\pi_{n+4}(A_n)$ for dimensional reasons. By Proposition 6.1, we have

$$s_0\circ\Sigma^{n-3}\alpha \sim \mathrm{in}_{A_n}\circ\Sigma^{n-3}\alpha : S^{n+4} \to B\vee A_n \subset B\vee \bigvee_{i\in\mathbb{Z}} \tau^i{\cdot}A_n,$$
$$s_0\circ\Sigma^{n-3}\beta \sim \Psi(\tau^{-1})\circ\mathrm{in}_{\tau\cdot A_n}\circ\widetilde{\psi(\tau)}\circ\Sigma^{n-3}\beta : S^{n+4} \to A_n \to \tau{\cdot}A_n \to B\vee\tau{\cdot}A_n \subset B\vee \bigvee_{i\in\mathbb{Z}} \tau^i{\cdot}A_n.$$

Hence we obtain that

$$s_0\circ(\Sigma^{n-3}\alpha + \psi(\tau)\circ\Sigma^{n-3}\beta) = s_0\circ\Sigma^{n-3}\alpha + \Psi(\tau)\circ s_0\circ\Sigma^{n-3}\beta$$
$$\sim \mathrm{in}_{A_n}\circ\Sigma^{n-3}\alpha + \mathrm{in}_{\tau\cdot A_n}\circ\widetilde{\psi(\tau)}\circ\Sigma^{n-3}\beta = \mathrm{in}_{A_n\vee\tau\cdot A_n}\circ(\Sigma^{n-3}\alpha + \widetilde{\psi(\tau)}\circ\Sigma^{n-3}\beta).$$

Thus the map $s_0\circ(\Sigma^{n-3}\alpha + \psi(\tau)\circ\Sigma^{n-3}\beta)$ is homotopic to the attaching map of the cell $1{\cdot}e^{n+5}$. Hence it induces a map $s : R_n \to B\vee\widetilde{R_n}$ so that $p\circ s$ is clearly the identity up to homotopy.

By Lemma 7.1, we obtain the following theorem.

Theorem 7.2 *R_n is a co-H-space.*

8 Unsplittability of R_n

In this section, we show that R_n is not standard. We state the following well-known result:

Proposition 8.1 *The set of invertible elements in the group ring $\mathbb{Z}\pi$ is $\pm\pi \subset \mathbb{Z}\pi$.*

Proof. Since π is the infinite cyclic group, $\mathbb{Z}\pi$ is isomorphic with $\mathbb{Z}[x, \frac{1}{x}]$ the ring of Laurent polynomials with coefficients in $\mathbb{Z}$. We can express each Laurent polynomial in the form $x^i(a_\ell x^\ell + a_{\ell-1}x^{\ell-1} + ... + a_1x^1 + a_0)$ with $a_\ell a_0 \neq 0$, $\ell \geq 0$ and $i \in \mathbb{Z}$. If the product of any two such Laurent polynomials, say $x^i(a_\ell x^\ell + ... + a_0)$ and $x^j(b_m x^m + ... + b_0)$, is equal to the unity, then we have that $i + j = \ell = m = 0$ and $a_0b_0 = 1$. Hence every invertible element can be expressed as $\pm x^i$ for some $i \in \mathbb{Z}$. *qed.*