Relative L-S category and categorical length

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ALGEBRAIC TOPOLOGY: OLD AND NEW
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L-S category

Definition (Lusternik-Schnirelmann)

$$\mathrm{cat}(M) = \mathrm{Min}\left\{ m \geq 0 \;\middle|\; \exists\{A_0, \ldots, A_m\,;\ \text{closed in } M\} \right\}$$

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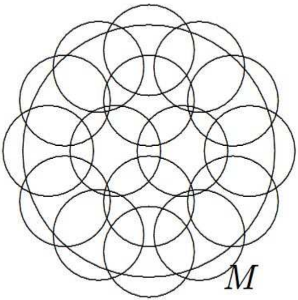

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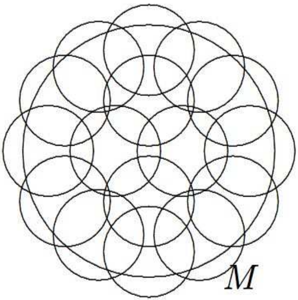

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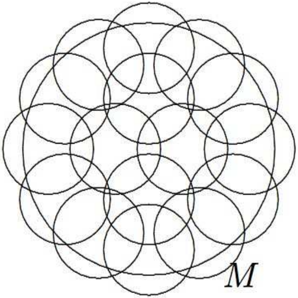

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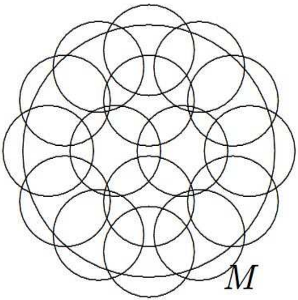

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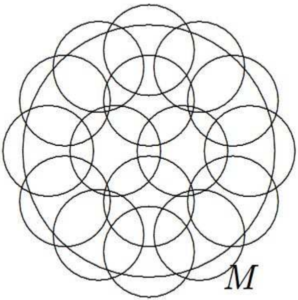

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$$\mathrm{cup}(X; R) = \mathrm{Min}\left\{ m{\geq}0 \,\middle|\, \forall_{u_0,...,u_m \in H^*(M;R)}\, u_0 \cdot u_1 \cdots u_m = 0 \right\}$$

Element of Hopf invariant one

Let us recall the following classical result: if an n-sphere is a Hopf space, then there must be a Hopf invariant one element in $\pi_{2n+1}(S^{n+1})$. The first non-trivial case, when $n = 15$ was solved in negative by Toda and

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Berstein-Hilton's criterion

Let us consider the following 2-cell complexes:

Example

1. $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$, $\mathbb{O}P^2$ — *(projective planes)*
2. $Q_2 = S^3 \cup_\omega e^7 \subset Sp(2)$ — *James' quasi-projective plane*

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Higher Hopf invariants

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For a map f from S^q to a space X with $\mathrm{cat}(X) = m$,

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For any X, the (based) loop space $\Omega(X)$ of X admits a natural A_∞-structure, a sequence of fibrations over projective spaces $P^m\Omega(X)$ with fibre $\Omega(X)$,

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Moreover, we can show the following theorem using this definition.

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$$H_m(f) \subset [\Sigma V, E^{m+1}\Omega(X)] \cong \ker\left\{(e^X_m)_* : [\Sigma V, P^m\Omega(X)] \to [\Sigma V, X]\right\}.$$

Moreover, we can show the following theorem using this definition.

Theorem (I)

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To understand these intricate ideas among relative L-S categories and a categorical sequence, we introduce a unified version of a relative L-S category, which explains when the categorical length goes up by one.

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$$\mathrm{cat}(X; K, L{:}A) \le \mathrm{cat}(X; K, A{:}A) \le \mathrm{cat}(X; L, A{:}A) + \mathrm{cat}(X; K, L{:}A).$$
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Corollary

1. *Let* $(X; K, L{:}*)$ *be a triad of maps from* $*$. *Then we obtain*
$$\mathrm{cat}(X; K, L{:}*) \le \mathrm{cat}^{\mathrm{BG}}(X, K) \le \mathrm{cat}^{\mathrm{BG}}(X, L) + \mathrm{cat}(X; K, L{:}*),$$
$$\mathrm{cat}(X; K, L{:}*) \le \mathrm{cat}^{\mathrm{AL}}(X, L) \le \mathrm{cat}^{\mathrm{AL}}(X, K) + \mathrm{cat}(X; K, L{:}*).$$
2. *For a pair* $(X, L{:}A)$ *of maps from* A, *we obtain*
$$\mathrm{cat}(X; X, L{:}A) \le \mathrm{cat}^{\mathrm{FH}}(X, A) \le \mathrm{cat}(X; L, A{:}A) + \mathrm{cat}^{\mathrm{FH}}(X, L).$$

A categorical sequence and a cone decomposition 1

Let us go back to Fox's paper on L-S category of a space X:

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Theorem

Let $(X; K, L{:}A)$ *be a triad of maps from* A*,* V *be a co-loop co-H-space and* $\alpha : V \to K$ *be a map such that* $X \supset \hat{K} = K \cup_\alpha CV \supset K$*. If* $\mathrm{cat}(X; K, L{:}A) \leq m$ *and* $H_m^{(X;K,L:A)}(\alpha) = 0$*, then* $\mathrm{cat}(X; \hat{K}, L{:}A) \leq m$*.*

Assume that there is a categorical sequence $\{F_i\,;\, 0 \leq i \leq m\}$ of X with $(\hat{K}, K, L) = (F_{m+1}, F_m, F_{m-1})$.

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Let $(X; K, L{:}A)$ be a triad of maps from A, V be a co-loop co-H-space and $\alpha : V \to K$ be a map such that $X \supset \hat{K} = K \cup_{\alpha} CV \supset K$. If $\mathrm{cat}(X; K, L{:}A) \leq m$ and $H_m^{(X;K,L:A)}(\alpha) = 0$, then $\mathrm{cat}(X; \hat{K}, L{:}A) \leq m$.

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So, using the inequalities among relative L-S categories, we see that the higher Hopf invariant determines when a cone decomposition gives a categorical sequence.

For example, we can obtain a sharper upper bound of L-S category of Spin(9).

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