Recent Progress on Lusternik-Schnirelmann Category

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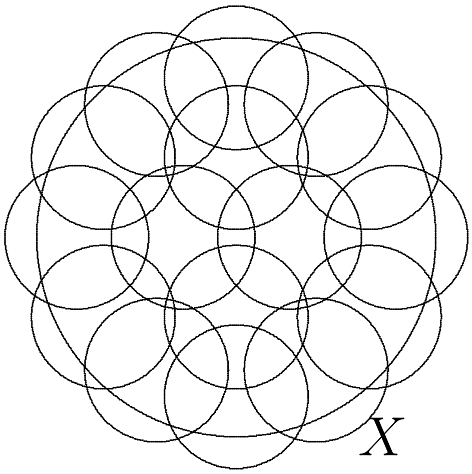
1 What is the L-S category of a space X



Definition 1.1 *Let X be a topological space.*

$$\operatorname{cat} X = \operatorname{Min}\left\{ m \geq 0 \;\middle|\; \begin{array}{l} \exists\{U_0, ..., U_m\,;\ \text{open in } X\} \\ X = \bigcup_{i=0}^{m} U_i,\ \text{each } U_i \text{ is contractible } \underline{\mathit{in}}\ X \end{array} \right\}$$

Let M be a compact closed manifold.

$$\operatorname{Crit} M = \operatorname{Min}\left\{ \#\{\textit{critical pts of } f\} \;\middle|\; \begin{array}{l} f : M \to \mathbb{R} \\ \textit{is a } C^{\infty}\textit{-map} \end{array} \right\}$$

Theorem 1.2 (Lusternik-Schnirelmann 1934)

$$\operatorname{Crit} M \geq \operatorname{cat} M + 1.$$

Definition 1.3 *A topological invariant* $g\mathrm{cat}\,X$ *is defined similarly but is not a homotopy invariant (R.H.Fox):*

$$g\mathrm{cat}\,X = \mathrm{Min}\left\{ m \geq 0 \;\middle|\; \begin{array}{l} \exists\{U_0,...,U_m\,;\ \text{open in } X\} \\ X = \bigcup_{i=0}^{m} U_i, \ \begin{array}{l}\text{each } U_i \text{ is con-} \\ \text{tractible}\end{array} \end{array} \right\}$$

Ganea modified $g\mathrm{cat}$ *and obtained the strong category:*

$$\mathrm{Cat}\,X = \mathrm{Min}\left\{ m \geq 0 \mid \exists_{\{Y(\simeq X)\}}\, g\mathrm{cat}\,Y = m \right\}$$

Theorem 1.4 (Ganea 1971)

$$\mathrm{Cat}\,X - 1 \leq \mathrm{cat}\,X \leq \mathrm{Cat}\,X \leq g\mathrm{cat}\,X.$$

Remark 1.5 *If* $g\mathrm{cat}\,X = \mathrm{Cat}\,X$ *were true in general, we could obtain Poincaré conjecture.*

Remark 1.6 *For* M *a manifold, Singhof pointed out that*

$$g\mathrm{cat}\,M + 1 \leq \mathrm{Crit}\,M.$$

2 Bar Spectral Sequence and Category Weight

For any space X and a cohomology theory h^*, there is a filtration $h^*(X) = F^{-1} \supset \widetilde{h}^*(X) = F^0 \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots$ and an associated bar spectral sequence

$$\{E_r^{*,*}, d_r^{*,*}\} \quad \text{s.t.} \quad \begin{cases} E_\infty^{s,*} \cong F^s/F^{s+1}, \\ E_2^{*,*} \cong \mathrm{Ext}_{h_*(\Omega X)}(h_*, h_*). \end{cases}$$

Theorem 2.1 (Whitehead 1957, Ginsburg 1963) *If* $\mathrm{cat}\, X \leq m$, *then* $E_\infty^{s,t} = 0$ *and* $d_r = 0$ *for any* $s, r > m$.

Fadell and Husseini (1992) introduced a topological invariant, *a category weight*, which is refined to be a homotopy invariant:

Definition 2.2 (Rudyak 1997, Strom 1997)

$$\mathrm{wgt}(u) = \mathrm{Max}\{m \geq 0 \,|\, u \in F_m\} \quad \textit{for } u \in \widetilde{h}^*(X).$$

which satisfies the following inequalities for $u, v \neq 0$.

(1) $\mathrm{wgt}(u) \leq \mathrm{cat}\, X$ *(2)* $\mathrm{wgt}(u) + \mathrm{wgt}(v) \leq \mathrm{wgt}(uv)$

Theorem 2.3 (Floer 1989, Hofer 1988, Rudyak 1999)

Let (M, ω) be a symplectic manifold with $\pi_2(M) = 0$ (or $I_\omega = 0 = I_c$) and $\phi : M \to M$ a Hamiltonian Symplectomorphism. Then $\mathrm{Fix}\, \phi \geq 1 + \text{cup-length}(M) = \mathrm{Crit}(M)$.

3 Higher Hopf invariants on the A_∞-structure

Berstein and Hilton (1960) introduced a notion of higher Hopf invariants, which is redefined using A_∞-structure of ΩX to see the relation between unstable and stable higher Hopf invariants:

Definition 3.1 (I 2002) *Let X be a space with* $\mathrm{cat}\, X = m$ *and V a co-H-space. The higher Hopf invariants are*

$$\begin{cases} H_m(\alpha) = \{H^\sigma_m(\alpha) \,|\, \sigma \textit{ is a structure of } \mathrm{cat}\, X = m\} \\ \mathcal{H}_m(\alpha) = \{\Sigma^\infty H^\sigma_m(\alpha) \,|\, \sigma \textit{ is a structure of } \mathrm{cat}\, X = m\} \end{cases}$$

where $H^\sigma_m : [\Sigma V, X] \to [\Sigma V, \Omega X * \cdots * \Omega X]$ *(I 1997) is a homomorphism depending on* σ *a structure of* $\operatorname{cat} X = m$.

Theorem 3.2 (I 1998) *There is a sequence of simply-connected two-cell complexes* $\{Q_\ell\,;\, \ell$ *a prime* $\geq 2\}$ *such that*

$$\begin{cases} \operatorname{cat}(Q_2 \times S^n) = \operatorname{cat} Q_2 \textit{ for all } n \geq 1 \\ \operatorname{cat}(Q_\ell \times S^n) = \operatorname{cat} Q_\ell \textit{ for all } n \geq 2 \textit{ and } \ell > 2 \end{cases}$$

Theorem 3.3 (I 2002) *Let* X *be* $(d-1)$*-connected with* $\dim X \leq d \operatorname{cat} X + d - 2$. *Let* $W = X \cup_\alpha D^{e+1}$ *with* $\operatorname{cat} W = \operatorname{cat} X + 1$ *and* $e \geq d$. *Then* $\operatorname{cat}(W \times S^n) = \operatorname{cat} W + 1$ *for all* $n \geq 1$ *if and only if* $\mathcal{H}_m(\alpha) \neq 0$, *where* $m = \operatorname{cat} X$.

Theorem 3.4 (I 2002) *There are simply-connected closed manifolds* M *and* N *such that*

$$\begin{cases} \operatorname{cat}(M \times S^n) = \operatorname{cat} M \textit{ for all } n \geq 2 \\ \operatorname{cat}(N \smallsetminus \{*\}) = \operatorname{cat} N \end{cases}$$

4 Ganea's problems

Problems 4.1 (T. Ganea, 1971, (15 problems))

[1] Compute L-S category for familiar manifolds.

[2] Is $\mathrm{cat}(X \times S^n) = \mathrm{cat}\, X + 1$ *true for any finite complex* X *and any* $n \geq 1$*?*

[4] Let $S^r \hookrightarrow E \to S^{t+1}$ *be a bundle. Describe* $\mathrm{cat}\, E$ *in terms of homotopy invariants of the characteristic map.*

[8] Let $X = S^3 \cup e^{2p+1}$*. Is* $\mathrm{Cat}(X \times X)$ *equal to* $\mathrm{cat}(X \times X)$*?*

[10] Is any co-H-space X *(i.e,* $\mathrm{cat}\, X = 1$*) of homotopy type of* $S^1 \vee \cdots \vee S^1 \vee Y$ *with* $\pi_1(Y) = 0$*?*

...

[O] For any closed manifold M, $\mathrm{cat}(M \smallsetminus \{*\}) = \mathrm{cat}\, M - 1$*?*

Theorem 4.2 (James 1978) *Let X be $(d-1)$-connected. Then* $\operatorname{cat} X \leq \dfrac{\dim X}{d}$ *(or $d \operatorname{cat} X \leq \dim X$).*

Theorem 4.3 (Singhof 1979, Rudyak 1997) *Let M be a closed manifold. If $\operatorname{cat} M \geq \dfrac{\dim M + 3}{2}$, then M satisfies $\operatorname{cat}(M \times S^n) = \operatorname{cat} M + 1$ for all $n \geq 1$.*

Gómez-Larrañaga and Gonzalez-Acuna (1992) and Oprea and Rudyak (to appear) give an answer to Problems 2 and O:

Theorem 4.4 *For M a closed 3-manifold, we have*

$$\begin{cases} \operatorname{cat}(M \times S^n) = \operatorname{cat} M + 1 \text{ for all } n \geq 1 \\ \operatorname{cat}(M \smallsetminus \{*\}) = \operatorname{cat} M \end{cases}$$

Example 4.5 *(1) Let $X = G_2$ the exceptional Lie group of rank 2. Then $H^*(G_2; \mathbb{F}_2) \cong P[x_3]/(x_3^4) \otimes \Lambda(x_5)$ with* $\operatorname{wgt}(x_3) = \operatorname{wgt}(x_5) = 1$. *Thus* $\operatorname{cat}(G_2) \geq \operatorname{wgt}(x_3^3 x_5) \geq 4$, *and hence $\operatorname{cat} G_2 = 4$ by Theorems 4.2 and 2.1.*

(2) Let $X = Sp(2)$. Then $H^*(Sp(2)) \cong \Lambda(x_3, x_7)$ with $\mathrm{wgt}(x_3) = \mathrm{wgt}(x_7) = 1$. Hence $\mathrm{cat}(Sp(2)) \geq \mathrm{wgt}(x_3 x_7) \geq 2$. But Schweitzer (1965) has shown using secondary cohomology operations that $\mathrm{cat}(Sp(2)) = 3 \neq 2$. Instead of using H^*, we might obtain $\mathrm{wgt}(x_3^2) = 2, \mathrm{wgt}(x_3^3) = 3$ using some other cohomology theory.

Question 4.6 *How can we know that $x_3^2 \neq 0$?*

5 Ganea's Problems and Hopf Invariants

Singhof answered to Ganea's Problem 1 as follows.

Theorem 5.1 (Singhof 1975)

$$\mathrm{cat}(SU(n)) = n - 1 \text{ and } \mathrm{cat}(U(n)) = n \text{ for } n \geq 1.$$

Extending the result of Schweitzer 1965, Singhof (1976) proved

Theorem 5.2 $\mathrm{cat}(Sp(n)) \geq n+1$ *for* $n \geq 2$.

Theorem 5.3 (I unpublished) *Let* $\alpha : S^6 \to S^3$ *be the attaching map of* 7*-cell in* $Sp(2)$*. Then* $x_3^2 = \mathcal{H}_1^h(\alpha)\cdot x_7$ *in* $h^*(Sp(2))$*, where* $\mathcal{H}_1^h$ *is given by*

$$\mathcal{H}_1^h : \pi_6(S^3) \xrightarrow{H_1} \pi_6(\Omega S^3 * \Omega S^3) \cong \pi_6(S^5) \xrightarrow{\Sigma^\infty} \pi_{\mathcal{S}}^{-1} \to h^{-1} \subset h^*,$$

where H_1 *is the Hopf invariant.*

This is an answer to Question 4.6. Extending the observation on generalised cohomology theory and Hopf invariants, we obtain

Theorem 5.4 (Mimura-I) $\mathrm{cat}(Sp(n)) \geq n+2$ *for* $n \geq 3$.

The following result is obtained independently to Theorem 5.4 by Fernández-Suárez, Gómez-Tato, Tanré and Strom.

Theorem 5.5 (F-G-T-S, M-I) $\mathrm{cat}(Sp(3)) = 5$.

Theorem 5.6 (Arkowitz-Stanley, to appear) *For a simply-connected co-H-space X, we have* $\mathrm{Cat}(X \times X) = 2 = \mathrm{cat}(X \times X)$*, which answers Problem 8.*

To Ganea's conjecture on co-H-spaces (Problem 10), we have

Theorem 5.7 (Saito-Sumi-I 1997) *Let X be a co-H-space. If $H_*(X)$ concentrated in dimensions 1, $n+1$ and $n+2$ and $H_{n+2}(X)$ has no torsion, then Ganea's conjecture on co-H-spaces (Problem 10) for X is true.*

Theorem 5.8 (I 1998) *There exists a sequence of co-H-spaces $\{R_n\,;\, n \geq 4\}$ each of which gives a counter-example to Ganea's conjecture on co-H-spaces.*

Theorem 5.9 (Hubbuck-I, to appear) *A p-completed version of Ganea's conjecture on co-H-spaces is true.*

Theorem 5.10 (I, to appear) *For E the total space of S^r-bundle over S^{t+1}, $\operatorname{cat} E$ is given as follows (Problem 4):*

<table>
<tr><th colspan="3">Conditions</th><th colspan="4">L-S category</th></tr>
<tr><th>r</th><th>t</th><th>α</th><th>$Q\times S^n$</th><th>Q</th><th>E</th><th>$E\times S^n$</th></tr>
<tr><td rowspan="5">$r=1$</td><td>$t=0$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="3">$t=1$</td><td>$\alpha=\pm 1$</td><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>$\alpha=0$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>otherwise</td><td>3</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>$t>1$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="6">$r>1$</td><td>$t<r$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="2">$t=r$</td><td>$\alpha=\pm 1$</td><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>$\alpha\neq\pm 1$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="3">$t>r$</td><td>$H_1(\alpha)=0$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>$H_1(\alpha)\neq 0$
$\Sigma^r H_1(\alpha)=0$</td><td rowspan="2">3 or 2
(1)</td><td rowspan="2">2</td><td>2</td><td>3</td></tr>
<tr><td>$\Sigma^r H_1(\alpha)\neq 0$</td><td>3</td><td>3 or 4
(2)</td></tr>
</table>

$$(1)\ \begin{cases} \Sigma^n H_1(\alpha)=0 \implies \operatorname{cat} Q\times S^n = 2, \\ \Sigma^{n+1} H_1(\alpha)\neq 0 \implies \operatorname{cat} Q\times S^n = 3. \end{cases}$$

$$(2)\ \begin{cases} \Sigma^{r+n} H_1(\alpha)=0 \implies \operatorname{cat} E\times S^n = 3, \\ \Sigma^{r+n+1} h_2(\alpha)\neq 0 \implies \operatorname{cat} E\times S^n = 4, \end{cases}$$

where α is the attaching map of $t+1$-cell of E and $Q = E \smallsetminus \{*\} \simeq S^r \cup_\alpha e^{t+1}$.

References

[1] M. Arkowitz, *Co-H-spaces*, In:"Handbook of algebraic topology", Ioan James; 1143-1173, North Holland, Amsterdam, 1995.

[2] I. Berstein and E. Dror, *On the homotopy type of non-simply connected co-H-space*, Ill. Jour. Math. **20** (1976), 528–534.

[3] I. Berstein and P. J. Hilton, *Category and generalised Hopf invariants*, Illinois. J. Math. **12** (1960), 421–432.

[4] J. M. Boardman and B. Steer, *On Hopf invariants*, Comment. Math. Helv. **42** (1967), 180–221.

[5] O. Cornea, Strong LS category equals cone-length, Topology **34** (1995), 377–381.

[6] O. Cornea, *Cone-length and Lusternik-Schnirelmann category*, Topology **33** (1994), 95–111.

[7] O. Cornea, *There is just one rational cone-length*, Trans. Amer. Math. Soc. **344** (1994), 835–848.

[8] C. R. Curjel, *A note on spaces of category* ≤ 2, Math. Z. **80** (1963), 293–299.

[9] B. Eckmann and P. Hilton, *Décomposition homologique d'un polyèdre simplement connexe*, C. R. Acad. Sci. Paris **248** (1959), 2054–2056.

[10] S. Eilenberg and T. Ganea, *On the Lusternik-Schnirelmann category of abstract groups*, Ann. of Math. **65** (1957), 517-518.

[11] A. Floer, *Cuplength estimates on Lagrangian intersections*, Comm. Pure Appl. Math. **42** (1989), 335–356.

[12] R. H. Fox, *On the Lusternik-Schnirelmann category*, Ann. of Math. (2) **42**, (1941), 333–370.

[13] T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois. J. Math. **11** (1967), 417–427.

[14] T. Ganea, *Cogroups and suspensions*, Invent. Math. **9** (1970), 185–197.

[15] T. Ganea, *Some problems on numerical homotopy invariants*, Symposium on Algebraic Topology, Lect. Notes in Math. **249**, Springer Verlag, Berlin (1971) 13–22.

[16] M. Ginsburg, *On the Lusternik-Schnirelmann category*, Ann. of Math. 77 (1963), 538–551.

[17] J. C. Gómez-Larrañaga and F. González-Acuña, *Lusternik-Schnirelmann category of 3-manifolds*, Topology **31** (1992), 791–800.

[18] W. Henn, *On almost rational co-H-spaces*, Proc. Amer. Math. Soc. **87**, (1983), 164–168.

[19] K. P. Hess, *A proof of Ganea's Conjecture for Rational Spaces*, Topology **30** (1991), 205–214.

[20] P. Hilton, G. Mislin and J. Roitberg, *On co-H-spaces*, Comment. Math. Helv. **53** (1978), 1–14.

[21] H. Hofer, *Lusternik-Schnirelman-theory for Lagrangian intersections*, Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), 465–499.

[22] J. R. Hubbuck and N. Iwase, *A p-complete version of the Ganea conjecture on co-H-spaces*, Proceedings of the Joint Summer Research Conference on Lusternik-Schnirelmann Category in the New Millennium (Mt.Holyoke, 2001), *Cont. Math.*, to appear.

[23] N. Iwase, *The Ganea conjecture on Co-H-spaces*, Topology, **40** (2001), 223–234.

[24] N. Iwase, *Projective spaces associated with an A_∞-structure and Ganea's conjecture on LS category*, GSM Kyushu U. preprint series **1997-17** (1997).

[25] N. Iwase, *Ganea's conjecture on Lusternik-Schnirelmann category*, Bull. Lon. Math. Soc., **30** (1998), 623–634.

[26] N. Iwase, *A_∞-method in Lusternik-Schnirelmann category*, Topology **41** (2002), 695–723.

[27] N. Iwase, *Lusternik-Schnirelmann category of a sphere-bundle over a sphere*, Topology, to appear.

[28] N. Iwase, S. Saito and T. Sumi, *Homology of the universal covering space of a co-H-space*, Trans. Amer. Math. Soc., **351** (1999), 4837–4846.

[29] I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17** (1978), 331-348.

[30] I. M. James, *Lusternik-Schnirelmann Category*, In: "Handbook of algebraic topology", Ioan James; 1293-1310, North Holland, Amsterdam, 1995.

[31] B. Jessup, *Rational L-S category and a conjecture of Ganea*, Trans. Amer. Math. Soc. **317** (1990), 655-660.

[32] K. Komatsu, *A boundary link is trivial if the Lusternik-Schnirelmann category of its complement is one*, Osaka J. Math. **29** (1992), 329–337.

[33] K. Komatsu and T. Matumoto, *Knot-link theory and Lusternik-Schnirelmann category*, Algebra and topology 1992 (Taejŏn), 211–216, Korea Adv. Inst. Sci. Tech., Taejŏn, 1992.

[34] P. Lambrechts, D. Stanley and L. Vandembroucq, *Embedding up to homotopy of two-cones into Euclidean space*, preprint.

[35] T. Matumoto, *Lusternik-Schnirelmann category and knot complement*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37** (1990), 103–107.

[36] T. Matumoto, *Lusternik-Schnirelmann category and knot complement. II*, Topology **34** (1995), 177–184.

[37] L. Montejano, *A quick proof of Singhof's* $\mathrm{cat}(M{\times}S^1) = \mathrm{cat}(M) + 1$ *theorem*, Manuscripta Math. **42** (1983), 49–52.

[38] J. Oprea and Y. Rudyak, *Detecting elements and Lusternik-Schnirelmann category of 3-manifolds*, Proceedings of the Joint Summer Research Conference on Lusternik-Schnirelmann Category in the New Millennium (Mt.Holyoke, 2001), *Cont. Math.*, to appear.

[39] Y. B. Rudyak, *On the Ganea conjecture for manifolds*, Proc. Amer. Math. Soc. **125** (1997), 2511–2512.

[40] Y. B. Rudyak, *On category weight and its applications*, Topology **38** (1999), 37–55.

[41] S. Saito, *On higher coassociativity*, Hiroshima Math. J. **6** (1976), 589–617.

[42] W. Singhof, *Minimal coverings of manifolds with balls*, Manuscripta Math. **29** (1979), 385–415.

[43] D. Stanley, *Spaces with Lusternik-Schnirelmann category n and cone length $n+1$*, preprint.

[44] J. D. Stasheff, *Homotopy associativity of H-spaces, I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312.

[45] L. Vandembroucq, *Adjunction spaces satisfying the Ganea conjecture*, preprint.

[46] G. W. Whitehead, *The homology suspension*, In: "Colloque de topologie algébrique, Louvain, 1956", Georges Thone, Liége; 89–95, Masson & Cie, Paris, 1957.