1 What is the L-S category of a space $X$

Definition 1.1  Let $X$ be a topological space.

\[
\text{cat } X = \min \left\{ m \geq 0 \mid \exists \{U_0, \ldots, U_m\}; \text{open in } X \right\}
\]
\[
X = \bigcup_{i=0}^{m} U_i, \text{ each } U_i \text{ is contractible in } X
\]

Let $M$ be a compact closed manifold.

\[
\text{Crit } M = \min \left\{ \# \{\text{critical pts of } f\} \mid f : M \to \mathbb{R} \text{ is a } C^\infty\text{-map} \right\}
\]

Theorem 1.2 (Lusternik-Schnirelmann 1934)

\[
\text{Crit } M \geq \text{cat } M + 1.
\]
Definition 1.3  A topological invariant $g\text{cat}X$ is defined similarly but is not a homotopy invariant (R.H. Fox):

$$g\text{cat}X = \min \left\{ m \geq 0 \mid \exists \{U_0, \ldots, U_m\}; \text{open in } X \right\}$$

$X = \bigcup_{i=0}^{m} U_i$, each $U_i$ is contractible

Ganea modified $g\text{cat}$ and obtained the strong category:

$$\text{Cat}X = \min \left\{ m \geq 0 \mid \exists \{Y(\simeq X)\}; g\text{cat}Y = m \right\}$$

Theorem 1.4 (Ganea 1971)

$$\text{Cat}X - 1 \leq \text{cat}X \leq \text{Cat}X \leq g\text{cat}X.$$ 

Remark 1.5  If $g\text{cat}X = \text{Cat}X$ were true in general, we could obtain Poincaré conjecture.

Remark 1.6  For $M$ a manifold, Singhof pointed out that

$$g\text{cat}M + 1 \leq \text{Crit}M.$$
2 Bar Spectral Sequence and Category Weight

For any space $X$ and a cohomology theory $h^*$, there is a filtration $h^*(X) = F^{-1} \supset \tilde{h}^*(X) = F^0 \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots$ and an associated bar spectral sequence

$$\{E_{r,*}, d_{r,*}\} \quad \text{s.t.} \quad \begin{cases} E_s^{s,*} & \cong F^s / F^{s+1}, \\ E_2^{*,*} & \cong \text{Ext}_{h_*(\Omega X)}(h_*, h_*). \end{cases}$$

**Theorem 2.1 (Whitehead 1957, Ginsburg 1963)** If $\text{cat} \; X \leq m$, then $E_{s,t}^{s,t} = 0$ and $d_r = 0$ for any $s, r > m$.

Fadell and Husseini (1992) introduced a topological invariant, a category weight, which is refined to be a homotopy invariant:

**Definition 2.2 (Rudyak 1997, Strom 1997)**

$$\text{wgt}(u) = \text{Max}\{m \geq 0 \mid u \in F_m\} \quad \text{for} \; u \in \tilde{h}^*(X).$$

which satisfies the following inequalities for $u, v \neq 0$. 
Theorem 2.3 (Floer 1989, Hofer 1988, Rudyak 1999)

Let $(M, \omega)$ be a symplectic manifold with $\pi_2(M) = 0$ (or $I_\omega = 0 = I_c$) and $\phi : M \to M$ a Hamiltonian Symplectomorphism. Then $\text{Fix} \phi \geq 1 + \text{cup-length}(M) = \text{Crit}(M)$.

3 Higher Hopf invariants on the $A_\infty$-structure

Berstein and Hilton (1960) introduced a notion of higher Hopf invariants, which is redefined using $A_\infty$-structure of $\Omega X$ to see the relation between unstable and stable higher Hopf invariants:

**Definition 3.1 (I 2002)** Let $X$ be a space with $\text{cat} X = m$ and $V$ a co-$H$-space. The higher Hopf invariants are

$$\begin{align*}
H_m(\alpha) &= \{ H_m^\sigma(\alpha) \mid \sigma \text{ is a structure of cat } X = m \} \\
\mathcal{H}_m(\alpha) &= \{ \Sigma^\infty H_m^\sigma(\alpha) \mid \sigma \text{ is a structure of cat } X = m \}
\end{align*}$$
where $H_m^\sigma : [\Sigma V, X] \to [\Sigma V, \Omega X \ast \cdots \ast \Omega X]$ (I 1997) is a homomorphism depending on $\sigma$ a structure of $\text{cat} X = m$.

**Theorem 3.2 (I 1998)** There is a sequence of simply-connected two-cell complexes $\{Q_\ell; \ell \text{ a prime } \geq 2\}$ such that

\[
\begin{align*}
\text{cat}(Q_2 \times S^n) &= \text{cat} Q_2 \text{ for all } n \geq 1 \\
\text{cat}(Q_\ell \times S^n) &= \text{cat} Q_\ell \text{ for all } n \geq 2 \text{ and } \ell > 2
\end{align*}
\]

**Theorem 3.3 (I 2002)** Let $X$ be $(d-1)$-connected with $\text{dim} \ X \leq d \text{ cat} X + d - 2$. Let $W = X \cup_{\alpha} D^{e+1}$ with $\text{cat} W = \text{cat} X + 1$ and $e \geq d$. Then $\text{cat}(W \times S^n) = \text{cat} W + 1$ for all $n \geq 1$ if and only if $H_m(\alpha) \neq 0$, where $m = \text{cat} X$.

**Theorem 3.4 (I 2002)** There are simply-connected closed manifolds $M$ and $N$ such that

\[
\begin{align*}
\text{cat}(M \times S^n) &= \text{cat} M \text{ for all } n \geq 2 \\
\text{cat}(N \setminus \{\ast\}) &= \text{cat} N
\end{align*}
\]
4 Ganea’s problems

Problems 4.1 (T. Ganea, 1971, (15 problems))


[2] Is $\text{cat}(X \times S^n) = \text{cat} X + 1$ true for any finite complex $X$ and any $n \geq 1$?

[4] Let $S^r \hookrightarrow E \rightarrow S^{t+1}$ be a bundle. Describe $\text{cat} E$ in terms of homotopy invariants of the characteristic map.

[8] Let $X = S^3 \cup e^{2p+1}$. Is $\text{Cat}(X \times X)$ equal to $\text{cat}(X \times X)$?

[10] Is any co-H-space $X$ (i.e, $\text{cat} X = 1$) of homotopy type of $S^1 \vee \cdots \vee S^1 \vee Y$ with $\pi_1(Y) = 0$?

\ldots

[O] For any closed manifold $M$, $\text{cat}(M \setminus \{\ast\}) = \text{cat} M - 1$?
**Theorem 4.2 (James 1978)** Let $X$ be $(d-1)$-connected. Then $\text{cat } X \leq \frac{\dim X}{d}$ (or $d \text{cat } X \leq \dim X$).

**Theorem 4.3 (Singhof 1979, Rudyak 1997)** Let $M$ be a closed manifold. If $\text{cat } M \geq \frac{\dim M + 3}{2}$, then $M$ satisfies $\text{cat}(M \times S^n) = \text{cat } M + 1$ for all $n \geq 1$.

Gómez-Larrañaga and Gonzalez-Acuna (1992) and Oprea and Rudyak (to appear) give an answer to Problems 2 and O:

**Theorem 4.4** For $M$ a closed 3-manifold, we have

\[
\begin{cases}
\text{cat}(M \times S^n) = \text{cat } M + 1 \text{ for all } n \geq 1 \\
\text{cat}(M \setminus \{*\}) = \text{cat } M
\end{cases}
\]

**Example 4.5** (1) Let $X = G_2$ the exceptional Lie group of rank 2. Then $H^*(G_2; \mathbb{F}_2) \cong P[x_3]/(x_3^4) \otimes \Lambda(x_5)$ with $\text{wgt}(x_3) = \text{wgt}(x_5) = 1$. Thus $\text{cat}(G_2) \geq \text{wgt}(x_3^3x_5) \geq 4$, and hence $\text{cat } G_2 = 4$ by Theorems 4.2 and 2.1.
(2) Let $X = Sp(2)$. Then $H^*(Sp(2)) \cong \Lambda(x_3, x_7)$ with $\text{wgt}(x_3) = \text{wgt}(x_7) = 1$. Hence $\text{cat}(Sp(2)) = \text{wgt}(x_3 x_7) \geq 2$. But Schweitzer (1965) has shown using secondary cohomology operations that $\text{cat}(Sp(2)) = 3 \neq 2$. Instead of using $H^*$, we might obtain $\text{wgt}(x_3^2) = 2, \text{wgt}(x_3^3) = 3$ using some other cohomology theory.

**Question 4.6** How can we know that $x_3^2 \neq 0$?

5 Ganea’s Problems and Hopf Invariants

Singhof answered to Ganea’s Problem 1 as follows.

**Theorem 5.1 (Singhof 1975)**

\[ \text{cat}(SU(n)) = n - 1 \text{ and } \text{cat}(U(n)) = n \text{ for } n \geq 1. \]

Extending the result of Schweitzer 1965, Singhof (1976) proved
Theorem 5.2 \( \text{cat}(Sp(n)) \geq n + 1 \) for \( n \geq 2 \).

Theorem 5.3 (I unpublished) \( \text{Let } \alpha : S^6 \to S^3 \text{ be the attaching map of 7-cell in } Sp(2). \text{ Then } x_3^2 = H_1^h(\alpha) \cdot x_7 \text{ in } h^*(Sp(2)), \text{ where } H_1^h \text{ is given by} \)

\[
H_1^h : \pi_6(S^3) \xrightarrow{H_1} \pi_6(\Omega S^3 \ast \Omega S^3) \cong \pi_6(S^5) \xrightarrow{\Sigma^\infty} \pi_S^{-1} \to h^{-1} \subset h^*,
\]

where \( H_1 \) is the Hopf invariant.

This is an answer to Question 4.6. Extending the observation on generalised cohomology theory and Hopf invariants, we obtain

Theorem 5.4 (Mimura-I) \( \text{cat}(Sp(n)) \geq n + 2 \) for \( n \geq 3 \).

The following result is obtained independently to Theorem 5.4 by Fernández-Suárez, Gómez-Tato, Tanré and Strom.

Theorem 5.5 (F-G-T-S, M-I) \( \text{cat}(Sp(3)) = 5 \).
Theorem 5.6 (Arkowitz-Stanley, to appear)  For a simply-connected co-H-space $X$, we have $\text{Cat}(X \times X) = 2 = \text{cat}(X \times X)$, which answers Problem 8.

To Ganea’s conjecture on co-H-spaces (Problem 10), we have

Theorem 5.7 (Saito-Sumi-I 1997)  Let $X$ be a co-H-space. If $H_\ast(X)$ concentrated in dimensions 1, $n + 1$ and $n + 2$ and $H_{n+2}(X)$ has no torsion, then Ganea’s conjecture on co-H-spaces (Problem 10) for $X$ is true.

Theorem 5.8 (I 1998)  There exists a sequence of co-H-spaces $\{R_n; n \geq 4\}$ each of which gives a counter-example to Ganea’s conjecture on co-H-spaces.

Theorem 5.9 (Hubbuck-I, to appear)  A $p$-completed version of Ganea’s conjecture on co-H-spaces is true.
Theorem 5.10 (I, to appear)  For $E$ the total space of $S^r$-bundle over $S^{t+1}$, $\text{cat } E$ is given as follows (Problem 4):

<table>
<thead>
<tr>
<th>Conditions</th>
<th>$L$-$S$ category</th>
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<tbody>
<tr>
<td>$r$</td>
<td>$Q \times S^n$</td>
</tr>
<tr>
<td>$t = 0$</td>
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</tr>
<tr>
<td>$r = 1$</td>
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<td>$t = 1$</td>
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<tr>
<td>$\alpha = 0$</td>
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<td>$t &gt; 1$</td>
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<td>$t &lt; r$</td>
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<tr>
<td>$t = r$</td>
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<tr>
<td>$\alpha \neq \pm 1$</td>
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<tr>
<td>$H_1(\alpha) = 0$</td>
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<tr>
<td>$t &gt; r$</td>
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<tr>
<td>$H_1(\alpha) \neq 0$</td>
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<tr>
<td>$\Sigma^r H_1(\alpha) = 0$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma^r H_1(\alpha) \neq 0$</td>
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</tbody>
</table>

(1) \( \Sigma^n H_1(\alpha) = 0 \iff \text{cat } Q \times S^n = 2, \)
\( \Sigma^{n+1} H_1(\alpha) \neq 0 \iff \text{cat } Q \times S^n = 3. \)

(2) \( \Sigma^{r+n} H_1(\alpha) = 0 \iff \text{cat } E \times S^n = 3, \)
\( \Sigma^{r+n+1} h_2(\alpha) \neq 0 \iff \text{cat } E \times S^n = 4, \)

where $\alpha$ is the attaching map of $t + 1$-cell of $E$ and $Q = E \setminus \{\ast\} \simeq S^r \cup_\alpha e^{t+1}$. 
References


