L-S Category of a Sphere-Bundle over a Sphere

AMS-IMS-SIAM Joint Summer Research Conference
Lusternik-Schnirelmann Category in the New Millennium
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1 What is the L-S category of a space $X$

Definition 1.1

$$\text{cat}(X) = \operatorname{Min} \left\{ m \geq 0 \mid \exists \{U_0, \ldots, U_m\}; \text{open in } X \right\}$$

$$X = \bigcup_{i=0}^{m} U_i; \text{ each } U_i \text{ is contractible in } X$$

Theorem 1.2 (Lusternik-Schnirelmann)

$$\#\{\text{critical points of a } C^\infty\text{-map } f : M \to \mathbb{R}\} > \text{cat } M.$$ 

But generally speaking, a simple definition does not suggest a simple way of calculation.
2 Ganea’s problems

Problems [T. Ganea, 1971, (15 problems)]

[1] Compute \( \text{cat } M \) for a closed manifold \( M \).

[2] \( \text{cat } X \times S^n = \text{cat } X + 1 \) ?

[4] Let \( S^r \hookrightarrow E \rightarrow S^{t+1} \) be a bundle. Describe \( \text{cat } E \) in terms of homotopy invariants of the characteristic map of the bundle. \( (E = S^r \cup \Psi D^{t+1} \times S^r, \; \Psi : S^t \times S^r \rightarrow S^r) \)

\[ \cdots \]

[O] For a closed manifold \( M \), \( \text{cat } M > \text{cat}(M - \{x\}) \) ?

Remark 2.1 If [O] is true, then so is [2] (Ganea's conjecture in L-S category) for closed manifolds.

Remark 2.2 For \( E = S^r \cup_\Psi D^{t+1} \times S^r, \; E - \{x\} \simeq S^r \cup_\alpha D^{t+1} \).
2.1 Hopf invariants

Classically, a Hopf invariant is defined to detect the existence of a multiplicative structure with unit on a sphere $S^{n-1}$:

$$H : [S^{2n-1}, S^n] \rightarrow \mathbb{Z}, \quad S^{2n-1} = S^{n-1} \ast S^{n-1}, \quad S^n = \Sigma S^{n-1},$$

where $\Sigma X = \{-1\} \ast X \ast \{1\}$ and $A_1 \ast A_2 = \{ta_1 + (1-t)a_2 \mid a_i \in A_i, \ t \in [0, 1]\}$. While the homotopy set $[\Sigma X, W]$ has a natural group structure, the induced map $f^* : [\Sigma B, W] \rightarrow [\Sigma A, W]$ from a map $f : \Sigma A \rightarrow \Sigma B$ is not a homomorphism, in general:

$$f^*(\alpha + \beta) \neq f^*(\alpha) + f^*(\beta), \quad \text{for } \alpha, \beta \in [\Sigma B, W].$$

The difference is given by the composition $[\alpha, \beta] \circ h_2(f)$ of the Whitehead product $[\alpha, \beta]$ and the Hopf invariant $H_1(f)$.

[Berstein-Hilton] $H_1(f)$ determines $\text{cat} \ (S^r \cup_f D^{t+1})$. 

2
2.2 (L-S category and Higher Hopf invariants)

\[
\text{cat } X \leq m \quad \xrightarrow{\Delta^{m+1}} \quad \prod^{m+1} X
\]

[Whitehead] (1957)

\[\exists h\]

[Berstein-Hilton] (1968)

(fat wedge) \(T^{m+1}X\)

\[H^S_m(f) \in \pi_{n+1}(\prod^{m+1} X, T^{m+1}X; A), (f: M(A, n) \to X)\]

\[\text{cat } S^r \cup_f D^{t+1} = 2 \iff H_1(f) = 0, (f: S^t \to S^r)\]

\[
\text{cat } X \leq m \quad \xrightarrow{\exists \{E^{r*}, d^r\}} \quad \text{Bar spectral sequence for } H_*(\Omega X) \text{ such that } d^r = 0 \text{ for } r > m.
\]

[Ginsburg] (1962)

[Ganea] (1967)

[Gilbert] (1968)

[Sakai] (2000)

\[\exists \sigma\]

\[\text{Counter Examples to } [2]\]
Definition 2.3  For $f \in \left[\Sigma V, X\right]$, we have

$$e^{X_{\sigma}}\Sigma \text{ad}(f) = ev^{\sigma}\Sigma \text{ad}(f) = f = 1_{X^\sigma}f = e_{m}^{X_{\sigma}}\sigma f,$$

Thus the difference $d_{m}(f) = \sigma f - \Sigma \text{ad}(f)$ defines $H^\sigma_{m}(f)$ by

$$p_{m+1}^{\Sigma V}H^\sigma_{m}(f) = d_{m}(f)$$

and $H^\sigma_{m}(f) = \Sigma^\infty H^\sigma_{m}(f)$.

$$H^S_{m}(f) = \left\{H^\sigma_{m}(f) \mid \sigma \text{ is a structure for cat } X = m \right\} \subset \left[\Sigma V, E^{m+1}(\Omega X)\right]$$

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Theorem 2.4  $H^S_{m} : \left[S^{n(m+1)-1}, P^{m}(S^{n-1})\right] \rightarrow \mathbb{Z}$ detects the $A_{m}$-structure of the multiplication of $S^{n-1}$ ($n=1,2,4,8$),

where $P^{m}(S^{n-1})$ is the $m$-th projective space.
3 LS category of a sphere-bundle over a sphere

Let $M$ be the total space of a sphere-bundle over a sphere:

$$S^r \hookrightarrow M \rightarrow S^{t+1} \quad \text{with structure group } G,$$

$$M = S^r \cup_{\Psi} D^{t+1} \times S^r, \quad \Psi = \mu \circ (\bar{\alpha} \times 1) : S^t \times S^r \rightarrow S^r,$$

where $\mu : G \times S^r \rightarrow S^r$ is the action of $G$ on the fibre $S^r$ and $\bar{\alpha} : S^t \rightarrow G$ is the characteristic map of the bundle. Thus

$$M = Q \cup_{\psi} D^{r+t+1}, \quad Q = M - \{x\} \simeq S^r \cup_{\alpha} D^{t+1},$$

$$\alpha = \Psi|_{S^t \times \{\ast\}}, \quad \psi \simeq [\iota_r, \chi_{t+1}]^T$$ a relative Whitehead product.

- $(\dim M \leq 3)$ cat $M$ is determined completely by Singhof, Montejano, Gomez-Gonzalez and Rudyak.

- $(\text{cat } Q \leq 1)$ cat $M$ is well-known by cup-length arguments.
4 An answer to Problem 4

Theorem 4.1  For $E = S^r \cup_\alpha D^t \cup_\psi D^{r+t+1}$ a closed manifold with $H_1(\alpha) \neq 0$, we have “$H_2^S(\psi) \neq 0 \iff \text{cat } E = 3$”:

<table>
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<tr>
<th>Conditions</th>
<th>L-S category</th>
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(1) \[ \sum^n H_1(\alpha) = 0 \implies \text{cat } Q \times S^n = 2, \]
\[ \sum^{n+1} H_1(\alpha) \neq 0 \implies \text{cat } Q \times S^n = 3. \]

(2) \[ \sum^{r+n} H_1(\alpha) = 0 \implies \text{cat } E \times S^n = 3, \]
\[ \sum^{r+n+1} h_2(\alpha) \neq 0 \implies \text{cat } E \times S^n = 4. \]
5 Manifold counter examples

Let us consider the Hopf (principal) fibrations

\[ S^1 \hookrightarrow S^3 \xrightarrow{\eta} S^2 \]  \quad \text{with structure group } U(1) \approx S^1,

\[ S^3 \hookrightarrow S^7 \xrightarrow{\nu} S^4 \]  \quad \text{with structure group } Sp(1) \approx S^3.

By taking orbit space of the action of \( U(1) \subset Sp(1) \), we obtain

\[ S^2 \hookrightarrow \mathbb{C}P^3 \xrightarrow{\pi} S^4 \]  \quad \text{with structure group } Sp(1).

**Definition 5.1** For \( \beta \in \pi_t(S^3) = [S^t, S^r] \), we have

\[ E(\beta) \xrightarrow{\Sigma \beta} S^{t+1} \]
\[ \mathbb{C}P^3 \xrightarrow{q} \text{Pull-Back} \]
\[ E(\beta) = Q(\beta) \cup_{\psi(\beta)} D^{t+3}, \quad Q(\beta) = E(\beta) - \{x\} \approx S^2 \cup_{\eta \circ \beta} D^{t+1}. \]

If \( H_1(\beta) = 0 \), then \( H_1(\eta \circ \beta) = \beta \), and hence we have

\[ \text{cat}(Q(\beta)) = 2 \iff \beta \neq 0. \]
Let $p$ be an odd prime. Then $\alpha_1(3) \circ \alpha_2(2p) : S^{2p} \to S^3$ satisfies $H_1(\alpha_1(3) \circ \alpha_2(2p)) = 0$. Using results of Toda and Oka on $p$-primary component of $\pi_*^S(S^0)$, we obtain the following lemma.

**Lemma 5.2** The set $\Sigma_* H_2^{SS}(\psi(\alpha_1(3) \circ \alpha_2(2p)))$ gives a map $\pm \Sigma^3(\alpha_1(3) \circ \alpha_2(2p))$ composed with an inclusion map.

Using this, we obtain the following results.

**Theorem 5.3** There is a closed manifold $M$ such that $\text{cat } (M \times S^n) = \text{cat } M$, for $n \geq 1$.

**Theorem 5.4** There is a closed manifold $N_p$ for each odd prime $p \geq 5$ such that $\text{cat } N_p = \text{cat } (N_p - \{x\})$.

**Remark 5.5** Pascal Lambrecht, Don Stanley and Lucile Vandembroucq have also obtained manifolds which satisfy the same property as $N_p$ in Theorem 5.4 does.
6 Outline of the proof of Lemma 5.2

By a concrete homotopy-theoretical observation, we can show

**Proposition 6.1** \( \exists_{H_2^{SS}(\psi(-)) \subset H_2^S(\psi(-))} \) such that

(1) \( \beta \circ \gamma = 0 \implies (\Sigma^2 \gamma)^* H_2^{SS}(\psi(\beta)) = \{0\} \) for all \( \beta, \gamma \).

(2) \( \ell \beta = 0 \implies \ell H_2^{SS}(\psi(\beta)) = \{0\} \) for all \( \beta \).

For the dimensional reasons, we have (for some \( a, b \in \mathbb{Z} \))

\[
\Sigma_* H_2^{SS}(\psi(\alpha_2(3))) = \{a\alpha_2(6) + b\alpha_1(2p + 4)\}
\]

with \( a = 1 \) by a result of Boardman-Steer, and hence

\[
\Sigma E(\Omega \alpha_1(4p - 2)_0)^* \Sigma_* H_2^{SS}(\psi(\alpha_1(3) \circ \alpha_2(2p)))
\]

\[
= \pm 2\alpha_1(4p + 1)^* \Sigma_* H_2^{SS}(\psi(\alpha_2(3))) = \{\pm \alpha_1(6) \circ \alpha_2(2p + 3)\}.
\]

Then the set \( \Sigma_* H_2^{SS}(\psi(\alpha_1(3) \circ \alpha_2(2p))) \) is described as

\[
\{\pm \alpha_1(6) \circ \alpha_2(2p + 3) + x(\nu_1 \circ \alpha_1(4p + 1) + \nu_1 \circ \alpha_2(2p + 3))\},
\]
for some $x \in \mathbb{Z}$. By a result of Oka, we know $\alpha_1 \circ \beta_1 \neq 0$ stably, while $\alpha_2 \circ \beta_1 = 0$ stably. Thus $x = 0$ and we obtain the lemma.

7 Proofs of Theorems 5.3 and 5.4

Let $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_2(6)$ with $p = 3$. Then $H_1(\alpha) = \alpha_1(3) \circ \alpha_2(6)$ $\neq 0$, $\Sigma^2 H_1(\alpha) \neq 0$ and $\Sigma^4 H_1(\alpha) = 0$ by Toda. Let $M_3$ be the $S^2$-bundle over $S^{14}$ induced by $\Sigma(\alpha_1(3) \circ \alpha_2(6)) : S^{14} \to S^4$ from the $S^2$-bundle $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$. Lemma 5.2 implies that $\text{cat}(M_3 \times S^n) = \text{cat} M_3 = 3$ for $n \geq 2$.

Let $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_2(2p)$ with $p$ odd $\geq 5$. Then $H_1(\alpha) = \alpha_1(3) \circ \alpha_2(2p) \neq 0$ and $\Sigma^2 H_1(\alpha) = 0$ by Toda. Let $N_p$ be the $S^2$-bundle over $S^{6p-4}$ induced by $\Sigma(\alpha_1(3) \circ \alpha_2(2p)) : S^{6p-4} \to S^4$ from the $S^2$-bundle $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$. Lemma 5.2 implies that $\text{cat} N_p = \text{cat}(N_p - \{x\}) = 2$. 

10
8 A new answer to Problem 4

Theorem 8.1  For $E$ an $S^r$-bundle over $S^{t+1}$ with $H_1(\alpha) \neq 0$, we have \( H^S_2(\psi) \ni 0 \iff \Sigma^r H_1(\alpha) = 0 \) and

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(1) \( \Sigma^{n+1} H_1(\alpha) \neq 0 \Rightarrow \text{cat } Q \times S^n = 3, \)
\( \Sigma^n H_1(\alpha) = 0 \Rightarrow \text{cat } Q \times S^n = 2. \)

(2) \( \Sigma^{r+n} H_1(\alpha) = 0 \Rightarrow \text{cat } E \times S^n = 3, \)
\( \Sigma^{r+n+1} h_2(\alpha) \neq 0 \Rightarrow \text{cat } E \times S^n = 4. \)
9 Outline of the proof of Theorem 8.1

Since $E$ is a bundle, we obtain that $H^S_2(\psi) \ni 1_{S^{r-1}}*H_1(\alpha) = \pm \Sigma^r H_1(\alpha)$. If $H^S_2(\psi) \ni H^\sigma_2(\psi) = 0$ for some $\sigma$, then it follows that $1_{S^{r-1}}*H_1(\alpha)$ is homotopic to a Whitehead product $[\nu_r, \delta]$ where $\nu_r : S^r \hookrightarrow \Sigma \Omega S^r \subset P^2(\Omega S^r)$ is the bottom-cell inclusion.

The key idea to proceed is obtained by looking at the first differential $d_1 : H_*(\Omega S^r \wedge \Omega S^r \wedge \Omega S^r) \to H_*(\Omega S^r \wedge \Omega S^r)$ of the Bar spectral sequence by Ginsburg. We observe that the subspace $S^{r-1} \ast \Omega S^r \ast \Omega S^r \subset \Omega S^r \ast \Omega S^r \ast \Omega S^r$ which contains the image of $1_{S^{r-1}}*H_1(\alpha)$ is a retract of $P^2(\Omega S^r)/\Sigma \Omega(S^r)$. Here, the Whitehead product $[\nu_r, \delta]$ vanishes after composing the retraction, since $S^r \subset \Sigma \Omega(S^r)$. Thus $\Sigma^r H_1(\alpha)$ must be trivial. The converse is clear by $H^S_2(\psi) \ni \pm \Sigma^r H_1(\alpha) = 0$. 

12