L-S CATEGORY OF PRINCIPAL FIBRE BUNDLES

"III JOINT MEETING JAPAN-MEXICO IN TOPOLOGY AND ITS APPLICATIONS"
6th~10st, December, 2004

Norio IWASE
(Kyushu University)

Lusternik-Schnirelmann category (L-S cat) is defined by Lusternik and Schnirelmann in 1934 as a numerical homotopy invariant of a manifold M which gives a lower-bound for the number of critical/stationary points of a smooth real-valued function on M.

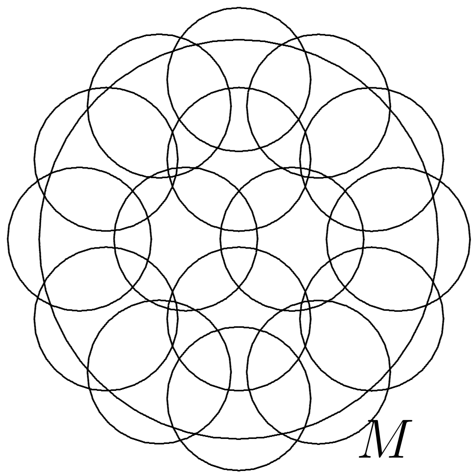

Figure 1

1 Lusternik-Schnirelmann category

Def 1.1

$$\mathrm{cat}(X) = \mathrm{Min}\left\{ m{\geq}0 \;\middle|\; \begin{array}{l} \exists\{A_0, ..., A_m\,;\ \text{closed in } X\} \\ X = \bigcup_{i=0}^{m} A_i,\ \text{each } A_i \text{ is contractible } \underline{\textit{in}}\ X \end{array} \right\}$$

$$= \mathrm{Min}\left\{ m{\geq}0 \;\middle|\; \begin{array}{l} \Delta^{m+1} : X \to \prod^{m+1} X \text{ is compressible into the fat wedge } \prod_{m}^{m+1} X. \end{array} \right\}$$

Thm 1.2 (Lusternik-Schnirelmann [15]) *Any C^∞-function on a closed manifold M has at least* $\mathrm{cat}(M)+1$ *critical points.*

1.1 "Strong" cats

The topological invariant $\mathrm{gcat}(X)$ was defined and shown not to be a homotopy invariant by Fox. Ganea altered the definition of strong category as a homotopy invariant $\mathrm{Cat}(X)$.

$$\mathrm{gcat}(X) = \mathrm{Min}\left\{ m \geq 0 \,\middle|\, \begin{array}{l} \exists\{A_0, \ldots, A_m\,;\ \text{closed in } X\} \\ X = \bigcup_{i=0}^{m} A_i,\ \text{each } A_i \text{ is contractible} \end{array} \right\}$$

$$\mathrm{Cat}(X) = \mathrm{Min}\left\{ m \geq 0 \,\middle|\, \exists_{\{Y(\simeq X)\}}\, \mathrm{gcat}(Y) = m \right\}$$

Def 1.3 (Ganea [5]) *For a space X, consider all the sequences $\{h_n : A_n \to Y_n \;\; m \geq n \geq 0\}$ such that $Y_0 = \{*\}$, $Y_{n+1} = C(h_n) \supset Y_n$ $(m-1 \geq n)$ and $Y_m \simeq X$ for some $m \geq 0$.* $\mathrm{Cone}(X)$ *is the least m minus 1 for all such sequences - a cone-decomposition of X.*

Thm 1.4 (Ganea [5]) $\mathrm{Cone}(X) = \mathrm{Cat}(X)$.

Thm 1.5 (Ganea [5]) $\mathrm{Cat}(X){-}1 \le \mathrm{cat}(X) \le \mathrm{Cat}(X)$.

Fact 1.6 *(1)* $\mathrm{cat}(\{*\}) = 0$.

(2) $\mathrm{cat}(S^n) = 1$. *More generally,* $\mathrm{cat}(\Sigma V) \le 1$.

(3) If X *dominates* Y*, then* $\mathrm{cat}(X) \ge \mathrm{cat}(Y)$. *In particular,* $\mathrm{cat}(X) = \mathrm{cat}(Y)$ *provided that* X *and* Y *are homotopy equivalent.*

(4) (Varadarajan [23], Hardie [7]) Fibre space (E, p, B, F) *satisfies for* cat*:* $\mathrm{cat}(E){+}1 \le (\mathrm{cat}(F){+}1)\cdot(\mathrm{cat}(B){+}1)$.

(5) Fibre space (E, p, B, F) *also satisfies for* Cat*:* $\mathrm{Cat}(E){+}1 \le (\mathrm{Cat}(F){+}1)\cdot(\mathrm{Cat}(B){+}1)$.

(6) (Fox [4]) $\mathrm{cat}(X{\times}Y) \le \mathrm{cat}(X){+}\,\mathrm{cat}(Y)$.

(7) (Takens [22]) $\mathrm{Cat}(X{\times}Y) \le \mathrm{Cat}(X){+}\,\mathrm{Cat}(Y)$.

1.2 "Weak" cats

Def 1.7 (Whitehead [24, 25])

$$w\mathrm{cat}(X) = \mathrm{Min}\left\{ m{\geq}0 \;\middle|\; \begin{array}{l} \bar{\Delta}^{m+1} : X \to \bigwedge^{m+1} X \\ \textit{is trivial.} \end{array} \right\}$$

where $\prod^{m+1} X / \prod_m^{m+1} X = \bigwedge^{m+1} X$ (smash product). By the definition, we obtain the following result for any X.

Thm 1.8 (Whitehead) *(1)* $w\mathrm{cat}(X) \leq \mathrm{cat}(X)$.

(2) Let h^* be a multiplicative cohomology. If a product of m elements in $\tilde{h}^*(X)$ is non-zero, then $w\mathrm{cat}(X) \geq m$.

Def 1.9 *cup-length is often denoted by* $\mathrm{c}(-)$, *but here we denote* $\mathrm{cup}(-)$ *to avoid confusion with Chern classes:*

(1) Let h be a multiplicative cohomology.

$$\mathrm{cup}(X;h) = \mathrm{Min}\left\{ m{\geq}0 \;\middle|\; \forall_{\{u_0,\cdots,u_m \in \tilde{h}^*(X)\}} u_0 {\cdot}{\cdots}{\cdot} u_m = 0 \right\}$$

(2) $\mathrm{cup}(X) = \mathrm{Max}_{h:\textit{a multiplicative cohomology}} \{\mathrm{cup}(X;h)\}$

Rem 1.10 $\mathrm{cup}(X; H^*(\ ;R))$ *is often denoted by* $\mathrm{cup}(X;R)$.

1.3 (Higher Hopf invariants)

Berstein-Hilton [1] defined their higher hopf invariants as follows, where s is a compression of Δ^{m+1} to the fat wedge:

$$H^s_m : \pi_q(X;A) \to \pi_{q+1}(\prod^{m+1} X, \prod^{m+1}_m X;A), \quad q \geq 1$$

Def 1.11 (unstable and stable Hopf invariants)

1. *For any X with $\mathrm{cat}(X) \leq m$ and a suspension ΣV,*

$$H_m : [\Sigma V, X] \to 2^{[\Sigma V, E^{m+1}(\Omega X)]},$$
$$H_m(f) = \left\{ H^{\sigma(X)}_m(f) \,\middle|\, \begin{array}{l}\sigma(X) \text{ is a structure map for} \\ \mathrm{cat}(X) = m\end{array} \right\}$$
$$\subset [\Sigma V, E^{m+1}(\Omega X)] \quad \textit{for } f \in [\Sigma V, X].$$

2. *Then we stabilise this by Σ^∞.*

$$\mathcal{H}_m : [\Sigma V, X] \overset{H_m}{\to} 2^{[\Sigma V, E^{m+1}(\Omega X)]} \overset{2^{\Sigma^\infty}*}{\longrightarrow} 2^{\{\Sigma V, E^{m+1}(\Omega X)\}}$$
$$\mathcal{H}_m(f) = \left\{ \begin{array}{l} \mathcal{H}^{\sigma(X)}_m(f) \\ = \Sigma^\infty H^{\sigma(X)}_m(f) \end{array} \,\middle|\, \begin{array}{l}\sigma(X) \text{ is a structure map} \\ \text{for } \mathrm{cat}(X) = m\end{array} \right\}$$
$$\subset \{\Sigma V, E^{m+1}(\Omega X)\} \quad \textit{for } f \in [\Sigma V, X]$$

Rem 1.12 *We can also define 'crude' Hopf invariants.*

2 New computable invariants

Rudyak and Strom altered the definition of Fadell-Husseini's topological invariant category weight (see [3]) as a homotopy invariant to give a new lower estimate for L-S category:

Def 2.1 (Rudyak [16, 17], Strom [21]) *For any element $u \in \tilde{h}^*(X)$, where h is a cohomology, one defines*

$$\mathrm{wgt}(u;h) = \mathrm{Min}\left\{ m{\geq}0 \,\middle|\, (e^X_m)_*(u) \neq 0 \right\}$$

Thm 2.2 (Rudyak [16, 17], Strom [21]) *Let h be a multiplicative cohomology.*

(1) $\mathrm{wgt}(u{+}v;h) \geq \mathrm{Min}\{\mathrm{wgt}(u;h), \mathrm{wgt}(v;h)\}$.

(2) $\mathrm{wgt}(uv;h) \geq \mathrm{wgt}(u;h) + \mathrm{wgt}(v;h)$.

(3) $\mathrm{wgt}(f^*(u);h) \geq \mathrm{wgt}(u;h)$ *for any map f.*

Def 2.3 (Rudyak [17])

$$r\mathrm{cat}(X) = \mathrm{Min}\{m{\geq}0 \mid \exists_{\sigma\in\{X,P^m(\Omega X)\}}\ e^X_m{\circ}\sigma \sim 1_X \ \textit{(stably)}\}.$$

In fact, for a symplectic mfd (M,ω), Rudyak shows that $\mathrm{rcat}(M)$ and $\dim M$ give the lower and upper bound for both $\mathrm{Fix}(M)$ and $\mathrm{Crit}(M)$ and that $\mathrm{rcat}(M) = \dim M$ under a suitable condition. We introduce versions of Toomer invariants by homomorphism $(e^X_m)_* : h^*(X) \to h^*(P^m(\Omega X))$.

Def 2.4 *(1) Let h be a cohomology theory.*

$$\text{i) } \mathrm{wgt}(X;h) = \mathrm{Min}\left\{ m{\geq}0 \;\middle|\; (e^X_m)_* \textit{ is a mono} \right\}$$

$$\text{ii) } \mathrm{Mwgt}(X;h) = \mathrm{Min}\left\{ m{\geq}0 \;\middle|\; \begin{array}{l}(e^X_m)_* \textit{ is a split mono of} \\ \textit{unstable } h^*h\textit{-modules}\end{array} \right\}$$

$$\text{(2) i) } \mathrm{wgt}(X) = \mathrm{Max}\left\{ \mathrm{wgt}(X;h) \;\middle|\; \begin{array}{l}h \textit{ is a multiplicative} \\ \textit{cohomology}\end{array} \right\}$$

$$\text{ii) } \mathrm{Mwgt}(X) = \mathrm{Max}\left\{ \mathrm{Mwgt}(X;h) \;\middle|\; \begin{array}{l}h \textit{ is a multiplica-} \\ \textit{tive cohomology}\end{array} \right\}$$

Thm 2.5 *Let h be a multiplicative cohomology.*

$$\mathrm{wgt}(X;h) = \mathrm{Max}\{\mathrm{wgt}(u;h) \mid u \neq 0 \textit{ in } \tilde{h}^*(X)\}$$

Thm 2.6 *The above formulae and Rudyak [17] imply*

$$\mathrm{cup}(X) \leq \mathrm{wgt}(X) = \mathrm{rcat}(X) \leq \mathrm{Mwgt}(X) \leq \mathrm{cat}(X).$$

2.1 Upper bounds for L-S cat of Lie groups

2.2 With Mimura and Nishimoto

Let $G \hookrightarrow E \xrightarrow{p} \Sigma A$ be a principal fibre bundle with structure group G and let $\mu : G{\times}G \to G$ be the multiplication.

Thm 2.7 *If a cofibre sequence* $K_i \xrightarrow{\rho_i} F_{i-1} \hookrightarrow F_i$, $1 \le i \le m$, *satisfies the following conditions, then* $\mathrm{Cat}(E) \le m{+}k$.

(1) $F_0 = \{*\}$, $F_m \simeq G$.

(2) The restriction of μ *to* $F_i{\times}F_j \subseteq F_m{\times}F_m \simeq G{\times}G$ *can be compressed into* F_{i+j}, $i{\ge}k$, $j{\ge}0$.

(3) $\alpha : A{\to}G$ *is compressible into* F_{k-1}, *for some* $k{\ge}m$.

This enables us to give a nice upper bound for L-S category of simply-connected compact Lie groups.

For non-simply-connected Lie groups, we need another result: Let $F \hookrightarrow X \to B$ be a fibre bundle with structure group G, where B is $(d{-}1)$-connected, $d{\ge}1$, and of finite dimension.

Thm 2.8 *If a cofibre sequence* $K_i \overset{\rho_i}{\to} F_{i-1} \hookrightarrow F_i$, $1 \le i \le m$, *satisfies the following conditions, then* $\mathrm{Cat}(E) \le m + \frac{\dim B}{d}$.

(1) $F_0 = \{*\}$, $F_m \simeq F$.

(2) The restriction of $\psi : G \times F \to F$, *the action of* G *on* F, *to* $G^{(d\cdot(i+2)-2)} \times F_j \subset G \times F_m \simeq G \times F$ *can be compressed into* F_{i+j}, $i \ge k$, $j \ge 0$.

These theorems produce determinations of L-S cat for a number of Lie groups of low rank.

2.3 With Kono To determine L-S cat of higher spinor groups, the above computations of cone-length are not strong enough. So, we try to reduce the value of L-S cat by using higher Hopf invariant.

Thm 2.9 *If a cofibre sequence* $K_i \overset{\rho_i}{\to} F_{i-1} \hookrightarrow F_i$, $1 \le i \le m$, *satisfies the following compatibility conditions, we have* $\mathrm{cat}(E) \le \mathrm{Max}(m+n, m+2)$. α *is compressible into* $F_n \subseteq$

$F_m \simeq G$ and $H_n^{\sigma_n}(\alpha) = 0$ for some $n \geq 1$, under the following compatibility condition.

(1) $F_0 = \{*\}$, $F_m \simeq F$.

(2) The restriction of μ to $F_i \times F_j \subseteq F_m \times F_m \simeq G \times G$ can be compressed into F_{i+j}, $i \geq k$, $j \geq 0$.

(3) $\alpha : A \to G$ is compressible into $F_k(G)$, $H_k^{\sigma_k}(\alpha) = 0$ for some $k \geq m$, where σ_k denotes the standard structure map of $\mathrm{cat}(F_k(G))$.

Proof ofTheorems 2.7, 2.8 and 2.9 : We can construct concretely a desired cone-decomposition. In particular, we compute the higher Hopf invariant of the attaching map of the top-cell, and obtain that it is 0. *QED.*

This result produce a determination of L-S cat of **Spin**(9).

Using the above results, we obtain the following table.

rank	1		2		3		4	
A_n	SU(2)	1	SU(3)	2	SU(4)	3	SU(5)	4
	PU(2)	3	PU(3)	6	PU(4)	9	PU(5)	12
B_n	Spin(3)	1	Spin(5)	3	Spin(7)	5	Spin(9)	?
	SO(3)	3	SO(5)	8	SO(7)	11	SO(9)	20
C_n	Sp(1)	1	Sp(2)	3	Sp(3)	5	Sp(4)	?
	PSp(1)	3	PSp(2)	8	PSp(3)	?	PSp(4)	?
D_n					Spin(6)	3	Spin(8)	6
					SO(6)	9	SO(8)	12
					PO(6)	9	PO(8)	18
excpt.			G_2	4			F_4	?

Prob 1 $\mathrm{cat}(\mathrm{Spin}(2n+1)) = \mathrm{cat}(\mathrm{Sp}(n))$ *?*

Prob 2 $\mathrm{cat}(\mathrm{PU}(n)) = 3(n-1)$ *?*

Rem 2.10 *(1) We know* $\mathrm{cat}(\mathrm{Sp}(4)) = 6$, 7 *or* 8.

(2) Kono-I. announced the following result:

$$\mathrm{cat}(\mathrm{Spin}(9)) = 8 = \mathrm{Mwgt}(\mathrm{Spin}(n); \mathbb{F}_2)$$

$$> 6 = \mathrm{wgt}(\mathrm{Spin}(n); \mathbb{F}_2).$$

(3) M.-N.-I. announced that $\mathrm{cat}(\mathrm{PU}(n)) \leq 3(n-1)$ *and*

$\mathrm{cat}(\mathrm{PU}(p^r)) = 3(p^r-1)$, *for any power of a prime* p^r.

References

[1] I. Berstein and P. J. Hilton, *Category and generalised Hopf invariants*, Illinois. J. Math. **12** (1968), 421–432.

[2] S. Eilenberg and T. Ganea, *On the Lusternik-Schnirelmann category of abstract groups*, Ann. of Math. **65** (1957), 517-518.

[3] E. Fadell and S. Husseini, *Category weight and Steenrod operations* (Spanish), Bol. Soc. Mat. Mexicana **37** (1992), 151–161.

[4] R. H. Fox, *On the Lusternik-Schnirelmann category*, Ann. of Math. (2) **42**, (1941), 333–370.

[5] T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois. J. Math. **11** (1967), 417–427.

[6] T. Ganea, *Some problems on numerical homotopy invariants*, Symposium on Algebraic Topology, Lect. Notes in Math. **249**, Springer Verlag, Berlin (1971) 13–22.

[7] K. A. Hardie, *A note on fibraitons and category*, Michigan Math. J. **17**, (1970), 351–352.

[8] N. Iwase, *Ganea's conjecture on Lusternik-Schnirelmann category*, Bull. London Math. Soc. **30** (1998), 623–634.

[9] N. Iwase, *A_∞-method in Lusternik-Schnirelmann category* (Kyushu Univ. preprint series in Math. **1998-13**, Kyushu Univ., 1998), Topology **41** (2002), 695–723.

[10] N. Iwase, *Lusternik-Schnirelmann category of a sphere-bundle over a sphere*, Topology **42** (2003), 701–713.

[11] N. Iwase and M. Mimura, *L-S categories of simply-connected compact simple Lie groups of low rank*, In: "Algebraic Topology: Categorical Decomposition Techniques", (Isle of Skye, 2001), 199–212, Progr. Math., **215**. Birkhäuser Verlag, Basel, 2003.

[12] N. Iwase, M. Mimura and T. Nishimoto, *On the cellular decomposition and the Lusternik-Schnirelmann category of $Spin(7)$*, Topology appl., **133** (2003), 1–14.

[13] I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17** (1978), 331-348.

[14] I. M. James, *Lusternik-Schnirelmann Category*, In: "Handbook of algebraic topology", 1293-1310, North Holland, Amsterdam, 1995.

[15] L. Lusternik and L. Schnirelmann, "Méthodes Topologiques dan les Problémes Variationnels", Actual. Sci. Ind., **88**, Hermann, Paris, 1934

[16] Y. B. Rudyak, *On the Ganea conjecture for manifolds*, Proc. Amer. Math. Soc. **125** (1997), 2511–2512.

[17] Y. B. Rudyak, *On category weight and its applications*, Topology **38** (1999), 37–55.

[18] W. Singhof, *Minimal coverings of manifolds with balls*, Manuscripta Math. **29** (1979), 385–415.

[19] D. Stanley, *Spaces with Lusternik-Schnirelmann category n and cone length $n+1$*, preprint.

[20] J. D. Stasheff, *Homotopy associativity of H-spaces, I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312.

[21] J. Strom, *Essential category weight and phantom maps*, Cohomological methods in homotopy theory (Bellaterra, 1998), 409–415, Progr. Math., 196, Birkhauser, Basel, 2001.

[22] F. Takens, *The Lusternik-Schnirelman categories of a product space*, Compositio Math. **22** (1970), 175–180.

[23] K. Varadarajan, *On fibrations and category*, Math. Z. **88** (1965), 267–273.

[24] G. W. Whitehead, *The homology suspension*, In: "Colloque de topologie algébrique, Louvain, 1956", 89–95, Georges Thone, Liége, Masson & Cie, Paris, 1957.

[25] G. W. Whitehead, "Elements of Homotopy Theory", GTM series **61**, Springer Verlag, Berlin, 1978.