1 Rational case

A graded module $H$ over a field $K$ is called a Hopf algebra if there are homomorphisms of $K$-modules

$$
\phi : H \otimes H \to H, \quad \eta : K \to H,
$$

$$
\psi : H \to H \otimes H, \quad \epsilon : H \to K,
$$

such that

1. $(H, \phi)$ is an algebra with two-sided unit $\eta(1)$,
2. $(H^*, \psi^*)$ is an algebra with two-sided unit $\epsilon^*(1)$,
3. $\epsilon$ and $\eta^*$ are homomorphisms of algebras and
4. $\psi$ and $\phi^*$ are homomorphisms of algebras.
A connected commutative associative Hopf algebra $H$ of finite type over $\mathbb{Q}$ is isomorphic as algebra to the tensor product of a polynomial algebra on even dimensional generators and an exterior algebra on odd:

$$H \cong P[x_1, x_2, \cdots] \otimes E(y_1, y_2, \cdots).$$

The resulting formula can be rewritten as

$$H \cong \bigotimes_{n=1}^{\infty} A[x_n]$$

where $x_n$ is a homogeneous generator of dimension $d_n \geq 1$ and $A[x]$ is defined as follows: $A[x] = P[x]$ if $d_n$ is even and $A[S] = E(S)$ if $d_n$ is odd.
A space $X$ is called a Hopf space if there is a map

$$\mu : X \times X \longrightarrow X$$

such that $\mu$ has two-sided homotopy unit.

(Scheerer [Sch85]) If a rational space $X_0$ is a Hopf space, then $X_0$ has the homotopy type of a generalized Eilenberg-Mac Lane space:

$$X_0 \simeq \bigoplus_{n=1}^{\infty} K(\pi_n(X_0); n)$$

where $\bigoplus$ denotes the weak product:

$$\bigoplus_{\lambda \in \Lambda} X_{\lambda} = \left\{ (x_\lambda) \in \prod_{\lambda \in \Lambda} X_\lambda \left| x_\lambda = * \text{ except for finitely many } \lambda \right. \right\}$$
Can it happen only for a Hopf space?
(Oprea [Op86]) If $X_0$ is a rational $G$-space of finite type, then $X_0$ has the homotopy type of a generalized Eilenberg-Mac Lane space.

(Aguadé [Ag87]) If $X_0$ is a rational $T$-space of finite type, then $X_0$ has the homotopy type of a generalized Eilenberg-Mac Lane space.

**Defn 1.1**

$$G(X, Y) = \left\{ [f] \in [X, Y] \bigg| \exists F: X \times Y \to Y \text{ s.t. } F \right\}$$

**Defn 1.2**

\begin{enumerate}
\item (Gottlieb [Go69]) A space $X$ is a $G$-space iff \[ \pi_n(X) = G(S^n, X) \text{ for all } n \geq 1. \]
\item (Aguadé [Ag87], Woo-Yoon [WY95]) A space $X$ is a $T$-space iff \[ [\Sigma A, X] = G(\Sigma A, X) \text{ for any space } A. \]
\end{enumerate}
What happens in non-rational case?

(L. Fuchs) Any abelian group $A$ is a direct sum of a divisible group and a reduced group:

$$A \cong (\text{divisible part}) \oplus (\text{reduced part})$$
2 Non-rational case

Let $\overline{\rho} : [S^n_Q, X] \to H_n(X)$ be a homomorphism defined by $\overline{\rho}(\alpha) = \alpha_*([S^n]\otimes 1)$, where we regard $H_n(S^n_Q) = H_n(S^n)\otimes Q$.

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Thm 2.1 Let $R$ be a finite or an infinite dimensional $\mathbb{Q}$-vector space. If $R \subset \overline{\rho}(G(S^n_Q, X)) \subseteq H_n(X)$ ($n \geq 2$), then we have

$$X \simeq Y \times K(R, n).$$

Cor 2.1.1 Let $R$ be a finite or an infinite dimensional $\mathbb{Q}$-vector space. Let $X$ be an $(n-1)$-connected $T$-space with $R \subseteq H_n(X)$ ($n \geq 2$). Then $X$ decomposes as

$$X \simeq Y \times K(R, n)$$
for a $T$-space $Y$. 
3 Rational case revisited

Thm 3.1 Let $R = \bigoplus_{\lambda \in \Lambda} \mathbb{Q}$ be a finite or an infinite dimensional $\mathbb{Q}$-vector space. If a rational space $X_0$ is an $(n-1)$-connected $G$-space with $H_n(X_0) \supseteq R$ for $n \geq 2$, then $X_0$ decomposes as

$$X_0 \cong Y_0 \times K(R, n),$$

where $Y_0$ is an rational $G$-space.

Cor 3.1.1 Let $X$ be a 0-connected CW complex with rationalization $X_{\mathbb{Q}}$, then the following conditions are equivalent:

(1) $X_{\mathbb{Q}}$ is a $G$-space.

(2) $X_{\mathbb{Q}}$ is a $T$-space.

(3) $X_{\mathbb{Q}}$ is a Hopf space.

(4) Every $k$-invariant of $X$ is of finite order.
Cor 3.1.2  If the rationalization $X_{\mathbb{Q}}$ of a 0-connected virtually nilpotent space $X$ is a $G$-space, then $X_{\mathbb{Q}}$ has the homotopy type of a weak product of Eilenberg-Mac Lane spaces:

$$X_{\mathbb{Q}} \simeq \bigoplus_{n=1}^{\infty} K(\pi_n(X_{\mathbb{Q}}); n)$$

Cor 3.1.3  Let $X$ be a 1-connected rational $G$-space. Then $X_{\mathbb{Q}}$ is a Hopf space and the Hopf algebra $H_\ast(X; \mathbb{Q})$ is isomorphic as co-algebra with a tensor product of a polynomial algebra on even dimensional generators and an exterior algebra on odd, where generators may be infinitely many:

$$H_\ast(X; \mathbb{Q}) \cong \bigotimes_{\lambda \in \Lambda} A[x_\lambda], \text{ as coalgebras,}$$

where $\Lambda$ denotes an index set of homogeneous generators.
References


