Nelson model

1. Fundamental facts
2. Nelson model with UV cutoff
3. Renormalized Nelson model

1 $\mathcal{F} = \bigoplus_{n=0}^{\infty} \underbrace{L^2_{sym}(\mathbb{R}^{3n})}_{=\mathcal{F}^{(n)}} \ni \bigoplus \Phi_n$ $\qquad \underline{n=0 \quad \mathbb{C}}$

$\sum_{n=0}^{\infty} \|\Phi_n\|^2 < \infty \iff \Phi = \bigoplus \Phi_n \in \mathcal{F}$, $\qquad \Omega = \{1, 0, 0, \dots\} \in \mathcal{F}$

creation, annihilation

$(a^{\dagger}(f)\Phi)^{(n)} = \frac{1}{\sqrt{n!}} \sum_{j=1}^{n} f(k_j)\, \Phi^{(n-1)}(k \cdots \hat{k_j} \cdots)$

$(a(f)\Phi)^{(n)} = \sqrt{(n+1)!} \int f(k)\, \Phi^{(n+1)}(k_1 \cdots k_n\, k)\, dk$

$a(f) : \mathcal{F}^{(n)} \to \mathcal{F}^{(n-1)}$, $\qquad a^{\dagger}(f)\ \mathcal{F}^{(n)} \to \mathcal{F}^{(n+1)}$,

- $[a(f), a^{\dagger}(g)] = (\bar{f}, g)$
- $a(f)^* = a^{\dagger}(\bar{f})$
- $\phi(\hat{f}) = \frac{1}{\sqrt{2}}\left[a^{\dagger}(\hat{f}) + a(\tilde{\hat{f}})\right]$ Field op.

  $\underline{\tilde{f}(k) = \hat{f}(-k)}$ $\qquad f$ real $\Rightarrow \tilde{\hat{f}} = \overline{\hat{f}}$ $\quad$ //

- $\left\{\prod_{j=1}^{m} \phi(f_j)\,\Omega\right\} = \mathcal{F}^{(n)}$

2nd quantization

Relativistic Schrödinger op $\sqrt{-\Delta+m}$ in $L^2(\mathbb{R}^3)$

N body R-Schrödinger op is $\sum_{j=1}^{N}\sqrt{-\Delta_j+m}$ on $L^2(\mathbb{R}^{3N})$

$$= \sum \mathbb{1}\cdots\otimes\sqrt{-\Delta_j+m^2}\otimes\cdots\mathbb{1}\otimes \qquad = \bigotimes_{j=1}^{N} L^2(\mathbb{R}^3)$$

$h : L^2 \to L^2$

$$d\Gamma(h) = \bigoplus_{n=0}^{\infty}\sum_{j=1}^{n}\mathbb{1}\otimes\cdots \underset{j\text{th}}{h}\cdots\otimes\mathbb{1}$$

- $d\Gamma\ a^+(f_1)\cdots a^+(f_n)\Omega = \sum a^+(f_1)\cdots a^+(hf)\cdots a^+(f_n)\Omega$
  $d\Gamma\ \Omega = 0$

- Let T be contraction on L^2

  $\Gamma(T) = \oplus\left[\overset{n}{\otimes} T\right]$ $\qquad \Gamma(e^{ith})$ one-parameter

  $\Gamma(T)\Gamma(S) = \Gamma(TS)$ $\qquad \therefore\ \Gamma(e^{ith}) = e^{itS}$

We can see that $S = \left(\sum_{j=1}^{m}\sqrt{-\Delta_j}+m\right)$

- $w = \sqrt{|u|^2+m^2}$ multiplication
  $d\Gamma(w) = H_f$ free field H.

- $d\Gamma(k) = P_f$ momentum op.

- $d\Gamma(\mathbb{1}) = N$ number op

  We see that

$$\left.\begin{aligned}[d\Gamma(h),\ a(f)] &= -a(hf)\\ [d\Gamma(h),\ a^+(f)] &= a^+(hf)\end{aligned}\right\} \odot$$

- $\| a^\#(f)\, \Phi \|_{\mathcal{F}} \leq \| (H_f + 1)^{1/2} \Phi \| \, \| f/\omega \|$

In particular $H_f + \phi(f)$ is s.a. $> -\infty$ if $f/\sqrt{\omega} \in L^2$.
by Kato-Rellich

- $e^{a^\#(f)}$ unb'dd op eg. $e^{a^+(g)} \Omega$ coherent vector

but

- $e^{a^+(f)} \, e^{-tH_f}$ is b'dd when $\|f/\sqrt{\omega}\| < \infty$

- BCH-formula

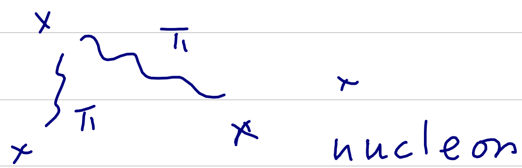
$$e^{\phi} = e^{\|h\|^2/4} \, e^{a^+(-\frac{1}{\sqrt{2}} g)} \, e^{a(\frac{1}{\sqrt{2}} \tilde{g})}$$

- $$e^{-2t(H_f + \phi(\hat{f}))} = e^{\frac{1}{4}\int_{-t}^{t}\int_{-t}^{t} w(t-s)\, dt ds} \, \underbrace{e^{a^+(U)}} \, \underbrace{e^{-2tH_f}} \, e^{a(\tilde{U})}$$

where $U = \int_{-t}^{t} -\frac{|\varphi|^2}{\omega} e^{-|s+T|\omega} ds \qquad \tilde{U} = \int_{-t}^{t} -\frac{|\varphi|^2}{\omega} e^{-|s-T|\omega} ds$

$$w(t-s) = \int \frac{|\hat{\varphi}|^2}{\omega} e^{-|t-s|\omega} dk$$

Nelson model = Linear interaction between nucleons & πions



$$\left(\sum_{j=1}^{N} -\frac{1}{2m_j}\Delta_j\right)+V \otimes \mathbb{1} + \mathbb{1}\otimes H_f$$

$$+\, g\sum_{j=1}^{N} \phi(x_j)$$

$$\phi(x) = \frac{1}{\sqrt{2}}\int\left\{ a^{+}\left(\frac{\hat{\varphi}}{\sqrt{\omega}}\, e^{-ikx}\right) + a\left(\frac{\hat{\varphi}}{\sqrt{\omega}}\, e^{+ikx}\right)\right\}$$

$N=1$, $m_j=1$, $g=1$

$$\left(-\frac{1}{2}\Delta+V\right)\otimes 1 + 1\otimes H_f + \phi(x)$$

$\hat{\varphi}$: UV cut-off function.

$$P_\mu := -i\nabla_{x_\mu}\otimes 1 + 1\otimes d\Gamma(k_\mu)$$ Total momentum

Let $V=0$. Then we can see that

$$[P_\mu, H]=0 \qquad \sigma(P_\mu)=\mathbb{R}$$

$$H=\int_{\mathbb{R}^3}^{\oplus} H(P)\,dP, \quad H(P)=\frac{1}{2}(P-P_f)^2+\phi(0)+H_f$$

on $\mathcal{F}$.

FKF

$(B_t)_{t\geq 0}$ 3D-BM i.e.

$(\mathcal{X}, \mathcal{B}, W^x)$ prob space $\mathbb{E}^0[B_t]=0$

$\mathbb{E}^0[B_tB_s] = t\wedge s$.

$$\mathbb{E}^x[f(B_t)] = \int f(y+x)P_t(y) = \left(e^{-t(-\frac{1}{2}\Delta)}f\right)(x)$$

$$\therefore \left(g, e^{-t(-\frac{1}{2}\Delta)}f\right) = \int g(x)\,\mathbb{E}^x[f(B_t)]$$

$$= \int \mathbb{E}^x[\overline{g(B_0)}f(B_T)]$$

• $$\left(g, e^{-t(-\frac{1}{2}\Delta+V)}f\right) = \int \mathbb{E}^x\left[\overline{g(B_0)}f(B_T)\,e^{-\int_0^t V(B_s)ds}\right]$$

$$\left(\Phi, e^{-2t(H_f+\phi(h))}\Psi\right) = e^{\frac{1}{2}S}\left(\Phi, e^{G^+(v)}e^{-2TH_f}e^{a(\bar{v})}\Psi\right)$$

• $$\left(g, e^{-2T(h)}f\right) = \int \mathbb{E}^x\left[\overline{g(B_{-T})}f(B_T)\,e^{-\int_{-T}^{T}V(B_s)ds}\right]$$

$(B_t)_{t\in\mathbb{R}}$: $B_{-t} \doteq B_t$, $B_t \perp\!\!\!\perp B_s$

for $t<0<s$ //

$$\mathbb{E}[f(B_t)] = \int f(y+x) P_t(y)\, dy$$

$$= \left(e^{-\frac{t}{2}(-\Delta)} f\right)(x), \{y, t\} \wedge \{\ .$$

$$(f, e^{-t h_0} g) = \int dx\, \overline{f(x)}\, E^x[g(B_t)]$$

$$(f, e^{-t(h_0+V)} g) = \int dx\, \mathbb{E}^x\left[\overline{f(B_0)}\, g(B_t)\, e^{-\int_0^t V(B_s) ds}\right]$$

$$h_0 + V = -\frac{1}{2}\Delta + V$$

we can extend this

$$\frac{1}{2}[\sigma \cdot (-i\nabla - a)]^2 + V \quad \text{or}$$

$$\sqrt{[\sigma \cdot (-i\nabla - a)]^2 + m^2} + V \quad \dots \text{etc}.$$

$$\Psi\left(\tfrac{1}{2}\sigma \cdot (-i\nabla - a)\right) + V \qquad \Psi : \text{Bernstein function}.$$

$$\Psi(x) = \sqrt{2x^2 + m^2}$$

Thm

$$\left(F, e^{-2TH} G\right) = \int dx\, \mathbb{E}^{x}\Big[\big(F(B_{-T}), \underbrace{e^{a^{+}(U)}}_{} e^{-2TH_f} e^{a(\tilde{U})} G(B_T)\big)\Big]$$

$$L\; e^{-\int_{-t}^{t} V(B_s)\,ds}\; e^{\frac{1}{2} S}$$

$$S = \int_{-T}^{T} dt \int_{-T}^{T} ds\, W(B_s - B_t, s-t) \qquad W(x,t) = \int \frac{\hat{\rho}}{\sqrt{2\omega}} e^{-t|\omega|} e^{-ikB_s} dk$$

$F = f \otimes \Omega$, $G = g \otimes \Omega$ のとき

$$\left(F, e^{-2TH} G\right) = \int dx\, \mathbb{E}^{x}\left[e^{\frac{1}{2}S} \overline{f(B_{-T})}\, g(B_T)\right]$$

$d = 3,\quad h = -\frac{1}{2}\Delta + V$

$H = h \otimes \mathbb{1} + \mathbb{1} \otimes H_f + \phi(x)$

$\phi(x) = \frac{1}{\sqrt{2}}\left\{a^+\left(\frac{\hat\varphi}{\sqrt{\omega}} e^{-ikx}\right) + a\left(\frac{\tilde{\hat\varphi}}{\sqrt{\omega}} e^{+ikx}\right)\right\}$

$\tilde{\hat\varphi} = \hat\varphi(-k)$ if φ is $\mathbb{R}$-valued, $\tilde{\hat\varphi} = \overline{\hat\varphi}$

Thus $\phi(x)^* = \phi(x)$ for each $x \in \mathbb{R}^3$

$\boxed{\hat\varphi/\sqrt{\omega} \in L^2}$

<u>Thm</u> $\hat\varphi/\omega \in L^2$. Then H is self-adjoint on $D(h\otimes \mathbb{1}) \cap D(\mathbb{1}\otimes H_f)$

☺ $\|\phi(x)F\| \le \varepsilon \|(h\otimes 1 + 1\otimes H_f)F\|(\|\varphi/\sqrt{\omega}\| + \|\varphi/\omega\|) + b_\varepsilon \|F\|$ ⌟

Groundstate

$$\begin{cases} \int |\hat{\varphi}|^2/\omega^3 \, dx < \infty & \text{IR} \\ & \text{Infrared regular} \\ \int |\hat{\varphi}|^2/\omega^2 \, dx = \infty & \text{IR singular} \end{cases}$$

$$\begin{cases} m > 0 & \text{massive} \\ m = 0 & \text{massless} \end{cases}$$

$$\hat{\varphi}(0) \neq 0 \Rightarrow \begin{cases} \text{massive} \to \text{IR regular} \\ \text{massless} \to \text{IR singular} \end{cases}$$

e) $V(x) = -\frac{1}{|x|}, \ -\frac{e^{-|x|}}{|x|}, \ |x|^2$ etc

Assumption V ① $\lim_{|x|\to\infty} V(x) = \infty$

② $V(x) \leq -\frac{a}{|x|^\alpha} \quad (0 < {}^\exists\alpha < 2)$

Thm. • Suppose that h has a ground state and IR regular $\Rightarrow$ H has a groundstate

• IR singular $\Rightarrow$ H has no groundstate.

I mentioned that way to make it easier for you all to understand

$m = 0$

1960 ~ 1970

$m = 0$

IR regular

$\exists$ ground state

IR singular

no ground state

1995 ~

- BFS, AH $|g| \ll 1$
- H. Spohn $\forall g$ 1998
- C. Gérard $\forall g$ 2000
- GLL 2001

Arai H H 1999

LMS 2001

- h has no ground state, but $|g| \gg 1$
  H has a ground state Enhanced binding

  H + Spohn 2001

- Nelson model on mfd

  Gérard + F.H. + Panatti + Suzuki 2011, 12, 13

Nelson on manifolds

$$H_{mfd} = \frac{1}{2}\,\partial^\mu a_{\mu\nu}(x)\,\partial^\nu + V$$
$$+\, d\Gamma(\hat{\omega}) + \hat{\phi}(x)$$

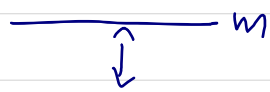
$$\phi(x) = \frac{1}{\sqrt{2}}\, a^\dagger\!\left(\frac{\hat{\varphi}}{\sqrt{\omega}}\, e^{-ikx}\right) + a^\dagger\!\left(\frac{\hat{\varphi}}{\sqrt{\omega}}\, e^{ikx}\right)$$

$$w = \sqrt{|k|^2 + m^2} \to \sqrt{-\Delta + m^2}$$
$$= \qquad \downarrow$$
$$\to \sqrt{\partial^\mu a_{\mu\nu}\partial^\nu + m^2(x)}$$

variable mass

$m^2(x)$

$$\underline{m^2(x) \sim \langle x\rangle^{-1}}$$



massive

massless

$m > 0$ $\quad \exists \varphi_m$ ground state $\quad \|\varphi_m\| = 1$

$\therefore \ \underset{m}{w\text{-}\lim}\, \varphi_m = \varphi_0 \qquad \varphi_0 \neq 0$

- $\exists K \varphi_m \not\to 0$ $\qquad \varphi_m = \bigoplus_{n=0}^{\infty} \varphi_m^{(n)}$

  cpt

- $\varphi_m^{(n)} \in W^{1,p}(U)$

  $i : W^{1,p}(U) \xrightarrow{\text{cpt}} L^q$

  $U \underset{\text{cpt}}{\subset} \mathbb{R}^3$ $\qquad q < \frac{dp}{d-p}$ $\quad \boxed{q < \frac{2p}{2-p}}$

  $\{\varphi_m^{(n)}\}_m \subset L^q$ cpt $\qquad \boxed{p > 1 \to q = 2}$

- Funtional integration

0) QFT - method Lattice app.

1) Functional integration

2) $\exists T$ cpt op st $T\varphi_m \not\to 0 \Rightarrow \varphi_0 \neq 0$

3) $\varphi_m^{(n)} \in W^{1,p}(U) \xrightarrow[i]{} L^q(U)$ $\qquad q < \frac{3p}{3-p}$

   $U \underset{\text{cpt}}{\subset} \mathbb{R}^{3n}$ $\qquad$ cpt $\qquad$ Rellich-K.

   $p > 6/5 \to \{\varphi_m^{(n)}\} \underset{\text{cpt}}{\subset} L^2$.

x Ground state by FH from Springer

ground state expectation (Localization)

$$\varphi = \varphi(x, \phi, n)$$

$$\varphi = \bigoplus_{n=0}^{\infty} \varphi^{(n)} \quad \text{s.t.} \quad \sum_{n=0}^{\infty} \| \varphi^{(n)} \|^2 < \infty$$

$$L^2 \otimes \mathcal{F} = \mathcal{H} \qquad \sum_{n=0}^{\infty} \| F_n(x) \|^2_{L^2 \otimes \mathcal{F}^{(n)}}$$

$$\| F \|^2_{\mathcal{H}} = \int \| F(x) \|^2_{\mathcal{F}} \, dx = \int \sum_{n=0}^{\infty} \| F_n(x) \|^2_{\mathcal{F}^{(n)}} \, dx$$

- $\| \varphi(x) \|_{\mathcal{F}} \sim e^{-|x|}$? $\quad C_1 e^{-C_2 |x|} \le \| \varphi(x) \|_{\mathcal{F}} \le C_3 e^{-C_4 |x|}$
  lower bound
- $\sum_{n=0}^{\infty} e^{+\beta n} \| \varphi_n(x) \|^2_{L^2 \otimes \mathcal{F}} < \infty$? $\quad \forall \beta$
- $\| e^{\beta \phi^2(g)} \varphi \|_{\mathcal{H}} < \infty \quad \exists \beta > 0 \ \exists g$

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$\frac{1}{\sqrt{2}}\, q + ip = a, \quad \frac{1}{\sqrt{2}}\, q - ip = a^{+} \qquad [a, a^{+}] = 1$

$h = a^{+}a = \frac{1}{2}(p^2 + q^2 - 1)$ harmonic oscillator

what is the ground state.

$\varphi_0 = \pi^{-1/4}\, e^{-\frac{1}{2}|x|^2}$

$\varphi_1 = x\,\varphi_1 \ \cdots \qquad \varphi_n = h_n(x)\,\varphi_0$

$f \geq 0$

$$\frac{e^{-th} f}{\| e^{-th} f \|} \longrightarrow \varphi_0 \quad (t \to \infty) \quad \because\ (f, \varphi_0) \neq 0$$

Actually $f = 1$

$$e^{-th} f = E^{x}\left[e^{-\frac{1}{2}\int_0^t |B_s|^2 ds} f(B_t) \right] = E^{x}\left[e^{-\frac{1}{2}\int_0^t |B_s|^2 ds} \right]$$

$$= e^{-\frac{1}{2}|x|^2 \frac{1}{\coth t}}$$

$$\therefore \frac{e^{-th} 1}{\| e^{-th} 1 \|^2} = \frac{e^{-\frac{1}{2}|x|^2 \frac{1}{\coth t}}}{\left(\int \left(e^{-\frac{1}{2}|x|^2 \frac{1}{\coth t}} \right)^2 dx \right)^{1/2}}$$

$\coth t \to 1 \quad t \to \infty$

$$= \pi^{-1/4}\, e^{-\frac{1}{2}|x|^2} \quad ✓$$

$$\| e^{\beta |x|^2} \varphi_0 \| \to \infty \quad \text{as } \beta \uparrow \tfrac{1}{2}$$

QM $\quad (-\frac{1}{2}\Delta + V)\varphi = E\varphi$

$\therefore \quad V(x) \sim |x|^{2n} \quad \therefore \quad \varphi(x) \sim e^{-|x|^{n+1}}$

$\therefore$

$$\varphi = e^{-t\mathcal{H} + tE}\varphi = e^{tE}\,\mathbb{E}^x\left[e^{-\int_0^t V}\varphi(B_t)\right]$$

$$\leq e^{tE}\,\mathbb{E}\left[e^{-\int_0^t V(B_s+x)\,ds}\right]\|\varphi\|_\infty$$

$\sigma_R(x) = \inf\{t > 0 \mid B_t + x \in B_R\}$ Carmona est

or

Agmon metric

$$\mathbb{E}\left[\mathbb{1}_{t<\sigma_R(x)}\, e^{-\int_0^t (V-E)}\right] \sim e^{-t(V(x)-E)}$$

$$+\mathbb{E}\left[\mathbb{1}_{t\geq\sigma_R(x)}\, e^{-\int_0^t V-E}\right] \sim e^{-\frac{|x|^2}{2t}}$$

$B_R(x) = \Gamma_R(x)$

$\frac{|x|^2}{2T} = (V(x) - E)t \quad \therefore \quad t = \sqrt{\frac{|x|^2}{2(V(x)-E)}}$

$\therefore \quad \varphi(x) \sim e^{-|x|\sqrt{V(x)-E}}$

φ ground state of H.

$$\varphi^T = e^{-TH} f\otimes\Omega \,/\, \| e^{-TH} f\otimes\Omega \| \to \varphi \quad (T\to\infty) \qquad \boxed{f \geqslant 0}$$

$$\| e^{\beta\,\phi(f)^2} \varphi \| \to ? \quad \beta\uparrow\ ?$$

- -

$N = d\Gamma(\mathbb{1})$ number op.

$$N\varphi = N \bigoplus_{n=0}^{\infty} \varphi_n = \bigoplus_{n=0}^{\infty} n\varphi_n$$

$$\therefore\ e^{\beta N}\varphi = \bigoplus e^{\beta n}\varphi_n$$

Here $(\varphi, e^{-\beta N}\varphi) = \sum(\varphi_n, e^{-\beta n}\varphi_n)$

$$\parallel$$

$$\lim_{T\to\infty} (\varphi^T, e^{-\beta N}\varphi^T) = \lim_{T\to\infty} \frac{(e^{-TH} f\otimes\Omega,\ e^{-\beta N} e^{-TH} f\otimes\Omega)}{(e^{-TH} f\otimes\Omega,\ e^{-TH} f\otimes\Omega)}$$

$$= \frac{\int \mathbb{E}^x\left[e^{S} f(B_{-T}) f(B_T)\, \underbrace{e^{-(1-e^{-\beta})\int_{-T}^{0}\int_{0}^{T} W(s-t,\, B_s - B_t)}}_{x\text{-indep}}\right] dx}{\int \mathbb{E}^x\left[e^{S} f(B_{-T}) f(B_T)\right] dx}$$

$$\Rightarrow \frac{\mathbb{E}^0\left[\int e^{S(x)} f(B_{-T}+x) f(B_T+x)\, dx\ e^{-(1-e^{-\beta})\int\int W}\right]}{\mathbb{E}^0\left[\int \cdots\cdots\cdots\ dx\right]}$$

$$\int e^{-(1-e^{-\beta})\int_{-T}^{0}\int_{0}^{T} W} d\mu_T \qquad \mathcal{H}$$

$$\bigcup_{s>0} \sigma(B_t \mid t\in[-s,s]) = \mathcal{B}$$

Thm (FH 2014)

μ_T is a measure on $(\mathcal{H}, \sigma(\mathcal{B}))$

s.t. $\exists \mu_\infty$ st $\mu_T(A) \to \mu_\infty(A)$ $\forall A \in \sigma(\mathcal{B})$

we assume

$$\left|\int_{-T}^{0}\int_{0}^{T} W\right| \le \int \frac{|\hat{\varphi}|^2}{\omega^3} < \infty$$

$$\therefore \int_{-\infty}^{0}\int_{0}^{\infty} W < \infty$$

$$\therefore \lim_{T} (\varphi^T, e^{-\beta N}\varphi^T) = (\varphi, e^{-\beta N}\varphi)$$

$$\| \qquad\qquad = \sum e^{-\beta n}\|\varphi_n\|^2$$

$$\int_{\mathcal{H}} e^{-(1-e^{-\beta})\int_{-\infty}^{0}\int_{0}^{\infty} W} d\mu_\infty \qquad \beta\in[0,\infty) \longmapsto \beta\in\mathbb{C}$$

$$\therefore \sum e^{+\beta n}\|\varphi_n\|^2 = \int e^{-(1-e^{+\beta})\int_{-\infty}^{0}\int_{0}^{\infty} W} d\mu_\infty < \infty$$

$$(\varphi, e^{-\beta\phi(g)^2}\varphi) = \lim_{T} (\varphi^T, e^{-\beta\phi(g)^2}\varphi^T)$$

$$= \frac{1}{\sqrt{1+\beta\|g\|^2}} \mathbb{E}_{\mu_0}\left[e^{\frac{-\beta K^2}{1+\beta\|g\|^2}} \right]$$

where

$$K = \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{-|s|\omega} e^{-iu\cdot B_s} \hat{\varphi}/\sqrt{\omega} ,\ g \right)_{L^2} ds$$

$$\therefore \ \| e^{+\frac{\beta}{2}\phi(g)^2}\varphi\|^2 \to \infty \qquad \beta \to \frac{1}{\|g\|^2}$$

$$H_\varepsilon = \left(-\frac{1}{2}\Delta + V\right) \otimes \mathbb{1} + \mathbb{1} \otimes H_f + \phi_\varepsilon(x)$$

$$\phi_\varepsilon(x) = \frac{1}{\sqrt{2}} \left\{ a^+\left(\frac{e^{-\varepsilon|k|^2/2}}{\sqrt{\omega}} e^{-ik\cdot x} \right) + a\left(\frac{e^{-\varepsilon|k|^2/2}}{\sqrt{\omega}} e^{+ik\cdot x} \right) \right\}$$

$\varepsilon > 0$ $\quad\quad$ $\omega = \sqrt{|k|^2 + m^2}$ $\quad\quad$ $m \geq 0$

Thm (E. Nelson 1964)

Let $E_\varepsilon = -\frac{1}{2} \int \frac{e^{-\varepsilon|k|^2}}{\omega} \cdot \frac{1}{|k|^2/2 + \omega} dk \to -\infty \ (\varepsilon \downarrow 0)$

$H_\varepsilon - E_\varepsilon \to \exists H_{ren} \ (\varepsilon \downarrow 0)$ in semi group.

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we had proven it by path measure.

$V = 0$

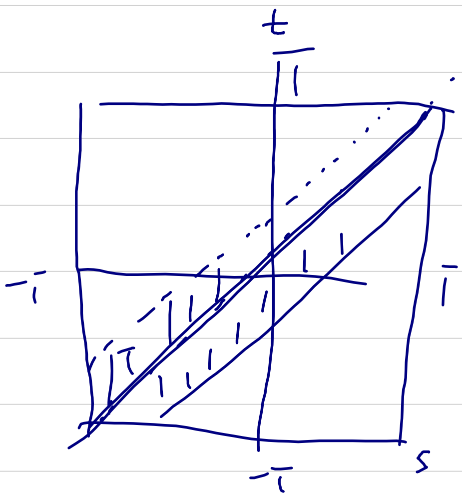
$$(f\otimes\Omega,\ e^{-2TH_\varepsilon} g\otimes\Omega) = \int dx\, \mathbb{E}^x\left[e^{\frac{1}{2}S} f(B_{-T})\, g(B_T) \right]$$

where

W

$$S = \int_{-T}^{T}\int_{-T}^{T}\left(\int \frac{e^{-\varepsilon|k|^2}}{2\omega}\, e^{-|t-s|\omega}\, e^{-ik(B_t-B_s)}\, dk\right) dt\, ds$$

$\varepsilon \to 0$



$+$

$$\int_{-T}^{T} ds \int_{s}^{s+\tau} W\, dt = 2T\rho(0,0) - \int_{-T}^{T} \rho(B_s - B_\tau,\ s-\tau)\, ds$$

$$+ \int_{-T}^{T}\int_{s}^{s+\tau} \nabla\rho(B_s - B_t,\ s-t)\, dB_t\, ds$$

$$\rho \doteq \rho_\varepsilon(x,t) = \int \frac{e^{-\varepsilon|k|^2}\, e^{-ikx}\, e^{-|t|\omega}}{2\omega}\, dk$$

1-dim

$$X_T = \int_0^T f(B_t)\,dB_t = \sum_{j=1}^{n} f(B_{\frac{t}{n}j-1})(B_{\frac{t}{n}j} - B_{\frac{t}{n}j-1})$$

$$g(X_T) - g(X_0) = dg(X_T)$$

$$\boxed{dB_t = \sqrt{dt}}$$

$$= g'\,dX_t + \frac{1}{2} g''\,dX_t\,dX_t$$

$$g(X_t) - \boxed{g(X_0)} = \int_0^T g'(X_t) f(B_t)\,dB_t + \frac{1}{2}\int_0^T g''(X_t) f^2\,dt$$

$$\frac{1}{2} S \quad \text{where} \quad S = \int_{-T}^{T}\int_{-T}^{T} W = 2\underbrace{\int_{-T}^{T}\int_{s}^{s+\tau}}_{D.} + 2\underbrace{\int_{-T}^{T}\int_{s+\tau}^{T}}_{OD}$$

$$\left.\begin{aligned}\int_{-T}^{T}\int_{s}^{s+T} W &= 2T\,P(0,0) - \int_{-T}^{T} P(B_s - B_{s+\tau} - \tau)\,ds \\ &+ \int_{-T}^{T}\int_{s}^{s+\tau} \nabla P(B_s - B_{t},\, s-t)\,dB_t\,ds\end{aligned}\right| *$$

$$\frac{1}{2} S \to \frac{1}{2} S - 2TP(0,0) = \underbrace{*}_{\substack{\text{D-term} \\ \text{- ren.}}} + \underbrace{\int_{-T}^{T}\int_{s+\tau}^{T} W}_{\text{OD-term}} = \frac{1}{2} S^{\varepsilon}_{ren}$$

Let $\tau = T$. OD-term disappeared.

$\therefore \ (f \otimes \Omega, \ e^{-2T(H+P(\varphi))} g \otimes \Omega)$

$= \int dx \ \mathbb{E}\left[\overline{f(B_T)}\, g(B_T)\, e^{\frac{1}{2} S^{\varepsilon}_{\mathrm{nen}}} \right] \longrightarrow \varepsilon \downarrow 0$ conv.

<u>STEP 2</u> $(f \otimes \Phi, \ e^{-2T(H+P(\varphi))} g \otimes \Phi)$

$\longrightarrow$ conv.

$F = F(\phi(f_1) \cdots \phi(f_m)) \quad F \in \mathcal{S}(\mathbb{R}^m)$

<u>STEP 3</u> $(F, \ e^{-2T(H+P(\varphi))} G)$

$\forall F, G$?

<u>Thm</u> (Key thm) $H_{\varepsilon} + P(\varphi) > -C$ $\exists$ ε-indep

$\forall F, \forall G \quad \exists F_n \ G_n \in D$ dense

$(F, \ e^{-2T H} G)$ limiting argument

$\therefore \ \exists H_{\mathrm{ren}}$ st $\lim_n (F, \ e^{-2T H_{\varepsilon} - P(\varphi)} G)$

(Mattee Møll 2017)

$$\left(F, e^{-2T H_{ren}} G\right)_{-\int_{-T}^{T} V}$$

$$= \int dx\, \mathbb{E}\left[e^{\frac{1}{2} S_{ren}^{V}} e^{-T}\left(F(B_{-T}), e^{a^{+}(U)} e^{-2TH_f} e^{a(\tilde{U})} G(B_T)\right)\right]$$

S_{ren} = complicated which include stochastic int.

$$U = -\frac{1}{\sqrt{2}} \int_{-T}^{T} \frac{1}{\sqrt{w}} e^{-|s-T|w} e^{-ik B_s}$$

$$\tilde{U} = -\frac{1}{\sqrt{2}} \int_{-T}^{T} \frac{1}{\sqrt{v}} e^{-|s+T|w} e^{\pm ik \cdot B_s}$$