Higher order phase transitions in the BCS model with imaginary magnetic field

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Abstract

In the BCS model with imaginary magnetic field at positive temperature we provide necessary and sufficient conditions for existence of a higher order phase transition driven by temperature. We define the order of the phase transition by regularity of the extended free energy density with temperature. More precisely we prove the following. There exist a non-vanishing free dispersion relation and a weak coupling constant such that a temperature-driven phase transition of order $n \in 4\mathbb{N} + 2$ $(= \{6, 10, 14, \cdots\})$ occurs if and only if the minimum of the magnitude of the free dispersion relation over the maximum is less than or equal to the critical value $\sqrt{17 - 12\sqrt{2}}$. These statements are also proved to be equivalent to that there exist a non-vanishing free dispersion relation and a weak coupling constant such that the phase boundary varying with the inverse temperature has a stationary point of inflection. Moreover, it follows that for any non-vanishing free dispersion relation and weak coupling constant the temperature-driven phase transition is of 2nd order if and only if the minimum of the magnitude of the free dispersion relation over the maximum is larger than $\sqrt{17 - 12\sqrt{2}}$. *

1 Introduction and main results

1.1 Introduction

The infinite-volume limit of the many-electron system governed by the Bardeen-Cooper-Schrieffer (BCS) model with imaginary magnetic field can be explicitly derived for any positive temperature and weak coupling constant if the free dispersion relation is non-vanishing, as shown in the preceding work [14]. While the temperature, the imaginary magnetic field and the coupling constant are largely

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restricted in [12], where the free Fermi surface is non-degenerate, and in [13], where the free Fermi surface is typically degenerate but non-empty, we have more freedom to choose these parameters in the framework of [14]. In particular, if the coupling constant is sufficiently small depending on the non-vanishing free dispersion relation, we can fully draw the phase boundary in the 2D plane of (inverse temperature, imaginary magnetic field) and justify the derivation of the infinite-volume limit of the free energy density at the same time. This means that we can reach a rigorous conclusion on the phase transitions happening in the infinite-volume limit of the many-electron system by means of mathematical analysis of the phase boundary. This is what we aim for in this paper.

The imaginary magnetic field can be considered as the real time variable in the context of dynamical phase transition ([1], [11]). This fact motivates us to work on this unconventional BCS model. However, we remark that our notion of phase diagram is different from the dynamical phase diagram defined in the physics literature (e.g. [26], [10], [20], [21]). Our phase diagram drawn in [14, Subsection 2.1] shows the boundary of a region where the gap equation has a positive solution in the plane of (inverse temperature, real time). On the other hand, the dynamical phase diagrams in [26], [10], [20], [21] show boundaries of different regions in a plane of 2 parameters, which does not include the real time variable. The 2 parameters plus the real time variable control a dynamical analogue of free energy density called return rate function. The 2 parameters belong to the inside of the boundaries if the return rate function depending on these parameters has a particular singularity with respect to the real time variable.

In [14, Section 2] we proved that the phase transition driven by the real time variable is of 2nd order and that driven by the temperature is also of 2nd order at most of the critical temperatures. Recall that we define the order of phase transition in terms of regularity of the extended free energy density, which is an analogy to the Ehrenfest classification. Moreover we gave a necessary and sufficient condition for the representative phase boundary to have only one local minimum point (LMiP). More precisely, the condition is that the minimum of the modulus of the free dispersion relation over the maximum is larger than the critical value $\sqrt{17-12\sqrt{2}}$. We did not relate the order of the phase transition to the uniqueness of LMiP, though in [14, Remark 2.6] we mentioned a possibility of the temperature-driven phase transition of higher order in case that the phase boundary has a stationary point of inflection.

The main results of this paper are obtained by pursuing the question raised in [14, Remark 2.6]. Admitting the free energy density of the BCS model with imaginary magnetic field characterized in [14, Theorem 1.3 (ii)], we prove that we can choose a non-vanishing free dispersion relation and a weak coupling constant so that the system has a temperature-driven phase transition of order n for some $n \in 4\mathbb{N}+2$ $(= \{6, 10, 14, \cdots\})$ if and only if the minimum of the modulus of the free dispersion relation over the maximum is less than or equal to $\sqrt{17-12\sqrt{2}}$. We also prove equivalence between existence of a higher order phase transition (HOPT) driven by temperature and existence of a stationary point of inflection (SPI) on the phase boundary. It follows in particular that the temperature-driven phase transition is of order $n \in 4\mathbb{N}+2$ if the critical inverse temperature is a SPI of the phase boundary, it is of 2nd order otherwise. In the previous work [14] we were unaware of the relation between the order of the phase transition and the critical value $\sqrt{17-12\sqrt{2}}$. The essential new finding in this paper is that the universal constant $\sqrt{17-12\sqrt{2}}$ is also a critical value for existence of a HOPT driven by temperature.

Once the equivalence between existence of a HOPT and existence of a SPI is established, we focus on the problem of existence / non-existence of a SPI. Our study on the uniqueness / non-uniqueness of a LMiP of the phase boundary in [14, Section 2] essentially helps us in this part. The proof of uniqueness of LMiP is technically close to the proof of non-existence of SPI. Specifically, we apply [14, Lemma 2.12] as the key lemma. When there are 2 LMiPs on the phase boundary, we can continuously transform the free dispersion relation until one of the LMiPs disappears. In the middle of this process a SPI appears on the phase boundary. This is how we prove the existence of a SPI and thus a HOPT. We remark that [14, Lemma 2.15] plays a key role in the proof of the existence in a critical case. After proving the main theorems we study specific models in terms of SPI and HOPT. There we also apply [14, Lemma 2.24] and admit the proof of [14, Proposition 2.26]. The critical value $\sqrt{17-12\sqrt{2}}$, whose original meaning is a root of the polynomial X^4-34X^2+1, is already involved in [14, Lemma 2.12], [14, Lemma 2.15] and [14, Lemma 2.24]. This work can certainly be seen as a continuation of [14, Section 2] from the technical viewpoint.

To the best of the author's knowledge, phase transition in the BCS model with imaginary magnetic field at positive temperature has not been reported in other articles except for [12], [13], [14]. Concerning the conventional BCS model without imaginary magnetic field, it is a general consensus that the temperature-driven transition between superconducting / normal phase is of 2nd order. Despite that there are many mathematical papers studying the BCS theory (see, e.g., the review articles [9], [2]), it seems that only a few have tried to prove the order of the phase transition. There are mathematical constructions toward the 2nd order phase transition in a BCS-type thermodynamic potential by Watanabe ([22], [23], [24], [25]).

We find more articles related to the present paper's theme, namely HOPT in superconductors, in physics literature. Cronström and Noga [3] obtained a mean field solution to the BCS model in thin films and a layered structure, which shows a 3rd order superconducting phase transition. There are attempts to explain experimentally observed anomalous superconducting phase transitions in terms of HOPT, especially of 3rd / 4th order, by extending the phenomenological Ginzburg-Landau theory. Kumar and the coauthors ([16], [18], [19], [17], [8]) initiated this approach. Later Ekuma and the coauthors ([4], [5], [7], [6]) continued in this line of research, aiming in particular to explain a 3rd order phase transition in iron-based superconductors.

This paper is organized as follows. In the rest of this section we prepare necessary concepts and state the main results of this paper. In Section 2 we prove the main theorems step by step by establishing various propositions ranging from the equivalence between HOPT and SPI to existence / non-existence of a SPI. In Section 3 we study whether HOPT is possible in multi-orbital non-hopping models and a one-dimensional nearest-neighbor hopping model. These are the same models as those analyzed in [14, Subsection 2.3] with regard to uniqueness / non-uniqueness of a LMiP of the phase boundary.

1.2 The main results

We keep using many of the notations introduced in [14, Section 1, Section 2]. Let us reintroduce the important ones for clarity of the present paper. With the dimension $d \in \mathbb{N}$ let $(\hat{\mathbf{v}}_j)_{j=1}^d$ denote a basis of $\mathbb{R}^d$. Define the subset Γ_∞^* of $\mathbb{R}^d$ by

$$\Gamma_\infty^* := \left\{ \sum_{j=1}^d \hat{k}_j \hat{\mathbf{v}}_j \ \middle|\ \hat{k}_j \in [0, 2\pi]\ (j = 1, \cdots, d) \right\}.$$

Originally the set Γ_∞^* is the continuum limit of a finite momentum lattice spanned by $(\hat{\mathbf{v}}_j)_{j=1}^d$, which is denoted by Γ^* below. Take $b \in \mathbb{N}$ and e_{min}, $e_{max} \in \mathbb{R}_{>0}$ satisfying $e_{min} \le e_{max}$. The set $\mathcal{E}(e_{min}, e_{max})$ of one-particle Hamiltonians in momentum space is defined as follows. $E \in \mathcal{E}(e_{min}, e_{max})$ if and only if

$$\begin{aligned}
&E \in C^\infty(\mathbb{R}^d, \mathrm{Mat}(b, \mathbb{C})),\\
&E(\mathbf{k}) = E(\mathbf{k})^*,\ \forall \mathbf{k} \in \mathbb{R}^d,\\
&E(\mathbf{k} + 2\pi\hat{\mathbf{v}}_j) = E(\mathbf{k}),\ \forall \mathbf{k} \in \mathbb{R}^d,\ j \in \{1, \cdots, d\},\\
&E(\mathbf{k}) = \overline{E(-\mathbf{k})},\ \forall \mathbf{k} \in \mathbb{R}^d, \qquad (1.1)\\
&\inf_{\mathbf{k} \in \mathbb{R}^d} \inf_{\substack{\mathbf{u} \in \mathbb{C}^b \\ \text{with } \|\mathbf{u}\|_{\mathbb{C}^b} = 1}} \|E(\mathbf{k})\mathbf{u}\|_{\mathbb{C}^b} = e_{min} (> 0),\\
&\sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b} = e_{max}.
\end{aligned}$$

Here $\mathrm{Mat}(b, \mathbb{C})$ is the complex Banach space of $b \times b$ complex matrices equipped with the operator norm $\|\cdot\|_{b \times b}$. Also, $\|\cdot\|_{\mathbb{C}^b}$ denotes the canonical norm of $\mathbb{C}^b$ induced by the Hermitian inner product.

Some of the properties assumed in $\mathcal{E}(e_{min}, e_{max})$ will not be used in this paper at all. For example, we do not need to assume that $\mathbf{k} \mapsto E(\mathbf{k})$ is infinitely differentiable and (1.1) to complete the proofs of the main results. We keep these conditions in this paper in order to emphasize that the free energy density analyzed in this paper is the same as that rigorously derived in [14, Theorem 1.3 (ii)] by assuming these conditions.

Our main theorems concern the free energy density which explicitly involves the solution Δ to the gap equation. Therefore we have to introduce the gap equation in advance. For $E \in \mathcal{E}(e_{min}, e_{max})$ the function $g_E : \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{aligned}
&g_E(x, t, z)\\
&:= -\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(x\sqrt{E(\mathbf{k})^2 + z^2})}{(\cos(t/2) + \cosh(x\sqrt{E(\mathbf{k})^2 + z^2}))\sqrt{E(\mathbf{k})^2 + z^2}} \right),\\
&D_d := |\det(\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_d)|^{-1}(2\pi)^{-d}.
\end{aligned}$$

The parameter U is real, negative and called coupling constant. Remind us that for any function $f : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ and $\mathbf{k} \in \mathbb{R}^d$ $f(E(\mathbf{k}))(\in \mathrm{Mat}(b, \mathbb{C}))$ is defined via the spectral decomposition of $E(\mathbf{k})$. The next lemma is essentially the same as [14, Lemma 1.1].

Lemma 1.1. *The following statements hold for any* $(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R}$. *The equation* $g_E(\beta, t, \Delta) = 0$ *has a solution* Δ *in* $[0, \infty)$ *if and only if* $g_E(\beta, t, 0) \ge 0$. *Moreover, if a solution exists in* $[0, \infty)$, *it is unique.*

This lemma enables us to define the function $\Delta : \mathbb{R}_{>0}\times\mathbb{R} \to \mathbb{R}_{\ge 0}$ as follows. For $(\beta,t) \in \mathbb{R}_{>0}\times\mathbb{R}$, if $g_E(\beta,t,0) \ge 0$, let $\Delta(\beta,t) \in \mathbb{R}_{\ge 0}$ be such that $g_E(\beta,t,\Delta(\beta,t)) = 0$. If $g_E(\beta,t,0) < 0$, let $\Delta(\beta,t) := 0$. Observe that

$$\Delta(\beta,t) = \Delta(\beta,\delta t + 4\pi m),\ \forall(\beta,t) \in \mathbb{R}_{>0}\times\mathbb{R},\ \delta \in \{1,-1\},\ m \in \mathbb{Z}.$$

Moreover, we define the function $F_E : \mathbb{R}_{>0}\times\mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}F_E(\beta,t) :=& \frac{\Delta(\beta,t)^2}{|U|} - \frac{D_d}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\log\Bigg(2\cos\left(\frac{t}{2}\right)e^{-\beta E(\mathbf{k})} \\ &+ e^{\beta(\sqrt{E(\mathbf{k})^2+\Delta(\beta,t)^2}-E(\mathbf{k}))} + e^{-\beta(\sqrt{E(\mathbf{k})^2+\Delta(\beta,t)^2}+E(\mathbf{k}))}\Bigg).\end{aligned}$$

We can see that

$$F_E(\beta,t) = F_E(\beta,\delta t + 4\pi m),\ \forall(\beta,t) \in \mathbb{R}_{>0}\times\mathbb{R},\ \delta \in \{1,-1\},\ m \in \mathbb{Z}. \tag{1.2}$$

According to [14, Theorem 1.3 (ii)], for any $E \in \mathcal{E}(e_{min}, e_{max})$ there exists $c' \in (0,1]$ such that for any $\beta \in \mathbb{R}_{>0}$, $t \in \mathbb{R}$,

$$U \in \left(-\frac{2c'}{b}\min\{e_{min}, e_{min}^{d+1}\}, 0\right), \tag{1.3}$$

$$F_E(\beta,t) = \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left(-\frac{1}{\beta L^d}\log\left(\mathrm{Tr}\,e^{-\beta \mathsf{H}+it\mathsf{S}_z}\right)\right),$$

where c' depends only on d, b, $(\hat{\mathbf{v}}_j)_{j=1}^d$ and the quantity

$$\sup_{\mathbf{k}\in\mathbb{R}^d}\ \sup_{\substack{m_j\in\mathbb{N}\cup\{0\}\\ (j=1,\cdots,d)}}\left\|\prod_{j=1}^d\frac{\partial^{m_j}}{\partial k_j^{m_j}}E(\mathbf{k})\right\|_{b\times b} 1_{\sum_{j=1}^d m_j\le d+2}. \tag{1.4}$$

For any proposition P $1_P := 1$ if P is true, $1_P := 0$ otherwise. The operator H is the BCS model with the reduced quartic interaction and the one-particle Hamiltonian $E(\cdot)$, and S_z is the z-component of the spin operator. The negative parameter U controls the strength of attractive interaction between Cooper pairs in the BCS model H. More precisely,

$$\begin{aligned}\mathsf{H} :=& \frac{1}{L^d}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\ \sum_{\sigma\in\{\uparrow,\downarrow\}}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}E(\mathbf{k})(\rho,\eta)\psi^*_{\rho\mathbf{x}\sigma}\psi_{\eta\mathbf{y}\sigma} \\ &+ \frac{U}{L^d}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\psi^*_{\rho\mathbf{x}\uparrow}\psi^*_{\rho\mathbf{x}\downarrow}\psi_{\eta\mathbf{y}\downarrow}\psi_{\eta\mathbf{y}\uparrow}, \\ \mathsf{S}_z :=& \frac{1}{2}\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}(\psi^*_{\rho\mathbf{x}\uparrow}\psi_{\rho\mathbf{x}\uparrow} - \psi^*_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\downarrow}),\end{aligned}$$

where $\mathcal{B} := \{1, 2, \cdots, b\}$,

$$\Gamma := \left\{\sum_{j=1}^d m_j\mathbf{v}_j\ \Bigg|\ m_j \in \{0, 1, \cdots, L-1\}\ (j = 1, \cdots, d)\right\},$$

$$\Gamma^* := \left\{ \sum_{j=1}^{d} \hat{m}_j \hat{\mathbf{v}}_j \ \Big| \ \hat{m}_j \in \left\{ 0, \frac{2\pi}{L}, \frac{4\pi}{L}, \cdots, 2\pi - \frac{2\pi}{L} \right\} \ (j = 1, \cdots, d) \right\},$$

$(\mathbf{v}_j)_{j=1}^d$ is a basis of $\mathbb{R}^d$, dual to $(\hat{\mathbf{v}}_j)_{j=1}^d$ and for $(\rho, \mathbf{x}, \sigma) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}$ $\psi_{\rho\mathbf{x}\sigma}$ $(\psi^*_{\rho\mathbf{x}\sigma})$ is the annihilation (creation) operator on the Fermionic Fock space $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$. The Fermionic operators appear only here in this paper. As we want to relate the present construction to the original definitions, U is taken as a negative parameter throughout this paper even though we essentially deal with $|U|$ in every estimate.

Next let us recall the notion of phase boundary. We define the subsets Q_+, Q_-, Q_0 of $\mathbb{R}_{>0} \times \mathbb{R}$ by

$$\begin{aligned} Q_+ &:= \{(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g_E(\beta, t, 0) > 0\}, \\ Q_- &:= \{(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g_E(\beta, t, 0) < 0\}, \\ Q_0 &:= \{(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g_E(\beta, t, 0) = 0\}. \end{aligned}$$

It follows that $\mathbb{R}_{>0} \times \mathbb{R} = Q_+ \sqcup Q_- \sqcup Q_0$ and $\Delta(\beta, t) > 0$ if and only if $(\beta, t) \in Q_+$. We call Q_0 phase boundary. The main theme of this paper is to study the regularity of F_E on the phase boundary Q_0. Because of the periodicity of $g_E(\beta, t, 0)$ with t, Q_0 is infinite union of copies of one representative curve. This paper's main problems can be solved by focusing on the representative curve. The next lemma is essentially the same as [14, Lemma 1.2] and supports the well-definedness of the representative curve.

Lemma 1.2. *Assume that* $|U| < \frac{2e_{min}}{b}$. *Then, there uniquely exists*

$$\beta_c \in \left(0, \frac{2}{e_{min}} \tanh^{-1} \left(\frac{b|U|}{2e_{min}} \right) \right]$$

such that

$$\begin{aligned} &g_E(\beta, \pi, 0) < 0, \ \forall \beta \in \mathbb{R}_{>0}, \\ &g_E(\beta, 2\pi, 0) > 0, \ \forall \beta \in (0, \beta_c), \\ &g_E(\beta_c, 2\pi, 0) = 0, \\ &g_E(\beta, 2\pi, 0) < 0, \ \forall \beta \in (\beta_c, \infty), \end{aligned}$$

where $\tanh^{-1} : (-1, 1) \to \mathbb{R}$ *is the inverse function of* $\tanh$.

From here we always assume that $U \in (-\frac{2e_{min}}{b}, 0)$ so that the existence of the critical inverse temperature β_c is guaranteed by Lemma 1.2. By the monotone increasing property of $t \mapsto g_E(\beta, t, 0)$ in $(\pi, 2\pi)$ for any $\beta \in (0, \beta_c)$ there uniquely exists $\tau(\beta) \in (\pi, 2\pi)$ such that $g_E(\beta, \tau(\beta), 0) = 0$. This defines the function $\tau : (0, \beta_c) \to (\pi, 2\pi)$. By [14, Lemma 2.2 (i)] $\tau \in C^\omega((0, \beta_c))$. Remind us that for any open set $O(\subset \mathbb{R}^n)$ $C^\omega(O)$ denotes the set of real analytic functions on O. Using the function τ, we can characterize the phase boundary Q_0 as follows.

$$\begin{aligned} Q_0 = &\{(\beta, \delta\tau(\beta) + 4\pi m) \mid \beta \in (0, \beta_c), \ \delta \in \{1, -1\}, \ m \in \mathbb{Z}\} \\ &\cup \{(\beta_c, 2\pi + 4\pi m) \mid m \in \mathbb{Z}\}. \end{aligned} \tag{1.5}$$

The above characterization was given in [14, (2.3)]. We can see that Q_0 is a union of copies of

$$\{(\beta, \tau(\beta)) \mid \beta \in (0, \beta_c)\} \cup \{(\beta, -\tau(\beta) + 4\pi) \mid \beta \in (0, \beta_c)\} \cup \{(\beta_c, 2\pi)\},$$

and thus we can consider the above set as the representative curve of the phase boundary. Moreover,

$$\begin{aligned} Q_+ &= \bigsqcup_{m \in \mathbb{Z}} \left\{ (\beta, t) \;\middle|\; \beta \in (0, \beta_c),\ t \in (\tau(\beta) + 4\pi m, -\tau(\beta) + 4\pi(m+1)) \right\}, \\ Q_- &= \bigsqcup_{m \in \mathbb{Z}} \left\{ (\beta, t) \;\middle|\; \beta \in (0, \beta_c),\ t \in (-\tau(\beta) + 4\pi m, \tau(\beta) + 4\pi m) \right\}. \end{aligned} \tag{1.6}$$

This interestingly suggests that in this weak coupling regime the gap equation has a positive solution only when the temperature is high.

To state the main theorems, we have to make clear our definition of phase transition. For $(\rho, \eta) = (+, -)$ or $(-, +)$ let us set

$$Q_{\rho,\eta} := \left\{ (\beta_0, t_0) \in Q_0 \;\middle|\; \exists \varepsilon \in \mathbb{R}_{>0} \text{ s.t. } \begin{array}{l} (\beta, t_0) \in Q_\rho,\ \forall \beta \in (\beta_0 - \varepsilon, \beta_0), \\ (\beta, t_0) \in Q_\eta,\ \forall \beta \in (\beta_0, \beta_0 + \varepsilon). \end{array} \right\}.$$

Here we should recall the fact that for any $E \in \mathcal{E}(e_{min}, e_{max})$

$$F_E|_{Q_+ \cup Q_-} \in C^\omega(Q_+ \cup Q_-),\ F_E \in C^1(\mathbb{R}_{>0} \times \mathbb{R}), \tag{1.7}$$

which was proved in [14, Proposition 2.5 (i)]. For $(\beta_0, t_0) \in \mathbb{R}_{>0} \times \mathbb{R}$, $n \in \mathbb{N}(= \{1, 2, 3, \cdots\})$, $(\rho, \eta) \in \{(+, -), (-, +)\}$ we define the properties $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_0, t_0)$, $(\mathrm{PT})_{n,(\rho,\eta)}$ as follows.

$(\mathrm{PT})_{n,(\rho,\eta)}(\beta_0, t_0)$

$$\begin{aligned} &(\beta_0, t_0) \in Q_{\rho,\eta}, \\ &\lim_{\beta \nearrow \beta_0} \frac{\partial^m F_E}{\partial \beta^m}(\beta, t_0),\ \lim_{\beta \searrow \beta_0} \frac{\partial^m F_E}{\partial \beta^m}(\beta, t_0) \text{ converge to finite values} \\ &\text{for any } m \in \{0, 1, \cdots, n\}, \text{ and} \\ &\lim_{\beta \nearrow \beta_0} \frac{\partial^m F_E}{\partial \beta^m}(\beta, t_0) = \lim_{\beta \searrow \beta_0} \frac{\partial^m F_E}{\partial \beta^m}(\beta, t_0),\ \forall m \in \{0, 1, \cdots, n-1\}, \\ &\lim_{\beta \nearrow \beta_0} \frac{\partial^n F_E}{\partial \beta^n}(\beta, t_0) \neq \lim_{\beta \searrow \beta_0} \frac{\partial^n F_E}{\partial \beta^n}(\beta, t_0). \end{aligned}$$

$(\mathrm{PT})_{n,(\rho,\eta)}$ There exists $(\beta_0, t_0) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_0, t_0)$ holds.

By analogy with the Ehrenfest classification we state that the system has a phase transition of order n driven by temperature when $(\mathrm{PT})_{n,(\rho,\eta)}$ holds. According to [14, Proposition 2.5 (ii)], $(\mathrm{PT})_{2,(+,-)}$, $(\mathrm{PT})_{2,(-,+)}$ hold. The question here is whether $(\mathrm{PT})_{n,(\rho,\eta)}$ holds for $n \geq 3$, or in other words, a phase transition of order $n(\geq 3)$ driven by temperature occurs. The following fact based on (1.2), (1.5), (1.6) will be useful later.

Lemma 1.3. *Let* $\beta_0 \in (0, \beta_c]$, $n \in \mathbb{N}$, $(\rho, \eta) \in \{(+, -), (-, +)\}$. *The following statements are equivalent to each other.*

- *There exists $t_0 \in \mathbb{R}$ such that $(PT)_{n,(\rho,\eta)}(\beta_0, t_0)$ holds.*
- *$\{t \in \mathbb{R} \mid (\beta_0, t) \in Q_{\rho,\eta}\} \neq \emptyset$ and for any $t_0 \in \mathbb{R}$ satisfying $(\beta_0, t_0) \in Q_{\rho,\eta}$ $(PT)_{n,(\rho,\eta)}(\beta_0, t_0)$ holds.*
- *If $\beta_0 < \beta_c$, $(PT)_{n,(\rho,\eta)}(\beta_0, \tau(\beta_0))$ holds. If $\beta_0 = \beta_c$, $(PT)_{n,(\rho,\eta)}(\beta_0, 2\pi)$ holds.*

In addition, we need to prepare the concept of stationary point of inflection (SPI).

Definition 1.4. Let $a, b, c \in \mathbb{R}$ satisfy $a < c < b$. Let $f \in C^1((a, b), \mathbb{R})$.

(1) We call c rising stationary point of inflection of f if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} &(c-\varepsilon, c+\varepsilon) \subset (a, b), \\ &\frac{df}{dx}(c) = 0, \\ &\frac{df}{dx}(x) > 0, \ \forall x \in (c-\varepsilon, c+\varepsilon)\backslash\{c\}. \end{aligned}$$

(2) We call c falling stationary point of inflection of f if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} &(c-\varepsilon, c+\varepsilon) \subset (a, b), \\ &\frac{df}{dx}(c) = 0, \\ &\frac{df}{dx}(x) < 0, \ \forall x \in (c-\varepsilon, c+\varepsilon)\backslash\{c\}. \end{aligned}$$

(3) We call c stationary point of inflection of f if c is either a rising stationary point of inflection or a falling stationary point of inflection of f.

We define the properties $(\mathrm{SPI})_\xi(\beta_0)$, $(\mathrm{SPI})_\xi$ for $\xi \in \{r, f\}$, $\beta_0 \in \mathbb{R}_{>0}$ as follows.

$(\mathrm{SPI})_r(\beta_0)$	β_0 is a rising stationary point of inflection of $\tau(\cdot) : (0, \beta_c) \to \mathbb{R}$.
$(\mathrm{SPI})_f(\beta_0)$	β_0 is a falling stationary point of inflection of $\tau(\cdot) : (0, \beta_c) \to \mathbb{R}$.
$(\mathrm{SPI})_\xi$	There exists $\beta_0 \in (0, \beta_c)$ such that $(\mathrm{SPI})_\xi(\beta_0)$ holds.

Using these terms, we can state our main theorems. Theorem 1.5 summarizes the equivalence between existence of a HOPT and existence of a SPI plus the fact that if a HOPT occurs, it must be of order $n \in 4\mathbb{N} + 2$.

Theorem 1.5. *Let $d, b \in \mathbb{N}$, $(\hat{\mathbf{v}}_j)_{j=1}^d$ be a basis of $\mathbb{R}^d$, e_{min}, $e_{max} \in \mathbb{R}_{>0}$ satisfy $e_{min} \le e_{max}$, $U \in (-\frac{2e_{min}}{b}, 0)$, $E \in \mathcal{E}(e_{min}, e_{max})$, $(\xi, \rho, \eta) \in \{(r, +, -), (f, -, +)\}$ and $\beta_0 \in (0, \beta_c)$. Then the following statements hold.*

(i) $(SPI)_\xi(\beta_0)$ holds if and only if there exists $n \in 4\mathbb{N}+2$ $(= \{6, 10, 14, \cdots\})$ such that $(PT)_{n,(\rho,\eta)}(\beta_0, \tau(\beta_0))$ holds.

(ii) $(SPI)_\xi$ does not hold if and only if $(PT)_{2,(\rho,\eta)}(\beta, t)$ holds for any $(\beta, t) \in Q_{\rho,\eta}$.

(iii) $(\beta, t) \in Q_{\rho,\eta}$ *and* $(PT)_{2,(\rho,\eta)}(\beta, t)$ *does not hold if and only if there exists* $n \in 4\mathbb{N} + 2$ *such that* $(PT)_{n,(\rho,\eta)}(\beta, t)$ *holds.*

In essence Theorem 1.6 gives a necessary and sufficient condition for existence of a HOPT and a SPI.

Theorem 1.6. *For any* $d, b \in \mathbb{N}$, *basis* $(\hat{\mathbf{v}}_j)_{j=1}^d$ *of* $\mathbb{R}^d$ *and* $e_{min}, e_{max} \in \mathbb{R}_{>0}$ *satisfying* $e_{min} \le e_{max}$ *the following statements are equivalent to each other.*

(i) For any $U_0 \in (0, \frac{2e_{min}}{b})$, $(\rho, \eta) \in \{(+,-), (-,+)\}$ *there exist* $U \in [-U_0, 0)$, $E \in \mathcal{E}(e_{min}, e_{max})$, $n \in 4\mathbb{N} + 2$ $(= \{6, 10, 14, \cdots\})$ *such that* $(PT)_{n,(\rho,\eta)}$ *holds.*

(ii) For any $U_0 \in (0, \frac{2e_{min}}{b})$, $\xi \in \{r, f\}$ *there exist* $U \in [-U_0, 0)$, $E \in \mathcal{E}(e_{min}, e_{max})$ *such that* $(SPI)_\xi$ *holds.*

(iii)

$$\frac{e_{min}}{e_{max}} \le \sqrt{17 - 12\sqrt{2}}.$$

Theorem 1.6 is not logically equivalent to the following theorem, which essentially gives necessary and sufficient conditions for the temperature-driven phase transition to be of 2nd order.

Theorem 1.7. *For any* $d, b \in \mathbb{N}$, *basis* $(\hat{\mathbf{v}}_j)_{j=1}^d$ *of* $\mathbb{R}^d$ *and* $e_{min}, e_{max} \in \mathbb{R}_{>0}$ *satisfying* $e_{min} \le e_{max}$ *the following statements are equivalent to each other.*

(i) There exists $U_0 \in (0, \frac{2e_{min}}{b})$ *such that for any* $U \in [-U_0, 0)$, $E \in \mathcal{E}(e_{min}, e_{max})$, $(\rho, \eta) \in \{(+,-), (-,+)\}$, $n \in \mathbb{N}_{\ge 3}$ $(= \{3, 4, 5, \cdots\})$ $(PT)_{n,(\rho,\eta)}$ *does not hold.*

(ii) There exists $U_0 \in (0, \frac{2e_{min}}{b})$ *such that for any* $U \in [-U_0, 0)$, $E \in \mathcal{E}(e_{min}, e_{max})$, $(\rho, \eta) \in \{(+,-), (-,+)\}$, $(\beta, t) \in Q_{\rho,\eta}$ $(PT)_{2,(\rho,\eta)}(\beta, t)$ *holds.*

(iii) There exists $U_0 \in (0, \frac{2e_{min}}{b})$ *such that for any* $U \in [-U_0, 0)$, $E \in \mathcal{E}(e_{min}, e_{max})$, $\xi \in \{r, f\}$ $(SPI)_\xi$ *does not hold.*

(iv)

$$\frac{e_{min}}{e_{max}} > \sqrt{17 - 12\sqrt{2}}.$$

Remark 1.8. According to Theorem 1.6, a HOPT driven by temperature exists in the case $\frac{e_{min}}{e_{max}} \le \sqrt{17 - 12\sqrt{2}}$. Strictly speaking, we cannot state that a HOPT exists in the BCS model with imaginary magnetic field unless the derivation of $F_E(\beta, t)$ from the many-electron system is justified. In the case $\frac{e_{min}}{e_{max}} < \sqrt{17 - 12\sqrt{2}}$ the existence of a HOPT is guaranteed while the derivation of $F_E(\beta, t)$ is justified by [14, Theorem 1.3 (ii)]. See Remark 2.13. In the case $\frac{e_{min}}{e_{max}} = \sqrt{17 - 12\sqrt{2}}$, however, we cannot prove existence of a HOPT while justifying the derivation of $F_E(\beta, t)$. See Remark 2.16.

2 Proof of the main results

In this section we will prove Theorem 1.5, Theorem 1.6 and Theorem 1.7. The proof of Theorem 1.5 will be completed in Subsection 2.1. We decompose Theorem 1.6, Theorem 1.7 into several claims. We will prove the claims step by step. Combination of them will complete the proof of Theorem 1.6, Theorem 1.7 in the end of this section.

2.1 HOPT and SPI

Here we prove Theorem 1.5, the equivalence between the claim (i) and the claim (ii) of Theorem 1.6 and the equivalence between the claim (i), the claim (ii) and the claim (iii) of Theorem 1.7. To this end, we define the functions $\widetilde{F}_E$, $\widetilde{g}_E$: $\mathbb{R}_{>0} \times \mathbb{R} \times (-e_{min}^2, \infty) \to \mathbb{R}$ for $E \in \mathcal{E}(e_{min}, e_{max})$ by

$$\widetilde{F}_E(x,t,z) := \frac{z}{|U|} - \frac{D_d}{x} \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr} \log \left(\cos\left(\frac{t}{2}\right) + \cosh(x\sqrt{E(\mathbf{k})^2 + z}) \right),$$

$$\widetilde{g}_E(x,t,z) := -\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr} \left(\frac{\sinh(x\sqrt{E(\mathbf{k})^2+z})}{(\cos(t/2) + \cosh(x\sqrt{E(\mathbf{k})^2+z}))\sqrt{E(\mathbf{k})^2+z}} \right).$$

Observe that

(2.1)
$$F_E(\beta,t) = \widetilde{F}_E(\beta,t,\Delta(\beta,t)^2) - \frac{D_d}{\beta} \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr} \log(2e^{-\beta E(\mathbf{k})}),\ \forall (\beta,t) \in \mathbb{R}_{>0} \times \mathbb{R},$$

(2.2)
$$g_E(x,t,z) = \widetilde{g}_E(x,t,z^2),$$

(2.3)
$$\frac{\partial \widetilde{F}_E}{\partial z}(x,t,z) = -\frac{1}{2}\widetilde{g}_E(x,t,z),$$

(2.4)
$$\frac{\partial \widetilde{g}_E}{\partial z}(x,t,z) < 0,\ \forall (x,t,z) \in \mathbb{R}_{>0} \times \mathbb{R} \times (-e_{min}^2, \infty).$$

The inequality (2.4) is based on the fact that

$$x \mapsto \frac{\sinh x}{(\varepsilon + \cosh x)x} : \mathbb{R}_{>0} \to \mathbb{R} \tag{2.5}$$

is strictly monotone decreasing for any $\varepsilon \in [-1,1]$. The equality (2.1) suggests that we can study the regularity of the function F_E by analyzing $\widetilde{F}_E(\beta,t,\Delta(\beta,t)^2)$ instead. It follows from (1.7), (2.1) that

(2.6)
$$(\beta,t) \mapsto \widetilde{F}_E(\beta,t,\Delta(\beta,t)^2) \text{ is real analytic in } Q_+ \cup Q_- \text{ and } C^1\text{-class in } \mathbb{R}_{>0} \times \mathbb{R}.$$

We can see from this fact and the inequality (2.4) that the statement of the next lemma makes sense.

Lemma 2.1. *For any $n \in \mathbb{N}_{\geq 2}$ $(= \{2, 3, 4, \cdots\})$ and $(\beta, t) \in Q_+$ the following equality holds.*

(2.7)

$$\begin{aligned}
&\left(\frac{\partial}{\partial \beta}\right)^n \widetilde{F}_E(\beta, t, \Delta(\beta, t)^2) \\
&= \frac{\partial^n \widetilde{F}_E}{\partial x^n}(\beta, t, \Delta(\beta, t)^2) \\
&\quad + \frac{1}{2\frac{\partial \widetilde{g}_E}{\partial z}(\beta, t, \Delta(\beta, t)^2)} \sum_{j=0}^{n-2} \left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial \widetilde{g}_E}{\partial x}(x, t, z) \frac{\partial^{n-1-j} \widetilde{g}_E}{\partial x^{n-1-j}}(x, t, z)\right)\Bigg|_{\substack{x=\beta, \\ z=\Delta(\beta,t)^2}} \\
&\quad + \sum_{\rho, \eta \in \mathbb{N}_{\geq 1}} 1_{\rho \leq \eta} 1_{\rho + \eta \leq n-1} P_{\rho, \eta} \left(\left(\frac{\partial \widetilde{g}_E}{\partial z}(x, t, z)\right)^{-1}, \left(\frac{\partial^{a+b} \widetilde{g}_E}{\partial x^a \partial z^b}(x, t, z)\right)_{\substack{a,b \in \mathbb{N} \cup \{0\} \\ 1 \leq a+b \leq n-1}}\right) \\
&\qquad \cdot \frac{\partial^\rho \widetilde{g}_E}{\partial x^\rho}(x, t, z) \frac{\partial^\eta \widetilde{g}_E}{\partial x^\eta}(x, t, z)\Bigg|_{\substack{x=\beta, \\ z=\Delta(\beta,t)^2}}.
\end{aligned}$$

Here $P_{\rho,\eta}$ is a polynomial with real coefficient for any $\rho, \eta \in \mathbb{N}_{\geq 1}$ satisfying $\rho \leq \eta$, $\rho + \eta \leq n - 1$. For $C_{a,b} \in \mathbb{C}$ $(a, b \in \mathbb{N} \cup \{0\}$, $1 \leq a + b \leq n - 1)$

$$(C_{a,b})_{\substack{a,b \in \mathbb{N} \cup \{0\} \\ 1 \leq a+b \leq n-1}} := (C_{0,1}, C_{0,2}, \cdots, C_{0,n-1}, C_{1,0}, C_{1,1}, \cdots, C_{1,n-2}, \cdots, C_{n-1,0}).$$

Proof. Take any $(\beta, t) \in Q_+$. By (2.3)

$$\frac{\partial}{\partial \beta} \widetilde{F}_E(\beta, t, \Delta(\beta, t)^2) = \frac{\partial \widetilde{F}_E}{\partial x}(\beta, t, \Delta(\beta, t)^2) - \Delta(\beta, t) \frac{\partial \Delta}{\partial \beta}(\beta, t) \widetilde{g}_E(\beta, t, \Delta(\beta, t)^2).$$

Here we remark that by the implicit function theorem for real analytic functions (see e.g. [15]) $\Delta \in C^\omega(Q_+ \cup Q_-)$. By (2.2) $\widetilde{g}_E(\beta, t, \Delta(\beta, t)^2) = 0$. Thus,

$$\frac{\partial}{\partial \beta} \widetilde{F}_E(\beta, t, \Delta(\beta, t)^2) = \frac{\partial \widetilde{F}_E}{\partial x}(\beta, t, \Delta(\beta, t)^2).$$

Moreover, by (2.3)

$$\begin{aligned}
\text{(2.8)} \qquad &\left(\frac{\partial}{\partial \beta}\right)^2 \widetilde{F}_E(\beta, t, \Delta(\beta, t)^2) \\
&= \frac{\partial^2 \widetilde{F}_E}{\partial x^2}(\beta, t, \Delta(\beta, t)^2) + 2\Delta(\beta, t) \frac{\partial \Delta}{\partial \beta}(\beta, t) \frac{\partial^2 \widetilde{F}_E}{\partial x \partial z}(\beta, t, \Delta(\beta, t)^2) \\
&= \frac{\partial^2 \widetilde{F}_E}{\partial x^2}(\beta, t, \Delta(\beta, t)^2) - \Delta(\beta, t) \frac{\partial \Delta}{\partial \beta}(\beta, t) \frac{\partial \widetilde{g}_E}{\partial x}(\beta, t, \Delta(\beta, t)^2).
\end{aligned}$$

We can derive from (2.2), (2.4) that

$$\text{(2.9)} \qquad \Delta(\beta, t) \frac{\partial \Delta}{\partial \beta}(\beta, t) = -\frac{\frac{\partial \widetilde{g}_E}{\partial x}(\beta, t, \Delta(\beta, t)^2)}{2\frac{\partial \widetilde{g}_E}{\partial z}(\beta, t, \Delta(\beta, t)^2)}.$$

By substituting (2.9) into (2.8) we obtain that

$$\left(\frac{\partial}{\partial \beta}\right)^2 \widetilde{F}_E(\beta, t, \Delta(\beta, t)^2) = \frac{\partial^2 \widetilde{F}_E}{\partial x^2}(\beta, t, \Delta(\beta, t)^2) + \frac{\left(\frac{\partial \widetilde{g}_E}{\partial x}(\beta, t, \Delta(\beta, t)^2)\right)^2}{2\frac{\partial \widetilde{g}_E}{\partial z}(\beta, t, \Delta(\beta, t)^2)},$$

which is (2.7) for $n = 2$.

Let us assume that (2.7) holds for some $n \in \mathbb{N}_{\geq 2}$. By differentiating both sides with β and using (2.3), (2.9) we have that

(2.10)

$$\begin{aligned}
&\left(\frac{\partial}{\partial\beta}\right)^{n+1} \widetilde{F}_E(\beta, t, \Delta(\beta, t)^2) \\
&= \frac{\partial^{n+1}\widetilde{F}_E}{\partial x^{n+1}}(\beta, t, \Delta(\beta, t)^2) + \frac{\frac{\partial \widetilde{g}_E}{\partial x}(\beta, t, \Delta(\beta, t)^2)\frac{\partial^n \widetilde{g}_E}{\partial x^n}(\beta, t, \Delta(\beta, t)^2)}{2\frac{\partial \widetilde{g}_E}{\partial z}(\beta, t, \Delta(\beta, t)^2)} \\
&\quad + \frac{1}{2\frac{\partial \widetilde{g}_E}{\partial z}(\beta, t, \Delta(\beta, t)^2)} \sum_{j=0}^{n-2} \left(\frac{\partial}{\partial x}\right)^{j+1} \left(\frac{\partial \widetilde{g}_E}{\partial x}(x, t, z)\frac{\partial^{n-1-j}\widetilde{g}_E}{\partial x^{n-1-j}}(x, t, z)\right)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}} \\
&\quad + \frac{\partial}{\partial x}\left(\frac{1}{2\frac{\partial \widetilde{g}_E}{\partial z}(x, t, z)}\right)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}} \\
&\quad\quad \cdot \sum_{j=0}^{n-2} \left(\frac{\partial}{\partial x}\right)^{j} \left(\frac{\partial \widetilde{g}_E}{\partial x}(x, t, z)\frac{\partial^{n-1-j}\widetilde{g}_E}{\partial x^{n-1-j}}(x, t, z)\right)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}} \\
&\quad - \frac{\frac{\partial \widetilde{g}_E}{\partial x}(\beta, t, \Delta(\beta, t)^2)}{\frac{\partial \widetilde{g}_E}{\partial z}(\beta, t, \Delta(\beta, t)^2)} \\
&\quad\quad \cdot \frac{\partial}{\partial z}\left(\frac{1}{2\frac{\partial \widetilde{g}_E}{\partial z}(x, t, z)} \sum_{j=0}^{n-2} \left(\frac{\partial}{\partial x}\right)^{j} \left(\frac{\partial \widetilde{g}_E}{\partial x}(x, t, z)\frac{\partial^{n-1-j}\widetilde{g}_E}{\partial x^{n-1-j}}(x, t, z)\right)\right)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}} \\
&\quad + \sum_{\rho,\eta\in\mathbb{N}_{\geq 1}} 1_{\rho\leq\eta}1_{\rho+\eta\leq n-1}\left(\frac{\partial}{\partial x} - \frac{\frac{\partial \widetilde{g}_E}{\partial x}(x, t, z)}{\frac{\partial \widetilde{g}_E}{\partial z}(x, t, z)}\frac{\partial}{\partial z}\right) \\
&\quad\quad \cdot \Bigg(P_{\rho,\eta}\left(\left(\frac{\partial \widetilde{g}_E}{\partial z}(x, t, z)\right)^{-1}, \left(\frac{\partial^{a+b}\widetilde{g}_E}{\partial x^a \partial z^b}(x, t, z)\right)_{\substack{a,b\in\mathbb{N}\cup\{0\}\\ 1\leq a+b\leq n-1}}\right) \\
&\quad\quad\quad \cdot \frac{\partial^\rho \widetilde{g}_E}{\partial x^\rho}(x, t, z)\frac{\partial^\eta \widetilde{g}_E}{\partial x^\eta}(x, t, z)\Bigg)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}}.
\end{aligned}$$

In more detail

$$\begin{aligned}
(2.11)\qquad &\sum_{j=0}^{n-2} \left(\frac{\partial}{\partial x}\right)^{j} \left(\frac{\partial \widetilde{g}_E}{\partial x}(x, t, z)\frac{\partial^{n-1-j}\widetilde{g}_E}{\partial x^{n-1-j}}(x, t, z)\right)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}} \\
&= \sum_{k=1}^{n-1}\sum_{j=k}^{n-1} \begin{pmatrix} j-1 \\ k-1 \end{pmatrix} \frac{\partial^k \widetilde{g}_E}{\partial x^k}(x, t, z)\frac{\partial^{n-k}\widetilde{g}_E}{\partial x^{n-k}}(x, t, z)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}},
\end{aligned}$$

$$\begin{aligned}
(2.12)\qquad &\sum_{j=0}^{n-2} \left(\frac{\partial}{\partial x}\right)^{j+1} \left(\frac{\partial \widetilde{g}_E}{\partial x}(x, t, z)\frac{\partial^{n-1-j}\widetilde{g}_E}{\partial x^{n-1-j}}(x, t, z)\right)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}} \\
&= \sum_{j=1}^{n-1} \left(\frac{\partial}{\partial x}\right)^{j} \left(\frac{\partial \widetilde{g}_E}{\partial x}(x, t, z)\frac{\partial^{n-j}\widetilde{g}_E}{\partial x^{n-j}}(x, t, z)\right)\Bigg|_{\substack{x=\beta,\\ z=\Delta(\beta,t)^2}}.
\end{aligned}$$

By using (2.11), (2.12) we can see that the 1st, 2nd, 3rd term of the right side of (2.10) can be organized into the 1st, 2nd term of the right side of (2.7) for $n+1$ and the 4th, 5th, 6th term of (2.10) can be summarized into the last term of the right side of (2.7) for $n+1$. Thus, (2.7) holds for $n+1$. The induction with n concludes the proof. □

To understand the following lemmas, let us recall that $\tau \in C^{\omega}((0,\beta_c))$, which is claimed in [14, Lemma 2.2 (i)].

Lemma 2.2. *(i)*

$$\frac{\partial \widetilde{g}_E}{\partial x}(\beta,\tau(\beta),0) = -\frac{\partial \widetilde{g}_E}{\partial t}(\beta,\tau(\beta),0)\frac{d\tau}{d\beta}(\beta),\ \forall \beta \in (0,\beta_c).$$

(ii) Assume that $\beta_0 \in (0,\beta_c)$, $n \in \mathbb{N}_{\ge 2}$ and

$$\frac{d^m\tau}{d\beta^m}(\beta_0) = 0,\ \forall m \in \{1,2,\cdots,n-1\}.$$

Then

$$\frac{\partial^m \widetilde{g}_E}{\partial x^m}(\beta_0,\tau(\beta_0),0) = 0,\ \forall m \in \{0,1,\cdots,n-1\},$$
$$\frac{\partial^n \widetilde{g}_E}{\partial x^n}(\beta_0,\tau(\beta_0),0) = -\frac{\partial \widetilde{g}_E}{\partial t}(\beta_0,\tau(\beta_0),0)\frac{d^n\tau}{d\beta^n}(\beta_0).$$

Proof. (i): The claim follows from the equality

$$\widetilde{g}_E(\beta,\tau(\beta),0) = 0,\ \forall \beta \in (0,\beta_c). \tag{2.13}$$

(ii): We can derive from (2.13) that

$$\left(\frac{\partial}{\partial x} + \frac{d\tau}{dx}(x)\frac{\partial}{\partial t}\right)^l \widetilde{g}_E(x,t,0)\Bigg|_{\substack{x=\beta,\\ t=\tau(\beta)}} = 0,\ \forall l \in \mathbb{N}\cup\{0\},\ \beta \in (0,\beta_c).$$

The result follows from this equality and the assumption. □

Lemma 2.3. *Assume that $\beta_0(\in (0,\beta_c))$ is a SPI of $\tau(\cdot)$. Then there exist $n \in 2\mathbb{N}+1$ $(=\{3,5,7,\cdots\})$ and $\varepsilon \in \mathbb{R}_{>0}$ such that $(\beta_0-\varepsilon,\beta_0+\varepsilon) \subset (0,\beta_c)$ and*

$$\frac{d^m\tau}{d\beta^m}(\beta_0) = 0,\ \forall m \in \{1,2,\cdots,n-1\},$$
$$\frac{d^n\tau}{d\beta^n}(\beta_0) \neq 0,$$
$$\frac{d\tau}{d\beta}(\beta) \neq 0,\ \forall \beta \in (\beta_0-\varepsilon,\beta_0+\varepsilon)\backslash\{\beta_0\}.$$

Moreover,

$$\frac{\partial^m \widetilde{g}_E}{\partial x^m}(\beta_0,\tau(\beta_0),0) = 0,\ \forall m \in \{0,1,\cdots,n-1\}, \tag{2.14}$$
$$\frac{\partial^n \widetilde{g}_E}{\partial x^n}(\beta_0,\tau(\beta_0),0) \neq 0.$$

Proof. The claims on $\tau(\cdot)$ are general properties of a real analytic function having a SPI. However, we provide the proof for clarity. By the assumption and the definition of SPI there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $(\beta_0 - \varepsilon, \beta_0 + \varepsilon) \subset (0, \beta_c)$, $\frac{d\tau}{d\beta}(\beta_0) = 0$ and

(2.15)
$$\frac{d\tau}{d\beta}(\beta) > 0,\ \forall \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)\backslash\{\beta_0\} \text{ or } \frac{d\tau}{d\beta}(\beta) < 0,\ \forall \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)\backslash\{\beta_0\}.$$

Since $\tau \in C^{\omega}((0, \beta_c))$, there exist $\varepsilon' \in (0, \varepsilon]$, $n \in \mathbb{N}_{\geq 2}$ such that $\frac{d^m\tau}{d\beta^m}(\beta_0) = 0$, $\forall m \in \{1, 2, \cdots, n-1\}$, $\frac{d^n\tau}{d\beta^n}(\beta_0) \neq 0$ and

$$\frac{d\tau}{d\beta}(\beta) = \sum_{m=n}^{\infty} \frac{1}{(m-1)!} \frac{d^m\tau}{d\beta^m}(\beta_0)(\beta - \beta_0)^{m-1},\ \forall \beta \in (\beta_0 - \varepsilon', \beta_0 + \varepsilon').$$

We can deduce from the property (2.15) and the above expansion that n must be odd. At this point the claims on $\tau(\cdot)$ have been proved. The claims on $\widetilde{g}_E$ follow from the above properties of $\tau(\cdot)$ and Lemma 2.2 (ii) plus the fact $\frac{\partial \widetilde{g}_E}{\partial t}(\beta_0, \tau(\beta_0), 0) > 0$ based on $\tau(\beta_0) \in (\pi, 2\pi)$. □

We can prove Theorem 1.5 by applying Lemma 2.1 and Lemma 2.3.

Proof of Theorem 1.5. (i): Assume that $(\text{SPI})_{\xi}(\beta_0)$ holds. We can see from (1.6) and the general behavior of $\tau(\cdot)$ proved in [14, Lemma 2.2] that $(\beta_0, \tau(\beta_0)) \in Q_{\rho,\eta}$. By Lemma 2.3 there exists $n_0 \in 2\mathbb{N} + 1$ such that (2.14) holds for $n = n_0$. We remark that

$$\left(\frac{\partial}{\partial \beta}\right)^l \widetilde{F}_E(\beta, t, \Delta(\beta, t)^2) = \frac{\partial^l \widetilde{F}_E}{\partial x^l}(\beta, t, 0),\ \forall (\beta, t) \in Q_-,\ l \in \mathbb{N} \cup \{0\}. \tag{2.16}$$

Bearing (2.11) in mind, we observe that for any $n \in \{2, 3, \cdots, 2n_0 - 1\}$ each of the 2nd, 3rd terms of the right-hand side of (2.7) contains $\frac{\partial^m \widetilde{g}_E}{\partial x^m}(\beta, t, \Delta(\beta, t)^2)$ for some $m \in \{1, 2, \cdots, n_0 - 1\}$. For $n = 2n_0$ the 2nd term contains $(\frac{\partial^{n_0} \widetilde{g}_E}{\partial x^{n_0}}(\beta, t, \Delta(\beta, t)^2))^2$ and each of the 3rd terms contains $\frac{\partial^m \widetilde{g}_E}{\partial x^m}(\beta, t, \Delta(\beta, t)^2)$ for some $m \in \{1, 2, \cdots, n_0 - 1\}$. This observation and the properties (2.4), (2.14), (2.16) imply that for any $n \in \{2, 3, \cdots, 2n_0 - 1\}$

$$\begin{aligned}
&\lim_{\substack{\beta \to \beta_0 \\ (\beta, \tau(\beta_0)) \in Q_+}} \left(\frac{\partial}{\partial \beta}\right)^n \widetilde{F}_E(\beta, \tau(\beta_0), \Delta(\beta, \tau(\beta_0))^2) \\
&= \lim_{\substack{\beta \to \beta_0 \\ (\beta, \tau(\beta_0)) \in Q_+}} \frac{\partial^n \widetilde{F}_E}{\partial x^n}(\beta, \tau(\beta_0), \Delta(\beta, \tau(\beta_0))^2) \\
&= \frac{\partial^n \widetilde{F}_E}{\partial x^n}(\beta_0, \tau(\beta_0), 0) \\
&= \lim_{\substack{\beta \to \beta_0 \\ (\beta, \tau(\beta_0)) \in Q_-}} \left(\frac{\partial}{\partial \beta}\right)^n \widetilde{F}_E(\beta, \tau(\beta_0), \Delta(\beta, \tau(\beta_0))^2), \\
&\lim_{\substack{\beta \to \beta_0 \\ (\beta, \tau(\beta_0)) \in Q_+}} \left(\frac{\partial}{\partial \beta}\right)^{2n_0} \widetilde{F}_E(\beta, \tau(\beta_0), \Delta(\beta, \tau(\beta_0))^2)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial^{2n_0}\widetilde{F}_E}{\partial x^{2n_0}}(\beta_0,\tau(\beta_0),0) + \frac{\sum_{j=n_0}^{2n_0-1}\begin{pmatrix} j-1 \\ n_0-1 \end{pmatrix}}{2\frac{\partial \widetilde{g}_E}{\partial z}(\beta_0,\tau(\beta_0),0)}\left(\frac{\partial^{n_0}\widetilde{g}_E}{\partial x^{n_0}}(\beta_0,\tau(\beta_0),0)\right)^2 \\
&< \frac{\partial^{2n_0}\widetilde{F}_E}{\partial x^{2n_0}}(\beta_0,\tau(\beta_0),0) \\
&= \lim_{\substack{\beta\to\beta_0 \\ (\beta,\tau(\beta_0))\in Q_-}} \left(\frac{\partial}{\partial\beta}\right)^{2n_0} \widetilde{F}_E(\beta,\tau(\beta_0),\Delta(\beta,\tau(\beta_0))^2).
\end{aligned}
$$

Combined with (2.1), the above argument concludes that $(\mathrm{PT})_{2n_0,(\rho,\eta)}(\beta_0,\tau(\beta_0))$ holds.

Assume that $(\mathrm{SPI})_\xi(\beta_0)$ does not hold and $(\beta_0,\tau(\beta_0))\in Q_{\rho,\eta}$. It follows from (1.5), (1.6) that $\frac{d\tau}{d\beta}(\beta_0)\ge 0$ if $\xi=r$, $\frac{d\tau}{d\beta}(\beta_0)\le 0$ if $\xi=f$. Consider the case that $\xi=r$ and $\frac{d\tau}{d\beta}(\beta_0)=0$. Since $\tau(\cdot)$ is real analytic and not constant, there exists $\varepsilon\in\mathbb{R}_{>0}$ such that $\frac{d\tau}{d\beta}(\beta)\neq 0$, $\forall\beta\in(\beta_0-\varepsilon,\beta_0+\varepsilon)\backslash\{\beta_0\}$. If $\frac{d\tau}{d\beta}(\beta)<0$, $\forall\beta\in(\beta_0-\varepsilon,\beta_0)$ or $\frac{d\tau}{d\beta}(\beta)<0$, $\forall\beta\in(\beta_0,\beta_0+\varepsilon)$, it contradicts that $(\beta_0,\tau(\beta_0))\in Q_{+,-}$. Thus $\frac{d\tau}{d\beta}(\beta)>0$, $\forall\beta\in(\beta_0-\varepsilon,\beta_0+\varepsilon)\backslash\{\beta_0\}$, which means that β_0 is a rising SPI, a contradiction. Therefore $\frac{d\tau}{d\beta}(\beta_0)>0$ if $\xi=r$. Similarly we can prove that $\frac{d\tau}{d\beta}(\beta_0)<0$ if $\xi=f$.

We can derive from this, (2.7) for $n=2$, Lemma 2.2 (i) and (2.16) that

$$
\begin{aligned}
&\lim_{\substack{\beta\to\beta_0 \\ (\beta,\tau(\beta_0))\in Q_+}} \left(\frac{\partial}{\partial\beta}\right)^{2} \widetilde{F}_E(\beta,\tau(\beta_0),\Delta(\beta,\tau(\beta_0))^2) \\
&= \frac{\partial^{2}\widetilde{F}_E}{\partial x^{2}}(\beta_0,\tau(\beta_0),0) + \frac{(\frac{\partial \widetilde{g}_E}{\partial t}(\beta_0,\tau(\beta_0),0)\frac{d\tau}{d\beta}(\beta_0))^2}{2\frac{\partial \widetilde{g}_E}{\partial z}(\beta_0,\tau(\beta_0),0)} \\
&< \frac{\partial^{2}\widetilde{F}_E}{\partial x^{2}}(\beta_0,\tau(\beta_0),0) \\
&= \lim_{\substack{\beta\to\beta_0 \\ (\beta,\tau(\beta_0))\in Q_-}} \left(\frac{\partial}{\partial\beta}\right)^{2} \widetilde{F}_E(\beta,\tau(\beta_0),\Delta(\beta,\tau(\beta_0))^2).
\end{aligned}
$$

Here we also used (2.4) and that $\frac{\partial \widetilde{g}_E}{\partial t}(\beta_0,\tau(\beta_0),0)>0$. This together with (2.1), (2.6) imply that $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_0,\tau(\beta_0))$ holds, and thus $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_0,\tau(\beta_0))$ does not hold for any $n\in 4\mathbb{N}+2$. If $(\beta_0,\tau(\beta_0))\notin Q_{\rho,\eta}$, $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_0,\tau(\beta_0))$ does not hold for any $n\in 4\mathbb{N}+2$ by definition. The claim (i) is proved.

(ii): Assume that $(\mathrm{SPI})_\xi$ does not hold. Take any $(\beta_1,t_1)\in Q_{\rho,\eta}$. First let us assume that $\beta_1\in(0,\beta_c)$. It follows from (1.5), (1.6) that $(\beta_1,\tau(\beta_1))\in Q_{\rho,\eta}$. The same argument as in the 2nd half of the proof of (i) leads to that

$$
\begin{aligned}
&\lim_{\substack{\beta\to\beta_1 \\ (\beta,\tau(\beta_1))\in Q_+}} \left(\frac{\partial}{\partial\beta}\right)^{2} \widetilde{F}_E(\beta,\tau(\beta_1),\Delta(\beta,\tau(\beta_1))^2) \\
&< \lim_{\substack{\beta\to\beta_1 \\ (\beta,\tau(\beta_1))\in Q_-}} \left(\frac{\partial}{\partial\beta}\right)^{2} \widetilde{F}_E(\beta,\tau(\beta_1),\Delta(\beta,\tau(\beta_1))^2).
\end{aligned}
$$

This property, (2.1) and (2.6) ensure that $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_1,\tau(\beta_1))$ holds. Then by Lemma 1.3 $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_1,t_1)$ holds.

Next let us assume that $\beta_1 = \beta_c$. In this case

$$\widetilde{g}_E(x,t_1,0) = -\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{1}{\tanh(\frac{x}{2}|E(\mathbf{k})|)|E(\mathbf{k})|}\right), \ \forall x \in \mathbb{R}_{>0},$$

and thus $\frac{\partial \widetilde{g}_E}{\partial x}(x,t_1,0) < 0$, $\forall x \in \mathbb{R}_{>0}$. Using this inequality, (2.4), (2.7) for $n = 2$ and (2.16), we deduce that

$$\begin{aligned}
&\lim_{\substack{\beta \to \beta_1 \\ (\beta,t_1) \in Q_+}} \left(\frac{\partial}{\partial \beta}\right)^2 \widetilde{F}_E(\beta,t_1,\Delta(\beta,t_1)^2) = \frac{\partial^2 \widetilde{F}_E}{\partial x^2}(\beta_1,t_1,0) + \frac{(\frac{\partial \widetilde{g}_E}{\partial x}(\beta_1,t_1,0))^2}{2\frac{\partial \widetilde{g}_E}{\partial z}(\beta_1,t_1,0)} \\
&< \frac{\partial^2 \widetilde{F}_E}{\partial x^2}(\beta_1,t_1,0) \\
&= \lim_{\substack{\beta \to \beta_1 \\ (\beta,t_1) \in Q_-}} \left(\frac{\partial}{\partial \beta}\right)^2 \widetilde{F}_E(\beta,t_1,\Delta(\beta,t_1)^2),
\end{aligned}$$

which together with (2.1), (2.6) imply that $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_1,t_1)$ holds. Thus we have proved that if $(\mathrm{SPI})_\xi$ does not hold, $(\mathrm{PT})_{2,(\rho,\eta)}(\beta,t)$ holds for any $(\beta,t) \in Q_{\rho,\eta}$.

If $(\mathrm{SPI})_\xi$ holds, by the claim (i) there exist $\beta_2 \in (0,\beta_c)$, $n \in 4\mathbb{N}+2$ such that $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_2,\tau(\beta_2))$ holds. This means that $(\beta_2,\tau(\beta_2)) \in Q_{\rho,\eta}$ and $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_2,\tau(\beta_2))$ does not hold. We have proved the claim (ii).

(iii): Assume that $(\beta_3,t_3) \in Q_{\rho,\eta}$ and $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_3,t_3)$ does not hold. If $\beta_3 = \beta_c$, by the 2nd half of the proof of (ii) $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_3,t_3)$ holds, which is a contradiction. Thus $\beta_3 \in (0,\beta_c)$. If $(\mathrm{SPI})_\xi(\beta_3)$ does not hold, by the 1st half of the proof of (ii) $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_3,t_3)$ holds, contradicting the assumption. Thus $(\mathrm{SPI})_\xi(\beta_3)$ must hold. Then by the 1st half of the proof of (i) there exists $n \in 4\mathbb{N}+2$ such that $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_3,\tau(\beta_3))$ holds. Moreover, by Lemma 1.3 $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_3,t_3)$ holds. The converse is obvious from the definition. □

As a corollary of Theorem 1.5, we can prove the following.

Corollary 2.4. *(1) The statements* (i), (ii) *of Theorem 1.6 are equivalent to each other.*

(2) The statements (i), (ii), (iii) *of Theorem 1.7 are equivalent to each other.*

Proof. (1): If $(\mathrm{PT})_{n,(\rho,\eta)}$ holds with $n \in 4\mathbb{N}+2$, by Lemma 1.3 there exists $\beta_0 \in (0,\beta_c]$ such that $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_0,\tau(\beta_0))$ holds if $\beta_0 < \beta_c$, $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_0,2\pi)$ holds if $\beta_0 = \beta_c$. If $\beta_0 = \beta_c$, it follows from the proof of Theorem 1.5 (ii) above that $(\mathrm{PT})_{2,(\rho,\eta)}(\beta_0,2\pi)$ holds, which is a contradiction. Thus $\beta_0 < \beta_c$ and $(\mathrm{PT})_{n,(\rho,\eta)}(\beta_0,\tau(\beta_0))$ holds. We can deduce the equivalence between (i) and (ii) of Theorem 1.6 from the above argument and Theorem 1.5 (i).

(2): Theorem 1.5 (ii) implies the equivalence between the statements (ii), (iii). We can deduce from the definition of $(\mathrm{PT})_{n,(\rho,\eta)}$ that the statement (ii) implies the statement (i). It suffices to show that the statement (i) implies the statement (iii). Suppose that for any $U_0 \in (0, \frac{2e_{min}}{b})$ there exist $U \in [-U_0,0)$, $E \in \mathcal{E}(e_{min},e_{max})$, $\xi \in \{r,f\}$ such that $(\mathrm{SPI})_\xi$ holds. By definition there exists $\beta_0 \in (0,\beta_c)$ such that $(\mathrm{SPI})_\xi(\beta_0)$ holds. Set $(\rho,\eta) := (+,-)$ if $\xi = r$, $(-,+)$ if $\xi = f$. By Theorem 1.5 (i) there exists $n \in \mathbb{N}_{\geq 3}$ such that $(\mathrm{PT})_{n,(\rho,\eta)}$ holds. This means that (i) implies (iii). Thus the claim holds true. □

The following corollary will be used in Subsection 2.4 and Subsection 2.5 to prove key propositions on which Theorem 1.6, Theorem 1.7 are based.

Corollary 2.5. *Under the same assumption of Theorem 1.5 the following statement holds.* $(SPI)_\xi$ *holds if and only if there exists* $n \in 4\mathbb{N}+2$ *such that* $(PT)_{n,(\rho,\eta)}$ *holds.*

Proof. By Theorem 1.5 (i), if $(\mathrm{SPI})_\xi$ holds, there exists $n \in 4\mathbb{N} + 2$ such that $(\mathrm{PT})_{n,(\rho,\eta)}$ holds. It follows from the 2nd half of the proof of Theorem 1.5 (ii) and (1.5), (1.6) that $(\mathrm{PT})_{2,(+,-)}(\beta_c, t_1)$ holds for any $t_1 \in \mathbb{R}$ satisfying $(\beta_c, t_1) \in Q_{+,-}$ and $(\beta_c, t) \notin Q_{-,+}$ for any $t \in \mathbb{R}$. This ensures that if $(\mathrm{PT})_{n,(\rho,\eta)}(\beta, t)$ holds for some $(\beta, t) \in \mathbb{R}_{>0} \times \mathbb{R}$, $n \in 4\mathbb{N} + 2$, then $\beta \in (0, \beta_c)$. We can deduce from this property, Lemma 1.3, Theorem 1.5 (i) that if there exists $n \in 4\mathbb{N} + 2$ such that $(\mathrm{PT})_{n,(\rho,\eta)}$ holds, $(\mathrm{SPI})_\xi$ holds. □

2.2 General lemmas

Here we prepare several lemmas in order to prove Theorem 1.6, Theorem 1.7 in the following subsections. For $E \in \mathcal{E}(e_{min}, e_{max})$ we define the function $F_\infty : \mathbb{R} \times (-1, 0) \to \mathbb{R}$ by

$$F_\infty(x, y) := D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(xE(\mathbf{k}))}{(y + \cosh(xE(\mathbf{k})))E(\mathbf{k})} \right).$$

In fact this function was defined in [14, (2.38)]. We keep using the same notation for consistency with the previous paper. First of all let us state a basic lemma which follows from Lemma 1.2 and is the same as [14, Lemma 2.1]. Presenting the whole statement here must be convenient for the readers to apply in the subsequent construction.

Lemma 2.6. *Assume that* $|U| < \frac{2e_{min}}{b}$, $y \in (-1, 0)$, $\beta \in \mathbb{R}_{>0}$, $E \in \mathcal{E}(e_{min}, e_{max})$ *and* $\frac{2}{|U|} = F_\infty(\beta, y)$. *Then* $\beta \in (0, \beta_c)$ *and* $y = \cos(\frac{\tau(\beta)}{2})$.

The next lemma gives a sufficient condition in terms of F_∞ for $\tau(\cdot)$ not to have any SPI.

Lemma 2.7. *Let* $S \subset \mathcal{E}(e_{min}, e_{max})$, $S \neq \emptyset$. *Assume that there exists* $y_0 \in (-1, 0)$ *such that for any* $y \in (-1, y_0]$ *and* $E \in S$ *there uniquely exists* $x_0 \in \mathbb{R}_{>0}$ *such that* $\frac{\partial F_\infty}{\partial x}(x_0, y) = 0$. *Then there exists* $U_0 \in (0, \frac{2e_{min}}{b})$ *such that for any* $U \in [-U_0, 0)$ *and* $E \in S$ $\tau(\cdot)$ *has no SPI in* $(0, \beta_c)$.

Proof. The first half of the proof is close to the initial part of the proof of [14, Proposition 2.8]. Take any $E \in \mathcal{E}(e_{min}, e_{max})$. It follows from Lemma 1.2 that for $U \in (-\frac{2e_{min}}{b}, 0)$

$$\beta_c \le \frac{2}{e_{min}} \tanh^{-1}\left(\frac{b|U|}{2e_{min}} \right) \le \frac{2\tanh^{-1}(1)}{e_{min}}.$$

By the monotone decreasing property of the function (2.5) and the above inequality

$$\frac{2}{|U|} \le \frac{b\sinh(\beta e_{min})}{e_{min}(\cos(\tau(\beta)/2) + \cosh(\beta e_{min}))} \le \frac{b\sinh(2\tanh^{-1}(1))}{e_{min}(\cos(\tau(\beta)/2) + 1)},$$

and thus

$$\cos\left(\frac{\tau(\beta)}{2}\right)+1\le\frac{b\sinh(2\tanh^{-1}(1))}{2e_{min}}|U|,\ \forall\beta\in(0,\beta_c).$$

This implies that there exists $U_0\in(0,\frac{2e_{min}}{b})$ such that for any $U\in[-U_0,0)$, $E\in\mathcal{E}(e_{min},e_{max})$

$$\cos\left(\frac{\tau(\beta)}{2}\right)\in(-1,y_0],\ \forall\beta\in(0,\beta_c). \tag{2.17}$$

Let us fix $U\in[-U_0,0)$ and $E\in S$. Suppose that β_0 ($\in(0,\beta_c)$) is a SPI of $\tau(\cdot)$. Let $\beta_1\in(0,\beta_c)$ be a global minimum point of $\tau(\cdot)$. Remark that by the behavior of $\tau(\cdot)$ summarized in [14, Lemma 2.2] a global minimum point exists. By the definition of SPI $\beta_1\neq\beta_0$. Let us assume that $\beta_1<\beta_0$. We can deduce from [14, Lemma 2.2] that there exists $\beta_2\in(0,\beta_1]$ such that $\tau(\beta_2)=\tau(\beta_0)$. It follows that

$$\frac{2}{|U|}=F_\infty\left(\beta_2,\cos\left(\frac{\tau(\beta_2)}{2}\right)\right)=F_\infty\left(\beta_2,\cos\left(\frac{\tau(\beta_0)}{2}\right)\right)=F_\infty\left(\beta_0,\cos\left(\frac{\tau(\beta_0)}{2}\right)\right).$$

By the mean value theorem there exists $\beta_3\in(\beta_2,\beta_0)$ such that

$$\frac{\partial F_\infty}{\partial x}\left(\beta_3,\cos\left(\frac{\tau(\beta_0)}{2}\right)\right)=0. \tag{2.18}$$

On the other hand, since β_0 is a SPI,

$$\begin{aligned}0&=\frac{\partial F_\infty}{\partial x}\left(\beta_0,\cos\left(\frac{\tau(\beta_0)}{2}\right)\right)-\frac{1}{2}\frac{d\tau}{d\beta}(\beta_0)\sin\left(\frac{\tau(\beta_0)}{2}\right)\frac{\partial F_\infty}{\partial y}\left(\beta_0,\cos\left(\frac{\tau(\beta_0)}{2}\right)\right)\\&=\frac{\partial F_\infty}{\partial x}\left(\beta_0,\cos\left(\frac{\tau(\beta_0)}{2}\right)\right).\end{aligned} \tag{2.19}$$

By (2.17) $\cos(\frac{\tau(\beta_0)}{2})\in(-1,y_0]$, which together with (2.18), (2.19) contradict the assumption. Similarly we can derive a contradiction by assuming that $\beta_1>\beta_0$. Therefore $\tau(\cdot)$ cannot have any SPI in $(0,\beta_c)$. □

The next lemma gives sufficient conditions in terms of F_∞ for $\tau(\cdot)$ to have a SPI.

Lemma 2.8. *Let $U_0\in(0,\frac{2e_{min}}{b})$, $y_0\in(-1,0)$.*

(i) Assume that x_0 is a rising SPI of the function $x\mapsto F_\infty(x,y_0):\mathbb{R}_{>0}\to\mathbb{R}$ and $F_\infty(x_0,y_0)\ge\frac{2}{U_0}$. Then there exists $U\in[-U_0,0)$ such that $\tau(\cdot)$ has a falling SPI in $(0,\beta_c)$.

(ii) Assume that x_0 is a falling SPI of the function $x\mapsto F_\infty(x,y_0):\mathbb{R}_{>0}\to\mathbb{R}$ and $F_\infty(x_0,y_0)\ge\frac{2}{U_0}$. Then there exists $U\in[-U_0,0)$ such that $\tau(\cdot)$ has a rising SPI in $(0,\beta_c)$.

Proof. We only give a proof to the claim (i). The claim (ii) can be proved similarly. By the assumption there exist $\varepsilon \in \mathbb{R}_{>0}$, $U \in [-U_0, 0)$ such that

$$
\begin{aligned}
&(x_0-\varepsilon, x_0+\varepsilon) \subset \mathbb{R}_{>0},\\
&\frac{\partial F_\infty}{\partial x}(x_0, y_0) = 0,\\
&\frac{\partial F_\infty}{\partial x}(x, y_0) > 0,\ \forall x \in (x_0-\varepsilon, x_0+\varepsilon)\backslash\{x_0\},\\
&-\frac{2}{|U|} + F_\infty(x_0, y_0) = 0.
\end{aligned} \tag{2.20}
$$

Here we use Lemma 2.6 to ensure that $x_0 \in (0, \beta_c)$ and $y_0 = \cos(\frac{\tau(x_0)}{2})$. We can derive from the equality $F_\infty(x, \cos(\frac{\tau(x)}{2})) = \frac{2}{|U|}$ $(x \in (0, \beta_c))$ that

$$
0 = \frac{\partial F_\infty}{\partial x}(x_0, y_0) - \frac{1}{2}\sin\left(\frac{\tau(x_0)}{2}\right)\frac{d\tau}{d\beta}(x_0)\frac{\partial F_\infty}{\partial y}(x_0, y_0).
$$

It follows from $\frac{\partial F_\infty}{\partial x}(x_0, y_0) = 0$, $\sin(\frac{\tau(x_0)}{2}) > 0$ and $\frac{\partial F_\infty}{\partial y}(x_0, y_0) < 0$ that

$$
\frac{d\tau}{d\beta}(x_0) = 0. \tag{2.21}
$$

By the analytic implicit function theorem (see e.g. [15]) there exist $\varepsilon_1 \in (0, \varepsilon]$ and a real analytic function $Y : (x_0 - \varepsilon_1, x_0 + \varepsilon_1) \to (-1, 0)$ such that

$$
\begin{aligned}
&-\frac{2}{|U|} + F_\infty(x, Y(x)) = 0,\ \forall x \in (x_0-\varepsilon_1, x_0+\varepsilon_1),\\
&Y(x_0) = y_0.
\end{aligned}
$$

Let us show that there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$
\begin{aligned}
&Y(x) < y_0,\ \forall x \in (x_0-\varepsilon_2, x_0),\\
&Y(x) > y_0,\ \forall x \in (x_0, x_0+\varepsilon_2).
\end{aligned} \tag{2.22}
$$

Suppose that for any $\varepsilon_3 \in (0, \varepsilon_1]$ there exists $x_1 \in (x_0-\varepsilon_3, x_0)$ such that $Y(x_1) \ge y_0$. By (2.20) and the fact $y \mapsto F_\infty(x_1, y) : (-1, 0) \to \mathbb{R}$ is strictly monotone decreasing

$$
\frac{2}{|U|} = F_\infty(x_1, Y(x_1)) \le F_\infty(x_1, y_0) < F_\infty(x_0, y_0) = \frac{2}{|U|},
$$

which is a contradiction. Thus there exists $\varepsilon_3 \in (0, \varepsilon_1]$ such that

$$
Y(x) < y_0,\ \forall x \in (x_0-\varepsilon_3, x_0).
$$

Similarly, suppose that for any $\varepsilon_4 \in (0, \varepsilon_1]$ there exists $x_2 \in (x_0, x_0+\varepsilon_4)$ such that $Y(x_2) \le y_0$. By (2.20) and the monotone decreasing property of the function $y \mapsto F_\infty(x_2, y) : (-1, 0) \to \mathbb{R}$

$$
\frac{2}{|U|} = F_\infty(x_2, Y(x_2)) \ge F_\infty(x_2, y_0) > F_\infty(x_0, y_0) = \frac{2}{|U|},
$$

which is again a contradiction. Therefore there exists $\varepsilon_4 \in (0, \varepsilon_1]$ such that

$$Y(x) > y_0, \ \forall x \in (x_0, x_0 + \varepsilon_4).$$

The above arguments conclude that the claim (2.22) holds true.

The property (2.22) implies that there exists $\varepsilon_5 \in (0, \varepsilon_2]$ such that

$$\frac{dY}{dx}(x) > 0, \ \forall x \in (x_0 - \varepsilon_5, x_0 + \varepsilon_5) \backslash \{x_0\}. \tag{2.23}$$

This can be confirmed by expanding the real analytic function $Y(\cdot)$ into the Taylor series around $x = x_0$. By applying Lemma 2.6 again we observe that $(x_0 - \varepsilon_5, x_0 + \varepsilon_5) \subset (0, \beta_c)$ and

$$Y(x) = \cos\left(\frac{\tau(x)}{2}\right), \ \forall x \in (x_0 - \varepsilon_5, x_0 + \varepsilon_5).$$

We can deduce from the above equality, (2.23) and the fact $\tau(x) \in (\pi, 2\pi)$, $\forall x \in (x_0 - \varepsilon_5, x_0 + \varepsilon_5)$ that

$$\frac{d\tau}{d\beta}(\beta) < 0, \ \forall \beta \in (x_0 - \varepsilon_5, x_0 + \varepsilon_5) \backslash \{x_0\}.$$

This combined with (2.21) concludes that x_0 is a falling SPI of $\tau(\cdot)$. $\square$

Let us prepare a key lemma to prove existence of a SPI of $\tau(\cdot)$ in Subsection 2.4, Subsection 2.5 under the assumption $\frac{e_{min}}{e_{max}} \le \sqrt{17 - 12\sqrt{2}}$. Let us recall the definition of the functions $W : \mathbb{R}_{>0} \times (-1, 0) \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$, $\widehat{W} : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$ given in [14, (2.62), Proof of Proposition 2.16].

$$\begin{aligned} W(x, y, z, s) &:= \frac{\sinh(x)}{y + \cosh(x)} + s\frac{\sinh(zx)}{(y + \cosh(zx))z}, \\ \widehat{W}(x, z, s) &:= \frac{x}{1 + \frac{x^2}{2}} + s\frac{x}{1 + z^2\frac{x^2}{2}}. \end{aligned} \tag{2.24}$$

Lemma 2.9. *For any $d, b \in \mathbb{N}$, basis $(\hat{\mathbf{v}}_j)_{j=1}^d$ of $\mathbb{R}^d$, e_{max}, $e_{min} \in \mathbb{R}_{>0}$ satisfying $0 < e_{min} < e_{max}$, $s_0 \in (0, 1)$ there exists*

$$\{E_{s,\delta}\}_{s \in (0, s_0), \delta \in (0, 1 - s_0^{\frac{1}{d}})} \subset \mathcal{E}(e_{min}, e_{max})$$

such that if we define $F_\delta : \mathbb{R}_{>0} \times (-1, 0) \times (0, s_0) \to \mathbb{R}$ by

$$F_\delta(x, y, s) := D_d \int_{\Gamma_\infty^*} d\mathbf{k} \operatorname{Tr}\left(\frac{\sinh(xE_{s,\delta}(\mathbf{k}))}{(y + \cosh(xE_{s,\delta}(\mathbf{k})))E_{s,\delta}(\mathbf{k})}\right) \tag{2.25}$$

for $\delta \in (0, 1 - s_0^{\frac{1}{d}})$, the following statements hold true.

(i) For any $\delta \in (0, 1 - s_0^{\frac{1}{d}})$

$$\sup_{\mathbf{k} \in \mathbb{R}^d} \sup_{\substack{m_j \in \mathbb{N} \cup \{0\} \\ (j = 1, \cdots, d)}} \left\| \prod_{j=1}^d \frac{\partial^{m_j}}{\partial k_j^{m_j}} E_{s,\delta}(\mathbf{k}) \right\|_{b \times b} 1_{\sum_{j=1}^d m_j \le d+2}$$

is constant with $s \in (0, s_0)$.

(ii)

$$F_\delta \in C^\infty(\mathbb{R}_{>0} \times (-1,0) \times (0,s_0)),\ \forall \delta \in (0, 1 - s_0^{\frac{1}{d}}).$$
$$F_\delta(\cdot, y, s) \in C^\omega(\mathbb{R}_{>0}),\ \forall (y,s) \in (-1,0) \times (0,s_0),\ \delta \in (0, 1 - s_0^{\frac{1}{d}}).$$

(iii)

$$\lim_{\substack{\delta \searrow 0 \\ \delta \in (0, 1 - s_0^{\frac{1}{d}})}} \frac{\partial^j F_\delta}{\partial x^j}(x,y,s) = bse_{max}^{j-1} \frac{\partial^j W}{\partial x^j}\left(e_{max}x, y, \frac{e_{min}}{e_{max}}, \frac{1-s}{s}\right)$$

locally uniformly with (x,y,s) *in* $\mathbb{R}_{>0} \times (-1,0) \times (0,s_0)$ *for* $j \in \{0,1,2\}$.

(iv)

$$\lim_{\substack{(y,\delta) \to (-1,0) \\ (y,\delta) \in (-1,0) \times (0, 1 - s_0^{\frac{1}{d}})}} (y+1)^{\frac{1}{2}(j+1)} \frac{\partial^j F_\delta}{\partial x^j}(\sqrt{y+1}x, y, s)$$
$$= bse_{max}^{j-1} \frac{\partial^j \widehat{W}}{\partial x^j}\left(e_{max}x, \frac{e_{min}}{e_{max}}, \frac{1-s}{s}\right)$$

locally uniformly with (x,s) *in* $\mathbb{R}_{>0} \times (0,s_0)$ *for* $j \in \{0,1\}$.

Remark 2.10. We will use the property (i) only to discuss the derivation of the free energy density from the many-electron system in Remark 2.13. The property (i) is not necessary to prove Theorem 1.6 and Theorem 1.7.

Proof of Lemma 2.9. We can construct $E_{s,\delta} \in \mathcal{E}(e_{min}, e_{max})$ in a way similar to the construction of "E" in [14, Lemma A.1]. Here let us describe the initial part of the construction in detail as it was skipped in the proof of [14, Lemma A.1]. Take any $\delta \in \mathbb{R}_{>0}$. Define the function $\phi_{1,\delta} : \mathbb{R} \to \mathbb{R}$ by

$$\phi_{1,\delta}(x) := \begin{cases} e^{\frac{1}{(x+\pi\delta)x}}, & x \in (-\pi\delta, 0), \\ 0, & x \in (-\infty, -\pi\delta] \cup [0, \infty). \end{cases}$$

Observe that $\phi_{1,\delta} \in C^\infty(\mathbb{R})$. Define the function $\phi_{2,\delta} : \mathbb{R} \to \mathbb{R}$ by

$$\phi_{2,\delta}(x) := \frac{\int_{-\infty}^x dt \phi_{1,\delta}(t)}{\int_{-\infty}^\infty dt \phi_{1,\delta}(t)}.$$

It follows that $\phi_{2,\delta} \in C^\infty(\mathbb{R})$. Then let us define the function $\phi_{3,\delta} : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\phi_{3,\delta}(x,s) := \phi_{2,\delta}(x + \pi s^{\frac{1}{d}}).$$

Observe that $\phi_{3,\delta} \in C^\infty(\mathbb{R} \times \mathbb{R}_{>0})$ and for any $s \in \mathbb{R}_{>0}$

$$\phi_{3,\delta}(x,s) = 0,\ \forall x \in (-\infty, -\pi(\delta + s^{\frac{1}{d}})],$$
$$\phi_{3,\delta}(x,s) = 1,\ \forall x \in [-\pi s^{\frac{1}{d}}, \infty),$$
$$\frac{\partial}{\partial x}\phi_{3,\delta}(x,s) > 0,\ \forall x \in (-\pi(\delta + s^{\frac{1}{d}}), -\pi s^{\frac{1}{d}}).$$

Moreover, define the function $\phi_{4,\delta} : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\phi_{4,\delta}(x,s) := \begin{cases} \phi_{3,\delta}(x,s), & x \in (-\infty, 0), \\ \phi_{3,\delta}(-x,s), & x \in [0,\infty). \end{cases}$$

Observe that $\phi_{4,\delta} \in C^\infty(\mathbb{R} \times \mathbb{R}_{>0})$ and for any $s \in \mathbb{R}_{>0}$

$$\begin{aligned}
&\phi_{4,\delta}(x,s) = 1 \text{ if } |x| \le \pi s^{\frac{1}{d}}, \\
&\phi_{4,\delta}(x,s) = 0 \text{ if } |x| \ge \pi(\delta + s^{\frac{1}{d}}), \\
&\phi_{4,\delta}(x,s) \in (0,1) \text{ if } \pi s^{\frac{1}{d}} < |x| < \pi(\delta + s^{\frac{1}{d}}), \\
&\phi_{4,\delta}(x,s) = \phi_{4,\delta}(-x,s),\ \forall x \in \mathbb{R}.
\end{aligned}$$

Furthermore we define the function $\phi_\delta : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\phi_\delta(x,s) := (e_{max} - e_{min})^{\frac{1}{d}} \phi_{4,\delta}(x - \pi, s),\ \forall (x,s) \in \mathbb{R} \times \mathbb{R}_{>0}.$$

It follows that $\phi_\delta \in C^\infty(\mathbb{R} \times \mathbb{R}_{>0})$ and for any $s \in \mathbb{R}_{>0}$

$$\begin{aligned}
&\phi_\delta(x,s) = (e_{max} - e_{min})^{\frac{1}{d}} \text{ if } |x - \pi| \le \pi s^{\frac{1}{d}}, \\
&\phi_\delta(x,s) = 0 \text{ if } |x - \pi| \ge \pi(\delta + s^{\frac{1}{d}}), \\
&\phi_\delta(x,s) \in (0, (e_{max} - e_{min})^{\frac{1}{d}}) \text{ if } \pi s^{\frac{1}{d}} < |x - \pi| < \pi(\delta + s^{\frac{1}{d}}), \\
&\phi_\delta(\pi + x, s) = \phi_\delta(\pi - x, s),\ \forall x \in \mathbb{R}.
\end{aligned}$$

Moreover, for any $n \in \mathbb{N} \cup \{0\}$, $c, c_0, c_1, \cdots, c_n \in \mathbb{R}$

$$\sup_{x \in \mathbb{R}} \left| c + \sum_{j=0}^{n} c_j \frac{\partial^j \phi_\delta}{\partial x^j}(x,s) \right| \text{ is constant with } s \in \mathbb{R}_{>0}. \tag{2.26}$$

Then by using ϕ_δ in place of "ϕ" we can construct $E_{s,\delta}$ in the same way as the construction of "E" in the proof of [14, Lemma A.1]. Let us sketch the construction for completeness. Let $s_0 \in (0,1)$, $\delta \in (0, 1 - s_0^{\frac{1}{d}})$. Define the function $\Phi_\delta : \mathbb{R}^d \times (0, s_0) \to \mathbb{R}$ by

$$\Phi_\delta(x_1, \cdots, x_d, s) := \prod_{j=1}^{d} \phi_\delta(x_j, s) + e_{min}.$$

Observe that $\Phi_\delta \in C^\infty(\mathbb{R}^d \times (0, s_0))$,

$$\begin{aligned}
&\Phi_\delta(x_1, \cdots, x_d, s) = e_{max} \text{ if } |x_j - \pi| \le \pi s^{\frac{1}{d}},\ \forall j \in \{1, \cdots, d\}, \\
&\Phi_\delta(x_1, \cdots, x_d, s) = e_{min} \text{ if } \exists j \in \{1, \cdots, d\} \text{ s.t. } |x_j - \pi| \ge \pi(\delta + s^{\frac{1}{d}}), \\
&\Phi_\delta(x_1, \cdots, x_d, s) \in (e_{min}, e_{max}) \text{ otherwise.}
\end{aligned}$$

Then we define the matrix-valued function $\hat{E}_{s,\delta} : \Gamma_\infty^* \to \mathrm{Mat}(b, \mathbb{C})$ by

$$\hat{E}_{s,\delta}(\mathbf{k}) := \Phi_\delta((\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_d)^{-1}\mathbf{k}, s) I_b,\ \mathbf{k} \in \Gamma_\infty^*.$$

Let $E_{s,\delta} : \mathbb{R}^d \to \mathrm{Mat}(b, \mathbb{C})$ be the periodic extension of $\hat{E}_{s,\delta}$ so that

$$E_{s,\delta}\left(\mathbf{k} + \sum_{j=1}^{d} 2\pi m_j \hat{\mathbf{v}}_j\right) = \hat{E}_{s,\delta}(\mathbf{k}),\ \forall \mathbf{k} \in \Gamma_\infty^*,\ (m_j)_{j=1}^d \in \mathbb{Z}^d.$$

One can check that $E_{s,\delta} \in \mathcal{E}(e_{min}, e_{max})$. In particular the property (1.1) can be confirmed in the same way as in the proof of [14, Lemma A.1].

Take any $m_j \in \mathbb{N} \cup \{0\}$ $(j = 1, \cdots, d)$ with $\sum_{j=1}^d m_j \le d + 2$. Set $V := (\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_d) \in \mathrm{Mat}(d, \mathbb{R})$. By (2.26) for any $s \in (0, s_0)$

$$
\begin{aligned}
&\sup_{\mathbf{k}\in\mathbb{R}^d} \left\| \prod_{j=1}^d \frac{\partial^{m_j}}{\partial k_j^{m_j}} E_{s,\delta}(\mathbf{k}) \right\|_{b\times b} = \sup_{\hat{\mathbf{k}}\in\mathbb{R}^d} \left\| \prod_{j=1}^d \frac{\partial^{m_j}}{\partial k_j^{m_j}} E_{s,\delta}(V\hat{\mathbf{k}}) \right\|_{b\times b} \\
&= \sup_{\hat{\mathbf{k}}\in[0,2\pi]^d} \left\| \prod_{j=1}^d \frac{\partial^{m_j}}{\partial k_j^{m_j}} \hat{E}_{s,\delta}(V\hat{\mathbf{k}}) \right\|_{b\times b} \\
&= \sup_{\hat{\mathbf{k}}\in\mathbb{R}^d} \left| \prod_{j=1}^d \left(\sum_{i=1}^d (V^{-1})_{i,j} \frac{\partial}{\partial \hat{k}_i} \right)^{m_j} \left(\prod_{j=1}^d \phi_\delta(\hat{k}_j, s) + e_{min} \right) \right| \\
&= \sup_{(\hat{k}_2,\cdots,\hat{k}_d)\in\mathbb{R}^{d-1}} \sup_{\hat{k}_1\in\mathbb{R}} \left| \prod_{j=1}^d \left(\sum_{i=1}^d (V^{-1})_{i,j} \frac{\partial}{\partial \hat{k}_i} \right)^{m_j} \left(\phi_\delta\left(\hat{k}_1, \frac{s_0}{2}\right) \prod_{j=2}^d \phi_\delta(\hat{k}_j, s) + e_{min} \right) \right| \\
&= \sup_{(\hat{k}_1,\hat{k}_3,\cdots,\hat{k}_d)\in\mathbb{R}^{d-1}} \sup_{\hat{k}_2\in\mathbb{R}} \left| \prod_{j=1}^d \left(\sum_{i=1}^d (V^{-1})_{i,j} \frac{\partial}{\partial \hat{k}_i} \right)^{m_j} \left(\prod_{l=1}^2 \phi_\delta\left(\hat{k}_l, \frac{s_0}{2}\right) \prod_{j=3}^d \phi_\delta(\hat{k}_j, s) + e_{min} \right) \right| \\
&\vdots \\
&= \sup_{\hat{\mathbf{k}}\in\mathbb{R}^d} \left| \prod_{j=1}^d \left(\sum_{i=1}^d (V^{-1})_{i,j} \frac{\partial}{\partial \hat{k}_i} \right)^{m_j} \left(\prod_{j=1}^d \phi_\delta\left(\hat{k}_j, \frac{s_0}{2}\right) + e_{min} \right) \right| \\
&= \sup_{\mathbf{k}\in\mathbb{R}^d} \left\| \prod_{j=1}^d \frac{\partial^{m_j}}{\partial k_j^{m_j}} E_{\frac{s_0}{2},\delta}(\mathbf{k}) \right\|_{b\times b},
\end{aligned}
$$

which implies the claim (i).

We can deduce the property (ii) from the equality

$$
F_\delta(x, y, s) = b(2\pi)^{-d} \int_{[0,2\pi]^d} d\hat{\mathbf{k}} \frac{\sinh(x\Phi_\delta(\hat{\mathbf{k}}, s))}{(y + \cosh(x\Phi_\delta(\hat{\mathbf{k}}, s)))\Phi_\delta(\hat{\mathbf{k}}, s)}.
$$

Let us define the function $\Phi : \mathbb{R}^d \times (0, s_0) \to \mathbb{R}$ by

$$
\Phi(x_1, \cdots, x_d, s) := \begin{cases} e_{max} & \text{if } |x_j - \pi| \le \pi s^{\frac{1}{d}},\ \forall j \in \{1, \cdots, d\}, \\ e_{min} & \text{if } \exists j \in \{1, \cdots, d\} \text{ s.t. } |x_j - \pi| > \pi s^{\frac{1}{d}}. \end{cases}
$$

Observe that

$$
\begin{aligned}
\lim_{\substack{\delta\searrow 0 \\ \delta\in(0,1-s_0^{\frac{1}{d}})}} F_\delta(x, y, s) &= b(2\pi)^{-d} \int_{[0,2\pi]^d} d\hat{\mathbf{k}} \frac{\sinh(x\Phi(\hat{\mathbf{k}}, s))}{(y + \cosh(x\Phi(\hat{\mathbf{k}}, s)))\Phi(\hat{\mathbf{k}}, s)} \\
&= bse_{max}^{-1} W\left(e_{max}x, y, \frac{e_{min}}{e_{max}}, \frac{1-s}{s} \right)
\end{aligned}
$$

locally uniformly with (x, y, s) in $\mathbb{R}_{>0} \times (-1, 0) \times (0, s_0)$. One can derive an upper bound on the right-hand side of the following equality to verify the claimed locally uniform convergence.

$$
F_\delta(x, y, s) - bse_{max}^{-1} W\left(e_{max}x, y, \frac{e_{min}}{e_{max}}, \frac{1-s}{s} \right)
$$

$$= b(2\pi)^{-d} \int_{Q(\pi(\delta+s^{\frac{1}{d}}))\setminus Q(\pi s^{\frac{1}{d}})} d\hat{\mathbf{k}}$$
$$\cdot \left(\frac{\sinh(x\Phi_\delta(\hat{\mathbf{k}}, s))}{(y + \cosh(x\Phi_\delta(\hat{\mathbf{k}}, s)))\Phi_\delta(\hat{\mathbf{k}}, s)} - \frac{\sinh(x\Phi(\hat{\mathbf{k}}, s))}{(y + \cosh(x\Phi(\hat{\mathbf{k}}, s)))\Phi(\hat{\mathbf{k}}, s)} \right),$$

where $Q(t) := [\pi - t, \pi + t]^d$ for $t \in (0, \pi)$. Moreover,

$$\lim_{\substack{(y,\delta)\to(-1,0) \\ (y,\delta)\in(-1,0)\times(0,1-s_0^{\frac{1}{d}})}} \sqrt{y+1}F_\delta(\sqrt{y+1}x, y, s) = b(2\pi)^{-d} \int_{[0,2\pi]^d} d\hat{\mathbf{k}} \frac{x}{1 + \frac{x^2}{2}\Phi(\hat{\mathbf{k}}, s)^2}$$
$$= bse_{max}^{-1}\widehat{W}\left(e_{max}x, \frac{e_{min}}{e_{max}}, \frac{1-s}{s}\right)$$

locally uniformly with (x, s) in $\mathbb{R}_{>0} \times (0, s_0)$. The convergent properties of the derivatives of F_δ can be confirmed similarly. □

2.3 Non-existence of SPI

Here we prove a proposition which ensures that the claim (iv) of Theorem 1.7 implies the claim (iii) of Theorem 1.7. In the proof we will use the function $u : \mathbb{R}_{>0} \times [-1, 1] \times \mathbb{R}_{>0} \to \mathbb{R}$ defined by

$$u(x, y, z) := \frac{\sinh(xz)}{(y + \cosh(xz))z}.$$

We essentially rely on [14, Lemma 2.12] to prove the next proposition.

Proposition 2.11. *Assume that* $\frac{e_{min}}{e_{max}} > \sqrt{17 - 12\sqrt{2}}$. *Then there exists* $U_0 \in (0, \frac{2e_{min}}{b})$ *such that for any* $U \in [-U_0, 0)$, $E \in \mathcal{E}(e_{min}, e_{max})$ $\tau(\cdot)$ *has no SPI in* $(0, \beta_c)$.

Proof. Let us prove the following statement.

(2.27)

There exists $y_0 \in (-1, 0)$ such that for any $y \in (-1, y_0]$, $E \in \mathcal{E}(e_{min}, e_{max})$
there uniquely exists $x_0 \in \mathbb{R}_{>0}$ such that $\frac{\partial F_\infty}{\partial x}(x_0, y) = 0.$

If (2.27) holds, then we can apply Lemma 2.7 with $S = \mathcal{E}(e_{min}, e_{max})$ to conclude the proof.

If $e_{min} = e_{max}$, $F_\infty(x, y) = bu(x, y, e_{max})$. For any $y \in (-1, 0)$, $\frac{\partial F_\infty}{\partial x}(x_0, y) = 0$ if and only if $x_0 = \frac{\cosh^{-1}(|y|^{-1})}{e_{max}}$, where $\cosh^{-1} : [1, \infty) \to \mathbb{R}_{\ge 0}$ is the inverse function of $\cosh|_{\mathbb{R}_{\ge 0}}$. Thus (2.27) holds.

Assume that $e_{min} < e_{max}$. Let us fix $E \in \mathcal{E}(e_{min}, e_{max})$. By applying Rouché's theorem one can prove that there are continuous functions $e_j : \Gamma_\infty^* \to \mathbb{R}$ $(j = 1, 2, \cdots, b)$ such that

$$e_1(\mathbf{k}) \le e_2(\mathbf{k}) \le \cdots \le e_b(\mathbf{k}),$$
$$\{\text{Eigenvalues of } E(\mathbf{k})\} = \{e_j(\mathbf{k})\}_{j=1}^b, \ \forall \mathbf{k} \in \Gamma_\infty^*.$$

It follows that

$$e_{min} = \min_{\mathbf{k}\in\Gamma_\infty^*}\min_{j\in\{1,\cdots,b\}}|e_j(\mathbf{k})|,\ e_{max} = \max_{\mathbf{k}\in\Gamma_\infty^*}\max_{j\in\{1,\cdots,b\}}|e_j(\mathbf{k})|,$$

$$F_\infty(x,y) = \sum_{j=1}^{b} D_d\int_{\Gamma_\infty^*} d\mathbf{k}u(x,y,|e_j(\mathbf{k})|),\ \forall (x,y)\in\mathbb{R}_{>0}\times(-1,0), \tag{2.28}$$

$$\frac{\partial F_\infty}{\partial x}(x,y) > 0,\ \forall x\in\left(0,\frac{\cosh^{-1}(|y|^{-1})}{e_{max}}\right], \tag{2.29}$$
$$\frac{\partial F_\infty}{\partial x}(x,y) < 0,\ \forall x\in\left[\frac{\cosh^{-1}(|y|^{-1})}{e_{min}},\infty\right),\ y\in(-1,0).$$

The inequalities (2.29) imply that for any $y\in(-1,-\frac{1}{2}]$ there exists $x_0(y)\in(\frac{\cosh^{-1}(|y|^{-1})}{e_{max}},\frac{\cosh^{-1}(|y|^{-1})}{e_{min}})$ such that $\frac{\partial F_\infty}{\partial x}(x_0(y),y)=0$. Observe that

$$\left|\frac{e_0}{\sqrt{y+1}}x_0(y)\right| \le c_{max}\frac{e_{max}}{e_{min}},\ \forall e_0\in[e_{min},e_{max}], \tag{2.30}$$

where

$$c_{max} := \sup_{y\in(-1,-\frac{1}{2}]}\frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}}.$$

Using the equality

$$\cosh^{-1}(|y|^{-1}) = \log\left(|y|^{-1}+\sqrt{|y|^{-2}-1}\right), \tag{2.31}$$

we can check that $0 < c_{max} < \infty$. By substituting $x = \frac{e_0}{\sqrt{y+1}}x_0(y)$ and using (2.30) we can deduce from [14, Lemma 2.12] that if $y\in(-1,-\frac{1}{2}]$ and

$$|y+1| < \frac{c_1\frac{e_{min}}{e_{max}}((\frac{e_{min}}{e_{max}})^2-17+12\sqrt{2})}{2\cosh^2(2c_{max}\frac{e_{min}}{e_{max}})\cosh^2(c_{max}\frac{e_{min}}{e_{max}})}, \tag{2.32}$$

then

$$\frac{\partial u}{\partial x}(x_0(y),y,e_0)\frac{\partial^2 u}{\partial x^2}(x_0(y),y,e_{min}) - \frac{\partial^2 u}{\partial x^2}(x_0(y),y,e_0)\frac{\partial u}{\partial x}(x_0(y),y,e_{min}) > 0,$$
$$\forall e_0\in(e_{min},e_{max}], \tag{2.33}$$

where $c_1\in\mathbb{R}_{>0}$ is the generic constant independent of any parameter, introduced in [14, Lemma 2.12]. We emphasis that c_1 is independent of E. We can derive from (2.28), (2.33) that

$$\frac{\partial u}{\partial x}(x_0(y),y,e_{min})\frac{\partial^2 F_\infty}{\partial x^2}(x_0(y),y) < \frac{\partial F_\infty}{\partial x}(x_0(y),y)\frac{\partial^2 u}{\partial x^2}(x_0(y),y,e_{min}) = 0.$$

Since $\frac{\partial u}{\partial x}(x_0(y),y,e_{min}) > 0$, $\frac{\partial^2 F_\infty}{\partial x^2}(x_0(y),y) < 0$. Essentially we have proved that if $y\in(-1,-\frac{1}{2}]$ satisfies (2.32) and $x_0\in(\frac{\cosh^{-1}(|y|^{-1})}{e_{max}},\frac{\cosh^{-1}(|y|^{-1})}{e_{min}})$ satisfies $\frac{\partial F_\infty}{\partial x}(x_0,y)=0$, then $\frac{\partial^2 F_\infty}{\partial x^2}(x_0,y)<0$. Take any $y\in(-1,-\frac{1}{2}]$ satisfying (2.32). Set

$$M := \left\{x\in\left(\frac{\cosh^{-1}(|y|^{-1})}{e_{max}},\frac{\cosh^{-1}(|y|^{-1})}{e_{min}}\right)\ \middle|\ \frac{\partial F_\infty}{\partial x}(x,y)=0\right\}.$$

We have already seen that $M \neq \emptyset$. Suppose that $\sharp M \geq 2$. Since $x \mapsto \frac{\partial F_\infty}{\partial x}(x,y) : \mathbb{R}_{>0} \to \mathbb{R}$ is real analytic, not identically zero, there exist $x_1, x_2 \in M$ such that $x_1 < x_2$ and $x \notin M$ for any $x \in (x_1, x_2)$. However, the property $\frac{\partial^2 F_\infty}{\partial x^2}(x_j, y) < 0$ $(j = 1, 2)$ implies that there exists $x_3 \in (x_1, x_2)$ such that $x_3 \in M$, which is a contradiction. Therefore $\sharp M = 1$. Combined with (2.29), the above argument ensures that the claim (2.27) holds with $y_0 = \min\{-\frac{1}{2}, -1 + \frac{c_2}{2}\}$, where $c_2 (\in \mathbb{R}_{>0})$ is the right-hand side of (2.32). Lemma 2.7 concludes the proof. □

2.4 Existence of SPI: non-critical case

Our purpose here is to prove existence of a SPI under the condition $\frac{e_{min}}{e_{max}} < \sqrt{17 - 12\sqrt{2}}$, or more precisely the following proposition. Remind us that the set $\{E_{s,\delta}\}_{s \in (0,s_0), \delta \in (0, 1 - s_0^{\frac{1}{d}})} \subset \mathcal{E}(e_{min}, e_{max})$ is constructed in Lemma 2.9.

Proposition 2.12. *Assume that $\frac{e_{min}}{e_{max}} < \sqrt{17 - 12\sqrt{2}}$. Then there exist $s_0 \in (0, 1)$ and $\delta \in (0, 1 - s_0^{\frac{1}{d}})$ such that the following statements hold.*

(i) For any $U_0 \in (0, \frac{2e_{min}}{b})$, $\xi \in \{r, f\}$ there exist $U \in [-U_0, 0)$, $s \in (0, s_0)$ such that $(SPI)_\xi$ holds with U and $E_{s,\delta} (\in \mathcal{E}(e_{min}, e_{max}))$.

(ii) For any $U_0 \in (0, \frac{2e_{min}}{b})$, $(\rho, \eta) \in \{(+, -), (-, +)\}$ there exist $U \in [-U_0, 0)$, $s \in (0, s_0)$, $n \in 4\mathbb{N}+2$ such that $(PT)_{n,(\rho,\eta)}$ holds with U and $E_{s,\delta} (\in \mathcal{E}(e_{min}, e_{max}))$.

Remark 2.13. The free energy density $F_E(\beta, t)$ was derived from the many-electron system in [14, Theorem 1.3 (ii)] for any $E \in \mathcal{E}(e_{min}, e_{max})$, $U \in \mathbb{R}_{<0}$ satisfying (1.3). It is not trivial if $(U, E_{s,\delta})$ introduced in Proposition 2.12 (i), (ii) satisfies (1.3). If so, the existence of SPI and HOPT is guaranteed by the proposition while the derivation of the free energy density is justified by [14, Theorem 1.3 (ii)]. According to the proof of Proposition 2.12, the choice of $s \in (0, s_0)$ depends on U_0. However, Lemma 2.9 (i) states that (1.4) with $E = E_{s,\delta}$ is independent of s. Assume $\frac{e_{min}}{e_{max}} < \sqrt{17 - 12\sqrt{2}}$ and let $s_0 \in (0, 1)$, $\delta \in (0, 1 - s_0^{\frac{1}{d}})$ be those introduced in Proposition 2.12. It follows in particular that

$$((1.4) \text{ with } E = E_{s,\delta}) = ((1.4) \text{ with } E = E_{\frac{s_0}{2},\delta})$$

for any $s \in (0, s_0)$. Take any

$$U_0 \in \left(0, \frac{2c'}{b} \min\{e_{min}, e_{min}^{d+1}\}\right),$$

where $c' \in (0, 1]$ is introduced in [14, Theorem 1.3] and depends only on d, b, $(\hat{\mathbf{v}}_j)_{j=1}^d$ and (1.4) with $E = E_{\frac{s_0}{2},\delta}$. Then the following statements hold true.

- For any $\xi \in \{r, f\}$ there exist $U \in [-U_0, 0)$, $s \in (0, s_0)$ such that $(\text{SPI})_\xi$ holds and $F_E(\beta, t)$ is derived from the many-electron system by [14, Theorem 1.3 (ii)] with U and $E_{s,\delta}$.

- For any $(\rho, \eta) \in \{(+, -), (-, +)\}$ there exist $U \in [-U_0, 0)$, $s \in (0, s_0)$, $n \in 4\mathbb{N} + 2$ such that $(\text{PT})_{n,(\rho,\eta)}$ holds and $F_E(\beta, t)$ is derived from the many-electron system by [14, Theorem 1.3 (ii)] with U and $E_{s,\delta}$.

Throughout this subsection we assume that $\frac{e_{min}}{e_{max}} < \sqrt{17 - 12\sqrt{2}}$. We need to introduce a function in order to construct the proof of the above proposition. The function $\tilde{w}(x, y, z)$ is defined in the open set $\tilde{D}$ of $\mathbb{C}^3$ as follows.

$$\tilde{D} :=$$
$$\left\{(x, y, z) \in \mathbb{C}^3 \;\middle|\; \left|1 + y \sum_{m=1}^{\infty} \frac{(y+1)^{m-1}}{(2m)!} 2^m z^m x^m\right| \left|1 + \sum_{n=1}^{\infty} \frac{(y+1)^{n-1}}{(2n)!} 2^n x^n\right| > 0\right\},$$
$$\tilde{w}(x, y, z) := -\frac{\left(1 + y \sum_{m=1}^{\infty} \frac{(y+1)^{m-1}}{(2m)!} 2^m x^m\right)\left(1 + \sum_{n=1}^{\infty} \frac{(y+1)^{n-1}}{(2n)!} 2^n z^n x^n\right)^2}{\left(1 + y \sum_{m=1}^{\infty} \frac{(y+1)^{m-1}}{(2m)!} 2^m z^m x^m\right)\left(1 + \sum_{n=1}^{\infty} \frac{(y+1)^{n-1}}{(2n)!} 2^n x^n\right)^2}.$$

In fact in [14, Subsection 2.2] the function $\tilde{w}$ was introduced as an analytic continuation of the function $w : D \to \mathbb{R}$ defined by

$$w(x, y, z) := -\frac{(1 + y\cosh(\sqrt{y+1}\sqrt{2x}))(y + \cosh(\sqrt{y+1}\sqrt{2zx}))^2}{(1 + y\cosh(\sqrt{y+1}\sqrt{2zx}))(y + \cosh(\sqrt{y+1}\sqrt{2x}))^2}, \tag{2.34}$$
$$D := \left\{(x, y, z) \in \mathbb{R}_{>0} \times (-1, 0) \times \mathbb{R}_{>0} \;\middle|\; x < \frac{1}{2z(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right\}.$$

Here we presented the full definition of these functions in order to make clear the continuity from the previous construction [14, Section 2]. The function w will be recalled in Subsection 2.5.

Set $\eta := (\frac{e_{min}}{e_{max}})^2 (\in (0, 17 - 12\sqrt{2}))$. Here we only need to use the function $x \mapsto \tilde{w}(x, -1, \eta) : (0, \eta^{-1}) \to \mathbb{R}$, which is characterized as

$$\tilde{w}(x, -1, \eta) = \frac{(x-1)(1+\eta x)^2}{(1-\eta x)(1+x)^2}, \quad x \in (0, \eta^{-1}).$$

Since $(\frac{1+\eta}{6\eta})^2 > \frac{1}{\eta}$, we can define the real numbers $a_+(\eta)$, $a_-(\eta)$ by

$$a_+(\eta) := \frac{1+\eta}{6\eta} + \left(\left(\frac{1+\eta}{6\eta}\right)^2 - \frac{1}{\eta}\right)^{\frac{1}{2}}, \quad a_-(\eta) := \frac{1+\eta}{6\eta} - \left(\left(\frac{1+\eta}{6\eta}\right)^2 - \frac{1}{\eta}\right)^{\frac{1}{2}}.$$

The behavior of the function $\tilde{w}(\cdot, -1, \eta)$ is the most important information to prove Proposition 2.12 and is summarized in [14, Lemma 2.18]. Here we restate it for readability of the present paper.

$$1 < a_-(\eta) < a_+(\eta) < \eta^{-1}, \tag{2.35}$$
$$\frac{\partial \tilde{w}}{\partial x}(x, -1, \eta) > 0, \ \forall x \in (0, a_-(\eta)),$$
$$\frac{\partial \tilde{w}}{\partial x}(a_-(\eta), -1, \eta) = 0, \tag{2.36}$$
$$\frac{\partial \tilde{w}}{\partial x}(x, -1, \eta) < 0, \ \forall x \in (a_-(\eta), a_+(\eta)),$$
$$\frac{\partial \tilde{w}}{\partial x}(a_+(\eta), -1, \eta) = 0, \tag{2.37}$$
$$\frac{\partial \tilde{w}}{\partial x}(x, -1, \eta) > 0, \ \forall x \in (a_+(\eta), \eta^{-1}),$$

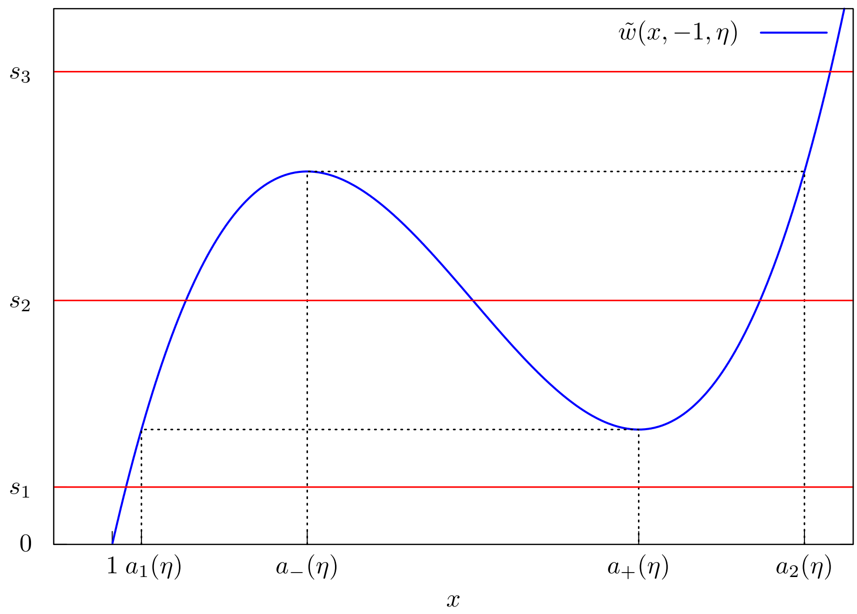

Figure 1: The schematic profile of $\tilde{w}(\cdot,-1,\eta)$ in $[1,\eta^{-1})$.

$$0<\tilde{w}(a_+(\eta),-1,\eta)<\tilde{w}(a_-(\eta),-1,\eta). \tag{2.38}$$

Since $\tilde{w}(1,-1,\eta)=0$ and $\lim_{x\nearrow\eta^{-1}}\tilde{w}(x,-1,\eta)=+\infty$, there uniquely exist $a_1(\eta)\in(1,a_-(\eta))$, $a_2(\eta)\in(a_+(\eta),\eta^{-1})$ such that

$$\tilde{w}(a_1(\eta),-1,\eta)=\tilde{w}(a_+(\eta),-1,\eta),\ \ \tilde{w}(a_2(\eta),-1,\eta)=\tilde{w}(a_-(\eta),-1,\eta).$$

In the following we fix

$$\begin{aligned}&s_1\in(0,\tilde{w}(a_+(\eta),-1,\eta)),\ \ s_2\in(\tilde{w}(a_+(\eta),-1,\eta),\tilde{w}(a_-(\eta),-1,\eta)),\\&s_3\in(\tilde{w}(a_-(\eta),-1,\eta),\infty).\end{aligned}$$

The schematic profile of the function $\tilde{w}(\cdot,-1,\eta)$ in $[1,\eta^{-1})$ is pictured in Figure 1. We remark that Figure 1 is a sketch, not the exact implementation of $\tilde{w}(\cdot,-1,\eta)$.

We can prove the next lemma by combining Lemma 2.9 with the above properties of $\tilde{w}(\cdot,-1,\eta)$. Recall that the function F_δ is defined in (2.25). Here we consider $(\frac{1}{2}s_1+1)^{-1}$ as s_0 introduced in Lemma 2.9.

Lemma 2.14. *There exist $y_0\in(-1,0)$, $\delta_0\in(0,1-(\frac{1}{2}s_1+1)^{-\frac{1}{d}})$ such that the following statements hold for any $y\in(-1,y_0]$, $\delta\in(0,\delta_0]$.*

(i)

$$\begin{aligned}\sqrt{2}<\frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}}&<\sqrt{2a_1(\eta)}<\sqrt{2a_-(\eta)}<\sqrt{2a_+(\eta)}<\sqrt{2a_2(\eta)}<\sqrt{2\eta^{-1}}\\&<\sqrt{\eta^{-1}}\frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}}.\end{aligned}$$

(ii)

(2.39)
$$\frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}x, y, \frac{1}{s_3+1}\right) > 0,\ \forall x \in [\sqrt{2a_1(\eta)}, \sqrt{2a_+(\eta)}].$$

(2.40)
$$\frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}\sqrt{2a_1(\eta)}, y, \frac{1}{s+1}\right) > 0,\ \frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}\sqrt{2a_+(\eta)}, y, \frac{1}{s+1}\right) > 0,$$
$\forall s \in [s_2, s_3]$.

(2.41)
$$\frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}\sqrt{2a_-(\eta)}, y, \frac{1}{s_2+1}\right) < 0.$$

(iii)

(2.42)
$$\frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}x, y, \frac{1}{s_1+1}\right) < 0,\ \forall x \in [\sqrt{2a_-(\eta)}, \sqrt{2a_2(\eta)}].$$

(2.43)
$$\frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}\sqrt{2a_-(\eta)}, y, \frac{1}{s+1}\right) < 0,\ \frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}\sqrt{2a_2(\eta)}, y, \frac{1}{s+1}\right) < 0,$$
$\forall s \in [s_1, s_2]$.

(2.44)
$$\frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}\sqrt{2a_+(\eta)}, y, \frac{1}{s_2+1}\right) > 0.$$

Proof. We can derive from (2.31) that

$$\lim_{y\searrow -1}\frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}} = \sqrt{2}. \tag{2.45}$$

It was remarked in the beginning of the proof of [14, Lemma 2.24] that for $y \in (-1, 0)$ sufficiently close to -1,

$$\frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}} > \sqrt{2}. \tag{2.46}$$

The claim (i) follows from (2.35), (2.45), (2.46) and that $a_1(\eta) \in (1, a_-(\eta))$, $a_2(\eta) \in (a_+(\eta), \eta^{-1})$. Recall the definition (2.24). Observe that

$$\frac{\partial \widehat{W}}{\partial x}(x, \sqrt{\eta}, s) = \frac{1-\eta\frac{x^2}{2}}{(1+\eta\frac{x^2}{2})^2}\left(s - \tilde{w}\left(\frac{x^2}{2}, -1, \eta\right)\right),\ \forall (x,s) \in (0, \sqrt{2\eta^{-1}}) \times \mathbb{R}_{>0},$$

and thus by (2.38) and the choice of s_1, s_2, s_3

$$\begin{aligned}&\frac{\partial \widehat{W}}{\partial x}(x, \sqrt{\eta}, s_3) > 0,\ \forall x \in [\sqrt{2a_1(\eta)}, \sqrt{2a_+(\eta)}].\\ &\frac{\partial \widehat{W}}{\partial x}(\sqrt{2a_1(\eta)}, \sqrt{\eta}, s) > 0,\ \frac{\partial \widehat{W}}{\partial x}(\sqrt{2a_+(\eta)}, \sqrt{\eta}, s) > 0,\ \forall s \in [s_2, s_3].\end{aligned} \tag{2.47}$$

$$\frac{\partial \widehat{W}}{\partial x}(\sqrt{2a_-(\eta)}, \sqrt{\eta}, s_2) < 0.$$
$$\frac{\partial \widehat{W}}{\partial x}(x, \sqrt{\eta}, s_1) < 0, \ \forall x \in [\sqrt{2a_-(\eta)}, \sqrt{2a_2(\eta)}].$$
$$\frac{\partial \widehat{W}}{\partial x}(\sqrt{2a_-(\eta)}, \sqrt{\eta}, s) < 0, \ \frac{\partial \widehat{W}}{\partial x}(\sqrt{2a_2(\eta)}, \sqrt{\eta}, s) < 0, \ \forall s \in [s_1, s_2].$$
$$\frac{\partial \widehat{W}}{\partial x}(\sqrt{2a_+(\eta)}, \sqrt{\eta}, s_2) > 0.$$

Figure 1 may help us understand the above inequalities. Lemma 2.9 (iv) implies that

$$\lim_{\substack{(y,\delta)\to(-1,0)\\ (y,\delta)\in(-1,0)\times(0,1-(\frac{1}{2}s_1+1)^{-\frac{1}{d}})}} (y+1)\frac{\partial F_\delta}{\partial x}\left(\frac{\sqrt{y+1}}{e_{max}}x, y, \frac{1}{s+1}\right) = \frac{b}{s+1}\frac{\partial \widehat{W}}{\partial x}(x, \sqrt{\eta}, s)$$

uniformly with (x, s) in $[\sqrt{2a_1(\eta)}, \sqrt{2a_2(\eta)}] \times [s_1, s_3]$. We can deduce the claims (ii), (iii) by combining the above convergent property with (2.47). □

The proof of Proposition 2.12 is based on Corollary 2.5, Lemma 2.8, Lemma 2.9 and Lemma 2.14.

Proof of Proposition 2.12. By Corollary 2.5 the claim (i) is equivalent to the claim (ii). Thus it suffices to give a proof to the claim (i). Let $y_0 \in (-1, 0)$, $\delta_0 \in (0, 1 - (\frac{1}{2}s_1 + 1)^{-\frac{1}{d}})$ be those introduced in Lemma 2.14. Observe that for any $x \in [\sqrt{2a_1(\eta)}, \sqrt{2a_2(\eta)}]$, $s \in [s_1, s_3]$, $\delta \in (0, \delta_0]$

$$\begin{aligned}
&\sqrt{y+1}F_\delta\left(\frac{\sqrt{y+1}}{e_{max}}x, y, \frac{1}{s+1}\right)\\
&\geq \frac{b}{(s_3+1)e_{max}} \inf_{\substack{x'\in[\sqrt{2a_1(\eta)},\sqrt{2a_2(\eta)}]\\ s'\in[s_1,s_3]}} \widehat{W}(x', \sqrt{\eta}, s')\\
&\quad - \sup_{\substack{x'\in[\sqrt{2a_1(\eta)},\sqrt{2a_2(\eta)}]\\ s'\in[s_1,s_3]}} \left|\sqrt{y+1}F_\delta\left(\frac{\sqrt{y+1}}{e_{max}}x', y, \frac{1}{s'+1}\right) - \frac{b}{(s'+1)e_{max}}\widehat{W}(x', \sqrt{\eta}, s')\right|.
\end{aligned}$$

We can apply Lemma 2.9 (iv) to ensure that there exist $y_1 \in (-1, y_0]$, $\delta_1 \in (0, \delta_0]$ such that

$$F_{\delta_1}\left(\frac{\sqrt{y+1}}{e_{max}}x, y, \frac{1}{s+1}\right) \geq \frac{b}{2\sqrt{y+1}(s_3+1)e_{max}} \inf_{\substack{x'\in[\sqrt{2a_1(\eta)},\sqrt{2a_2(\eta)}]\\ s'\in[s_1,s_3]}} \widehat{W}(x', \sqrt{\eta}, s')$$

for any $y \in (-1, y_1]$. Take any $U_0 \in (0, \frac{2e_{max}}{b})$. By the above inequality there exists $y_2 \in (-1, y_1]$ such that

$$F_{\delta_1}\left(\frac{\sqrt{y_2+1}}{e_{max}}x, y_2, \frac{1}{s+1}\right) \geq \frac{2}{U_0}, \ \forall x \in [\sqrt{2a_1(\eta)}, \sqrt{2a_2(\eta)}], \ s \in [s_1, s_3]. \tag{2.48}$$

Here we apply the inequalities given in Lemma 2.14 (ii) with $\delta = \delta_1$, $y = y_2$. By (2.39), (2.41) and the fact that

$$s \mapsto \min_{x\in\left[\frac{\sqrt{y_2+1}}{e_{max}}\sqrt{2a_1(\eta)}, \frac{\sqrt{y_2+1}}{e_{max}}\sqrt{2a_+(\eta)}\right]} \frac{\partial F_{\delta_1}}{\partial x}\left(x, y_2, \frac{1}{s+1}\right)$$

is continuous in $[s_2, s_3]$ there exists $\hat{s} \in (s_2, s_3)$ such that

$$\min_{x \in \left[\frac{\sqrt{y_2+1}}{e_{max}} \sqrt{2a_1(\eta)}, \frac{\sqrt{y_2+1}}{e_{max}} \sqrt{2a_+(\eta)} \right]} \frac{\partial F_{\delta_1}}{\partial x} \left(x, y_2, \frac{1}{\hat{s}+1} \right) = 0.$$

Moreover, by (2.40) there exists

$$\hat{x} \in \left(\frac{\sqrt{y_2+1}}{e_{max}} \sqrt{2a_1(\eta)}, \frac{\sqrt{y_2+1}}{e_{max}} \sqrt{2a_+(\eta)} \right)$$

such that

$$\frac{\partial F_{\delta_1}}{\partial x} \left(\hat{x}, y_2, \frac{1}{\hat{s}+1} \right) = 0.$$

Furthermore, since $x \mapsto \frac{\partial F_{\delta_1}}{\partial x}(x, y_2, \frac{1}{\hat{s}+1}) : \mathbb{R}_{>0} \to \mathbb{R}$ is real analytic and not identically zero, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$(\hat{x} - \varepsilon, \hat{x} + \varepsilon) \subset \left(\frac{\sqrt{y_2+1}}{e_{max}} \sqrt{2a_1(\eta)}, \frac{\sqrt{y_2+1}}{e_{max}} \sqrt{2a_+(\eta)} \right),$$
$$\frac{\partial F_{\delta_1}}{\partial x} \left(x, y_2, \frac{1}{\hat{s}+1} \right) > 0, \ \forall x \in (\hat{x} - \varepsilon, \hat{x} + \varepsilon) \backslash \{\hat{x}\}.$$

This means that $\hat{x}$ is a rising SPI of $F_{\delta_1}(\cdot, y_2, \frac{1}{\hat{s}+1})$. Since $F_{\delta_1}(\cdot, y_2, \frac{1}{\hat{s}+1}) = F_\infty(\cdot, y_2)$ with $E_{\frac{1}{\hat{s}+1},\delta_1} \in \mathcal{E}(e_{min}, e_{max})$, the above property and (2.48) enable us to apply Lemma 2.8 (i) to conclude that there exists $U \in [-U_0, 0)$ such that $\tau(\cdot)$ has a falling SPI in $(0, \beta_c)$.

Using Lemma 2.14 (iii), Lemma 2.8 (ii) in place of Lemma 2.14 (ii), Lemma 2.8 (i) respectively, we can argue in a way parallel to the above argument to prove existence of a rising SPI of $\tau(\cdot)$ for some $U \in [-U_0, 0)$, $E_{\frac{1}{\tilde{s}+1},\delta_1} \in \mathcal{E}(e_{min}, e_{max})$ with $\tilde{s} \in (s_1, s_2)$.

We have proved the claims with $s_0 = (\frac{1}{2}s_1 + 1)^{-1}$, $\delta = \delta_1$. □

2.5 Existence of SPI: critical case

Here we prove existence of a SPI when $\frac{e_{min}}{e_{max}} = \sqrt{17 - 12\sqrt{2}}$.

Proposition 2.15. *Assume that* $\frac{e_{min}}{e_{max}} = \sqrt{17 - 12\sqrt{2}}$. *Then the following statements hold.*

(i) For any $U_0 \in (0, \frac{2e_{min}}{b})$, $\xi \in \{r, f\}$ *there exist* $U \in [-U_0, 0)$, $s \in (0, 1)$, $\delta \in (0, 1 - s^{\frac{1}{d}})$ *such that* $(SPI)_\xi$ *holds with* U *and* $E_{s,\delta} (\in \mathcal{E}(e_{min}, e_{max}))$.

(ii) For any $U_0 \in (0, \frac{2e_{min}}{b})$, $(\rho, \eta) \in \{(+, -), (-, +)\}$ *there exist* $U \in [-U_0, 0)$, $s \in (0, 1)$, $\delta \in (0, 1 - s^{\frac{1}{d}})$, $n \in 4\mathbb{N} + 2$ *such that* $(PT)_{n,(\rho,\eta)}$ *holds with* U *and* $E_{s,\delta} (\in \mathcal{E}(e_{min}, e_{max}))$.

Remark 2.16. As we can see from the proof, we have to choose $s \in (0, 1)$, $\delta \in (0, 1 - s^{\frac{1}{d}})$ after fixing U_0. We cannot prove that the condition (1.3) holds for the pair $(U, E_{s,\delta})$ introduced in the proposition. Accordingly we cannot prove existence of a SPI of $\tau(\cdot):(0,\beta_c)\to\mathbb{R}$ or existence of a HOPT driven by temperature while justifying the derivation of $F_E(\beta,t)$ from the many-electron system in the case $\frac{e_{min}}{e_{max}}=\sqrt{17-12\sqrt{2}}$. In the case $\frac{e_{min}}{e_{max}}<\sqrt{17-12\sqrt{2}}$ we can choose δ before fixing U_0 as claimed in Proposition 2.12, and thus we can reach the positive conclusions stated in Remark 2.13.

Set $\eta_0:=17-12\sqrt{2}$, $a_0:=3+2\sqrt{2}$. As a preliminary, let us recall properties of the function

$$\tilde{w}(x,-1,\eta_0)=\frac{(x-1)(1+\eta_0 x)^2}{(1-\eta_0 x)(1+x)^2},\ x\in(0,\eta_0^{-1}),$$

which form the basis of the proof. Observe that

$$\frac{\partial\tilde{w}}{\partial x}(x,-1,\eta_0)=\frac{3\eta_0(1-\eta_0)(1+\eta_0 x)}{(1-\eta_0 x)^2(1+x)^3}\left(x^2-\frac{\eta_0+1}{3\eta_0}x+\frac{1}{\eta_0}\right),$$

which is equal to [14, (2.47)], and

$$x^2-\frac{\eta_0+1}{3\eta_0}x+\frac{1}{\eta_0}=(x-a_0)^2.$$

These imply that

$$\begin{aligned}&\frac{\partial\tilde{w}}{\partial x}(x,-1,\eta_0)>0,\ \forall x\in(0,\eta_0^{-1})\backslash\{a_0\},\\&\frac{\partial\tilde{w}}{\partial x}(a_0,-1,\eta_0)=0,\end{aligned}\tag{2.49}$$

and thus

$$\begin{aligned}&\frac{1}{2}\tilde{w}(a_0,-1,\eta_0)<\tilde{w}(a_0,-1,\eta_0)<\tilde{w}\left(\frac{1}{2}\eta_0^{-1},-1,\eta_0\right),\\&\sup_{x\in[1,a_0]}\tilde{w}(x,-1,\eta_0)<2\tilde{w}(a_0,-1,\eta_0),\\&\inf_{x\in[a_0,\frac{1}{2}\eta_0^{-1}]}\tilde{w}(x,-1,\eta_0)>\frac{1}{2}\tilde{w}(a_0,-1,\eta_0).\end{aligned}\tag{2.50}$$

In the proof of Proposition 2.15 we essentially use [14, Lemma 2.15], which concerns properties of the function $w(x,y,\eta_0)$ defined in (2.34).

Proof of Proposition 2.15. By Corollary 2.5 the claim (i) is equivalent to the claim (ii). Thus it suffices to prove the claim (i). We apply Lemma 2.9 (iv) with $s_0=(1+\frac{1}{3}\tilde{w}(a_0,-1,\eta_0))^{-1}$ to ensure that there exist $\delta_1\in(0,1-(1+\frac{1}{3}\tilde{w}(a_0,-1,\eta_0))^{-\frac{1}{d}})$, $y_1\in(-1,0)$ such that

$$\begin{aligned}&\sqrt{y+1}F_\delta\left(\frac{\sqrt{y+1}}{e_{max}}x,y,\frac{1}{s+1}\right)\\&\geq\frac{b}{2(2\tilde{w}(a_0,-1,\eta_0)+1)e_{max}}\inf_{x'\in[\sqrt{2},\sqrt{2\eta_0^{-1}}]}\widehat{W}\left(x',\sqrt{\eta_0},\frac{1}{2}\tilde{w}(a_0,-1,\eta_0)\right),\\&\forall x\in[\sqrt{2},\sqrt{2\eta_0^{-1}}],\ y\in(-1,y_1],\ s\in\left[\frac{1}{2}\tilde{w}(a_0,-1,\eta_0),2\tilde{w}(a_0,-1,\eta_0)\right],\ \delta\in(0,\delta_1].\end{aligned}$$

Take any $U_0 \in (0, \frac{2e_{min}}{b})$. The above property guarantees that there exists $y_2 \in (-1, y_1]$ such that

(2.51)
$$F_\delta\left(\frac{\sqrt{y+1}}{e_{max}}x, y, \frac{1}{s+1}\right) \geq \frac{2}{U_0},$$
$$\forall x \in [\sqrt{2}, \sqrt{2\eta_0^{-1}}],\ y \in (-1, y_2],\ s \in \left[\frac{1}{2}\tilde{w}(a_0, -1, \eta_0), 2\tilde{w}(a_0, -1, \eta_0)\right],\ \delta \in (0, \delta_1].$$

It follows from [14, Lemma 2.15] that there exists $y_3 \in (-1, y_2]$ such that for any $y \in (-1, y_3]$

$$\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2 < a_0 < \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2,$$
$$0 < w(a_0, y, \eta_0) < 1.$$

Moreover, there exist

$$x_1(y) \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, a_0\right),\ x_2(y) \in \left(a_0, \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)$$

such that

$$\begin{aligned} &w(x_1(y), y, \eta_0) = w(a_0, y, \eta_0) = w(x_2(y), y, \eta_0), \\ &w(x, y, \eta_0) > w(a_0, y, \eta_0),\ \forall x \in (x_1(y), a_0), \\ &w(x, y, \eta_0) < w(a_0, y, \eta_0),\ \forall x \in (a_0, x_2(y)). \end{aligned} \tag{2.52}$$

We can deduce from (2.46), (2.50) and the property

$$\lim_{y \searrow -1} \sup_{x \in [1, \frac{1}{2}\eta_0^{-1}]} |w(x, y, \eta_0) - \tilde{w}(x, -1, \eta_0)| = 0 \tag{2.53}$$

that there exists $y_4 \in (-1, y_3]$ such that

$$1 < \frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2,$$
$$\frac{1}{2}\tilde{w}(a_0, -1, \eta_0) < w(a_0, y, \eta_0) < \tilde{w}\left(\frac{1}{2}\eta_0^{-1}, -1, \eta_0\right),$$
$$\sup_{x \in [1, a_0]} w(x, y, \eta_0) < 2\tilde{w}(a_0, -1, \eta_0),\quad \inf_{x \in [a_0, \frac{1}{2}\eta_0^{-1}]} w(x, y, \eta_0) > \frac{1}{2}\tilde{w}(a_0, -1, \eta_0) \tag{2.54}$$

for any $y \in (-1, y_4]$.

Let us prove that there exists $\hat{y} \in (-1, y_4]$ such that

$$x_2(\hat{y}) < \frac{1}{2}\eta_0^{-1}. \tag{2.55}$$

Set $\xi := \frac{1}{2}(\tilde{w}(\frac{1}{2}\eta_0^{-1}, -1, \eta_0) - \tilde{w}(a_0, -1, \eta_0))$. It follows that

$$\tilde{w}(a_0, -1, \eta_0) + \xi < \tilde{w}\left(\frac{1}{2}\eta_0^{-1}, -1, \eta_0\right). \tag{2.56}$$

By (2.53) there exists $y_5 \in (-1, y_4]$ such that

$$w(a_0, y, \eta_0) < \tilde{w}(a_0, -1, \eta_0) + \xi, \ \forall y \in (-1, y_5]. \tag{2.57}$$

Let us take $\varepsilon \in \mathbb{R}_{>0}$ so that

$$\eta_0^{-1} - \varepsilon > \frac{1}{2}\eta_0^{-1}, \quad \frac{\eta_0^{-1} - \varepsilon - 1}{\varepsilon\eta_0(1 + \eta_0^{-1})^2} \geq \tilde{w}\left(\frac{1}{2}\eta_0^{-1}, -1, \eta_0\right).$$

We define $T : (-1, y_5] \to \mathbb{R}$ by

$$T(y) := -\left.\frac{1 + y\cosh(\sqrt{y+1}\sqrt{2x})}{1 + y\cosh(\sqrt{y+1}\sqrt{2\eta_0 x})}\right|_{x=\eta_0^{-1}-\varepsilon} \cdot \left.\frac{(y+1)^2}{(y + \cosh(\sqrt{y+1}\sqrt{2x}))^2}\right|_{x=\frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2}.$$

Observe that

$$\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2 < \eta_0^{-1} - \varepsilon < \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2,$$

$$w(x, y, \eta_0) \geq T(y), \ \forall x \in \left[\eta_0^{-1} - \varepsilon, \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right), \ y \in (-1, y_5],$$

$$\lim_{\substack{y \searrow -1 \\ y \in (-1, y_5]}} T(y) = \frac{\eta_0^{-1} - \varepsilon - 1}{\varepsilon\eta_0(1 + \eta_0^{-1})^2} \geq \tilde{w}\left(\frac{1}{2}\eta_0^{-1}, -1, \eta_0\right).$$

These properties plus (2.56) imply that there exists $y_6 \in (-1, y_5]$ such that

$$\begin{aligned} &w(x, y, \eta_0) \geq \tilde{w}(a_0, -1, \eta_0) + \xi, \\ &\forall x \in \left[\eta_0^{-1} - \varepsilon, \ \frac{1}{2\eta_0(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right), \ y \in (-1, y_6]. \end{aligned} \tag{2.58}$$

On the other hand, since

$$\lim_{\substack{y \searrow -1 \\ y \in (-1, y_6]}} \sup_{x \in [\frac{1}{2}\eta_0^{-1}, \eta_0^{-1} - \varepsilon]} |w(x, y, \eta_0) - \tilde{w}(x, -1, \eta_0)| = 0,$$

by (2.49) and (2.56) there exists $\hat{y} \in (-1, y_6]$ such that

$$w(x, \hat{y}, \eta_0) \geq \tilde{w}(a_0, -1, \eta_0) + \xi, \ \forall x \in \left[\frac{1}{2}\eta_0^{-1}, \eta_0^{-1} - \varepsilon\right]. \tag{2.59}$$

By combining (2.57), (2.58) with (2.59) we obtain that

$$w(x, \hat{y}, \eta_0) > w(a_0, \hat{y}, \eta_0), \ \forall x \in \left[\frac{1}{2}\eta_0^{-1}, \frac{1}{2\eta_0(\hat{y}+1)}(\cosh^{-1}(|\hat{y}|^{-1}))^2\right).$$

If $x_2(\hat{y}) \geq \frac{1}{2}\eta_0^{-1}$,

$$w(x_2(\hat{y}), \hat{y}, \eta_0) > w(a_0, \hat{y}, \eta_0) = w(x_2(\hat{y}), \hat{y}, \eta_0),$$

which is a contradiction. Therefore $x_2(\hat{y}) < \frac{1}{2}\eta_0^{-1}$.

Let us set

$$s_1 := \frac{1}{2}\tilde{w}(a_0,-1,\eta_0),$$
$$s_2 := \frac{1}{2}\left(\inf_{x\in[a_0,x_2(\hat{y})]} w(x,\hat{y},\eta_0) + w(a_0,\hat{y},\eta_0)\right),$$
$$s_3 := \frac{1}{2}\left(\sup_{x\in[x_1(\hat{y}),a_0]} w(x,\hat{y},\eta_0) + w(a_0,\hat{y},\eta_0)\right),$$
$$s_4 := 2\tilde{w}(a_0,-1,\eta_0).$$

We can see from (2.52), (2.54), (2.55) that

(2.60)
$$s_1 < \inf_{x\in[a_0,x_2(\hat{y})]} w(x,\hat{y},\eta_0) < s_2 < w(a_0,\hat{y},\eta_0) = w(x_1(\hat{y}),\hat{y},\eta_0) = w(x_2(\hat{y}),\hat{y},\eta_0)$$
$$< s_3 < \sup_{x\in[x_1(\hat{y}),a_0]} w(x,\hat{y},\eta_0) < s_4.$$

Observe that

$$\frac{\partial W}{\partial x}(\sqrt{y+1}x, y, \sqrt{z}, s) = \frac{1+y\cosh(\sqrt{z(y+1)}x)}{(y+\cosh(\sqrt{z(y+1)}x))^2}\left(s - w\left(\frac{x^2}{2}, y, z\right)\right) \quad (2.61)$$

for any $(x,y,z)\in\mathbb{R}_{>0}\times(-1,0)\times\mathbb{R}_{>0}$ satisfying $x<\frac{1}{\sqrt{z(y+1)}}\cosh^{-1}(|y|^{-1})$. Let $\hat{x}_1\in(x_1(\hat{y}),a_0)$, $\hat{x}_2\in(a_0,x_2(\hat{y}))$ be such that

$$w(\hat{x}_1,\hat{y},\eta_0) = \max_{x\in[x_1(\hat{y}),a_0]} w(x,\hat{y},\eta_0),\ w(\hat{x}_2,\hat{y},\eta_0) = \min_{x\in[a_0,x_2(\hat{y})]} w(x,\hat{y},\eta_0). \quad (2.62)$$

Combination of (2.60), (2.61), (2.62) implies that

$$\frac{\partial W}{\partial x}(\sqrt{\hat{y}+1}x,\hat{y},\sqrt{\eta_0},s_1) < 0,\ \forall x\in[\sqrt{2a_0},\sqrt{2x_2(\hat{y})}],$$
$$\frac{\partial W}{\partial x}(\sqrt{\hat{y}+1}\sqrt{2a_0},\hat{y},\sqrt{\eta_0},s) < 0,\ \frac{\partial W}{\partial x}(\sqrt{\hat{y}+1}\sqrt{2x_2(\hat{y})},\hat{y},\sqrt{\eta_0},s) < 0,\ \forall s\in[s_1,s_2],$$
$$\frac{\partial W}{\partial x}(\sqrt{\hat{y}+1}\sqrt{2\hat{x}_2},\hat{y},\sqrt{\eta_0},s_2) > 0,$$
$$\frac{\partial W}{\partial x}(\sqrt{\hat{y}+1}x,\hat{y},\sqrt{\eta_0},s_4) > 0,\ \forall x\in[\sqrt{2x_1(\hat{y})},\sqrt{2a_0}],$$
$$\frac{\partial W}{\partial x}(\sqrt{\hat{y}+1}\sqrt{2x_1(\hat{y})},\hat{y},\sqrt{\eta_0},s) > 0,\ \frac{\partial W}{\partial x}(\sqrt{\hat{y}+1}\sqrt{2a_0},\hat{y},\sqrt{\eta_0},s) > 0,\ \forall s\in[s_3,s_4],$$
$$\frac{\partial W}{\partial x}(\sqrt{\hat{y}+1}\sqrt{2\hat{x}_1},\hat{y},\sqrt{\eta_0},s_3) < 0.$$

Here we apply Lemma 2.9 (ii), (iii) with $s_0=(1+\frac{1}{3}\tilde{w}(a_0,-1,\eta_0))^{-1}$ to derive from the above inequalities that there exists $\hat{\delta}\in(0,\delta_1]$ such that

$$\frac{\partial F_{\hat{\delta}}}{\partial x}(\cdot,\hat{y},\cdot)\in C(\mathbb{R}_{>0}\times[(s_4+1)^{-1},(s_1+1)^{-1}]),$$
$$F_{\hat{\delta}}(\cdot,\hat{y},s)\in C^\omega(\mathbb{R}_{>0}),\ \forall s\in[(s_4+1)^{-1},(s_1+1)^{-1}],$$

(2.63)
$$\frac{\partial F_{\hat{\delta}}}{\partial x}\left(\frac{\sqrt{\hat{y}+1}}{e_{max}}x,\hat{y},\frac{1}{s_1+1}\right) < 0,\ \forall x\in[\sqrt{2a_0},\sqrt{2x_2(\hat{y})}],$$

(2.64)
$$\frac{\partial F_{\hat\delta}}{\partial x}\left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2a_0}, \hat y, \frac{1}{s+1}\right) < 0,\ \frac{\partial F_{\hat\delta}}{\partial x}\left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2x_2(\hat y)}, \hat y, \frac{1}{s+1}\right) < 0,\ \forall s \in [s_1, s_2],$$
(2.65)
$$\frac{\partial F_{\hat\delta}}{\partial x}\left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2\hat x_2}, \hat y, \frac{1}{s_2+1}\right) > 0,$$
(2.66)
$$\frac{\partial F_{\hat\delta}}{\partial x}\left(\frac{\sqrt{\hat y+1}}{e_{max}}x, \hat y, \frac{1}{s_4+1}\right) > 0,\ \forall x \in [\sqrt{2x_1(\hat y)}, \sqrt{2a_0}],$$
(2.67)
$$\frac{\partial F_{\hat\delta}}{\partial x}\left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2x_1(\hat y)}, \hat y, \frac{1}{s+1}\right) > 0,\ \frac{\partial F_{\hat\delta}}{\partial x}\left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2a_0}, \hat y, \frac{1}{s+1}\right) > 0,\ \forall s \in [s_3, s_4],$$
(2.68)
$$\frac{\partial F_{\hat\delta}}{\partial x}\left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2\hat x_1}, \hat y, \frac{1}{s_3+1}\right) < 0.$$

By (2.63), (2.65), (2.66), (2.68) and the fact that $s \mapsto \max_{x\in I} \frac{\partial F_{\hat\delta}}{\partial x}(x, \hat y, \frac{1}{s+1})$, $s \mapsto \min_{x\in I} \frac{\partial F_{\hat\delta}}{\partial x}(x, \hat y, \frac{1}{s+1})$ are continuous in $[s_1, s_4]$ for any closed interval $I \subset \mathbb{R}_{>0}$ there exist $\hat s_1 \in (s_1, s_2)$, $\hat s_2 \in (s_3, s_4)$ such that

$$\max_{x\in\left[\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2a_0}, \frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2x_2(\hat y)}\right]} \frac{\partial F_{\hat\delta}}{\partial x}\left(x, \hat y, \frac{1}{\hat s_1+1}\right) = 0,$$
$$\min_{x\in\left[\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2x_1(\hat y)}, \frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2a_0}\right]} \frac{\partial F_{\hat\delta}}{\partial x}\left(x, \hat y, \frac{1}{\hat s_2+1}\right) = 0.$$

Moreover, by (2.64), (2.67) there exist

$$\zeta_1 \in \left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2x_1(\hat y)}, \frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2a_0}\right),\ \zeta_2 \in \left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2a_0}, \frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2x_2(\hat y)}\right)$$

such that

$$\frac{\partial F_{\hat\delta}}{\partial x}\left(\zeta_2, \hat y, \frac{1}{\hat s_1+1}\right) = 0, \tag{2.69}$$
$$\frac{\partial F_{\hat\delta}}{\partial x}\left(\zeta_1, \hat y, \frac{1}{\hat s_2+1}\right) = 0. \tag{2.70}$$

Furthermore, since $\frac{\partial F_{\hat\delta}}{\partial x}(\cdot, \hat y, \frac{1}{\hat s_j+1}) \in C^\omega(\mathbb{R}_{>0})$ $(j = 1, 2)$ and these functions are not identically zero, there exists $\hat\varepsilon \in \mathbb{R}_{>0}$ such that

$$(\zeta_1 - \hat\varepsilon, \zeta_1 + \hat\varepsilon) \subset \left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2x_1(\hat y)}, \frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2a_0}\right),$$
$$\frac{\partial F_{\hat\delta}}{\partial x}\left(x, \hat y, \frac{1}{\hat s_2+1}\right) > 0,\ \forall x \in (\zeta_1 - \hat\varepsilon, \zeta_1 + \hat\varepsilon)\backslash\{\zeta_1\}, \tag{2.71}$$
$$(\zeta_2 - \hat\varepsilon, \zeta_2 + \hat\varepsilon) \subset \left(\frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2a_0}, \frac{\sqrt{\hat y+1}}{e_{max}}\sqrt{2x_2(\hat y)}\right),$$

$$(2.72)\qquad \frac{\partial F_{\hat{\delta}}}{\partial x}\left(x, \hat{y}, \frac{1}{\hat{s}_1+1}\right) < 0,\ \forall x \in (\zeta_2 - \hat{\varepsilon}, \zeta_2 + \hat{\varepsilon})\backslash\{\zeta_2\}.$$

Finally the properties (2.51), (2.69), (2.72) enable us to apply Lemma 2.8 (ii) to ensure that for $E_{\frac{1}{\hat{s}_1+1},\hat{\delta}}$ $(\in \mathcal{E}(e_{min}, e_{max}))$ and some $U \in [-U_0, 0)$ $\tau(\cdot)$ has a rising SPI in $(0, \beta_c)$. Similarly by (2.51), (2.70), (2.71) we can apply Lemma 2.8 (i) to conclude that for $E_{\frac{1}{\hat{s}_2+1},\hat{\delta}}$ $(\in \mathcal{E}(e_{min}, e_{max}))$ and some $U \in [-U_0, 0)$ $\tau(\cdot)$ has a falling SPI in $(0, \beta_c)$. Thus the claim (i) holds true. □

2.6 Proof of Theorem 1.6 and Theorem 1.7

We can complete the proof of Theorem 1.6 and Theorem 1.7 by applying Proposition 2.11, Proposition 2.12 and Proposition 2.15.

Proof of Theorem 1.6. The equivalence between the claim (i) and the claim (ii) was proved in Corollary 2.4 (1). By Proposition 2.12 and Proposition 2.15 the claim (iii) implies the claim (ii). If the claim (iii) does not hold, by Proposition 2.11 the claim (ii) does not hold. Therefore the claim (iii) is equivalent to the claim (ii). The proof is complete. □

Proof of Theorem 1.7. Corollary 2.4 (2) ensures the equivalence between the claims (i), (ii), (iii). By Proposition 2.11 the claim (iv) implies the claim (iii). It follows from Proposition 2.12, Proposition 2.15 that if the claim (iv) does not hold, the claim (iii) does not hold. Thus the claim (iv) is equivalent to the claim (iii), which concludes the proof. □

3 Specific models

Our main theorems are claimed for the general set of free dispersion relations $\mathcal{E}(e_{min}, e_{max})$. One natural question is whether HOPT occurs in a specific model belonging to $\mathcal{E}(e_{min}, e_{max})$ by varying parameters on which the model depends. We focus on the following 2 models of $\mathcal{E}(e_{min}, e_{max})$.

(1) For $d \in \mathbb{N}$, $b \in \mathbb{N}_{\ge 2}$, $b' \in \{1, 2, \cdots, b-1\}$, a basis $(\hat{\mathbf{v}}_j)_{j=1}^d$ of $\mathbb{R}^d$, $e_{min}, e_{max} \in \mathbb{R}_{>0}$ with $e_{min} \le e_{max}$

$$E_b(\mathbf{k}) := \begin{pmatrix} e_{max} I_{b'} & 0 \\ 0 & e_{min} I_{b-b'} \end{pmatrix},\ \mathbf{k} \in \mathbb{R}^d.$$

(2) For $t \in \mathbb{R}_{\ge 0}$, $e_{min} \in \mathbb{R}_{>0}$

$$E_1(k) := t(\cos k + 1) + e_{min},\ k \in \mathbb{R}.$$

The model (1) is actually independent of the variable $\mathbf{k}$. It is a one-particle Hamiltonian of non-hopping multi-orbital electron. In the model (2) $d = b = 1$, $e_{max} = 2t + e_{min}$. It is the dispersion relation of a free electron hopping between nearest neighbor sites in the 1-dimensional lattice $\mathbb{Z}$. In fact these models were studied in [14, Subsection 2.3] in terms of uniqueness of local minimum point of the phase boundary. Our aim here is to study these models in terms of SPI and

HOPT. It is advantageous that we can use the technical lemma [14, Lemma 2.24] to analyze the model (1). Also we can deduce non-existence of SPI in the model (2) from the proof of [14, Proposition 2.26].

Concerning the model (1), we want to prove the following proposition.

Proposition 3.1. *(i) Assume that* $\frac{b-b'}{b'} \in [3-2\sqrt{2}, \infty)$*. Then for any* e_{min}*,* $e_{max} \in \mathbb{R}_{>0}$ *satisfying* $e_{min} \le e_{max}$ *there exists* $U_0 \in (0, \frac{2e_{min}}{b})$ *such that for any* $U \in [-U_0, 0)$ $\tau(\cdot)$ *has no SPI in* $(0, \beta_c)$*.*

(ii) Assume that $\frac{b-b'}{b'} \in (\frac{1}{8}, 3-2\sqrt{2})$*. Then for any* $e_{min} \in \mathbb{R}_{>0}$*,* $U_0 \in (0, \frac{2e_{min}}{b})$ *there exist* $e_1, e_2 \in (0, \sqrt{17-12\sqrt{2}})$*,* $U_1, U_2 \in [-U_0, 0)$ *such that* $e_2 < e_1$ *and if* $\frac{e_{min}}{e_{max}} = e_1$*,* $U = U_1$*,* $\tau(\cdot)$ *has a rising SPI in* $(0, \beta_c)$*, if* $\frac{e_{min}}{e_{max}} = e_2$*,* $U = U_2$*,* $\tau(\cdot)$ *has a falling SPI in* $(0, \beta_c)$*.*

(iii) Assume that $\frac{b-b'}{b'} \in (0, \frac{1}{8}]$*. Then for any* $e_{min} \in \mathbb{R}_{>0}$*,* $U_0 \in (0, \frac{2e_{min}}{b})$ *there exist* $e_3 \in (0, \sqrt{17-12\sqrt{2}})$*,* $U \in [-U_0, 0)$ *such that if* $\frac{e_{min}}{e_{max}} = e_3$*,* $\tau(\cdot)$ *has a rising SPI in* $(0, \beta_c)$*.*

We can derive the following corollary from the above proposition and Theorem 1.5.

Corollary 3.2. *(i) Assume that* $\frac{b-b'}{b'} \in [3-2\sqrt{2}, \infty)$*. Then for any* e_{min}*,* $e_{max} \in \mathbb{R}_{>0}$ *satisfying* $e_{min} \le e_{max}$ *there exists* $U_0 \in (0, \frac{2e_{min}}{b})$ *such that for any* $U \in [-U_0, 0)$*,* $(\rho, \eta) \in \{(+,-), (-,+)\}$*,* $(\beta, t) \in Q_{\rho,\eta}$ $(PT)_{2,(\rho,\eta)}(\beta, t)$ *holds.*

(ii) Assume that $\frac{b-b'}{b'} \in (\frac{1}{8}, 3-2\sqrt{2})$*. Then for any* $e_{min} \in \mathbb{R}_{>0}$*,* $U_0 \in (0, \frac{2e_{min}}{b})$ *there exist* $e_1, e_2 \in (0, \sqrt{17-12\sqrt{2}})$*,* $U_1, U_2 \in [-U_0, 0)$ *such that* $e_2 < e_1$ *and if* $\frac{e_{min}}{e_{max}} = e_1$*,* $U = U_1$*,* $(PT)_{n,(+,-)}$ *holds for some* $n \in 4\mathbb{N}+2$*, if* $\frac{e_{min}}{e_{max}} = e_2$*,* $U = U_2$*,* $(PT)_{n,(-,+)}$ *holds for some* $n \in 4\mathbb{N}+2$*.*

(iii) Assume that $\frac{b-b'}{b'} \in (0, \frac{1}{8}]$*. Then for any* $e_{min} \in \mathbb{R}_{>0}$*,* $U_0 \in (0, \frac{2e_{min}}{b})$ *there exist* $e_3 \in (0, \sqrt{17-12\sqrt{2}})$*,* $U \in [-U_0, 0)$ *such that if* $\frac{e_{min}}{e_{max}} = e_3$*,* $(PT)_{n,(+,-)}$ *holds for some* $n \in 4\mathbb{N}+2$*.*

The proof of Proposition 3.1 is based on Lemma 3.3 below. Recall the definition of the functions $w(x, y, z)$, $\tilde{w}(x, y, z)$ and their properties summarized in front of Lemma 2.14 to understand the statements and the proof of the lemma. In addition we will use the following properties.

$$\lim_{\eta \nearrow 17-12\sqrt{2}} \tilde{w}(a_+(\eta), -1, \eta) = \lim_{\eta \nearrow 17-12\sqrt{2}} \tilde{w}(a_-(\eta), -1, \eta) = 3-2\sqrt{2}, \tag{3.1}$$

$$\lim_{\eta \searrow 0} \tilde{w}(a_-(\eta), -1, \eta) = \frac{1}{8}, \ \lim_{\eta \searrow 0} \tilde{w}(a_+(\eta), -1, \eta) = 0, \tag{3.2}$$

which can be derived from the facts that

$$\lim_{\eta \nearrow 17-12\sqrt{2}} a_+(\eta) = \lim_{\eta \nearrow 17-12\sqrt{2}} a_-(\eta) = 3+2\sqrt{2},$$

$$\lim_{\eta \searrow 0} a_-(\eta) = 3, \ \lim_{\eta \searrow 0} a_+(\eta) = +\infty, \ \lim_{\eta \searrow 0} \eta a_+(\eta) = \frac{1}{3}.$$

Moreover we need that

$$\frac{d}{d\eta}\tilde{w}(a_\delta(\eta),-1,\eta) > 0,\ \forall \delta \in \{+,-\},\ \eta \in (0, 17-12\sqrt{2}). \tag{3.3}$$

This can be confirmed as follows.

$$\frac{\partial \tilde{w}}{\partial z}(x,-1,z) = \frac{(x-1)x(1+zx)(3-zx)}{(x+1)^2(1-zx)^2} > 0,\ \forall z \in (0,1),\ x \in (1, z^{-1}), \tag{3.4}$$

and thus by (2.35), (2.36), (2.37)

$$\begin{aligned}\frac{d}{d\eta}\tilde{w}(a_\delta(\eta),-1,\eta) &= \frac{\partial \tilde{w}}{\partial x}(a_\delta(\eta),-1,\eta)\frac{da_\delta}{d\eta}(\eta) + \frac{\partial \tilde{w}}{\partial z}(a_\delta(\eta),-1,\eta)\\ &= \frac{\partial \tilde{w}}{\partial z}(a_\delta(\eta),-1,\eta) > 0,\ \forall \eta \in (0,17-12\sqrt{2}),\ \delta \in \{+,-\}.\end{aligned}$$

Lemma 3.3. *(i) For any $s \in (\frac{1}{8}, 3-2\sqrt{2})$ there exist η_1, η_2, η_3, $\eta_4 \in (0, 17-12\sqrt{2})$, $y_1 \in (-1,0)$ such that $\eta_4 < \eta_3 < \eta_2 < \eta_1$, $a_+(\eta_2) < \eta_1^{-1}$ and for any $y \in (-1, y_1]$*

$$\begin{aligned}&\frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}} > \sqrt{2},\\ &w(x,y,\eta_1) > s,\ \forall x \in \left[\frac{1}{2}(a_-(\eta_2)+a_+(\eta_2)), \frac{1}{2}(a_+(\eta_2)+\eta_1^{-1})\right],\\ &w\left(\frac{1}{2}(a_-(\eta_2)+a_+(\eta_2)), y, \eta\right) > s,\ w\left(\frac{1}{2}(a_+(\eta_2)+\eta_1^{-1}), y, \eta\right) > s,\ \forall \eta \in [\eta_2, \eta_1],\\ &w(a_+(\eta_2), y, \eta_2) < s,\\ &w(x,y,\eta_4) < s,\ \forall x \in \left[\frac{1}{2}(1+a_-(\eta_3)), \frac{1}{2}(a_-(\eta_3)+a_+(\eta_3))\right],\\ &w\left(\frac{1}{2}(1+a_-(\eta_3)), y, \eta\right) < s,\ w\left(\frac{1}{2}(a_-(\eta_3)+a_+(\eta_3)), y, \eta\right) < s,\ \forall \eta \in [\eta_4, \eta_3],\\ &w(a_-(\eta_3), y, \eta_3) > s.\end{aligned}$$

(ii) For any $s \in (0, \frac{1}{8}]$ there exist η_5, $\eta_6 \in (0, 17-12\sqrt{2})$, $y_2 \in (-1,0)$ such that $\eta_6 < \eta_5$, $a_+(\eta_6) < \eta_5^{-1}$ and for any $y \in (-1, y_2]$

$$\begin{aligned}&\frac{\cosh^{-1}(|y|^{-1})}{\sqrt{y+1}} > \sqrt{2},\\ &w(x,y,\eta_5) > s,\ \forall x \in \left[\frac{1}{2}(a_-(\eta_6)+a_+(\eta_6)), \frac{1}{2}(a_+(\eta_6)+\eta_5^{-1})\right],\\ &w\left(\frac{1}{2}(a_-(\eta_6)+a_+(\eta_6)), y, \eta\right) > s,\ w\left(\frac{1}{2}(a_+(\eta_6)+\eta_5^{-1}), y, \eta\right) > s,\ \forall \eta \in [\eta_6, \eta_5],\\ &w(a_+(\eta_6), y, \eta_6) < s.\end{aligned}$$

Remark 3.4. By using the inequality $\cosh^{-1}(|y|^{-1})/\sqrt{y+1} > \sqrt{2}$ we can check that the variable (x,y,η) belongs to the domain D where the function w is defined in the statement of the above lemma.

Proof. (i): Take any $s \in (\frac{1}{8}, 3-2\sqrt{2})$. We can deduce from the properties (2.38), (3.1), (3.2), (3.3) that there uniquely exist $\hat{\eta}_1, \hat{\eta}_2 \in (0, 17-12\sqrt{2})$ such that $\hat{\eta}_2 < \hat{\eta}_1$,

$$\tilde{w}(a_+(\hat{\eta}_1), -1, \hat{\eta}_1) = \tilde{w}(a_-(\hat{\eta}_2), -1, \hat{\eta}_2) = s.$$

Moreover, by the profile of $\tilde{w}(\cdot, -1, \eta)$ described in Subsection 2.4, (3.3) and (3.4) there exists small $\varepsilon \in \mathbb{R}_{>0}$ such that the following inequalities hold.

$$a_+(\hat{\eta}_1) < \hat{\eta}_1^{-1}.$$

$$\tilde{w}(x, -1, \eta) > s,\ \forall x \in \left[\frac{1}{2}(a_-(\hat{\eta}_1) + a_+(\hat{\eta}_1)) - \varepsilon, \frac{1}{2}(a_+(\hat{\eta}_1) + \hat{\eta}_1^{-1}) + \varepsilon\right],\ \eta \in (\hat{\eta}_1, \hat{\eta}_1 + \varepsilon].$$

$$\tilde{w}\left(\frac{1}{2}(a_-(\hat{\eta}_1) + a_+(\hat{\eta}_1)), -1, \hat{\eta}_1\right) > s,\ \tilde{w}\left(\frac{1}{2}(a_+(\hat{\eta}_1) + \hat{\eta}_1^{-1}), -1, \hat{\eta}_1\right) > s.$$

$$\tilde{w}(a_+(\eta), -1, \eta) < s,\ \forall \eta \in [\hat{\eta}_1 - \varepsilon, \hat{\eta}_1).$$

$$\tilde{w}(x, -1, \eta) < s,\ \forall x \in \left[\frac{1}{2}(1 + a_-(\hat{\eta}_2)) - \varepsilon, \frac{1}{2}(a_-(\hat{\eta}_2) + a_+(\hat{\eta}_2)) + \varepsilon\right],\ \eta \in [\hat{\eta}_2 - \varepsilon, \hat{\eta}_2).$$

$$\tilde{w}\left(\frac{1}{2}(1 + a_-(\hat{\eta}_2)), -1, \hat{\eta}_2\right) < s,\ \tilde{w}\left(\frac{1}{2}(a_-(\hat{\eta}_2) + a_+(\hat{\eta}_2)), -1, \hat{\eta}_2\right) < s.$$

$$\tilde{w}(a_-(\eta), -1, \eta) > s,\ \forall \eta \in (\hat{\eta}_2, \hat{\eta}_2 + \varepsilon].$$

Then we can choose $\eta_1 \in (\hat{\eta}_1, 17-12\sqrt{2})$, $\eta_2 \in (0, \hat{\eta}_1)$ to be close to $\hat{\eta}_1$ and $\eta_3 \in (\hat{\eta}_2, 17-12\sqrt{2})$, $\eta_4 \in (0, \hat{\eta}_2)$ to be close to $\hat{\eta}_2$ so that $\eta_4 < \eta_3 < \eta_2 < \eta_1$,

(3.5)

$$a_+(\eta_2) < \eta_1^{-1},$$

(3.6)

$$\tilde{w}(x, -1, \eta_1) > s,\ \forall x \in \left[\frac{1}{2}(a_-(\eta_2) + a_+(\eta_2)), \frac{1}{2}(a_+(\eta_2) + \eta_1^{-1})\right],$$

(3.7)

$$\tilde{w}\left(\frac{1}{2}(a_-(\eta_2) + a_+(\eta_2)), -1, \eta\right) > s,\ \tilde{w}\left(\frac{1}{2}(a_+(\eta_2) + \eta_1^{-1}), -1, \eta\right) > s,\ \forall \eta \in [\eta_2, \eta_1],$$

(3.8)

$$\tilde{w}(a_+(\eta_2), -1, \eta_2) < s,$$

$$\tilde{w}(x, -1, \eta_4) < s,\ \forall x \in \left[\frac{1}{2}(1 + a_-(\eta_3)), \frac{1}{2}(a_-(\eta_3) + a_+(\eta_3))\right],$$

$$\tilde{w}\left(\frac{1}{2}(1 + a_-(\eta_3)), -1, \eta\right) < s,\ \tilde{w}\left(\frac{1}{2}(a_-(\eta_3) + a_+(\eta_3)), -1, \eta\right) < s,\ \forall \eta \in [\eta_4, \eta_3],$$

$$\tilde{w}(a_-(\eta_3), -1, \eta_3) > s.$$

The claimed inequalities follow from (2.46), the above inequalities and the uniform convergence properties

$$\lim_{y \searrow -1} \sup_{\substack{x \in [\frac{1}{2}(a_-(\eta_2) + a_+(\eta_2)), \frac{1}{2}(a_+(\eta_2) + \eta_1^{-1})] \\ \eta \in [\eta_2, \eta_1]}} |w(x, y, \eta) - \tilde{w}(x, -1, \eta)| = 0, \tag{3.9}$$

$$\lim_{y \searrow -1} \sup_{\substack{x \in [\frac{1}{2}(1 + a_-(\eta_3)), \frac{1}{2}(a_-(\eta_3) + a_+(\eta_3))] \\ \eta \in [\eta_4, \eta_3]}} |w(x, y, \eta) - \tilde{w}(x, -1, \eta)| = 0.$$

(ii): Take any $s \in (0, \frac{1}{8}]$. By (3.1), (3.2) there exists $\hat{\eta}_3 \in (0, 17 - 12\sqrt{2})$ such that $\tilde{w}(a_+(\hat{\eta}_3), -1, \hat{\eta}_3) = s$. We can choose $\eta_1 \in (\hat{\eta}_3, 17 - 12\sqrt{2})$, $\eta_2 \in (0, \hat{\eta}_3)$ sufficiently close to $\hat{\eta}_3$ so that the same inequalities as (3.5), (3.6), (3.7), (3.8) hold. Then by applying the uniform convergence property of the form (3.9) we obtain the claimed inequalities. □

Proof of Proposition 3.1. We set

$$s := \frac{b - b'}{b'}, \quad \eta := \left(\frac{e_{min}}{e_{max}}\right)^2$$

during the proof. First of all we note that

(3.10)
$$F_\infty(x, y) = D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(xE_b(\mathbf{k}))}{(y + \cosh(xE_b(\mathbf{k})))E_b(\mathbf{k})}\right) = \frac{b'}{e_{max}} W(e_{max}x, y, \sqrt{\eta}, s),$$

which together with (2.61) implies that

(3.11)
$$\frac{\partial F_\infty}{\partial x}(\sqrt{y+1}x, y) = b' \frac{1 + y\cosh(\sqrt{\eta(y+1)}e_{max}x)}{(y + \cosh(\sqrt{\eta(y+1)}e_{max}x))^2}\left(s - w\left(\frac{e_{max}^2 x^2}{2}, y, \eta\right)\right),$$
$$\forall y \in (-1, 0),\ x \in \left(0, \frac{\cosh^{-1}(|y|^{-1})}{e_{min}\sqrt{y+1}}\right).$$

We will also use the following convergence property.

$$\lim_{y \searrow -1} \sqrt{y+1} W(\sqrt{y+1}x, y, \sqrt{\xi}, s) = \widehat{W}(x, \sqrt{\xi}, s) \tag{3.12}$$

locally uniformly with $(x, \xi) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$.

(i): Assume that $s \in [3 - 2\sqrt{2}, \infty)$. If $\frac{e_{min}}{e_{max}} > \sqrt{17 - 12\sqrt{2}}$, Proposition 2.11 ensures the result. Assume that $\frac{e_{min}}{e_{max}} = \sqrt{17 - 12\sqrt{2}}$. Here we apply [14, Lemma 2.24 (i)] to guarantee that

(3.13)

$\exists y_0 \in (-1, 0)$ s.t.

$\forall y \in (-1, y_0]\ \exists x_0(y) \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right)$ s.t.

$w(x, y, \eta) < s,\ \forall x \in \left(\frac{1}{2(y+1)}(\cosh^{-1}(|y|^{-1}))^2, x_0(y)\right),$

$w(x_0(y), y, \eta) = s,$

$w(x, y, \eta) > s,\ \forall x \in \left(x_0(y), \frac{1}{2\eta(y+1)}(\cosh^{-1}(|y|^{-1}))^2\right).$

Since

$$\frac{\partial F_\infty}{\partial x}(\sqrt{y+1}x, y) > 0,\ \forall x \in \left(0, \frac{\cosh^{-1}(|y|^{-1})}{e_{max}\sqrt{y+1}}\right],$$

$$\frac{\partial F_\infty}{\partial x}(\sqrt{y+1}x, y) < 0, \ \forall x \in \left[\frac{\cosh^{-1}(|y|^{-1})}{e_{min}\sqrt{y+1}}, \infty\right)$$

for any $y \in (-1, 0)$, combination of (3.11) and (3.13) proves that for any $y \in (-1, y_0]$ there exists $\hat{x}_0 \in (\frac{\cosh^{-1}(|y|^{-1})}{e_{max}}, \frac{\cosh^{-1}(|y|^{-1})}{e_{min}})$ such that

$$\begin{aligned}
&\frac{\partial F_\infty}{\partial x}(x, y) > 0, \ \forall x \in (0, \hat{x}_0),\\
&\frac{\partial F_\infty}{\partial x}(\hat{x}_0, y) = 0,\\
&\frac{\partial F_\infty}{\partial x}(x, y) < 0, \ \forall x \in (\hat{x}_0, \infty).
\end{aligned}$$

Now the assumption of Lemma 2.7 with $S = \{E_b\}$ is satisfied and thus the claim follows from the lemma in this case.

Assume that $\frac{e_{min}}{e_{max}} < \sqrt{17 - 12\sqrt{2}}$. By (3.1) and (3.3) $s \in (\tilde{w}(a_-(\eta), -1, \eta), \infty)$. Thus we can apply [14, Lemma 2.24 (ii)] to ensure that the property (3.13) holds. Then by repeating the same argument as above and using Lemma 2.7 we can deduce the claim in this case as well. The proof of (i) is complete.

(ii): Assume that $s \in (\frac{1}{8}, 3 - 2\sqrt{2})$. Take any $e_{min} \in \mathbb{R}_{>0}$ and $U_0 \in (0, \frac{2e_{min}}{b})$. Let η_1, η_2, η_3, $\eta_4 \in (0, 17 - 12\sqrt{2})$, $y_1 \in (-1, 0)$ be those introduced in Lemma 3.3 (i). We can see from (3.10) that for any

$$e_{max} \in \left[\frac{e_{min}}{\sqrt{\eta_1}}, \frac{e_{min}}{\sqrt{\eta_2}}\right], \ x \in \left[\frac{\sqrt{1 + a_-(\eta_2)}}{e_{max}}, \frac{\sqrt{a_+(\eta_2) + \eta_1^{-1}}}{e_{max}}\right], \ y \in (-1, 0)$$

$$\begin{aligned}
&\sqrt{y+1}F_\infty(\sqrt{y+1}x, y)\\
&\geq \frac{b'\sqrt{\eta_2}}{e_{min}} \inf_{\substack{x \in [\sqrt{1+a_-(\eta_2)}, \sqrt{a_+(\eta_2)+\eta_1^{-1}}]\\ \xi \in [\eta_2, \eta_1]}} \sqrt{y+1}W(\sqrt{y+1}x, y, \sqrt{\xi}, s).
\end{aligned} \tag{3.14}$$

By the convergence property (3.12) there exists $y_2 \in (-1, y_1]$ such that for any $y \in (-1, y_2]$

$$\begin{aligned}
&\inf_{\substack{x \in [\sqrt{1+a_-(\eta_2)}, \sqrt{a_+(\eta_2)+\eta_1^{-1}}]\\ \xi \in [\eta_2, \eta_1]}} \sqrt{y+1}W(\sqrt{y+1}x, y, \sqrt{\xi}, s)\\
&\geq \frac{1}{2} \inf_{\substack{x \in [\sqrt{1+a_-(\eta_2)}, \sqrt{a_+(\eta_2)+\eta_1^{-1}}]\\ \xi \in [\eta_2, \eta_1]}} \widehat{W}(x, \sqrt{\xi}, s).
\end{aligned} \tag{3.15}$$

We can derive from (2.46), (3.14), (3.15) that there exists $\hat{y} \in (-1, y_2]$ such that

$$F_\infty(\sqrt{\hat{y}+1}x, \hat{y}) \geq \frac{2}{U_0}, \ \forall e_{max} \in \left[\frac{e_{min}}{\sqrt{\eta_1}}, \frac{e_{min}}{\sqrt{\eta_2}}\right], \ x \in \left[\frac{\sqrt{1 + a_-(\eta_2)}}{e_{max}}, \frac{\sqrt{a_+(\eta_2) + \eta_1^{-1}}}{e_{max}}\right], \tag{3.16}$$

$$\frac{\cosh^{-1}(|\hat{y}|^{-1})}{\sqrt{\hat{y}+1}} > \sqrt{2}. \tag{3.17}$$

It follows from the inequalities claimed in Lemma 3.3 (i) that there exists $\hat{\eta}_1 \in (\eta_2, \eta_1)$ such that

$$\min_{x\in[\frac{1}{2}(a_-(\eta_2)+a_+(\eta_2)),\frac{1}{2}(a_+(\eta_2)+\eta_1^{-1})]} w(x,\hat{y},\hat{\eta}_1) = s,$$
$$w\left(\frac{1}{2}(a_-(\eta_2)+a_+(\eta_2)),\hat{y},\hat{\eta}_1\right) > s,\ w\left(\frac{1}{2}(a_+(\eta_2)+\eta_1^{-1}),\hat{y},\hat{\eta}_1\right) > s.$$

Let $x_0 \in (\frac{1}{2}(a_-(\eta_2)+a_+(\eta_2)), \frac{1}{2}(a_+(\eta_2)+\eta_1^{-1}))$ be a minimizer. Set

$$e_{max} := \frac{e_{min}}{\sqrt{\hat{\eta}_1}},\ \hat{x} := \frac{\sqrt{2(\hat{y}+1)x_0}}{e_{max}}.$$

By (3.16)

$$F_\infty(\hat{x},\hat{y}) \geq \frac{2}{U_0}. \tag{3.18}$$

Observe that by (3.17)

$$\frac{\sqrt{2x_0}}{e_{max}} < \frac{\sqrt{a_+(\eta_2)+\eta_1^{-1}}}{e_{max}} < \frac{\sqrt{2\eta_1^{-1}}}{e_{max}} < \frac{\sqrt{2\hat{\eta}_1^{-1}}}{e_{max}} = \frac{\sqrt{2}}{e_{min}} < \frac{\cosh^{-1}(|\hat{y}|^{-1})}{e_{min}\sqrt{\hat{y}+1}},$$

and thus by (3.11)

$$\frac{\partial F_\infty}{\partial x}(\hat{x},\hat{y}) = b'\frac{1+\hat{y}\cosh(\sqrt{2\hat{\eta}_1(\hat{y}+1)x_0})}{(\hat{y}+\cosh(\sqrt{2\hat{\eta}_1(\hat{y}+1)x_0}))^2}(s - w(x_0,\hat{y},\hat{\eta}_1)) = 0. \tag{3.19}$$

We remark that by (3.17)

$$\sqrt{2\hat{\eta}_1(\hat{y}+1)x_0} < \sqrt{\hat{\eta}_1(\hat{y}+1)(a_+(\eta_2)+\eta_1^{-1})} < \sqrt{2\hat{\eta}_1(\hat{y}+1)\eta_1^{-1}} < \sqrt{2(\hat{y}+1)}$$
$$< \cosh^{-1}(|\hat{y}|^{-1}),$$

and thus

$$1+\hat{y}\cosh(\sqrt{2\hat{\eta}_1(\hat{y}+1)x_0}) > 0.$$

We can deduce from this inequality and the definition of x_0 that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $\frac{\partial F_\infty}{\partial x}(x,\hat{y}) < 0$ for any $x \in (\hat{x}-\varepsilon, \hat{x}+\varepsilon)\backslash\{\hat{x}\}$. This together with (3.18), (3.19) enables us to apply Lemma 2.8 (ii) to conclude that there exists $U \in [-U_0, 0)$ such that $\tau(\cdot)$ has a rising SPI in $(0,\beta_c)$. Remind us that $\frac{e_{min}}{e_{max}} = \sqrt{\hat{\eta}_1} \in (\sqrt{\eta_2}, \sqrt{\eta_1})$. The existence of a rising SPI is now proved with $e_1 = \sqrt{\hat{\eta}_1}$.

The existence of a falling SPI can be proved similarly. However, we provide the proof for completeness. We can derive from (3.10) that for any

$$e_{max} \in \left[\frac{e_{min}}{\sqrt{\eta_3}}, \frac{e_{min}}{\sqrt{\eta_4}}\right],\ x \in \left[\frac{\sqrt{1+a_-(\eta_3)}}{e_{max}}, \frac{\sqrt{a_-(\eta_3)+a_+(\eta_3)}}{e_{max}}\right],\ y \in (-1,0)$$
$$\sqrt{y+1}F_\infty(\sqrt{y+1}x,y)$$
$$\geq \frac{b'\sqrt{\eta_4}}{e_{min}} \inf_{\substack{x\in[\sqrt{1+a_-(\eta_3)},\sqrt{a_-(\eta_3)+a_+(\eta_3)}]\\ \xi\in[\eta_4,\eta_3]}} \sqrt{y+1}W(\sqrt{y+1}x,y,\sqrt{\xi},s).$$

Application of (3.12) yields that there exists $y_3 \in (-1, y_1]$ such that for any $y \in (-1, y_3]$

$$\begin{aligned}&\inf_{\substack{x\in[\sqrt{1+a_-(\eta_3)},\sqrt{a_-(\eta_3)+a_+(\eta_3)}]\\ \xi\in[\eta_4,\eta_3]}} \sqrt{y+1}W(\sqrt{y+1}x, y, \sqrt{\xi}, s)\\ &\geq \frac{1}{2}\inf_{\substack{x\in[\sqrt{1+a_-(\eta_3)},\sqrt{a_-(\eta_3)+a_+(\eta_3)}]\\ \xi\in[\eta_4,\eta_3]}} \widehat{W}(x, \sqrt{\xi}, s).\end{aligned}$$

We can deduce from these inequalities and (2.46) that there exists $\tilde{y} \in (-1, y_3]$ such that

(3.20)
$$F_\infty(\sqrt{\tilde{y}+1}x, \tilde{y}) \geq \frac{2}{U_0},\ \forall e_{max} \in \left[\frac{e_{min}}{\sqrt{\eta_3}}, \frac{e_{min}}{\sqrt{\eta_4}}\right],\ x \in \left[\frac{\sqrt{1+a_-(\eta_3)}}{e_{max}}, \frac{\sqrt{a_-(\eta_3)+a_+(\eta_3)}}{e_{max}}\right],$$

(3.21)
$$\frac{\cosh^{-1}(|\tilde{y}|^{-1})}{\sqrt{\tilde{y}+1}} > \sqrt{2}.$$

The inequalities of Lemma 3.3 (i) imply that there exists $\hat{\eta}_2 \in (\eta_4, \eta_3)$ such that

$$\begin{aligned}&\max_{x\in[\frac{1}{2}(1+a_-(\eta_3)),\frac{1}{2}(a_-(\eta_3)+a_+(\eta_3))]} w(x, \tilde{y}, \hat{\eta}_2) = s,\\ &w\left(\frac{1}{2}(1+a_-(\eta_3)), \tilde{y}, \hat{\eta}_2\right) < s,\ w\left(\frac{1}{2}(a_-(\eta_3)+a_+(\eta_3)), \tilde{y}, \hat{\eta}_2\right) < s.\end{aligned}$$

Let $\tilde{x}_0 \in (\frac{1}{2}(1+a_-(\eta_3)), \frac{1}{2}(a_-(\eta_3)+a_+(\eta_3)))$ be a maximizer and set

$$e_{max} := \frac{e_{min}}{\sqrt{\hat{\eta}_2}},\ \tilde{x} := \frac{\sqrt{2(\tilde{y}+1)\tilde{x}_0}}{e_{max}}.$$

By (3.20)

$$F_\infty(\tilde{x}, \tilde{y}) \geq \frac{2}{U_0}. \tag{3.22}$$

Moreover, by (3.21)

$$\frac{\sqrt{2\tilde{x}_0}}{e_{max}} < \frac{\sqrt{a_-(\eta_3)+a_+(\eta_3)}}{e_{max}} < \frac{\sqrt{2\eta_3^{-1}}}{e_{max}} < \frac{\sqrt{2\hat{\eta}_2^{-1}}}{e_{max}} = \frac{\sqrt{2}}{e_{min}} < \frac{\cosh^{-1}(|\tilde{y}|^{-1})}{e_{min}\sqrt{\tilde{y}+1}},$$

and thus by (3.11)

$$\frac{\partial F_\infty}{\partial x}(\tilde{x}, \tilde{y}) = b'\frac{1+\tilde{y}\cosh(\sqrt{2\hat{\eta}_2(\tilde{y}+1)\tilde{x}_0})}{(\tilde{y}+\cosh(\sqrt{2\hat{\eta}_2(\tilde{y}+1)\tilde{x}_0}))^2}(s - w(\tilde{x}_0, \tilde{y}, \hat{\eta}_2)) = 0. \tag{3.23}$$

By using (3.21) again we can derive that

$$\begin{aligned}\sqrt{2\hat{\eta}_2(\tilde{y}+1)\tilde{x}_0} < \sqrt{\hat{\eta}_2(\tilde{y}+1)(a_-(\eta_3)+a_+(\eta_3))} < \sqrt{2\hat{\eta}_2(\tilde{y}+1)\eta_3^{-1}} < \sqrt{2(\tilde{y}+1)}\\ < \cosh^{-1}(|\tilde{y}|^{-1}),\end{aligned}$$

and thus

$$1 + \tilde{y} \cosh(\sqrt{2\hat{\eta}_2(\tilde{y}+1)\tilde{x}_0}) > 0.$$

By considering this inequality we can deduce from (3.23) and the definition of $\tilde{x}_0$ that there exists $\tilde{\varepsilon} \in \mathbb{R}_{>0}$ such that $\frac{\partial F_\infty}{\partial x}(x, \tilde{y}) > 0$ for any $x \in (\tilde{x} - \tilde{\varepsilon}, \tilde{x} + \tilde{\varepsilon}) \backslash \{\tilde{x}\}$. This coupled with (3.23) means that $\tilde{x}$ is a rising SPI of $F_\infty(\cdot, \tilde{y})$. Since we have (3.22), we can apply Lemma 2.8 (i) to ensure that there exists $U \in [-U_0, 0)$ such that $\tau(\cdot)$ has a falling SPI in $(0, \beta_c)$. Here $\frac{e_{min}}{e_{max}} = \sqrt{\hat{\eta}_2} \in (\sqrt{\eta_4}, \sqrt{\eta_3})$.

Now we can see that the claim (ii) holds with $e_1 = \sqrt{\hat{\eta}_1}$, $e_2 = \sqrt{\hat{\eta}_2}$.

(iii): By using Lemma 3.3 (ii) in place of Lemma 3.3 (i) we can repeat the same argument as the 1st half of the proof of (ii) to prove the claim. □

In [14, Proposition 2.25] we derived $\tau(\beta)$ exactly. Let us numerically implement the exact solution to observe that $\tau(\cdot)$ has SPIs as suggested by Proposition 3.1. We set $b = 8$, $b' = 7$, $e_{min} = 1$, $U = -\frac{1}{8}$ so that $\frac{b-b'}{b'} \in (\frac{1}{8}, 3-2\sqrt{2})$, $|U| \in (0, \frac{2e_{min}}{b})$. In fact these parameters take the same values as in the numerical example in [14, Sub-subsection 2.3.1]. Based on Proposition 3.1 (ii), we expect that we can find $e_1, e_2 \in (0, \sqrt{17-12\sqrt{2}})$ such that $e_2 < e_1$ and if $e_{max} = \frac{1}{e_1}$, $\tau(\cdot)$ has a rising SPI, if $e_{max} = \frac{1}{e_2}$, $\tau(\cdot)$ has a falling SPI. In Figure 2 we plot the graphs of $\tau(\beta)$, $\frac{d\tau}{d\beta}(\beta)$ for $e_{max} = 6.643, 8.342$. We can see that $\tau(\cdot)$ has a rising SPI when $e_{max} = 6.643$ and $\tau(\cdot)$ has a falling SPI when $e_{max} = 8.342$. This means that our expectation is realized with $e_1 = \frac{1}{6.643}(\approx 0.1505)$, $e_2 = \frac{1}{8.342}(\approx 0.1199) \in (0, \sqrt{17-12\sqrt{2}})$ $(\approx (0, 0.1716))$.

Concerning the model (2), we claim the following proposition. In fact it is an immediate consequence of Lemma 2.7 and the proof of [14, Proposition 2.26].

Proposition 3.5. *For any $t \in \mathbb{R}_{\geq 0}$, $e_{min} \in \mathbb{R}_{>0}$ there exists $U_0 \in (0, 2e_{min})$ such that for any $U \in [-U_0, 0)$ $\tau(\cdot)$ has no SPI in $(0, \beta_c)$.*

Proof. We have shown in the proof of [14, Proposition 2.26] that there exists $y_0 \in (-1, 0)$ such that for any $y \in (-1, y_0]$ there uniquely exists $x_0 \in \mathbb{R}_{>0}$ such that $\frac{\partial F_\infty}{\partial x}(x_0, y) = 0$. See around the equation "(2.101)" in [14]. Then Lemma 2.7 with $S = \{E_1\}$ ensures the result. □

Data availability

The data that supports the findings of this study are available within the article.

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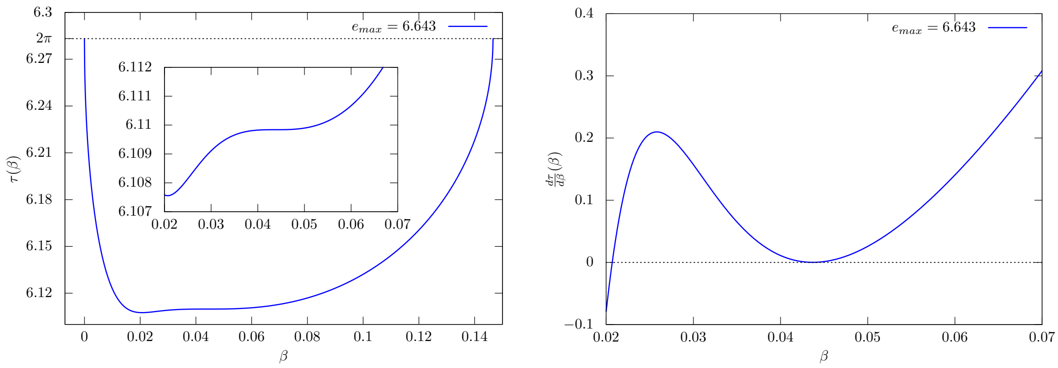
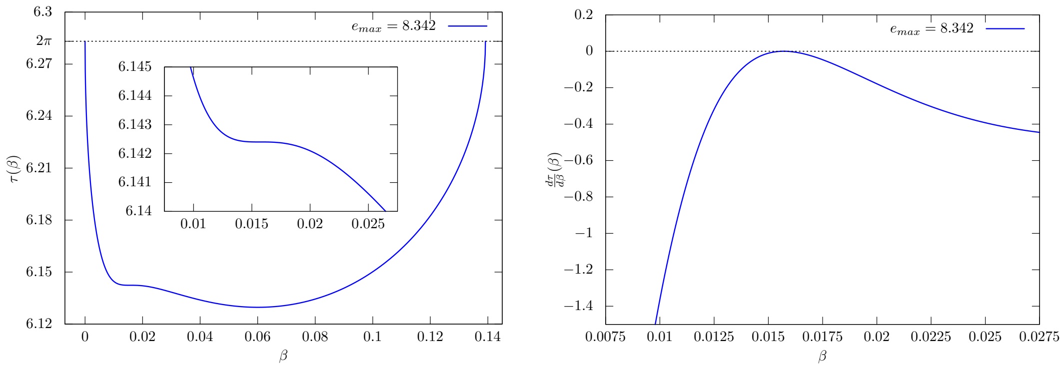


$\tau(\beta)$ (left), $\frac{d\tau}{d\beta}(\beta)$ (right) with $e_{max} = 6.643$



$\tau(\beta)$ (left), $\frac{d\tau}{d\beta}(\beta)$ (right) with $e_{max} = 8.342$

Figure 2: Parts of the graphs $\{(\beta, \tau(\beta)) \mid \beta \in (0, \beta_c)\}$, $\{(\beta, \frac{d\tau}{d\beta}(\beta)) \mid \beta \in (0, \beta_c)\}$ for $b = 8$, $b' = 7$, $U = -\frac{1}{8}$, $e_{min} = 1$, $e_{max} = 6.643, 8.342$. The exact solution was implemented.

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