Representations of Pauli-Fierz type models by path measures

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Dedicated to Professor Toru Ozawa on his sixtieth birthday

Abstract

Functional integral representations of the semigroups generated by Pauli-Fierz type Hamiltonians in quantum field theory are reviewed. Firstly we introduce functional integral representations for Schrödinger type operators. Secondly those for Pauli-Fierz type Hamiltonians are shown. Finally inequalities derived from functional integral representations are shown.

Contents

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1 Introduction

Congratulations on Prof.Toru Ozawa's sixtieth birthday. It is my great pleasure to be able to publish a paper on this occasion. I was a bit surprised to hear that he was "already" 60 years old, as he always seems to be young and full of mathematical power. On the other hand, Prof. Ozawa has been well known since he was quite young and has trained many excellent students, so I was also surprised to hear that he was "only" 60 years old. I first met Prof. Ozawa around 1994 or 1995 at Hokkaido University, when he was already well known for his scattering theory for nonlinear Schrödinger operators, and it is astonishing to think that he was only 32 or 33 years old at the time.

I also thank Prof. Ozawa for always giving me words of encouragement whenever we were together in conferences, and for inviting me to be a lecturer at SGU workshop to be held at Waseda University in 2020. I was also excited to be invited to Prof. Ozawa's sixtieth birthday conference in 2021, but both were cancelled due to COVID 19, which I regret. In 2022, after COVID19 subsided somewhat, I was grateful to be again invited to be a lecturer at SGU workshop of Waseda University.

As an impressive memory of Prof. Ozawa, he gave me a question on the regularity of an integral kernel appearing in Theorem 3.3 in this paper at the Ph.D. dissertation defense of mine in 1996 in Sapporo. In one fourth of a century since then, the research of mine has been fortunately progressed considerably and a massive book [73] has been published in 2020.

I wish that Prof. Ozawa will continue to take good care of himself and work hard on his research for ever.

Our motivation of the investigation of functional integral representations of the semigroups generated by self-adjoint Hamiltonians was to solve problems in the statistical mechanics. Our interest has been then shifted to the construction of functional integral representations themselves, not only for Schrödinger operators, but also for relativistic Schrödinger operators and Schrödinger operators with spin 1/2, etc. In this period we found papers [28]-[30] in physics discussed functional integral representations for Schrödinger type operators, which were very useful.

Furthermore, we were able to obtain functional integral representations for models coupled to quantum fields. Fortunately, around 2014, we could prove that functional integral representations were useful tools for non-perturbative analysis of Hamiltonians in quantum field theory, and our research were accelerated since then.

The following is a concrete explanation. The analysis of point spectra embedded in continuous spectrum in quantum field theory had been investigated, and the research had turned to analyzing the properties of their eigenvalues and eigenvectors. In particular a special attention was payed to ground states. We refer to see [66] and references therein for the development of the investigation of eigenvectors and ground states of models in quantum field theory. However, since the corresponding eigenvalues are embedded eigenvalues, there was no established method to analyze their eigenvectors.

From functional integral representations, we can define a measure on a path space, which is called the Gibbs measure [73, p.195, p.378, p.256, p.491]. The Gibbs measure $\mu_{\rm Gibbs}$ can be used to express the expectation value of a certain observable with respect to a ground state in a non-perturbative way:

$$(\Psi_{\rm g}, \mathcal{O}\Psi_{\rm g}) = \int F_{\mathcal{O}}(q) {\rm d}\mu_{\rm Gibbs}(q),$$

where $\mathcal{O}$ denotes an observable and $\Psi_{\rm g}$ a ground state of a model. The basic models of quantum field theory we are concerned are the Nelson model [111, 110, 112], the spin-boson model, and the Pauli-Fierz model in non-relativistic QED [116]. The Pauli-Fierz model is the main object in this paper. The Nelson model is an interaction model of charged particles linearly coupled with a scalar bose field and has been greatly analyzed. The analysis of the renormalized Nelson model has been also completed in a non-perturbative manner by using functional integral representations in [16, 49, 74]. The spin-boson model describes a linear interaction between spin (or two excited states) and a scalar bose field. The spectral properties of the spin-boson model has been studied in [58] in terms of a functional integration. On the other hand, for the Pauli-Fierz model, although a functional integral representation exists, we feel that its application is still insufficient. We hope that some young researchers will read this paper and become interested in this research.

In this article, we will introduce functional integral representations for the Pauli-Fierz type models [64, 72, 65]. See Table 1 for the list of models we consider in this paper.

	Pauli-Fierz type operator	Hilbert space
H	$\frac{1}{2}(-{\rm i}\nabla - A)^2 + V + H_{\rm f}$	$L^2(\mathbb{R}^3) \otimes L^2({\rm Q})$
$H(p)$	$\frac{1}{2}(p - {\rm P}_{\rm f} - A(0))^2 + H_{\rm f}$	$L^2({\rm Q})$
$H_{\rm R}$	$((-{\rm i}\nabla - A)^2 + m^2)^{1/2} - m + H_{\rm f} + V$	$L^2(\mathbb{R}^3) \otimes L^2({\rm Q})$
$H_{\rm R}(p)$	$\left((p - {\rm P}_{\rm f} - A(0))^2 + m^2\right)^{1/2} - m + H_{\rm f}$	$L^2({\rm Q})$
$H_{\rm S}$	$\frac{1}{2}(-{\rm i}\nabla - A)^2 + V + H_{\rm f} - \frac{1}{2}\sigma \cdot B$	$L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2({\rm Q})$
$H_{\rm S}(p)$	$\frac{1}{2}(p - {\rm P}_{\rm f} - A(0))^2 + H_{\rm f} - \frac{1}{2}\sigma \cdot B(0)$	$\mathbb{Z}_2 \otimes L^2({\rm Q})$

Table 1: Pauli-Fierz type operators

A formal integration on the set of continuous paths due to Richard Feynman played an important role in the modern quantum physics. It has already existed due to Wiener's work on Brownian motion initiated in 1923 when the quantum mechanics was not yet found!, and it was Kac in 1949 who first showed a suitable framework, however, not for the path integral directly, see [82, 83]. In contrast to Feynman integrals, a Feynman-Kac formula offers an integral representation of semigroup e^{-tH} instead of unitary group $e^{-\mathrm{i}tH}$ on a path space endowed with a probability measure. Here we show below the reason why this improves the situation. Let

$$A(q,\dot{q};0,t)=\int_0^t\left(\frac{1}{2}\dot{q}^2(s)-V(q(s))\right)\mathrm{d}s,$$

where $\frac{1}{2}\dot{q}^2-V(q)$ is a Lagrangian derived from the Legendre transform of Hamiltonian

$$h=-\frac{1}{2}\Delta+V.$$

The formal expression below

$$(e^{-\mathrm{i}tH})(x,y)=C\int_{\mathfrak{C}_{xy}}e^{\mathrm{i}A(q,\dot{q};0,t)}\prod_{0\le s\le t}\mathrm{d}q(s) \tag{1.1}$$

gives an integral kernel of the operator $e^{-\mathrm{i}tH}$ for all t. Here $\mathrm{d}q(s)$ is Lebesgue measure for each s, and $\mathfrak{C}_{xy}$ the set of continuous paths $q(\cdot):[0,t]\to\mathbb{R}^3$ such that $q(0)=x$ and $q(t)=y$. The right-hand side of (1.1) is Feynman's integral, and $e^{\mathrm{i}A(q,\dot{q};0,t)}$ is a phase factor introduced by Feynman. See Feynman's paper [38, 39]. Also we refer to see e.g., [2, 46, 41, 50, 80] for Feynman's integrals. Analytic continuation $s\to-\mathrm{i}s$, $\mathrm{d}s\to-\mathrm{i}\mathrm{d}s$ and the replacement $\dot{q}(s)^2\to-\dot{q}(s)^2$ in (1.1) lead to the kernel

$$(e^{-tH})(x,y)=C\int_{\mathfrak{C}_{xy}}e^{-\int_0^t V(q(s))\mathrm{d}s}e^{-\frac{1}{2}\int_0^t\dot{q}(s)^2\mathrm{d}s}\prod_{0\le s\le t}\mathrm{d}q(s). \tag{1.2}$$

It is possible to define a mathematically meaningful measure $\mathrm{d}\mathcal{W}^{x,y}_{[0,t]}(q)$ whose formal expression is given by

$$\exp\left(-\int_0^t\frac{1}{2}\dot{q}(s)^2\mathrm{d}s\right)\prod_{0\le s\le t}\mathrm{d}q(s).$$

The paths of a random process $(B_t)_{t\ge0}$ called Brownian motion just have the required properties and the Feynman-Kac formula

$$(e^{-tH})(x,y)=\int_{\mathfrak{C}}e^{-\int_0^t V(B_s(w))\mathrm{d}s}\mathrm{d}\mathcal{W}^{x,y}_{[0,t]}(w),\quad\forall t\ge0,\forall x,y\in\mathbb{R}^3 \tag{1.3}$$

with $\mathfrak{C}=C([0,\infty),\mathbb{R}^3)$ rigorously holds. In particular, there is a measure supported on the space $C([0,\infty),\mathbb{R}^3)$ of continuous functions $[0,\infty)\to\mathbb{R}^3$, and it can be identified as Wiener measure $\mathcal{W}^{x,y}_{[0,t]}$ conditional on paths leaving from x at time 0 and ending in y at time t. Kac has actually proved [81] that the heat equation with an initial data $\phi(x)$:

$$\begin{cases}-\dfrac{\partial f}{\partial t}=-\dfrac{1}{2}\Delta f+Vf,\\[2ex] f(x,0)=\phi(x),\end{cases} \tag{1.4}$$

is solved by the function

$$f(x,t)=\int_{\mathfrak{C}}e^{-\int_0^t V(B_s(w))\mathrm{d}s}\phi(B_t(w))\mathrm{d}\mathcal{W}^x(w). \tag{1.5}$$

Here $\mathcal{W}^x$ is Wiener measure starting from x at $t = 0$. Equation (1.4) is actually the same as the imaginary time Schrödinger equation: See e.g., [18, 110]. The Feynman-Kac formula for the Schrödinger operator h derived from the heat equation can be written as (1.5):

$$e^{-th} f(x) = f(x, t) = (1.5).$$

We then construct functional integral representations for Schrödinger type operators [70, 71].

This paper includes a short review of [73, Chapter 3] and we add some new results. We can construct functional integral representations of other operators, derived from quantum field theory [60, 64, 72, 54, 65, 49]. Both functional integral representations are related each other. A Schrödinger operator with a vector potential $a = a(x) = (a_1(x), a_2(x), a_3(x))$ is given by

$$\frac{1}{2}(\sigma \cdot (-\mathrm{i}\nabla - a))^2 + V, \tag{1.6}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 2×2 Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

On the other hand the Pauli-Fierz Hamiltonian with spin 1/2 is given by

$$\frac{1}{2}(\sigma \cdot (-\mathrm{i}\nabla - A))^2 + V + H_{\mathrm{f}}, \tag{1.7}$$

where the quantized radiation field $A = A(x) = (A_1(x), A_2(x), A_3(x))$ and the free Hamiltonian H_{f} are defined by

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \int_{\mathbb{R}^3} e^j_\mu(k) \left(\frac{\hat{\varphi}(k)}{\sqrt{|k|}} e^{-\mathrm{i}kx} a^*(k, j) + \frac{\hat{\varphi}(-k)}{\sqrt{|k|}} e^{+\mathrm{i}kx} a(k, j) \right) \mathrm{d}k$$

and

$$H_{\mathrm{f}} = \sum_{j=\pm 1} \int |k| a^*(k, j) a(k, j) \mathrm{d}k.$$

Here $a^*(k, j)$ and $a(k, j)$ are the creation and annihilation operators, respectively. They formally satisfy $[a(k, j), a^*(k', j')] = \delta(k-k')\delta_{jj'}$. Physically $\hat{\varphi}$ denotes the Fourier transform of the charge distribution φ. If $\varphi(x) = \delta(x)$, then

$$\hat{\varphi} = \text{ constant.}$$

Since (1.6) and (1.7) are of similar forms, it would be easy to predict that the functional integral representations would also be similar. On the other hand, we can know the effect of quantum field by comparing a functional integral representation for (1.6) with that of (1.7).

This paper is organized as follows. In Section 2 we prepare stochastic tools and introduce functional integral representations for Schrödinger type operators. See the list of operators in Table 2. Section 3 is devoted to investigating the Pauli-Fierz Hamiltonian and its translation invariant version. Section 4 is devoted to investigating the relativistic Pauli-Fierz Hamiltonian and its translation invariant version. In Section 5 the Pauli-Fierz Hamiltonian with spin 1/2 is studied. This section is rather complicated and several statements are shown in Appendices B and C. In Section 6 we derive energy comparisofn inequalities for the Pauli-Fierz type Hamiltonians. Section 7 is the concluding remarks.

2 Schrödinger operators by path measures

In this section we introduce functional integral representations of Schrödinger type operators.

	Schrödinger operator
h	$-\frac{1}{2}\Delta + V$
$h(a)$	$\frac{1}{2}(-\mathrm{i}\nabla - a)^2 + V$
$h_{\mathrm{R}}(a)$	$((-\mathrm{i}\nabla - a)^2 + m^2)^{1/2} - m + V$
$h_{\mathrm{S}}(a,b)$	$\frac{1}{2}(-\mathrm{i}\nabla - a)^2 + V - \frac{1}{2}\sigma \cdot b$
$h_{\mathrm{SR}}(a,b)$	$((-\mathrm{i}\nabla - a)^2 - \sigma \cdot b + m^2)^{1/2} - m + V$

Table 2: Schrödinger operators, a:vector field, b:magnetic field, V:potential, σ:spin

2.1 Stochastic preparations

In order to construct functional integral representations of the semigroups generated by Schrödinger type operators we will need several independent stochastic processes $(B_t)_{t\geq 0}$, $(N_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$. See Table 3. Here we explain properties of these stochastic processes. We denote the expectation with respect to path measure W^x starting at x by

$$\int_{\substack{\text{path}\\ \text{space}}} f \mathrm{d}W^x = \mathbb{E}_W^x[f].$$

(Brownian motion) We give the definition of Brownian motion. $(B_t)_{t\geq 0} = (B_t^1, B_t^2, B_t^3)_{t\geq 0}$ is 3D-Brownian motion on a probability space $(\mathscr{X}, \mathcal{B}, \mathcal{W}^x)$ iff

(1) $\mathcal{W}^x(B_0 = x) = 1$, i.e., $B_0(w) = x$ for almost surely $w \in \mathscr{X}$;

(2) the increments $(B_{t_i} - B_{t_{i-1}})_{1\leq i\leq n}$ are independent Gaussian random variables for every collection $0 = t_0 < t_1 \ldots < t_n$ with the mean and covariance are given by

$$\mathbb{E}_{\mathcal{W}}^0[B_t^\mu] = 0,$$
$$\mathbb{E}_{\mathcal{W}}^0[B_s^\mu B_t^\nu] = \delta_{\mu\nu}(s \wedge t),$$

respectively;

(3) the map $t \mapsto B_t(w)$ is continuous almost surely $w \in \mathscr{X}$.

By the definition of Brownian motion, $B_t - B_s$ and B_{t-s} have the same distribution on $\mathbb{R}^3$, and $B_t - B_s$ and $B_u - B_v$ for $v \le u \le s \le t$ are independent. The distribution of $B_t - B_s$ for $t > s$ is given by

$$\Pi_{t-s}(x) = (2\pi(t-s))^{-3/2} e^{-|x|^2/2(t-s)}.$$

Thus it follows that

$$\mathbb{E}^x_{\mathcal{W}}[f(B_{t_1}, \ldots, B_{t_n})] = \int_{\mathbb{R}^{3N}} f(y_1, \ldots, y_n) \prod_{j=1}^n \Pi_{t_j - t_{j-1}}(y_j - y_{j-1}) \mathrm{d}y_1 \ldots . \mathrm{d}y_n \tag{2.1}$$

with $y_0 = x$ and $t_0 = 0$. Brownian motion is also Markov and then

$$\mathbb{E}^x_{\mathcal{W}}[f(B_{t+s}) \mid \mathscr{F}_s] = \mathbb{E}^x_{\mathcal{W}}[f(B_{t+s}) \mid \sigma(B_s)] = \mathbb{E}^{B_s}_{\mathcal{W}}[f(B_t)]$$

for any $x \in \mathbb{R}^3$, where $\mathbb{E}^{B_s}_{\mathcal{W}}[\ldots]$ is $\mathbb{E}^y_{\mathcal{W}}[\ldots]$ evaluated $y = B_s$, $\mathscr{F}_s = \sigma(B_u \mid 0 \le u \le s)$ is the minimal sigma-filed generated by $\{B_u \mid 0 \le u \le s\}$, and $\sigma(B_s)$ by B_s.

Remark 2.1 *One concrete realisation of Brownian motion is as follows. There exists the so-called Wiener measure $\mathcal{W}^x$ on $C([0,\infty); \mathbb{R}^3)$ such that $B_t(w) = w(t)$ for $w \in C([0,\infty); \mathbb{R}^3)$ becomes Brownian motion under $\mathcal{W}^x$. In this paper we do not choose any special Brownian motion unless otherwise stated.*

By (2.1) Brownian motion satisfies that

$$\mathbb{E}^x_{\mathcal{W}}[f(B_t)] = e^{\frac{t}{2}\Delta} f(x) \quad a.e.\ x \in \mathbb{R}^3$$

and hence the generator of Brownian motion is

$$\frac{1}{2}\Delta.$$

Let $M(0,t)$ be the set of measurable functions $f : [0,\infty) \times \mathscr{X} \to \mathbb{R}^3$ such that

(1) $f(t, \cdot)$ is $\mathbb{C}^3$-valued $\mathscr{F}_t$-measurable for each t,

(2) $\mathbb{E}^0_{\mathcal{W}}[\int_0^t |f(s,w)|^2 \mathrm{d}s] < \infty$.

Then for $f \in M(0,t)$ one can define the stochastic integral X_t presented as

$$X_t = \int_0^t f(s,w) \cdot \mathrm{d}B_s.$$

X_t satisfies that $\mathbb{E}^x[X_t] = 0$ and Itô-isometry $\mathbb{E}^x[|X_t|^2] = \int_0^t \mathbb{E}^x[|f(s,\cdot)|^2] \mathrm{d}s$. The stochastic integral plays an important role through this article.

Now we define the so-called Lévy process. We refer to see e.g., [4, 120] and [99, Section 3.1]. A stochastic process $(X_t)_{t \ge 0}$ on a probability space $(\mathscr{Y}, \mathcal{B}, \mathcal{W})$ is a Lévy process iff

(1) $\mathcal{W}(X_0 = 0) = 1$;

(2) the increments $(X_{t_i} - X_{t_{i-1}})_{1 \le i \le n}$ are independent random variables for every collection $0 = t_0 < t_1 \ldots < t_n$;

(3) $X_t - X_s$ and X_{t-s} have the same distribution for every $0 \le s \le t$;

(4) The map $t \mapsto X_t(w)$ is stochastically continuous, i.e.,

$$\lim_{s \to t} \mathcal{W}(|X_s - X_t| > \varepsilon) = 0$$

for all $t \ge 0$ and $\varepsilon > 0$.

Brownian motion is a Lévy process. Let $(X_t)_{t \ge 0}$ be a Lévy process on $(\mathscr{Y}, \mathcal{B}, \mu)$. It is known that there exists a Lévy process $(Y_t)_{t \ge 0}$ such that $Y_t(w) = X_t(w)$ for $w \in \mathscr{Y} \setminus N_t$, where $\mu(N_t) = 0$, and $t \mapsto Y_t(w)$ is right-continuous and has left-limits for almost surely w. See e.g., [99, Corollary 3.5]. $(Y_t)_{t \ge 0}$ is called the càdlàg version of $(X_t)_{t \ge 0}$. In what follows we assume that Lévy processes are right-continuous and have left-limits for almost surely w. Note also that a Lévy process is a Markov process.

We introduce two Lévy processes below.

(Poisson process) A nonnegative integer-valued Lévy process $(N_t)_{t \ge 0}$ on a probability space $(\mathscr{X}_\mu, \mathcal{B}_\mu, \mu)$ is called a Poisson process iff $\mathbb{E}_\mu[e^{iuN_t}] = e^{t(e^{iu}-1)}$ for $u \in \mathbb{R}$.

The distribution of $N_t - N_s$ on $\mathbb{R}$ is given by $\rho(u) = e^{(t-s)(e^{iu}-1)}$, $u \in \mathbb{R}$. It holds that

$$\mu(N_t = n) = \frac{t^n}{n!} e^{-t}, \quad n \in \mathbb{N} \cup \{0\}.$$

Since $\sum_{n=0}^\infty \mu(N_t = n) = 1$, we have

$$\mathbb{E}_\mu^0[f(N_t)] = \sum_{n=0}^\infty \frac{t^n}{n!} f(n) e^{-t}.$$

The poisson process $(N_t)_{t \ge 0}$ is also Markov and its generator is L given by

$$Lf(x) = f(x+1) - f(x).$$

We define integrals with respect to $(N_t)_{t \ge 0}$ in terms of the sum of evaluations at jumping times, i.e., for g we write

$$\int_a^b g(s, N_s) \mathrm{d}N_s = \sum_{\substack{a \le r \le b \\ N_{r+} \ne N_{r-}}} g(r, N_r).$$

Here $N_{r+} = \lim_{s \downarrow r} N_s$ and $N_{r-} = \lim_{s \uparrow r} N_s$. Then

$$\int_a^{b+} g(s, N_{s-}) \mathrm{d}N_s = \begin{cases} \displaystyle\sum_{\substack{a \le r < b \\ N_{r+} \ne N_{r-}}} g(r, N_{r-}), & N_{b+} = N_{b-}, \\ \displaystyle\sum_{\substack{a \le r < b \\ N_{r+} \ne N_{r-}}} g(r, N_{r-}) + g(b, N_b), & N_{b+} \ne N_{b-}. \end{cases}$$

The expectation of $\int_a^{b+} g(s, N_{s-}) \mathrm{d}N_s$ satisfies that

$$\mathbb{E}\left[\int_a^{b+} g(s, N_{s-}) \mathrm{d}N_s\right] = \mathbb{E}\left[\int_a^{b+} g(s, N_{s-}) \mathrm{d}s\right].$$

Let

$$\mathbb{Z}_2 = \{-1, +1\}. \tag{2.2}$$

By the Poisson process we also define a $\mathbb{Z}_2$-valued stochastic process $(\theta_t)_{t\geq 0}$ on $(\mathscr{X}_\mu, \mathcal{B}_\mu, \mu)$ by

$$\theta_t = (-1)^{N_t}. \tag{2.3}$$

This is called the spin process in this paper. Stochastic process $(\theta_t)_{t\geq 0}$ is useful to study functional integral representations for Schrödinger operators with spin 1/2.

(Subordinator) A 1D-Lévy process $(T_t)_{t\geq 0}$ is called the subordinator iff whenever $s \leq t$ implies that $T_s \leq T_t$ almost surely.

Let $(T_t)_{t\geq 0}$ be a subordinator on a probability space $(\mathscr{X}_\nu, \mathcal{B}_\nu, \nu)$. $(T_t)_{t\geq 0}$ is nonnegative and non-decreasing. Thus it seems to be "random time". Let us consider a subordinator $(T_t)_{t\geq 0}$ such that its Laplace transform is given by

$$\mathbb{E}_\nu^0[e^{-uT_t}] = e^{-t((2u+m^2)^{1/2}-m)}, \quad u \geq 0, \tag{2.4}$$

where $m \geq 0$ plays a role of the mass of quanta in physics. The existence of the subordinator (2.4) is established. See e.g., [99, Example 3.90]. Comparing $\mathbb{E}_{\mathcal{W}}^x[f(B_t)] = e^{\frac{t}{2}\Delta}f(x)$ and (2.4), we can see that

$$\mathbb{E}_\nu^0\mathbb{E}_{\mathcal{W}}^x[f(B_{T_t})] = e^{-t((-\Delta+m^2)^{1/2}-m)}f(x) \quad a.e.\ x \in \mathbb{R}^3.$$

Hence the generator of the compound process $(B_{T_t})_{t\geq 0}$ is the free relativistic Schrödinger operator:

$$-(-\Delta + m^2)^{1/2} + m.$$

The distribution of T_t on $\mathbb{R}$ is given by

$$\rho(r,t) = \frac{t}{\sqrt{2\pi r^3}} \exp\left(-\frac{1}{2}\left(\frac{t^2}{r} + m^2 r\right) + mt\right) \mathbb{1}_{[0,\infty)}(r).$$

Furthermore the subordinate spin process $(\theta_{T_t})_{t\geq 0}$ is used for constructing functional integral representations for relativistic Schrödinger operators with spin 1/2.

We use the shorthand: $\mathbb{E}_{\mathcal{W}}^x\mathbb{E}_\mu^\alpha\mathbb{E}_\nu^0 = \mathbb{E}^{x,\alpha,0}$, $\mathbb{E}_{\mathcal{W}}^x\mathbb{E}_\mu^\alpha = \mathbb{E}^{x,\alpha}$ and $\mathbb{E}_{\mathcal{W}}^x = \mathbb{E}^x$ etc. The role of these three stochastic processes is as follows. Clearly, Schrödinger operator $-\frac{1}{2}\Delta + V$ can be described by $(B_t)_{t\geq 0}$ under V. The spin process $(\theta_t)_{t\geq 0}$ results from Schrödinger operators with spin 1/2. Finally, the subordinator $(T_t)_{t\geq 0}$ appears in relativistic Schrödinger operators. A particular combination of these three independent stochastic processes then yields functional integral representations of $(f, e^{-tK}g)$, where $K = h, h(a), h_{\mathrm{R}}(a), h_{\mathrm{S}}(a,b), h_{\mathrm{SR}}(a,b)$. See Table 2. For each K we shall show that

$$(f, e^{-tK}g) = \int_{\text{space}} \mathbb{E}_W^y[\bar{f}(\xi_0)g(\xi_t)e^{Z_t}]\mathrm{d}\rho(y),$$

where $(\xi_t)_{t\geq 0}$ is a stochastic process, W a path measure and e^{Z_t} an integral kernel. The generator $-G$ of ξ_t satisfies that

$$(f, e^{-tG}g) = \int_{\text{space}} \mathbb{E}_W^y[\bar{f}(\xi_0)g(\xi_t)]\mathrm{d}\rho(y).$$

	process	space	measure
Brownian Motion	$(B_t)_{t\geq 0}$	$\mathscr{X}$	$\mathcal{W}$
Poisson process	$(N_t)_{t\geq 0}$	$\mathscr{X}_\mu$	μ
Spin process	$(\theta_t)_{t\geq 0}$	$\mathscr{X}_\mu$	μ
Subordinator	$(T_t)_{t\geq 0}$	$\mathscr{X}_\nu$	ν
Subordinate Brownian motion	$(B_{T_t})_{t\geq 0}$	$\mathscr{X} \times \mathscr{X}_\nu$	$\mathcal{W} \times \nu$
Subordinate spin process	$(\theta_{T_t})_{t\geq 0}$	$\mathscr{X}_\mu \times \mathscr{X}_\nu$	$\mu \times \nu$

Table 3: Stochastic processes

As is mentioned above, we then have examples of generators below:

$$(f, e^{\frac{t}{2}\Delta} g) = \int_{\mathbb{R}^3} \mathbb{E}^x[\bar{f}(B_0) g(B_t)] \mathrm{d}x,$$
$$(f, e^{-t((-\Delta+m^2)^{1/2}-m)} g) = \int_{\mathbb{R}^3} \mathbb{E}^{x,0}[\bar{f}(B_0) g(B_{T_t})] \mathrm{d}x.$$

Here we assumed that the space dimension is three, it can be however straightforwardly extended to any dimension.

2.2 Schrödinger operators $h(a)$

We begin with showing function integral representations of $e^{-th(a)}$ with Schrödinger operators $h(a)$ without proofs as an introduction. We refer to see e.g., [110, 127, 128, 1] of functional integral representations of Schrödinger operators and its applications. Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a measurable function and a Schrödinger operator is defined by

$$h = -\frac{1}{2}\Delta + V$$

acting in $L^2(\mathbb{R}^3)$. The following proposition is basic for functional integral representations.

Proposition 2.1 *Let $V \in L^\infty(\mathbb{R}^3)$. Then for $f, g \in L^2(\mathbb{R}^3)$,*

$$(f, e^{-th} g) = \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f(B_0)} g(B_t) e^{-\int_0^t V(B_s)\mathrm{d}s}] \mathrm{d}x. \tag{2.5}$$

In particular,

$$(e^{-th} g)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s} g(B_t)]. \tag{2.6}$$

Several proofs of this proposition are known. It involves (1) application of Itô formula [79]:

$$\begin{aligned}
&\mathrm{d}e^{-\int_0^t V(B_s)\mathrm{d}s} f(B_t) \\
&= \mathrm{d}e^{-\int_0^t V(B_s)\mathrm{d}s} \cdot f(B_t) + e^{-\int_0^t V(B_s)\mathrm{d}s} \cdot \mathrm{d}f(B_t) + \mathrm{d}e^{-\int_0^t V(B_s)\mathrm{d}s} \cdot \mathrm{d}f(B_t) \\
&= e^{-\int_0^t V(B_s)\mathrm{d}s} \left(-V(B_t) f(B_t) + \frac{1}{2}\Delta f(B_t) \right) \mathrm{d}t + e^{-\int_0^t V(B_s)\mathrm{d}s} \nabla f(B_t) \cdot \mathrm{d}B_t,
\end{aligned}$$

and (2) application of the Trotter product formula [133, 22, 86, 87, 88]:

$$e^{-th} = \lim_{n\to\infty} \left(e^{-\frac{t}{n}(-\frac{1}{2}\Delta)} e^{-\frac{t}{n}V} \right)^n.$$

Here $\mathrm{d}X_t$ denotes $\mathrm{d}X_t = X_t - X_0$. Thus

$$\mathrm{d}e^{-\int_0^t V(B_s)\mathrm{d}s} f(B_t) = e^{-\int_0^t V(B_s)\mathrm{d}s} f(B_t) - f(B_0).$$

This formula can be extended for general potentials V. Furthermore introducing a vector potential $a = (a_1, a_2, a_3) : \mathbb{R}^3 \to \mathbb{R}^3$, we define Schrödinger operators with a vector potential a by

$$h(a) = \frac{1}{2}(-\mathrm{i}\nabla - a)^2 + V.$$

Proposition 2.2 *Suppose that $a \in (C_\mathrm{b}^2(\mathbb{R}^3))^3$ and $V \in L^\infty(\mathbb{R}^3)$. Then for $f, g \in L^2(\mathbb{R}^3)$,*

$$(f, e^{-th(a)} g) = \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f(B_0)} g(B_t) e^{Z_t}] \mathrm{d}x. \tag{2.7}$$

Here

$$Z_t = -\mathrm{i}\int_0^t a(B_s) \circ \mathrm{d}B_s - \int_0^t V(B_s)\mathrm{d}s \tag{2.8}$$

with

$$\int_0^t a(B_s) \circ \mathrm{d}B_s = \int_0^t a(B_s) \cdot \mathrm{d}B_s + \frac{1}{2}\int_0^t \nabla \cdot a(B_s)\mathrm{d}s.$$

In particular,

$$(e^{-th(a)} f)(x) = \mathbb{E}^x[e^{Z_t} f(B_t)]. \tag{2.9}$$

We give a comment on this formula. Since $h(a)$ is self-adjoint, it follows that

$$(f, e^{-th(a)} g) = \overline{(g, e^{-th(a)} f)}.$$

Integral kernel Z_t however has the purely imaginary part $-\mathrm{i}\int_0^t a(B_s) \circ \mathrm{d}B_s$. For a moment, we may feel odd. We are often asked a question on it by probabilists. We can see that

$$\begin{aligned}
\overline{(g, e^{-th(a)} f)} &= \int_{\mathbb{R}^3} \mathbb{E}^x[g(B_0) \bar{f}(B_t) e^{\bar{Z}_t}] \mathrm{d}x = \int_{\mathbb{R}^3} \mathbb{E}^x[g(\dot{B}_0) \bar{f}(\dot{B}_t) e^{\dot{\bar{Z}}_t}] \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \mathbb{E}^0[g(x) \bar{f}(\dot{B}_t + x) e^{\dot{\bar{Z}}_t(x)}] \mathrm{d}x,
\end{aligned}$$

where $(\dot{B}_s)_{0\le s\le t} = (B_{t-s} - B_t)_{0\le s\le t}$ is also Brownian motion, and

$$\dot{Z}_t(x) = +\mathrm{i}\int_0^t a(\dot{B}_s + x)\circ \mathrm{d}\dot{B}_s - \int_0^t V(\dot{B}_s + x)\mathrm{d}s.$$

Chang variables as $x \to y - \dot{B}_t = y + B_t$. Hence we can compute as

$$-\int_0^t V(y - \dot{B}_t + \dot{B}_s)\mathrm{d}s = -\int_0^t V(y + B_u)\mathrm{d}u,$$

while that of the stochastic integral is

$$+\mathrm{i}\int_0^t a(x + \dot{B}_s)\circ \mathrm{d}\dot{B}_s = +\mathrm{i}\int_0^t a(y - \dot{B}_t + \dot{B}_s)\circ \mathrm{d}\dot{B}_s = -\mathrm{i}\int_0^t a(y + B_s)\circ \mathrm{d}B_s.$$

Note that signature $+$ is changed to signature $-$, because of the fact

$$\int_0^t a(B_s)\circ \mathrm{d}\dot{B}_s = \lim_{n\to\infty}\sum_{j=1}^{2^n}\frac{1}{2}\left(a(B_{\frac{jt}{2^n}}) + a(B_{\frac{(j-1)t}{2^n}})\right)\left(B_{\frac{(j-1)t}{2^n}} - B_{\frac{jt}{2^n}}\right) = -\int_0^t a(B_s)\circ \mathrm{d}B_s.$$

We conclude that

$$\overline{(g, e^{-th(a)}f)} = \int_{\mathbb{R}^3}\mathbb{E}^y[g(y + B_t)\bar{f}(y)e^{Z_t}]\mathrm{d}y = (f, e^{-th(a)}g).$$

The self-adjointness of $h(a)$ and the existence of the purely imaginary part $-\mathrm{i}\int_0^t a(B_s)\circ \mathrm{d}B_s$ become compatible.

2.3 Schrödinger operators with Kato-class potentials

We consider Schrödinger operators with singular potentials. Kato-class is a class of singular potentials but we can define Schrödinger operators with Kato-class potentials as self-adjoint operators by functional integral representations.

Definition 2.1 (1) *$V : \mathbb{R}^d \to \mathbb{R}$ is called a Kato-class potential whenever*

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{B_r(x)}|g(x-y)V(y)|\mathrm{d}y = 0 \tag{2.10}$$

holds, where $B_r(x)$ is the closed ball of radius r centered at x, and $g(x)$ is the function given by

$$g(x) = \begin{cases} |x|, & d = 1, \\ -\log|x|, & d = 2, \\ |x|^{2-d}, & d \ge 3. \end{cases} \tag{2.11}$$

We denote the set of Kato-class potentials by $\mathcal{K}(\mathbb{R}^d)$.

(2) *V is a local Kato-class potential whenever $V\lceil_K \in \mathcal{K}(\mathbb{R}^d)$ for any compact set $K \subset \mathbb{R}^d$. We denote the set of local Kato-class potentials by $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$.*

(3) *V is Kato-decomposable whenever $V = V_+ - V_-$ with $V_+ \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ and $V_- \in \mathcal{K}(\mathbb{R}^d)$.*

We give non-trivial examples of Kato-class potentials for $d = 3$. Notice that

$$\left| \int_{B_r(x)} \frac{1}{|x-y|} |V(y)| \mathrm{d}y \right| \leq \left(\int_{B_r(x)} \frac{1}{|x-y|^p} \mathrm{d}y \right)^{1/p} \left(\int_{B_r(x)} |V(y)|^q \mathrm{d}y \right)^{1/q},$$

with $1/p + 1/q = 1$. When $p < 3$, the first factor goes to zero as $r \to 0$. Thus V is Kato-class if $\sup_{x\in\mathbb{R}^3} \int_{B_r(x)} |V(y)|^q \mathrm{d}y < \infty$ for some $r > 0$ and some $q > 3/2$. Thus

$$V(x) = \frac{1}{|x|^{2-\varepsilon}}, \quad \varepsilon > 0$$

is Kato-class.

We are in the position to prove exponential integrability of integrals over some potentials. When $V_- \in \mathcal{K}(\mathbb{R}^3)$, it can be seen that the exponent $e^{\int_0^t V(B_s)\mathrm{d}s}$ is integrable with respect to Wiener measure so that $\mathbb{E}^x[e^{\int_0^t V(B_s)\mathrm{d}s}]$ is finite for all x.

Lemma 2.3 ([73, Lemma 4.105],[19]) *Let $V \in \mathcal{K}(\mathbb{R}^d)$ and $V(x) \geq 0$ a.e. $x \in \mathbb{R}^d$. Then there exist $\beta, \gamma > 0$ such that*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x[e^{\int_0^t V(B_s)\mathrm{d}s}] < \gamma e^{\beta t}. \tag{2.12}$$

Furthermore, if $V \in L^p(\mathbb{R}^d)$ with $p > d/2$ and $1 \leq p < \infty$, then $\beta \leq c(p)^{1/\varepsilon}\Gamma(\varepsilon)^{1/\varepsilon}\|V\|_p^{1/\varepsilon}$, where $\varepsilon = 1 - \frac{d}{2p}$ and $c(p) = \begin{cases} (2\pi)^{-d/2} & p = 1, \\ (2\pi)^{-d/2p} q^{d/(2q)} & p > 1 \end{cases}$ with $\frac{1}{p} + \frac{1}{q} = 1$. In particular $L^p(\mathbb{R}^d) \subset \mathcal{K}(\mathbb{R}^d)$ for $p > d/2$ and $1 \leq p < \infty$.

Let V be Kato-class. Define the map K_t on $L^2(\mathbb{R}^3)$ by

$$(K_t f)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s} f(B_t)].$$

Lemma 2.4 *$\{K_t : t \geq 0\}$ is a symmetric C_0-semigroup on $L^2(\mathbb{R}^3)$.*

Proof: The boundedness of K_t follows from Lemma 2.3. Define $\tilde{B}_s = B_{t-s} - B_t$, $0 \leq s \leq t$, for a fixed $t > 0$. We see that $\tilde{B}_s$ is also a Brownian motion. Thus

$$(f, K_t g) = \mathbb{E}^0 \left[\int_{\mathbb{R}^3} \overline{f(x)} e^{-\int_0^t V(\tilde{B}_s + x)\mathrm{d}s} g(\tilde{B}_t + x) \mathrm{d}x \right].$$

Changing the variable x to $y = \tilde{B}_t + x$, we obtain

$$\begin{aligned} (f, K_t g) &= \mathbb{E}^0 \left[\int_{\mathbb{R}^3} \overline{f(y - \tilde{B}_t)} e^{-\int_0^t V(\tilde{B}_s - \tilde{B}_t + y)\mathrm{d}s} g(y) \mathrm{d}y \right] \\ &= \int_{\mathbb{R}^3} \mathbb{E}^0 [\overline{f(y + B_t)} e^{-\int_0^t V(B_{t-s} + y)\mathrm{d}s} g(y)] \mathrm{d}y = (K_t f, g), \end{aligned} \tag{2.13}$$

i.e., K_t is symmetric. Write now $Z_t = e^{-\int_0^t V(B_s)\mathrm{d}s}$. The semigroup property follows directly from the Markov property of Brownian motion:

$$\begin{aligned}(K_sK_tf)(x) &= \mathbb{E}^x[Z_s\mathbb{E}^{B_s}(Z_tf(B_t))]\\ &= \mathbb{E}^x[\mathbb{E}^x[Z_se^{-\int_0^t V(B_{s+u})\mathrm{d}u}f(B_{s+t})|\mathcal{F}_s]]\\ &= \mathbb{E}^x\left[Z_{s+t}f(B_{s+t})\right] = K_{s+t}f(x).\end{aligned}$$

The strong continuity of $t \mapsto K_t$ is implied by

$$\|K_tf - f\| \le \mathbb{E}^0[\|e^{-\int_0^t V(\cdot+B_s)\mathrm{d}s}f(\cdot+B_t) - f\|] \to 0$$

as $t \to 0$, by the Lebesgue dominated convergence theorem.

QED

By the Stone's theorem for semigroups there exists a unique self-adjoint operator K such that

$$K_t = e^{-tK}$$

for any $t \ge 0$. For example we can define a self-adjoint operator $K = -\frac{1}{2}\Delta - \frac{1}{|x|^\alpha}$ for $0 \le \alpha < 2$, since $\frac{1}{|x|^\alpha} \in \mathcal{K}(\mathbb{R}^3)$ for $0 \le \alpha < 2$.

$-\frac{1}{2}\Delta$ is defined as a self-adjoint operator in the Hilbert space $L^2(\mathbb{R}^3)$ and $e^{\frac{t}{2}\Delta}$ by the spectral measure associated with $-\Delta$. Hence $e^{\frac{t}{2}\Delta}f$ is well-defined for any $f \in L^2(\mathbb{R}^3)$, while it can be represented as $e^{\frac{t}{2}\Delta}f(x) = \mathbb{E}^x[f(B_t)]$ for each $x \in \mathbb{R}^3$. We note that the right-hand side $\mathbb{E}^x[f(B_t)]$ can be defined not only for $f \in L^2(\mathbb{R}^3)$ but also $f \in L^p(\mathbb{R}^3)$ for $1 \le p \le \infty$. More precisely $e^{\frac{t}{2}\Delta}f$ for $f \in L^p(\mathbb{R}^3)$ is defined by $\mathbb{E}^x[f(B_t)]$. We can show that $e^{\frac{t}{2}\Delta}$ maps $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ for every $1 \le p \le q \le \infty$. For $-\frac{1}{2}\Delta + V$ with Kato-decomposable V we can show the statements below:

(1) Let $V = V_+ - V_-$. Suppose that $V_+ \in L^1_{loc}(\mathbb{R}^3)$ and $V_- \in \mathcal{K}(\mathbb{R}^3)$. Then for each $x \in \mathbb{R}^3$, $\mathbb{E}^x[e^{-\int_0^t V(B_u)\mathrm{d}u}f(B_t)]$ is well-defined. We can however say more strong statements for kato-decomposable V. Let V be Kato-decomposable and $f \in L^p(\mathbb{R}^3)$ for $1 \le p \le \infty$. Then for every $t > 0$, $x \mapsto K_tf(x)$ is continuous [1].

(2) Let V be Kato-decomposable. Then for every $1 \le p \le q \le \infty$, K_t maps $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ as bounded operators, i.e., $\|K_tf\|_{L^q(\mathbb{R}^3)} \le C\|f\|_{L^p(\mathbb{R}^3)}$.

2.4 Schrödinger operators with spin $1/2$ $h_{\mathrm{S}}(a,b)$

In this section we discuss functional integral representations for Schrödinger operators with spin 1/2. We refer to see [31, 29, 42, 30, 70, 71] for functional integral representations of Schrödinger operators with spin 1/2. Since the Trotter product formula holds even when there is spin, it is easy to see that a functional integral representation can be obtained like Schrödinger operators, but an infinite product of matrices appears in the integral kernel. It may be possible to have an exact form of the infinite product of matrices, but we will introduce another way.

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be the 2×2 Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

σ_js are traceless symmetric matrices. Consider the $\mathbb{R}^3 \times \mathbb{Z}_2$-valued joint Brownian and jump process

$$\mathscr{X} \times \mathscr{X}_\mu \ni (w, m) \mapsto X_t(w, m) = (B_t(w), \theta_t(m)) \in \mathbb{R}^3 \times \mathbb{Z}_2$$

with initial value X_0. The generator of $(X_t)_{t\geq 0}$ is

$$G_0 = \frac{1}{2}\Delta - \sigma_{\mathrm{F}},$$

where σ_{F} is the fermionic harmonic oscillator defined by

$$\sigma_{\mathrm{F}} = \frac{1}{2}(\sigma_3 + \mathrm{i}\sigma_2)(\sigma_3 - \mathrm{i}\sigma_2) = -\sigma_1 + \mathbb{1}_{2\times 2}.$$

Let $b = (b_1, b_2, b_3)$ be a magnetic potential, and

$$h_{\mathrm{S}}(a, b) = \frac{1}{2}(-\mathrm{i}\nabla - a)^2 - \frac{1}{2}\sigma \cdot b + V \tag{2.14}$$

be a Schrödinger operator with spin 1/2 defined in $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$. In physics a Schrödinger operator with spin 1/2 is defined by

$$\frac{1}{2}(\sigma \cdot (-\mathrm{i}\nabla - a))^2 + V.$$

We can see that

$$\frac{1}{2}(\sigma \cdot (-\mathrm{i}\nabla - a))^2 + V = \frac{1}{2}(-\mathrm{i}\nabla - a)^2 - \frac{1}{2}\sigma \cdot \mathrm{rot} a + V$$

and $b = \mathrm{rot} a$. In this section however we assume that a and b are independent vectors in (2.14).

We identify $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3) = L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \ni \binom{f_1}{f_2} \mapsto f(x, \cdot) \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, where $f_j(x) = f(x, j)$. See Table 5 in Section 5. The proposition below is established.

Proposition 2.5 ([31, 70]) *Let $a \in (C_{\mathrm{b}}^2(\mathbb{R}^3))^3$, $b \in (L^\infty(\mathbb{R}^3))^3$ and $V \in L^\infty(\mathbb{R}^3)$. Suppose that*

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \left| \log \frac{1}{2}\sqrt{b_1(y)^2 + b_2(y)^2} \right| \frac{e^{-|y-x|^2/(2s)}}{(2\pi s)^{3/2}} \mathrm{d}y < \infty$$

for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$. Then

$$(f, e^{-th_{\mathrm{S}}(a,b)} g) = e^t \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[\overline{f(X_0)} g(X_t) e^{Z_t}] \mathrm{d}x. \tag{2.15}$$

Here

$$\begin{aligned} Z_t = &-\mathrm{i}\int_0^t a(B_s) \circ \mathrm{d}B_s - \int_0^t V(B_s)\mathrm{d}s \\ &+ \int_0^t \frac{1}{2}\theta_{N_s} b_3(B_s)\mathrm{d}s + \int_0^{t+} \log\left(\frac{1}{2}(b_1(B_s) - \mathrm{i}\theta_{N_{s-}} b_2(B_s))\right) \mathrm{d}N_s. \end{aligned} \tag{2.16}$$

Note that inserting $V = 0$, $a = (0, 0, 0)$ and $b = (2, 0, 0)$ yields that $h_{\mathrm{S}}(a, b) = -\frac{1}{2}\Delta - \sigma_1$ and

$$Z_t = \int_0^{t+} \log 1 \mathrm{d}N_s = 0.$$

Hence it follows that

$$e^t \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[\overline{f(X_0)} g(X_t) e^{Z_t}] \mathrm{d}x = (f, e^t e^{-t(-\frac{1}{2}\Delta + \sigma_F)} g) = (f, e^{-t(-\frac{1}{2}\Delta - \sigma_1)} g).$$

2.5 Relativistic Schrödinger operators $h_{\mathrm{R}}(a)$

We consider relativistic Schrödinger operators. We refer to see [28, 20, 70] for functional integral representations for relativistic Schrödinger operators. In the relativistic case, the subordinator explained above appears in addition. Let

$$h_{\mathrm{R}}(a) = ((-\mathrm{i}\nabla - a)^2 + m^2)^{1/2} - m + V.$$

When $V = 0$, we can see that

$$\mathbb{E}_\nu^0[e^{-T_t h(a)}] = e^{-th_{\mathrm{R}}(a)}$$

by the definition of the subordinator $(T_t)_{t\geq 0}$. Hence by

$$(f, e^{-th(a)}g) = \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f(B_0)}g(B_t)e^{-\mathrm{i}\int_0^t a(B_s)\circ \mathrm{d}B_s}]\mathrm{d}x,$$

we have

$$(f, e^{-th_{\mathrm{R}}(a)}g) = \int_{\mathbb{R}^3} \mathbb{E}^{x,0}[\overline{f(B_0)}g(B_{T_t})e^{-\mathrm{i}\int_0^{T_t} a(B_s)\circ \mathrm{d}B_s}]\mathrm{d}x. \tag{2.17}$$

We have the proposition below.

Proposition 2.6 ([28, 70]) *Suppose that $V \in L^\infty(\mathbb{R}^3)$, $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$ and $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Then*

$$(f, e^{-th_{\mathrm{R}}(a)}g) = \int_{\mathbb{R}^3} \mathbb{E}^{x,0}[\overline{f(B_0)}g(B_{T_t})e^{Z_t}]\mathrm{d}x, \tag{2.18}$$

where

$$Z_t = -\mathrm{i}\int_0^{T_t} a(B_s)\circ \mathrm{d}B_s - \int_0^t V(B_{T_s})\mathrm{d}s. \tag{2.19}$$

Proof: The proof is an application of the Trotter product formula and (2.17).

QED

2.6 Relativistic Schrödinger operators with spin 1/2 $h_{\mathrm{SR}}(a, b)$

We consider functional integral representations for relativistic Schrödinger operators with spin 1/2. Combining the relativistic case and the spin case well, we can get functional integral representations for relativistic Schrödinger operators with spin 1/2. We refer to see [32, 70, 71] for functional integral representations for relativistic Schrödinger operators with spin 1/2. We define the subordinate process $(q_t)_{t\geq 0}$ in terms of the $\mathbb{R}^3 \times \mathbb{Z}_2$-valued stochastic process $(X_t)_{t\geq 0} = ((B_t, \theta_t))_{t\geq 0}$:

$$\mathscr{X} \times \mathscr{X}_\nu \times \mathscr{X}_\nu \ni (w, m, w_2) \mapsto q_t(w, m, w_2) = (B_{T_t(w_2)}(w), \theta_{T_t(w_2)}(m)) \in \mathbb{R}^3 \times \mathbb{Z}_2.$$

In a similar manner to $(X_t)_{t\geq 0}$, we can identify the generator of $(q_t)_{t\geq 0}$. The generator of $(q_t)_{t\geq 0}$ is

$$G = -(-\Delta + 2\sigma_{\mathrm{F}} + m^2)^{1/2} + m. \tag{2.20}$$

This is obtained through the equalities

$$\sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}^3}\mathbb{E}^{x,\alpha,0}\left[\overline{f(q_0)}g(q_t)\right]\mathrm{d}x=\mathbb{E}^0_\nu\left[\sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}^3}\mathbb{E}^{x,\alpha}_{\mathcal{W}\times\mu}\left[\overline{f(q_0)}g(q_t)\right]\mathrm{d}x\right]$$
$$=\mathbb{E}^0_\nu[(f,e^{-T_t(-\frac{1}{2}\Delta+\sigma_F)}g)]=(f,e^{-tG}g).$$

Hence it follows that (2.20) is the generator of $(q_t)_{t\geq 0}$.

Let

$$h_{\mathrm{SR}}(a,b)=((-\mathrm{i}\nabla-a)^2-\sigma\cdot b+m^2)^{1/2}-m+V$$

be a relativistic Schrödinger operator with vector potential a, magnetic potential b and spin 1/2 defined on $\mathbb{C}^2\otimes L^2(\mathbb{R}^3)$. Let $\rho(r,t)$ be the distribution of T_t on $\mathbb{R}$.

Proposition 2.7 ([32, 70]) *Let $V\in L^\infty(\mathbb{R}^3)$, $a\in(L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$ and $\nabla\cdot a\in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Suppose that $b\in(L^\infty(\mathbb{R}^3))^3$ and*

$$\int_0^\infty \mathrm{d}r\rho(r,t)\int_0^r\mathrm{d}s\int_{\mathbb{R}^3}\left|\log\left(\frac{1}{2}\sqrt{b_1(y)^2+b_2(y)^2}\right)\right|\frac{e^{-|y-x|^2/2s}}{(2\pi s)^{3/2}}\mathrm{d}y<\infty$$

for all $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$. Then

$$(f,e^{-th_{\mathrm{SR}}(a,b)}g)=\sum_{\alpha=0,1}\int_{\mathbb{R}^3}\mathbb{E}^{x,\alpha,0}[e^{T_t}\overline{f(q_0)}g(q_t)e^{Z_t}]\mathrm{d}x,\tag{2.21}$$

where

$$Z_t=-\mathrm{i}\int_0^{T_t}a(B_s)\circ\mathrm{d}B_s-\int_0^t V(B_{T_s})\mathrm{d}s+\int_0^{T_t}\frac{1}{2}\theta_{N_s}b_3(B_s)\mathrm{d}s$$
$$+\int_0^{T_t+}\log\left(\frac{1}{2}(b_1(B_s)-\mathrm{i}\theta_{N_{s-}}b_2(B_s))\right)\mathrm{d}N_s.\tag{2.22}$$

Note that $V=0$, $a=(0,0,0)$ and $b=(2,0,0)$. Then $h_{\mathrm{SR}}(a,b)=(-\Delta-2\sigma_1+m^2)^{1/2}-m$ and $Z_t=\int_0^{T_t+}\log 1\mathrm{d}N_s=0$. Hence it follows that

$$\sum_{\alpha=0,1}\int_{\mathbb{R}^3}\mathbb{E}^{x,\alpha,0}[e^{T_t}\overline{f(q_0)}g(q_t)e^{Z_t}]\mathrm{d}x=\mathbb{E}^0[e^{T_t}(f,e^{-T_t(-\frac{1}{2}\Delta+\sigma_F)}g)]$$
$$=(f,e^{-t((-\Delta-2\sigma_1+m^2)^{1/2}-m)}g).$$

We exhibit all the results mentioned above in Table 4.

2.7 Brief summaries of applications

There are many applications of functional integral representations to spectral analysis of Schrödinger operators. We refer to see e.g., [127]. Here are some of them.

(Singular potentials) By functional integral representations we can define a Schrödinger operator with a singular potential V. E.g., Kato-class potentials (see Definition 2.1). We define $\mathrm{K}_t f(x)=\mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s}f(B_t)]$ for a Kato-class potential V. One can show that K_t defines a

Schrödinger operator	space	path space	process ξ_t	generator G	measure W	kernel Z_t
$h(a)$	$\mathbb{R}^3$	$\mathscr{X}$	B_t	$\frac{1}{2}\Delta$	$\mathcal{W}$	(2.8)
$h_{\mathrm{R}}(a)$	$\mathbb{R}^3$	$\mathscr{X}\times\mathscr{X}_\nu$	B_{T_t}	$-(-\Delta+m^2)^{1/2}+m$	$\mathcal{W}\times\nu$	(2.19)
$h_{\mathrm{S}}(a,b)$	$\mathbb{R}^3\times\mathbb{Z}_2$	$\mathscr{X}\times\mathscr{X}_\mu$	$X_t=(B_t,\theta_t)$	$\frac{1}{2}\Delta-\sigma_F$	$\mathcal{W}\times\mu$	(2.16)
$h_{\mathrm{SR}}(a,b)$	$\mathbb{R}^3\times\mathbb{Z}_2$	$\mathscr{X}\times\mathscr{X}_\mu\times\mathscr{X}_\nu$	$q_t=(B_{T_t},\theta_{T_t})$	$-(-\Delta+2\sigma_F+m^2)^{1/2}+m$	$\mathcal{W}\times\mu\times\nu$	(2.22)

Table 4: Schrödinger operators

symmetric C_0-semigroup on $L^2(\mathbb{R}^3)$. Thus there exists a unique self-adjoint operator K such that $\mathrm{K}_t=e^{-tK}$ by the Stone's theorem.

Functional integral representations of e^{-th} are also useful to study properties of Schrödinger operators with singular potentials. It includes Klauder phenomena which yields

$$\lim_{\varepsilon\to 0}-\frac{1}{2}\Delta+\varepsilon V\neq-\frac{1}{2}\Delta$$

for some singular V. See [33, 34, 124].
(Ergodic properties) [36, 37, 117] From the formula

$$(f,e^{-th}g)=\int_{\mathbb{R}^3}\mathbb{E}^x[\ldots]\mathrm{d}x,$$

one can immediately see that $(f,e^{-th}g)>0$ for any $f,g\geq 0$. This derives that e^{-th} is positivity improving or ergodic.
(Spatial decay) [1, 19, 20, 71] If h admits an eigenfunction ϕ at eigenvalue E, then $e^{-th}\phi=e^{-Et}\phi$ and hence the identity

$$\phi=e^{-t(h-E)}\phi \tag{2.23}$$

follows for any $t\geq 0$. Thus this eigenfunction can be represented in terms of an average over the paths of a stochastic process. This makes possible to obtain information on the spectral properties of h by probabilistic means. Let $h=-\frac{1}{2}\Delta+V$. Then we have

$$\phi(x)=e^{tE}\mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s}\phi(B_t)]. \tag{2.24}$$

From this we can see e.g., the spatial decay of $\phi(x)$ from both upper and lower. While we define

$$X_t(x)=e^{tE}e^{-\int_0^t V(B_s+x)\mathrm{d}s}\phi(B_t+x).$$

It can be shown that $(X(x)_t)_{t\geq 0}$ is a martingale and then

$$\mathbb{E}^0[X_{t\wedge\tau}(x)]=\mathbb{E}^0[X_0(x)]=\phi(x)$$

for any stopping time τ. Choosing a suitable stopping time τ, we can also estimate the spatial decay of $\phi(x)$ from both upper and lower. We refer to see e.g., [20].

(Hypercontractivity) [27, 25, 26] It is not hard to establish that $e^{\Delta/2}: L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$ for every $1 \le p \le q \le \infty$. We can extend this to e^{-th}. From (2.24) and the Riesz-Thorin interpolation [73, Lemma 4.108] we can also show that e^{-th} is hypercontractive. I.e., e^{-th} : $L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$ for every $1 \le p \le q \le \infty$.
(Smoothing effect) [1] For suitable V we can see that

$$x \mapsto \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s}\phi(B_t)]$$

is continuous. E.g., $V = V_+ - V_-$ with Kato-class V_+ and local Kato-class V_-. From this the spatial continuity of $x \mapsto \phi(x)$ can be obtained by (2.23).
(Lieb-Thirring inequality) [97, 98, 93, 94, 24, 95] The number of non-positive eigenvalues

$$N = \#\{e \mid e \in \sigma_p(h), e \le 0\}$$

of h can be also estimated by using the functional integral representation of e^{-th}, which is known as Lieb-Thirring inequality:

$$N \le a_3 \int_{\mathbb{R}^3} |V_-(x)|^{3/2}\mathrm{d}x.$$

This inequality can be extended to general Schrödinger type operators. In general for

$$h^\Psi = \Psi(-\Delta) + V$$

with a Bernstein function Ψ [121], it can be seen under some conditions that

$$N \le a_{3,\Psi} \int_{\mathbb{R}^3} \Psi^{-1}(|V_-(x)|)^{3/2}\mathrm{d}x.$$

Here Ψ^{-1} is the inverse function of Ψ. The examples of Bernstein functions include $\Psi(x) = x^\alpha$ for $0 < \alpha < 1$ and $\Psi(x) = 1 - e^{-\beta x}$ for $\beta > 0$. Let $\Psi(x) = \sqrt{x}$. Then $N \le a_{3,\Psi} \int_{\mathbb{R}^3} |V_-(x)|^3\mathrm{d}x$ follows. See [24].
(Non-relativistic limit) [67] Let c be the velocity of the light. Let $T_t(c)$ be the subordinator such that

$$\mathbb{E}_\nu^0[e^{-uT_t(c)}] = e^{-t(\sqrt{2c^2u+m^2c^4}-mc^2)}.$$

Thus one can show that

$$\lim_{c\to\infty} \mathbb{E}_\nu^0[f(T_t(c))] = f(\frac{t}{m})$$

for any bounded continuous function f. The right-hand side $f(\frac{t}{m})$ is deterministic. Let

$$h_c = \sqrt{-\Delta + m^2c^4} - mc^2 + V.$$

Thus by (2.18) one can see that

$$\begin{aligned}\lim_{c\to\infty}(f, e^{-th_c}g) &= \lim_{c\to\infty}\int_{\mathbb{R}^3} \mathbb{E}^{x,0}[\overline{f(B_0)}g(B_{T_t(c)})e^{-\int_0^t V(B_{T_s(c)})\mathrm{d}s}]\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f(B_0)}g(B_{\frac{t}{m}})e^{-\int_0^t V(B_{\frac{s}{m}})\mathrm{d}s}]\mathrm{d}x = (f, e^{-t(-\frac{1}{2m}\Delta+V)}g).\end{aligned}$$

Applications mentioned above are available not only for h but also other Schrödinger operators $h(a), h_{\mathrm{R}}(a), h_S(a,b)$ and $h_{\mathrm{SR}}(a,b)$.

3 Pauli-Fierz model

3.1 Newton-Maxwell equation

Consider a single classical particles in the configuration space $\mathbb{R}^3$, interacting with an electromagnetic field. The particle has an assigned mass $m = 1$ and a charge distribution $\hat{\varphi}$, and their dynamics is completely characterized by their momentum $p \in \mathbb{R}^3$ and position $q \in \mathbb{R}^3$. On the other hand, it is convenient for our purposes to describe the electromagnetic field in the Coulomb gauge by a complex vector field:

$$\alpha = (\alpha_\lambda)_{\lambda=1,2} : \mathbb{R}^3 \longrightarrow \mathbb{C}.$$

Thus the Hamiltonian of the Newton–Maxwell system takes the form

$$H(p,q,\alpha) = \frac{1}{2}\big(p - A(q,\alpha)\big)^2 + V(q) + \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \bar{\alpha}_\lambda(k)|k|\alpha_\lambda(k)\mathrm{d}k.$$

This leads to the Newton–Maxwell equations:

$$\begin{cases} \nabla_t p = -\dfrac{\partial H}{\partial q}, \\ \nabla_t q = \dfrac{\partial H}{\partial p}, \\ \mathrm{i}\nabla_t \alpha_\lambda(k) = \dfrac{\partial H}{\partial \bar{\alpha}(k)}. \end{cases}$$

Hence we obtain

$$\text{(NM)} \begin{cases} \nabla_t p = (p - A(q,\alpha)) \cdot \nabla_q A(q,\alpha) - \nabla_q V(q), \\ \nabla_t q = p - A(q,\alpha), \\ \mathrm{i}\nabla_t \alpha_\lambda(k) = |k|\alpha_\lambda(k) - \dfrac{\hat{\varphi}(k)}{\sqrt{2|k|}}(p - A(q,\alpha)) \cdot e_\lambda(k)e^{-\mathrm{i}kq}. \end{cases}$$

The model considered in this paper is the so-called Pauli-Fierz model [116], which is the quantized version of $H(p,q,\alpha)$. In the classical limit, $\hbar \to 0$, of the Pauli-Fierz model leads to the Newton–Maxwell equations. We refer to see [3, 131].

3.2 Pauli-Fierz Hamiltonian

The Hamiltonian of non-relativistic QED is defined as a self-adjoint operator on a Hilbert space. The Pauli-Fierz Hamiltonian describes the minimal interaction between electrons and a quantized radiation field, where electrons are treated as quantum mechanical matters and the number of electrons is fixed at one. Hence the Pauli-Fierz Hamiltonian can be interpreted as a Schrödinger operator of one electron coupled with a quantized radiation field. We refer to see [125, 44, 7] for tools of quantum field theory. Let

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}$$

be the Hilbert space describing the joint electron-photon state vectors. Here

$$\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^3 \times \mathbb{Z}_2))$$

is the boson Fock space over $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. Here the boson Fock space $\mathcal{F}(\mathcal{K})$ over $\mathcal{K}$ is defined by

$$\mathcal{F}(\mathcal{K}) = \bigoplus_{n=0}^{\infty} [\otimes_s^n \mathcal{K}],$$

where $\otimes_s^n$ denotes the n-fold symmetric tensor product. See Appendix A for the fundamental facts on boson Fock spaces. The elements of the set $\mathbb{Z}_2$ account for the fact that a photon is a transversal wave perpendicular to the direction of its propagation, thus it has two components. See Figure 1. The Fock vacuum in $\mathcal{F}$ is defined by

$$\Omega = \mathbb{1} \oplus 0 \oplus 0 \cdots .$$

We identify $\mathcal{H}$ as the set of $\mathcal{F}$-valued L^2-functions on $\mathbb{R}^3$:

$$\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} \mathrm{d}x. \tag{3.1}$$

This will be used without further notice in what follows. Let $a(f)$ and $a^*(f)$ be the annihilation operator and the creation operator on $\mathcal{F}$ smeared by $f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, respectively. Then

$$\begin{aligned} [a(f), a^*(g)] &= \sum_{j=\pm 1} (\bar{f}_j, g_j), \\ [a(f), a(g)] &= 0 = [a^*(f), a^*(g)]. \end{aligned}$$

We use the identification $L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ and set

$$a^\sharp(f, +1) = a^\sharp(f \oplus 0), \quad a^\sharp(f, -1) = a^\sharp(0 \oplus f),$$

where $a^\sharp$ stands for either operator. The finite particle subspace of $\mathcal{F}$ is given by

$$\mathcal{F}_{\rm fin} = \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(m)} = 0 \text{ for all } m > N \text{ with some } N\}.$$

Next we define the quantized radiation field with a cutoff function $\hat{\varphi}$. Put

$$\begin{aligned} \wp_\mu^j(x) &= \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} e_\mu^j(k) e^{-\mathrm{i}kx} \in L^2(\mathbb{R}_k^3), \\ \tilde{\wp}_\mu^j(x) &= \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} e_\mu^j(k) e^{\mathrm{i}kx} \in L^2(\mathbb{R}_k^3), \end{aligned}$$

for $x \in \mathbb{R}^3$, $j = \pm 1$ and $\mu = 1, 2, 3$, where ω is the dispersion relation defined by

$$\omega(k) = |k|.$$

Here $\hat{\varphi}$ is the Fourier transform of the charge distribution φ. The vectors $e^{+1}(k)$ and $e^{-1}(k)$ are called polarization vectors, that is, $e^{+1}(k)$, $e^{-1}(k)$ and $k/|k|$ form a right-hand system at $k \in \mathbb{R}^3$;

$$e^i(k) \cdot e^j(k) = \delta_{ij}, \quad e^j(k) \cdot k = 0, \quad e^{+1}(k) \times e^{-1}(k) = k/|k|.$$

The quantized radiation field with cutoff function $\hat{\varphi}$ is defined by

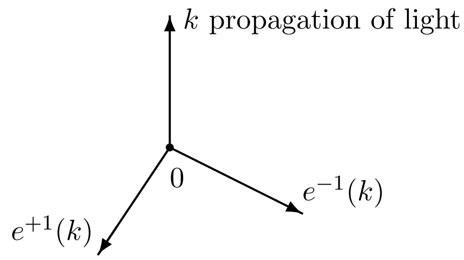

Figure 1: Polarization vector

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{j=\pm 1} (a^*(\wp_\mu^j(x), j) + a(\tilde{\wp}_\mu^j(x), j)), \quad \mu = 1, 2, 3.$$

In the case of $\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$ and $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$, $A_\mu(x)$ is symmetric, and moreover essentially self-adjoint on $\mathcal{F}_{\rm fin}$ for each $x \in \mathbb{R}^3$. We denote the closure of $A_\mu(x)\lceil_{\mathcal{F}_{\rm fin}}$ by the same symbol. A_μ is a self-adjoint operator on

$$D(A_\mu) = \left\{ F \in \mathcal{H} \middle| F(x) \in D(A_\mu(x)) \text{ a.e. } x \text{ and } \int_{\mathbb{R}^3} \|A_\mu(x)F(x)\|_{\mathcal{F}}^2 \mathrm{d}x < \infty \right\}$$

and acts as $(A_\mu F)(x) = A_\mu(x)F(x)$ for $F \in D(A_\mu)$. Since $k \cdot e^j(k) = 0$, it implies the Coulomb gauge condition

$$\nabla_x \cdot A = 0.$$

This in turn yields $\sum_{\mu=1}^3 [\nabla_\mu, A_\mu] = 0$. The formally it is written as

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \int_{\mathbb{R}^3} e_\mu^j(k) \left(\frac{\hat{\varphi}(k)}{\sqrt{|k|}} e^{-\mathrm{i}kx} a^*(k,j) + \frac{\hat{\varphi}(-k)}{\sqrt{|k|}} e^{+\mathrm{i}kx} a(k,j) \right) \mathrm{d}k.$$

Let us explain the term $\frac{\hat{\varphi}(k)}{\sqrt{|k|}}\mathrm{d}k$ in $A_\mu(x)$. Let

$$H_m = \{k \in \mathbb{R}^4 \mid k \cdot \tilde{k} = m^2, k_0 > 0\}$$

be called mass hyperboloids. Here $\tilde{k} = (k_0, -k_1, -k_2, -k_3)$. The Lorentz group $\mathscr{L}$ is the set of linear transformations on $\mathbb{R}^4$ that preserve $k \cdot \tilde{l}$, i.e.,

$$\Lambda \in \mathscr{L} \text{ iff } k \cdot \tilde{l} = \Lambda k \cdot \widetilde{\Lambda l} \quad \forall k, l \in \mathbb{R}^4.$$

The restricted Lorentz group $\mathscr{L}_+^\uparrow$ is the subgroup of $\mathscr{L}$ such that $\Lambda = (\Lambda_{\mu\nu})_{0 \le \mu,\nu \le 3} \in \mathscr{L}_+^\uparrow$ iff $\det\Lambda = 1$ and $\Lambda_{00} > 0$. The mass hyperboloids is invariant under the restricted Lorentz group $\mathscr{L}_+^\uparrow$. Let $V_m : H_m \to \mathbb{R}^3$ for $m > 0$ and $V_0 : H_0 \to \mathbb{R}^3 \setminus \{0\}$ for $m = 0$ be the homeomorphisms defined by $V_m(k_0, k_1, k_2, k_3) = (k_1, k_2, k_3)$. Define

$$\rho(E) = \int_{V_m(E)} \frac{1}{\sqrt{k_1^2 + k_2^2 + k_3^2 + m^2}} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3, \quad E \subset H_m.$$

Then ρ is a measure on H_m. The measure ρ is invariant with respect to all $\Lambda \in \mathscr{L}_+^\uparrow$, i.e., $\rho(\Lambda E) = \rho(E)$. From this the measure

$$\frac{1}{\sqrt{|k|}}\mathrm{d}k$$

on $\mathbb{R}^3$ is called the relativistically covariant measure for $m = 0$. If $\hat{\varphi} \neq 1$, then

$$\frac{\hat{\varphi}(k)}{\sqrt{|k|}}\mathrm{d}k$$

breaks relativistic covariance.

Next we introduce the free field Hamiltonian on $\mathcal{F}$. Let us consider the massless and free relativistic Schrödinger operator $\sqrt{-\Delta}$ on $L^2(\mathbb{R}^3)$. Then the N-body massless and free relativistic Schrödinger operator is given by

$$\sum_{j=1}^{N} \sqrt{-\Delta_j} \tag{3.2}$$

on $L^2(\mathbb{R}^{3N}) \cong \otimes^N L^2(\mathbb{R}^3)$. The free field Hamiltonian H_f on $\mathcal{F}$ is the direct sum of (3.2), and then given in terms of the second quantization of $\omega(k)$:

$$H_\mathrm{f} = \mathrm{d}\Gamma(\omega).$$

It leaves the n-particle subspace $\mathcal{F}^{(n)} = \otimes_s^n L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ invariant, i.e., for $f \in \mathcal{F}^{(n)}$,

$$H_\mathrm{f} f(k_1, \ldots, k_n, j_1, \ldots, j_n) = \left(\sum_{j=1}^{N} \omega(k_j)\right) f(k_1, \ldots, k_n, j_1, \ldots, j_n).$$

See Appendix A.2 for second quantizations.

The Pauli-Fierz model describes the minimal interaction between an electron and the quantized radiation field. The electron is described by the Schrödinger operator

$$H_\mathrm{p} = -\frac{1}{2}\Delta + V$$

in $L^2(\mathbb{R}^3)$. The Hamiltonian for the electron decoupled with the quantized radiation field is given by

$$H_0 = -\frac{1}{2}\Delta \otimes \mathbb{1} + \mathbb{1} \otimes H_\mathrm{f}$$

with domain

$$D(H_0) = D\left(-\frac{1}{2}\Delta \otimes \mathbb{1}\right) \cap D(\mathbb{1} \otimes H_\mathrm{f}).$$

The interaction is obtained by the minimal coupling $-\mathrm{i}\nabla_\mu \otimes \mathbb{1} \mapsto -\mathrm{i}\nabla_\mu \otimes \mathbb{1} - A_\mu$. Then the Pauli-Fierz Hamiltonian is defined by

$$H = \frac{1}{2}(-\mathrm{i}\nabla \otimes \mathbb{1} - A)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_\mathrm{f}. \tag{3.3}$$

We give a remark on the definition of H. When defining the sum of operators, each operator should be defined on the same Hilbert space. For the Pauli-Fierz Hamiltonian, however, $-\mathrm{i}\nabla \otimes \mathbb{1}, V \otimes \mathbb{1}$ and $\mathbb{1} \otimes H_{\mathrm{f}}$ are defined on Hilbert space $L^2(\mathbb{R}^3) \otimes \mathcal{F}$, but A on $\int_{\mathbb{R}^3}^{\oplus} \mathcal{F} \mathrm{d}x$. It confuses readers who see it for the first time. However we respect and follow the traditional definition of the Pauli-Fierz Hamiltonian under the identification of $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ and $\int_{\mathbb{R}^3}^{\oplus} \mathcal{F} \mathrm{d}x$ in (3.1).

It is also known that Hamiltonians defined through different polarizations are unitary equivalent each others. Thus we may fix polarization vectors as they are the most convenient.

In what follows we neglect tensor notation $\otimes$ unless confusions may arise. Then H is simply written as

$$H = \frac{1}{2}(-\mathrm{i}\nabla - A)^2 + V + H_{\mathrm{f}}.$$

We say $V \in \mathcal{R}_{\mathrm{Kato}}$ iff $D(\Delta) \subset D(V)$ and there exist $0 \le a < 1$ and $0 \le b$ such that

$$\|Vf\| \le a \left\| -\frac{1}{2}\Delta f \right\| + b\|f\|$$

for $f \in D(\Delta)$. Moreover we say that $\varphi \in \mathcal{U}$ iff $\varphi \in \mathscr{S}'(\mathbb{R}^3)$ satisfies that $\hat{\varphi}$ is a local L^1-function, $\hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$ and

$$\sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3),\ \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3),\ \hat{\varphi}/\omega \in L^2(\mathbb{R}^3).$$

We give comments on the class $\mathcal{U}$. $\hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$ and $\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$ ensure that A is well-defined and symmetric. $\hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$ implies that A is relatively bounded with respect to $H_{\mathrm{f}}^{1/2}$, and $\sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$ yields $\nabla_\mu A_\nu$ is well-defined.

In this paper unless otherwise stated we suppose the following assumptions.

Assumption 3.1 *$\varphi \in \mathcal{U}$ and $V \in \mathcal{R}_{\mathrm{Kato}}$.*

Under this assumption on V, by the Kato-Rellich theorem [84] one can see that H_{p} is self-adjoint on $D(\Delta)$.

Proposition 3.1 ([61, 63, 105, 52, 35]) *Suppose that Assumption 3.1 holds. Then H is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of H_0.*

Note that condition for the self-adjointness is weakend in [105].

3.3 Pauli-Fierz Hamiltonian in Schrödinger representation

We introduce a Q-space associated with the quantized radiation field and reformulate the Pauli-Fierz Hamiltonian on $L^2(\mathbb{R}^3) \otimes L^2(\mathrm{Q})$ instead of $\mathcal{H}$. This is called Schrödinger representation. Furthermore, we introduce Euclidean quantum fields associated with the Pauli-Fierz Hamiltonian to derive a functional integral representation of e^{-tH}. We refer to see [122, 123] for a Q-space representation of a Fock space and [114, 125] for Euclidean fields.

The following setting is taken from [61]. For a real-valued $f \in L^2(\mathbb{R}^3)$ we set

$$A_\mu(f) = \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \int_{\mathbb{R}^3} e_\mu^j(k)(\hat{f}(k)a^*(k,j) + \hat{f}(-k)a(k,j))\mathrm{d}k.$$

With this notation we write $A_\mu(x) = A_\mu(\tilde{\varphi}(\cdot - x))$, where $\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})\check{}$. The relations

$$(\Omega, A_\mu(f)\Omega) = 0, \tag{3.4}$$

$$(\Omega, A_\mu(f)A_\nu(g)\Omega) = \frac{1}{2}(\hat{f}, \delta^\perp_{\mu\nu}\hat{g}) \tag{3.5}$$

are immediate. Here $\delta^\perp_{\mu\nu}(k)$ is the transversal delta function defined by

$$\delta^\perp_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}, \quad \mu, \nu = 1, 2, 3,$$

and we used the identity:

$$\sum_{j=1,2} e^j_\mu(k) e^j_\nu(k) + \frac{k_\mu k_\nu}{|k|^2} = \delta_{\mu\nu}.$$

Let $\delta^\perp = (\delta^\perp_{\mu\nu})_{1 \le \mu,\nu \le 3}$ be the 3×3 matrix:

$$\delta^\perp(k) = \begin{pmatrix} 1 - \dfrac{k_1^2}{|k|^2} & -\dfrac{k_1 k_2}{|k|^2} & -\dfrac{k_1 k_3}{|k|^2} \\ -\dfrac{k_2 k_1}{|k|^2} & 1 - \dfrac{k_2^2}{|k|^2} & -\dfrac{k_2 k_3}{|k|^2} \\ -\dfrac{k_3 k_1}{|k|^2} & -\dfrac{k_3 k_2}{|k|^2} & 1 - \dfrac{k_3^2}{|k|^2} \end{pmatrix}.$$

In order to construct the functional integral representation of $(F, e^{-tH}G)$ we prepare probability spaces $(\mathrm{Q}_\#, \Sigma_\#, \mu_\#)$, $\# = 0, 1$, and Gaussian random variables $\hat{A}_\#(f)$ indexed by $f \in \bigoplus^3 L^2_{\mathbb{R}}(\mathbb{R}^{3+\#})$ with mean zero:

$$\mathbb{E}_{\mu_\#}[\hat{A}_\#(f)] = 0,$$

and the covariance:

$$\mathbb{E}_{\mu_\#}[\hat{A}_\#(f)\hat{A}_\#(g)] = \mathrm{q}_\#(f, g).$$

Here the bilinear forms $\mathrm{q}_\#$ on $(\bigoplus^3 L^2_{\mathbb{R}}(\mathbb{R}^{3+\#})) \times (\bigoplus^3 L^2_{\mathbb{R}}(\mathbb{R}^{3+\#}))$ are defined by

$$\mathrm{q}_0(f, g) = \frac{1}{2}(\hat{f}, \delta^\perp \hat{g}), \tag{3.6}$$

$$\mathrm{q}_1(F, G) = \frac{1}{2}(\hat{F}, \delta^\perp \hat{G}). \tag{3.7}$$

Note that $\hat{F} = \hat{F}(k_0, k)$, $\hat{G} = \hat{G}(k_0, k)$ but $\delta^\perp = \delta^\perp(k)$ for $(k_0, k) \in \mathbb{R} \times \mathbb{R}^3$. The definitions of (3.6) and (3.7) are motivated by (3.4) and (3.5). It is established that there exist a probability space $(\mathrm{Q}_\#, \Sigma_\#, \mu_\#)$ and a family of Gaussian random variables $(\hat{A}_\#(f), f \in \bigoplus^3 L_{\mathbb{R}}(\mathbb{R}^{3+\#}))$ such that the mean is zero and the covariance is given by $\mathrm{q}_\#$. Define the μth component of $\hat{A}_\#$ by

$$\hat{A}_{\#,\mu}(f) = \hat{A}_\#\left(\bigoplus_{\nu=1}^3 \delta_{\mu\nu} f\right), \quad f \in L^2(\mathbb{R}^{3+\#}).$$

We shall set $\hat{A} = \hat{A}_0$, $\mathrm{q} = \mathrm{q}_0$, $\mathrm{Q} = \mathrm{Q}_0$, $\mu_M = \mu_0$, and $\mathbb{A} = \hat{A}_1$, $\mathrm{q_E} = \mathrm{q}_1$, $\mathrm{Q_E} = \mathrm{Q}_1$, $\mu_\mathrm{E} = \mu_1$, It is known that

$$\mathrm{LH}\left\{ :\prod_{j=1}^{n} \hat{A}_{\#}(f_j):, \mathbb{1} \;\middle|\; f_j \in \bigoplus^{3} L^2(\mathbb{R}^{3+\#}), j=1,...,n, n \in \mathbb{N} \right\} \tag{3.8}$$

is dense in $L^2(\mathrm{Q}_{\#})$ Here $:\ldots:$ denotes the Wick product. Thus

$$\int_{\mathrm{Q}_{\#}} :\prod_{j=1}^{n} \hat{A}_{\#}(f_j)::\prod_{i=1}^{m} \hat{A}_{\#}(g_i): \mathrm{d}\mu_{\#} = 0, \quad n \neq m.$$

We set (3.8) as $L^2_{\mathrm{fin}}(\mathrm{Q}_{\#})$. We define the second quantization $\Gamma_{\#\#'}(T)$. Here and in what

$$\begin{tikzcd}
L^2(\mathbb{R}^{3+\#}) \arrow[r, "T"] \arrow[d, dotted] & L^2(\mathbb{R}^{3+\#'}) \arrow[d, dotted] \\
L^2(\mathrm{Q}_{\#}) \arrow[r, "\Gamma_{\#\#'}(T)"'] & L^2(\mathrm{Q}_{\#'})
\end{tikzcd}$$

Figure 2: Fanctor $\Gamma_{\#\#'}$

follows for the operator $S : L^2(\mathbb{R}^{3+\#}) \to L^2(\mathbb{R}^{3+\#'})$ we use the same notation S as for the operator $\bigoplus^3 S : \bigoplus^3 L^2(\mathbb{R}^{3+\#}) \to \bigoplus^3 L^2(\mathbb{R}^{3+\#'})$, $(f_1, f_2, f_3) \mapsto (Sf_1, Sf_2, Sf_3)$ for notational convenience, and we write simply $\hat{A}_{\#}(Tf)$ for $\hat{A}_{\#}((\bigoplus^3 T)f)$ etc. Let $\Gamma_{\#\#'}(T)\mathbb{1} = \mathbb{1}$ and

$$\Gamma_{\#\#'}(T):\prod_{i=1}^{n} \hat{A}_{\#}(f_i): = :\prod_{i=1}^{n} \hat{A}_{\#'}(Tf_i):.$$

If T is a contraction operator, then so is $\Gamma_{\#\#'}(T)$. Thus $\Gamma_{\#\#'}$ is a functor between sets of contraction operators $\{T : L^2(\mathbb{R}^{3+\#}) \to L^2(\mathbb{R}^{3+\#'})\}$. We write $\Gamma_{00} = \Gamma : L^2(\mathrm{Q}) \to L^2(\mathrm{Q})$, $\Gamma_{11} = \Gamma_{\mathrm{E}} : L^2(\mathrm{Q}_{\mathrm{E}}) \to L^2(\mathrm{Q}_{\mathrm{E}})$ and $\Gamma_{01} = \Gamma_{\mathrm{Int}} : L^2(\mathrm{Q}) \to L^2(\mathrm{Q}_{\mathrm{E}})$.

$$\begin{tikzcd}
L^2(\mathbb{R}^3) \arrow[r, "\mathrm{j}_t"] \arrow[d, "i_{-1/2}"'] \arrow[dr, phantom, "\circlearrowright"] & L^2(\mathbb{R}^4) \arrow[d, "i_{-1}"] \\
\dot{H}_{-1/2}(\mathbb{R}^3) \arrow[r, "\tau_t = \delta_t \otimes \cdot"'] & \dot{H}_{-1}(\mathbb{R}^4)
\end{tikzcd}$$

Figure 3: Isometries j_t between $L^2(\mathbb{R}^3)$ and $L^2(\mathbb{R}^4)$

We introduce a family of isometries connecting the Minkowski quantum field and the Euclidean quantum field. Let $\dot{H}_{\nu}(\mathbb{R}^n)$ be the homogeneous Sobolev space:

$$\dot{H}_{\nu}(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^3) \mid \hat{f}(k)|k|^{\nu/2} \in L^2(\mathbb{R}^3)\}.$$

Thus

$$\tau_t : \dot{H}_{-1/2}(\mathbb{R}^3) \to \dot{H}_{-1}(\mathbb{R}^4)$$

is defined by $\tau_t f = \delta_t \otimes f$. Then $\|\tau_t f\|_{\dot{H}_{-1}(\mathbb{R}^4)} = \|f\|_{\dot{H}_{-1/2}(\mathbb{R}^3)}$. Let $i_{-1/2} : L^2(\mathbb{R}^3) \to \dot{H}_{-1/2}(\mathbb{R}^3)$ and $i_{-1} : L^2(\mathbb{R}^4) \to \dot{H}_{-1}(\mathbb{R}^4)$ be given by

$$\begin{aligned}\widehat{i_{-1/2}f}(k) &= \sqrt{\omega(k)}\hat{f}(k),\\ \widehat{i_{-1}f}(k_0,k) &= \sqrt{\omega(k)^2+|k_0|^2}\hat{f}(k_0,k).\end{aligned}$$

In [61] we define the family of isometries $\mathrm{j}_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^4)$ by

$$\mathrm{j}_t = (i_{-1})^{-1} \circ \tau_t \circ i_{-1/2}, \quad t \in \mathbb{R}.$$

See Figure 3. We see that

$$\mathrm{j}_s^*\mathrm{j}_t = e^{-|s-t|\hat{\omega}},$$

where $\hat{\omega} = \omega(-\mathrm{i}\nabla)$. See Figure 4.

$$\begin{tikzcd} L^2(\mathbb{R}^3) \arrow[r, "\mathrm{j}_t"] \arrow[d, "e^{-|s-t|\hat{\omega}}"'] & L^2(\mathbb{R}^4) \arrow[dl, "\mathrm{j}_s^*"] \arrow[dl, phantom, "\circlearrowright", shift right=6] \\ L^2(\mathbb{R}^3) & \end{tikzcd}$$

Figure 4: Decomposition of $e^{-|s-t|\hat{\omega}}$

$$\begin{tikzcd} L^2(\mathrm{Q}) \arrow[r, "\mathrm{J}_t"] \arrow[d, "e^{-|s-t|H_{\mathrm{f}}}"'] & L^2(\mathrm{Q}_{\mathrm{E}}) \arrow[dl, "\mathrm{J}_s^*"] \arrow[dl, phantom, "\circlearrowright", shift right=6] \\ L^2(\mathrm{Q}) & \end{tikzcd}$$

Figure 5: Decomposition of $e^{-|s-t|H_{\mathrm{f}}}$

Define the family of isometries $\mathrm{J}_t : L^2(\mathrm{Q}) \to L^2(\mathrm{Q}_{\mathrm{E}})$ by the second quantization of j_t, i.e.,

$$\mathrm{J}_t = \Gamma_{\mathrm{Int}}(\mathrm{j}_t).$$

Let K be a self-adjoint operator in $L^2(\mathbb{R}^3)$. Then $\{\Gamma(e^{\mathrm{i}tK}) : t \in \mathbb{R}\}$ turns to be the strongly continuous one-parameter unitary group. Then the Stone's theorem tells us that there exists a unique self-adjoint operator $\mathrm{d}\Gamma(K)$ such that

$$\Gamma(e^{\mathrm{i}tK}) = e^{\mathrm{i}t\mathrm{d}\Gamma(K)}, \quad t \in \mathbb{R}.$$

Let

$$\mathrm{d}\Gamma(\hat{\omega}) = \hat{H}_{\mathrm{f}}.$$

Since $e^{-tH_{\mathrm{f}}} = \Gamma(e^{-t\hat{\omega}})$, $e^{-t\hat{H}_{\mathrm{f}}}$ can be decomposed as

$$\mathrm{J}_t^*\mathrm{J}_s = e^{-|s-t|\hat{H}_{\mathrm{f}}}.$$

See Figure 5. Isometry J_t plays an important role in functional integral representations, which connects Minkowskian quantum fields with Euclidean quantum fields. Let $\hat{h} = h(-\mathrm{i}\nabla)$ be the self-adjoint operator with a real-valued symbol h. Then $\mathrm{J}_t\Gamma(e^{-\mathrm{i}\hat{h}}) = \Gamma_{\mathrm{E}}(e^{-\mathrm{i}\hat{h}\otimes\mathbb{1}})\mathrm{J}_t$ and $\mathrm{J}_t\mathrm{d}\Gamma(\hat{h}) = \mathrm{d}\Gamma_{\mathrm{E}}(\hat{h}\otimes\mathbb{1})\mathrm{J}_t$ hold. See Figure 6.

$$\begin{tikzcd} L^2(\mathrm{Q}) \arrow[r, "\mathrm{J}_t"] \arrow[d, "\Gamma(e^{-\mathrm{i}\hat{h}})"'] \arrow[dr, phantom, "\circlearrowright"] & L^2(\mathrm{Q}_\mathrm{E}) \arrow[d, "\Gamma_\mathrm{E}(e^{-\mathrm{i}\hat{h}\otimes\mathbb{1}})"] \\ L^2(\mathrm{Q}) \arrow[r, "\mathrm{J}_t"'] & L^2(\mathrm{Q}_\mathrm{E}) \end{tikzcd}$$

Figure 6: Intertwining properties

3.4 Wiener-Itô-Segal isomorphism between $\mathcal{F}$ and $L^2(\mathrm{Q})$

The Wiener-Itô-Segal isomorphism U is an isomorphism between $L^2(\mathrm{Q})$ and $\mathcal{F}$. Let us define $U : \mathcal{F} \to L^2(\mathrm{Q})$ by

$$\begin{aligned} &U\Omega = \mathbb{1}, \\ &U{:}\prod_{i=1}^{n} A(f_i){:}\Omega = {:}\prod_{i=1}^{n} \hat{A}(f_i){:}. \end{aligned}$$

Thus U becomes a unitary operator from $\mathcal{F}$ to $L^2(\mathrm{Q})$. We denote $\mathbb{1}\otimes U : \mathcal{H} \to L^2(\mathbb{R}^3)\otimes L^2(\mathrm{Q})$ by U for simplicity. Note that the inverse Fourier transform of $g(k,x) = e^{-\mathrm{i}kx}\hat{\varphi}(k)/\sqrt{\omega(k)}$ equals $\check{g}(y,x) = \tilde{\varphi}(y-x)$, where $\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})\check{} \in L^2(\mathbb{R}^3)$. Notice that the test function of $A_\mu(f)$ is $\hat{f}$ but not f.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "A(f)"] \arrow[d, "U"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F} \arrow[d, "U"] \\ L^2(\mathrm{Q}) \arrow[r, "\hat{A}(f)"'] & L^2(\mathrm{Q}) \end{tikzcd}$$

Figure 7: Wiener-Itô-Segal isomorphism U

As seen above, the isometry J_t connects $L^2(\mathrm{Q})$ with $L^2(\mathrm{Q}_\mathrm{E})$. We can also see the intertwining properties:

$$\mathrm{J}_t\hat{A}(f) = \mathbb{A}(\mathrm{j}_t f)\mathrm{J}_t.$$

See Figure 8. This property is very important to construct functional integral representations for Pauli-Fierz type Hamiltonians.

Now we define the Pauli-Fierz Hamiltonian in the Schrödinger representation. The relations

$$\begin{aligned} &UA_\mu(x)U^{-1} = \hat{A}_\mu(\tilde{\varphi}(\cdot - x)), \\ &UH_\mathrm{f}U^{-1} = \hat{H}_\mathrm{f}, \end{aligned}$$

follow directly. As a result

$$UHU^{-1} = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + V + \hat{H}_\mathrm{f},$$

$$\begin{tikzcd}
L^2(\mathrm{Q}) \arrow[r, "\hat{A}(f)"] \arrow[d, "\mathrm{J}_t"'] \arrow[dr, phantom, "\circlearrowright"] & L^2(\mathrm{Q}) \arrow[d, "\mathrm{J}_t"] \\
L^2(\mathrm{Q}_\mathrm{E}) \arrow[r, "\mathbb{A}(\mathrm{j}_t f)"'] & L^2(\mathrm{Q}_\mathrm{E})
\end{tikzcd}$$

Figure 8: $\hat{A}(f)$ and $\mathbb{A}(f)$

with $\hat{A} = (\hat{A}_1, \hat{A}_2, \hat{A}_3)$. In what follows we use notation H for UHU^{-1} and H_f for $\hat{H}_\mathrm{f}$. The Pauli-Fierz Hamiltonian in Schrödinger representation is defined by

$$H = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + V + H_\mathrm{f}$$

in $L^2(\mathbb{R}^3) \otimes L^2(\mathrm{Q})$. Unless confusions arise we also denote $L^2(\mathbb{R}^3) \otimes L^2(\mathrm{Q})$ by $\mathcal{H}$, and we set $\mathcal{H}_\mathrm{E} = L^2(\mathbb{R}^3) \otimes L^2(\mathrm{Q}_\mathrm{E})$.

3.5 Hilbert space-valued stochastic integrals

We define a Hilbert space-valued stochastic integral. It will be first explained in some generality and then applied to the Pauli-Fierz Hamiltonian. Let $\mathcal{K}$ be a Hilbert space and define

$$C^n(\mathbb{R}^3; \mathcal{K}) = \big\{ f : \mathbb{R}^3 \to \mathcal{K} \,|\, f \text{ is } n \text{ times strongly continuously differentiable} \big\}$$

and

$$C^n_\mathrm{b}(\mathbb{R}^3; \mathcal{K}) = \big\{ f \in C^n(\mathbb{R}^3; \mathcal{K}) \big| \sup_{|z| \leq n, x \in \mathbb{R}^3} \|\nabla^z f(x)\|_\mathcal{K} < \infty \big\},$$

where $|z| = z_1 + z_2 + z_3$ for $z = (z_1, z_2, z_3)$ and $\nabla^z = \nabla^{z_1}_{x_1} \nabla^{z_2}_{x_2} \nabla^{z_3}_{x_3}$ denotes strong derivative. We set $L^2(\mathscr{X}) = L^2(\mathscr{X}, \mathrm{d}\mathcal{W}^x)$. The proof of the following lemma is straightforward and similar to the case of real-valued processes. Let $f \in C^1_\mathrm{b}(\mathbb{R} \times \mathbb{R}^3; \mathcal{K})$. The sequence defined by

$$\mathrm{J}^\mu_n(f) = \sum_{j=1}^{2^n} f\left(\frac{j-1}{2^n} t, B_{\frac{j-1}{2^n} t} \right) \left(B^\mu_{\frac{j}{2^n} t} - B^\mu_{\frac{j-1}{2^n} t} \right)$$

is a Cauchy sequence in $L^2(\mathscr{X}) \otimes \mathcal{K}$. For $f \in C^1_\mathrm{b}(\mathbb{R} \times \mathbb{R}^3; \mathcal{K})$ the limit

$$\int_0^t f(s, B_s) \mathrm{d}B^\mu_s = s\!-\!\lim_{n \to \infty} \mathrm{J}^\mu_n(f)$$

defines a $\mathcal{K}$-valued stochastic integral. By the above definition

$$\mathbb{E}\left[\left(\int_0^t f(s, B_s) \mathrm{d}B^\mu_s, \int_0^t g(s, B_s) \mathrm{d}B^\nu_s \right)_\mathcal{K} \right] = \delta_{\mu\nu} \mathbb{E}\left[\int_0^t (f(s, B_s), g(s, B_s))_\mathcal{K} \mathrm{d}s \right]$$

holds. $\mathrm{J}_n^\mu(f)$ is defined by the sum of f evaluated at the left endpoints. Then $f(\frac{j-1}{2^n}t, B_{\frac{j-1}{2^n}t})$ and $B^\mu_{\frac{j}{2^n}t} - B^\mu_{\frac{j-1}{2^n}t}$ are independent, while we define $S_n^\mu(f)$ below by evaluating at both of the left and right endpoints. Let $f \in C_{\mathrm{b}}^2(\mathbb{R}^3;\mathcal{K})$ and

$$S_n^\mu(f) = \sum_{j=1}^{2^n} \left(f(B_{\frac{j}{2^n}t}) + f(B_{\frac{j-1}{2^n}t}) \right) \left(B^\mu_{\frac{j}{2^n}t} - B^\mu_{\frac{j-1}{2^n}t} \right).$$

Similarly to the case of real-valued processes we can see that

$$s-\lim_{n\to\infty} S_n^\mu(f) = \int_0^t f(B_s)\mathrm{d}B_s^\mu + \frac{1}{2}\int_0^t \nabla_\mu f(B_s)\mathrm{d}s$$

holds in $L^2(\mathscr{X}) \otimes \mathcal{K}$. Thus extra term $\frac{1}{2}\int_0^t \nabla_\mu f(B_s)\mathrm{d}s$ appears.

We will construct the functional integral through the Euclidean quantum field, and will use an $L^2(\mathbb{R}^4)$-valued stochastic integral of the form

$$\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu. \tag{3.9}$$

However, since

$$\frac{\|\mathrm{j}_s f - \mathrm{j}_t f\|^2}{|s-t|^2} = 2\left(\hat{f}, \frac{(1-e^{-|s-t|\omega})}{|s-t|}\hat{f}\right)\frac{1}{|s-t|}$$

diverges as $t \to s$ for $\hat{f} \in D(\omega)$, $\mathbb{R}\times\mathbb{R}^3 \ni (s,x) \mapsto \mathrm{j}_s\tilde{\varphi}(\cdot - x) \in L^2(\mathbb{R}^4)$ is not strongly differentiable in $s \in \mathbb{R}$. Then $\mathrm{j}_s\tilde{\varphi}(\cdot - x) \notin C_{\mathrm{b}}^1(\mathbb{R}_s \times \mathbb{R}_x^3; L^2(\mathbb{R}^4))$. Therefore we need to give a proper definition of (3.9).

Lemma 3.2 *If $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbb{R}^4)$, then for each $\mu = 1,2,3$,*

$$S_n^\mu(\lambda) = \sum_{j=1}^{2^n} \mathrm{j}_{\frac{(j-1)t}{2^n}}\lambda\left(\cdot - B_{\frac{(j-1)t}{2^n}}\right)\left(B^\mu_{\frac{jt}{2^n}} - B^\mu_{\frac{(j-1)t}{2^n}}\right), \quad n = 1,2,3,\ldots,$$

is a Cauchy sequence in $L^2(\mathscr{X}) \otimes L^2(\mathbb{R}^4)$.

Proof: Fix an μ. Write $S_n = S_n^\mu(\lambda)$ and $\eta_* = \mathrm{j}_*\lambda(\cdot - B_*)$. Then

$$S_{n+1} - S_n = \sum_{m=1}^{2^n}\left(\eta_{\frac{(2m-1)t}{2^{n+1}}} - \eta_{\frac{(2m-2)t}{2^{n+1}}}\right)\left(B^\mu_{\frac{2mt}{2^{n+1}}} - B^\mu_{\frac{(2m-1)t}{2^{n+1}}}\right).$$

Hence

$$\mathbb{E}^x[\|S_{n+1} - S_n\|^2] = \sum_{m=1}^{2^n}\mathbb{E}^x\left[\left\|\eta_{\frac{(2m-1)t}{2^{n+1}}} - \eta_{\frac{(2m-2)t}{2^{n+1}}}\right\|^2\right]\frac{t}{2^{n+1}}.$$

We have

$$\left\|\eta_{\frac{(2m-1)t}{2^{n+1}}} - \eta_{\frac{(2m-2)t}{2^{n+1}}}\right\|^2 \le \|\omega\hat{\lambda}\|^2\left|B^\mu_{\frac{(2m-1)t}{2^{n+1}}} - B^\mu_{\frac{(2m-2)t}{2^{n+1}}}\right|^2 + \frac{t}{2^n}\|\lambda\|\|\omega\hat{\lambda}\|.$$

We conclude that

$$\left(\mathbb{E}^x[\|S_m - S_n\|^2]\right)^{1/2} \le \left(\frac{2\|\lambda\|\|\omega\hat{\lambda}\| + \|\omega\hat{\lambda}\|^2}{2}\right)^{1/2} \sum_{j=n+1}^{m} \frac{t}{2^{(j+1)/2}},$$

thus $\{S_n\}_{n=1}^\infty$ is a Cauchy sequence.

QED

Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbb{R}^4)$. Then

$$\int_0^t \mathrm{j}_s \lambda(\cdot - B_s)\mathrm{d}B_s^\mu = s\text{-}\lim_{n\to\infty} S_n^\mu, \quad \mu = 1,2,3,$$

defines an $L^2(\mathbb{R}^4)$-valued stochastic integral, where the strong limit is in the strong topology of $L^2(\mathscr{X}) \otimes L^2(\mathbb{R}^4)$. By the definition it is seen that

$$\mathbb{E}^x\left[\left(\int_0^t \mathrm{j}_s\lambda(\cdot - B_s)\mathrm{d}B_s^\mu, \int_0^t \mathrm{j}_s\rho(\cdot - B_s)\mathrm{d}B_s^\nu\right)\right] = t\delta_{\mu\nu}(\lambda,\rho).$$

3.6 Functional integral representations for the Pauli-Fierz Hamiltonian

Using the Trotter product formula and the factorization formula $e^{-|s-t|H_{\mathrm{f}}} = \mathrm{J}_t^*\mathrm{J}_s$, we derive a functional integral representation of e^{-tH}. A combination of the functional integral representation of $e^{-tH_{\mathrm{p}}}$ and the equality $e^{-tH_{\mathrm{f}}} = \mathrm{J}_0^*\mathrm{J}_t$ gives

$$(F, e^{-tH_0}G)_{\mathcal{H}} = (\mathrm{J}_0F, e^{-tH_{\mathrm{p}}}\mathrm{J}_tG)_{\mathcal{H}_{\mathrm{E}}} = \int_{\mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s}(\mathrm{J}_0F(B_0), \mathrm{J}_tG(B_t))_{L^2(\mathrm{Q_E})}]\mathrm{d}x.$$

The next theorem is due to [60]. See also [40, 130], and [104] for great extensions.

Theorem 3.3 *We suppose that $V \in L^\infty(\mathbb{R}^3)$. Then*

$$(F, e^{-tH}G) = \int_{\mathbb{R}^3} \mathbb{E}^x\left[e^{-\int_0^t V(B_s)\mathrm{d}s}(\mathrm{J}_0F(B_0), e^{-\mathrm{i}\mathbb{A}(K_t)}\mathrm{J}_tG(B_t))_{L^2(\mathrm{Q_E})}\right]\mathrm{d}x. \tag{3.10}$$

Here K_t denotes the $\bigoplus^3 L^2(\mathbb{R}^3)$-valued stochastic integral given by

$$K_t = \bigoplus_{\mu=1}^3 \int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu.$$

Proof: Firstly we assume that $V = 0$, and we write $\mathbb{A}_s(x) = \mathbb{A}(\bigoplus^3 \mathrm{j}_s\tilde{\varphi}(\cdot - x))$ in this proof. Define the family of symmetric contraction operators $P_s : \mathcal{H} \to \mathcal{H}$ by

$$P_sF(x) = \int_{\mathbb{R}^3} \Pi_s(x-y)e^{\mathrm{i}H(x,y)}F(y)\mathrm{d}y, \quad s > 0,$$

with $P_0F = F$, where $\Pi_s(x) = \frac{1}{(2\pi s)^{3/2}}\exp(-|x|^2/2s)$ is the heat kernel and

$$H(x,y) = \frac{1}{2}(\hat{A}(x) + \hat{A}(y)) \cdot (x-y).$$

By a direct computation

$$(F,(P_{\frac{t}{2^n}})^{2^n}G)=\int_{\mathbb{R}^3}\mathrm{d}x\int_{(\mathbb{R}^3)^{2^n}}\overline{F(x)}e^{\mathrm{i}\sum_{j=1}^{2^n}H_j}G(x_{2^n})\left(\prod_{j=1}^{2^n}\Pi_{\frac{t}{2^n}}(x_{j-1}-x_j)\right)\prod_{j=1}^{2^n}\mathrm{d}x_j$$

with $x=x_0$, $H_j=H(x_{j-1},x_j)$. This can be expressed by using Brownian motion as

$$(F,(P_{\frac{t}{2^n}})^{2^n}G)=\int_{\mathbb{R}^3}\mathbb{E}^x[(F(B_0),e^{-\mathrm{i}\hat{A}(L_n)}G(B_t))]\mathrm{d}x,$$

where $L_n=\bigoplus_{\mu=1}^{3}\sum_{m=0}^{2^n}(\tilde{\varphi}(\cdot-B_{t\frac{m}{2^n}})+\tilde{\varphi}(\cdot-B_{t\frac{m-1}{2^n}}))(B^{\mu}_{t\frac{m}{2^n}}-B^{\mu}_{t\frac{m-1}{2^n}})$. It is seen that

$$L_n\to L_t=\bigoplus_{\mu=1}^{3}\int_0^t\tilde{\varphi}(\cdot-B_s)\mathrm{d}B_s^{\mu}$$

as $n\to\infty$ strongly in $\bigoplus^3\left(L^2(\mathscr{X})\otimes L^2(\mathbb{R}^3)\right)$. Here we used the Coulomb gauge condition. This implies that for $F,G\in\mathcal{H}_{\mathrm{E}}$,

$$\lim_{n\to\infty}(F,(P_{\frac{t}{2^n}})^{2^n}G)=\int_{\mathbb{R}^3}\mathbb{E}^x[(F(B_0),e^{-\mathrm{i}\hat{A}(L_t)}G(B_t))]\mathrm{d}x. \tag{3.11}$$

From (3.11) it follows that $|\lim_{n\to\infty}(F,(P_{\frac{t}{2^n}})^{2^n}G)|\le\|F\|\|G\|$. Hence for each $t\ge0$ there exists a symmetric bounded operator S_t such that

$$\lim_{n\to\infty}(F,(P_{\frac{t}{2^n}})^{2^n}G)=(F,S_tG).$$

Since $(P_{\frac{t}{2^n}})^{2^n}$ is uniformly bounded as $\|(P_{\frac{t}{2^n}})^{2^n}\|\le1$, the above weak convergence improves to

$$s\!-\!\lim_{n\to\infty}(P_{\frac{t}{2^n}})^{2^n}=S_t,\quad t\ge0. \tag{3.12}$$

Furthermore, by (3.11)

$$(F,S_tG)=\int_{\mathbb{R}^3}\mathbb{E}^x[(F(B_0),e^{-\mathrm{i}\hat{A}(L_t)}G(B_t))]\mathrm{d}x. \tag{3.13}$$

Putting these together we can show that $\{S_t:t\ge0\}$ is a symmetric C_0-semigroup, thus there exists a unique self-adjoint operator K such that $S_t=e^{-tK}$, $t\ge0$. Let

$$H(\hat{A})=\frac{1}{2}(-\mathrm{i}\nabla-\hat{A})^2.$$

We have

$$\lim_{t\to\infty}(F,t^{-1}(\mathbb{1}-P_t)G)=(F,H(\hat{A})G) \tag{3.14}$$

for $F,G\in C_0^\infty(\mathbb{R}^3)\otimes L^2_{\mathrm{fin}}(\mathrm{Q})$ in Lemma 3.4 below. This leads to

$$(t^{-1}(e^{-tK}-\mathbb{1})F,G)=\lim_{n\to\infty}(t^{-1}((P_{\frac{t}{2^n}})^{2^n}-\mathbb{1})F,G)=-\int_0^1(H(\hat{A})F,e^{-tsH(\hat{A})}G)\mathrm{d}s. \tag{3.15}$$

In the second equality above we used (3.14). (3.15) can be immediately extended to vectors $F \in D(H_0)$ and $G \in D(K)$. Take $t \downarrow 0$ on both sides of (3.15). Then it holds that $(F, KG) = (H(\hat{A})F, G)$ for $F \in D(H_0)$ and $G \in D(K)$, which implies that $K \supset H(\hat{A})\lceil_{D(H_0)}$ as $H(\hat{A})$ is self-adjoint. Define

$$\hat{H} = K \dotplus H_{\mathrm{f}},$$

where $\dotplus$ denotes the quadratic form sum [85]. The Trotter product formula for quadratic form sums [88, 87] and the factorization $e^{-\frac{t}{n}H_{\mathrm{f}}} = \mathrm{J}^*_{\frac{kt}{n}}\mathrm{J}_{\frac{(k+1)t}{n}}$ yield that

$$(F, e^{-t\hat{H}}G) = \lim_{n\to\infty}\left(\mathrm{J}_0 F, \left(\prod_{i=0}^{n-1} R_i\right)\mathrm{J}_t G\right),$$

where $R_j = \mathrm{J}_{\frac{jt}{n}} e^{-\frac{t}{n}K}\mathrm{J}^*_{\frac{jt}{n}}$. Using the definition of e^{-tK} we get

$$\mathrm{J}_s e^{-tK}\mathrm{J}_s^* G(x) = s\text{-}\lim_{n\to\infty}\int_{(\mathbb{R}^3)^{2^n}} \mathrm{J}_s e^{\mathrm{i}\sum_{j=1}^{2^n} H_j}\mathrm{J}_s^* G(x_{2^n})\left(\prod_{j=1}^{2^n}\Pi_{\frac{t}{2^n}}(x_{j-1}-x_j)\right)\prod_{j=1}^{2^n}\mathrm{d}x_j$$

with $x = x_0$. Write $\delta_j = \delta_j(n, t, n_j) = t/(n2^{n_j})$ for $j = 0, 1, \ldots, n-1$ and define $P_{s,j} : \mathcal{H}_{\mathrm{E}} \to \mathcal{H}_{\mathrm{E}}$ by P_s with $H(x, y)$ replaced by the Euclidean version $\mathbb{H}_{\frac{jt}{n}}(x, y)$ given by

$$\mathbb{H}_s(x, y) = \frac{1}{2}\left(\mathbb{A}_s(x) + \mathbb{A}_s(y)\right)\cdot(x - y)$$

with $\mathbb{A}_s(x) = \mathbb{A}(\mathrm{j}_s\tilde{\varphi}(\cdot - x))$ as

$$P_{s,j}F(x) = \int_{\mathbb{R}^3}\Pi_s(x-y)e^{\mathrm{i}\mathbb{H}_{\frac{jt}{n}}(x,y)}F(y)\mathrm{d}y.$$

We have

$$\left(\mathrm{J}_0 F, \left(\prod_{i=0}^{n-1} R_i\right)\mathrm{J}_t G\right) = \lim_{n_0\to\infty}\cdots\lim_{n_{n-1}\to\infty}\left(\mathrm{J}_0 F, \left(\prod_{i=0}^{n-1}(P_{\delta_i,i})^{2^{n_i}}\right)\mathrm{J}_t G\right).$$

Here we used the Markov property [115, 113, 125] of the projection $E_s = \mathrm{J}_0\mathrm{J}_s^*$. As a result we have

$$\left(\mathrm{J}_0 F, \left(\prod_{i=0}^{n-1} R_i\right)\mathrm{J}_t G\right) = \lim_{n_0\to\infty}\cdots\lim_{n_{n-1}\to\infty}\int_{\mathbb{R}^3}\mathbb{E}^x[(\mathrm{J}_0 F(B_0), e^{-\mathrm{i}\mathbb{A}(K)}\mathrm{J}_t G(B_t))]\mathrm{d}x$$

with $K = K(n_0, n_1, \ldots, n_{n-1}, n)$ given by

$$K = \bigoplus_{\mu=1}^{3}\sum_{j=0}^{n-1}\sum_{m=1}^{2^{n_j}}\mathrm{j}_{\frac{jt}{n}}(\tilde{\varphi}(\cdot - B_{\frac{t}{n}(\frac{m}{2^{n_j}}+j)}) + \tilde{\varphi}(\cdot - B_{\frac{t}{n}(\frac{m-1}{2^{n_j}}+j)}))(B^\mu_{\frac{t}{n}(\frac{m}{2^{n_j}}+j)} - B^\mu_{\frac{t}{n}(\frac{m-1}{2^{n_j}}+j)}).$$

Note that

$$K \to \bigoplus_{\mu=1}^{3}\sum_{j=0}^{n-1}\int_{\frac{jt}{n}}^{t(j+1)/n}\mathrm{j}_{\frac{jt}{n}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu$$

as $n_0, n_1, \ldots, n_{n-1} \to \infty$ in $\bigoplus^3(L^2(\mathscr{X}) \otimes L^2(\mathbb{R}^3))$. Finally as $n \to \infty$ we have

$$(F, e^{-t\hat{H}}G) = \int_{\mathbb{R}^3} \mathbb{E}^x[(\mathrm{J}_0 F(B_0), e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t))]\mathrm{d}x. \tag{3.16}$$

By the construction of $\hat{H}$,

$$\hat{H} \supset \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + H_{\mathrm{f}}\lceil_{D(H_0)}, \tag{3.17}$$

but Proposition 3.1 yields that

$$\hat{H} = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + H_{\mathrm{f}}$$

follows, and $\hat{H}$ is self-adjoint on $D(H_0)$. A functional integral representation of $e^{-t\hat{H}}$ including non-zero V can be obtained by the Trotter product formula

$$(F, e^{-t\hat{H}}G) = \lim_{n\to\infty}(F, (e^{-\frac{t}{n}K} e^{-\frac{t}{n}V} e^{-\frac{t}{n}H_{\mathrm{f}}})^n G).$$

This completes the proof.

QED

It remains to show (3.14).

Lemma 3.4 *It follows (3.14), i.e.,*

$$\lim_{t\to\infty}(F, t^{-1}(\mathbb{1} - P_t)G) = (F, H(\hat{A})G)$$

for $F, G \in C_0^\infty(\mathbb{R}^3) \otimes L^2_{\mathrm{fin}}(\mathrm{Q})$.

Proof: It is directly seen that

$$\frac{d}{ds}(F, P_s G) = \int_{\mathbb{R}^3} \mathrm{d}x \int_{\mathbb{R}^3} \Pi_s(x-y) \left(F(x), \frac{1}{2}\Delta_y e^{\mathrm{i}H(x,y)} G(y) \right) \mathrm{d}y.$$

We have

$$(F(x), \Delta_{y_\mu} e^{\mathrm{i}H(x,y)} G(y)) = \mathrm{I}^+_\mu(x,y) + \mathrm{I}^-_\mu(x,y),$$

where

$$\begin{aligned}
\mathrm{I}^+_\mu(x,y) = &\left(F(x), e^{\mathrm{i}H(x,y)} \left(\hat{A}^\mu(y)\mathrm{i}\nabla_{y_\mu} + \frac{1}{2}\mathrm{i}\hat{A}^{\mu\mu}(y) + \frac{1}{2}\hat{A}^\mu(y)(\hat{A}_\mu(x) + \hat{A}_\mu(y)) \right) G(y) \right)(x-y) \\
&- \frac{1}{4}(F(x), e^{\mathrm{i}H(x,y)}(\hat{A}^\mu(y)(x-y))^2 G(y)), \\
\mathrm{I}^-_\mu(x,y) = &\left(F(x), e^{\mathrm{i}H(x,y)} \left(\Delta_{y_\mu} - (\hat{A}_\mu(x) + \hat{A}_\mu(y))\mathrm{i}\nabla_{y_\mu} - \frac{1}{4}(\hat{A}_\mu(x) + \hat{A}_\mu(y))^2 \right) G(y) \right).
\end{aligned}$$

Here $\hat{A}^\mu(y) = \nabla_{y_\mu}\hat{A}(y)$ and $\hat{A}^{\mu\mu}(y) = \nabla^2_{y_\mu}\hat{A}(y)$. We have

$$\lim_{t\to 0} \int_{\mathbb{R}^3} \mathrm{d}x \int_{\mathbb{R}^3} \Pi_t(x-y)\mathrm{I}^+_\mu(x,y)\mathrm{d}y = 0.$$

It can be also seen that the map $y \mapsto \mathrm{I}_\mu^-(x,y)$ is continuous for each $x \in \mathbb{R}^3$ and it follows that $\lim_{y\to x} \mathrm{I}_\mu^-(x,y) = -(F(x), (-\mathrm{i}\nabla_{x_\mu} - A_\mu(x))^2 G(x))$. Hence we have

$$\lim_{t\to 0} \int_{\mathbb{R}^3} \mathrm{d}x \int_{\mathbb{R}^3} \Pi_t(x-y)\mathrm{I}_\mu^-(x,y)\mathrm{d}y = -\int_{\mathbb{R}^3} (F(x), (-\mathrm{i}\nabla_{x_\mu} - A_\mu)^2 G(x))\mathrm{d}x.$$

Together with them we have

$$\lim_{t\to\infty} \frac{1}{t}(F, (\mathbb{1} - P_t)G) = -\lim_{t\to 0+} \frac{d}{dt}(F, P_t G) = (F, H(\hat{A})G).$$

Thus the lemma follows.

QED

In Theorem 3.3 we assumed boundedness of external potentials. We can offer an extension of the functional integral representation to a wider potential class.

Theorem 3.5 ([60]) *Take*

$$H = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + H_\mathrm{f} \dot{+} V_+ \dot{-} V_-$$

for $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *and* V_- *which is* $-\frac{1}{2}\Delta$ *form bounded with relative bound* < 1. *Then* (3.10) *holds for* H.

Proof: The proof is a minor modification of [127, Theorem 6.2]. The following proof is taken from [60]. Let

$$V_{+n}(x) = \begin{cases} V_+(x), & V_+(x) < n, \\ n, & V_+(x) \geq n, \end{cases} \quad V_{-m}(x) = \begin{cases} V_-(x), & V_-(x) < m, \\ m, & V_-(x) \geq m. \end{cases}$$

Set $V_{n,m} = V_{+n} - V_{-m}$ and $h = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + H_\mathrm{f}$. Then

$$(F, e^{-t(H+V_{n,m})}G) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V_{n,m}(B_s)\mathrm{d}s} \Big(F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t) \Big) \right] \mathrm{d}x. \tag{3.18}$$

Define the closed quadratic forms

$$\begin{aligned} \mathrm{q}_{n,m}(F,F) &= (h^{1/2}F, h^{1/2}F) + (V_{+n}^{1/2}F, V_{+n}^{1/2}F) - (V_{-m}^{1/2}F, V_{-m}^{1/2}F), \\ \mathrm{q}_{n,\infty}(F,F) &= (h^{1/2}F, h^{1/2}F) + (V_{+n}^{1/2}F, V_{+n}^{1/2}F) - (V_-^{1/2}F, V_-^{1/2}F), \\ \mathrm{q}_{\infty,\infty}(F,F) &= (h^{1/2}F, h^{1/2}F) + (V_+^{1/2}F, V_+^{1/2}F) - (V_-^{1/2}F, V_-^{1/2}F), \end{aligned}$$

whose form domains are $Q(\mathrm{q}_{n,m}) = Q(h)$, $Q(\mathrm{q}_{n,\infty}) = Q(h)$ and $Q(\mathrm{q}_{\infty,\infty}) = Q(h) \cap Q(V_+)$. Note that

$$\mathrm{q}_{n,m} \downarrow \mathrm{q}_{n,\infty}, \quad m \uparrow \infty$$

and

$$\mathrm{q}_{n,\infty} \uparrow \mathrm{q}_{\infty,\infty}, \quad n \uparrow \infty.$$

By [85, VIII. Theorem 3.11], $\mathrm{q}_{n,\infty}$ is a closed quadratic form, and we can conclude that for all $t \geq 0$,

$$\exp\left(-t\left(h + V_{+n} - V_{-m}\right)\right) \to \exp\left(-t\left(h + V_{+n} \dot{-} V_{-}\right)\right) \tag{3.19}$$

as $m \to \infty$. By [85, VIII. Theorem 3.13] and [126], $\mathrm{q}_{\infty,\infty}$ is also a closed quadratic form, and

$$\exp(-t(h \dot{+} V_{+n} \dot{-} V_{-})) \to \exp(-t(h \dot{+} V_{+} \dot{-} V_{-})) \tag{3.20}$$

as $n \to \infty$ strongly. By taking first $n \to \infty$ and then $m \to \infty$ it can be proven that both sides of (3.18) converge. I.e., the left-hand side of (3.18) converges by (3.19) and (3.20). We have the inequality

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V_{n,m}(B_s)\mathrm{d}s} \left| \left(F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t) \right) \right| \right] \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V_{n,\infty}(B_s)\mathrm{d}s} (|F(B_0)|, e^{-tH_\mathrm{f}} |G(B_t)|) \right] \mathrm{d}x < \infty.
\end{aligned}$$

Here the finiteness of the second term can be derived. Since

$$|e^{-\int_0^t V_{n,m}(B_s)\mathrm{d}s} (F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t))| \leq e^{-\int_0^t V_{n,\infty}(B_s)\mathrm{d}s} (|F(B_0)|, e^{-tH_\mathrm{f}} |G(B_t)|),$$

the dominated convergence theorem yields that

$$\begin{aligned}
&\lim_{m\to\infty} \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V_{n,m}(B_s)\mathrm{d}s} \left(F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t) \right) \right] \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V_{n,\infty}(B_s)\mathrm{d}s} (F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t)) \right] \mathrm{d}x.
\end{aligned}$$

Furthermore since

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V_{n,\infty}(B_s)\mathrm{d}s} \left| \left(F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t) \right) \right| \right] \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V_{n,\infty}(B_s)\mathrm{d}s} (|F(B_0)|, e^{-tH_\mathrm{f}} |G(B_t)|) \right] \mathrm{d}x < \infty
\end{aligned}$$

and

$$\left| e^{-\int_0^t V_{n',\infty}(B_s)\mathrm{d}s} \left(F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t) \right) \right| \leq e^{-\int_0^t V_{n,\infty}(B_s)\mathrm{d}s} (|F(B_0)|, e^{-tH_\mathrm{f}} |G(B_t)|)$$

for $n \leq n'$, the dominated convergence theorem again yields that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V_{n,\infty}(B_s)\mathrm{d}s} \left(F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t) \right) \right] \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} (F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t)) \right] \mathrm{d}x.
\end{aligned}$$

Together with them the right-hand side of (3.18) converges to

$$\int_{\mathbb{R}^3} \mathbb{E}^x [e^{-\int_0^t V(B_s)\mathrm{d}s} (F(B_0), \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t))] \mathrm{d}x$$

for any $F, G \in D(H_0)$ as first $m \to \infty$ and then $n \to \infty$. Then the proof is complete.

QED

3.7 Pauli-Fierz Hamiltonian with Kato-class potential

We consider the Pauli-Fierz Hamiltonian with Kato-class potential V. This section is due to [54]. We introduce the assumption below.

Assumption 3.2 *$\varphi \in \mathcal{U}$ and V is Kato-decomposable.*

Firstly we are interested in defining H with a Kato-class potential as a self-adjoint operator. This will be done through the functional integral representation established in the previous section. We have

$$(e^{-tH}G)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s} \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t G(B_t)]. \tag{3.21}$$

Conversely, we shall show that a sufficient condition to define the right-hand side of (3.21) is that V is of Kato-class. The idea used for Schrödinger operators with Kato-class potentials [23] can be extended to the Pauli-Fierz Hamiltonian.

Let V be a Kato-decomposable potential and define the family of operators

$$(\mathrm{K}_t F)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_r)\mathrm{d}r} \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t F(B_t)].$$

Lemma 3.6 *Suppose Assumption 3.2. Then K_t is bounded on $\mathcal{H}$.*

Proof: Let $F \in \mathcal{H}$. By Schwarz inequality we have

$$\|\mathrm{K}_t F\|_{\mathcal{H}}^2 \le \int_{\mathbb{R}^3} \mathbb{E}^0[e^{-2\int_0^t V(B_r+x)\mathrm{d}r}] \mathbb{E}^0[\|F(B_t+x)\|^2]\mathrm{d}x.$$

Since V is of Kato-class, we have $\sup_{x\in\mathbb{R}^3} \mathbb{E}^0[e^{-2\int_0^t V(B_r+x)\mathrm{d}r}] = C < \infty$, and thus

$$\|\mathrm{K}_t F\|_{\mathcal{H}}^2 \le C\|F\|_{\mathcal{H}}^2.$$

QED

We shall show that $\{\mathrm{K}_t : t \ge 0\}$ is a symmetric C_0-semigroup. To do that we introduce a time shift operator u_t on $L^2(\mathbb{R}^4)$ by

$$u_t f(x_0, x) = f(x_0 - t, x), \quad x = (x_0, x) \in \mathbb{R} \times \mathbb{R}^3.$$

It is straightforward that $u_t^* = u_{-t}$ and $u_t^* u_t = 1$. We denote the second quantization of u_t by $U_t = \Gamma_{\mathrm{E}}(u_t)$ which acts on $L^2(\mathrm{Q_E})$ and is unitary. It follows that $u_t \mathrm{j}_s = \mathrm{j}_{s+t}$ for every $t, s \in \mathbb{R}$. It derives the formula $U_t \mathrm{J}_s = \mathrm{J}_{s+t}$ (Figure 9).

Lemma 3.7 *Suppose Assumption 3.2. Then $\mathrm{K}_s\mathrm{K}_t = \mathrm{K}_{s+t}$ holds true for $s, t \ge 0$. Moreover $t \mapsto \mathrm{K}_t$ is strongly continuous and $\mathrm{K}_0 = 1\!\!1$.*

Proof: By the definition of K_t we have

$$\mathrm{K}_s\mathrm{K}_t F = \mathbb{E}^x \left[e^{-\int_0^s V(B_r)\mathrm{d}r} \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_s)} \mathrm{J}_s \mathbb{E}^{B_s}[e^{-\int_0^t V(B_r)\mathrm{d}r} \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t F(B_t)] \right]. \tag{3.22}$$

By the formulae $\mathrm{J}_s \mathrm{J}_0^* = \mathrm{E}_s U_{-s}^*$ (Figure 11) and $\mathrm{J}_t = U_{-s}\mathrm{J}_{t+s}$, we see that (3.22) is equal to

$$\mathbb{E}^x \left[e^{-\int_0^s V(B_r)\mathrm{d}r} \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_s)} \mathrm{E}_s \mathbb{E}^{B_s}[e^{-\int_0^t V(B_r)\mathrm{d}r} U_{-s}^* e^{-\mathrm{i}\mathbb{A}(K_t)} U_{-s} \mathrm{J}_{t+s} F(B_t)] \right]. \tag{3.23}$$

$$
\begin{tikzcd}
L^2(\mathrm{Q}) \arrow[r, "\mathrm{J}_s"] \arrow[d, "\mathrm{J}_{s+t}"'] & L^2(\mathrm{Q}_\mathrm{E}) \arrow[dl, "U_t"] \arrow[dl, phantom, bend right, "\circlearrowright"] \\
L^2(\mathrm{Q}_\mathrm{E}) &
\end{tikzcd}
$$

Figure 9: Shift by U_t

Since U_s is unitary, we have

$$U_{-s}^* e^{-\mathrm{i}\mathbb{A}(K_t)} U_{-s} = e^{-\mathrm{i}\mathbb{A}(u_{-s}^* K_t)}$$

as an operator. The test function of the exponent $u_{-s}^* K_t$ is given by

$$u_{-s}^* K_t = \bigoplus_{\mu=1}^{3} \int_0^t \mathrm{j}_{r+s} \tilde{\varphi}(\cdot - B_r) \mathrm{d}B_r^\mu.$$

Moreover by the Markov property of E_t, $t \in \mathbb{R}$, we can neglect E_s in (3.23), and by the Markov property of $(B_t)_{t\geq 0}$ we have

$$
\begin{aligned}
(\mathrm{K}_s \mathrm{K}_t F)(x) &= \mathbb{E}^x[e^{-\int_0^s V(B_r)\mathrm{d}r} \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_s)} \mathbb{E}^x[e^{-\int_s^{s+t} V(B_r)\mathrm{d}r} e^{-\mathrm{i}\mathbb{A}(K_s^{s+t})} \mathrm{J}_{s+t} F(B_{s+t}) | \mathscr{F}_s]] \\
&= \mathbb{E}^x[e^{-\int_0^{s+t} V(B_r)\mathrm{d}r} \mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_{s+t})} \mathrm{J}_{s+t} F(B_{s+t})] = K_{s+t} F,
\end{aligned}
$$

where we recall that $(\mathscr{F}_t)_{t\geq 0}$ denotes the natural filtration of $(B_t)_{t\geq 0}$. The second statement can be proven immediately.

QED

$$
\begin{tikzcd}
L^2(\mathrm{Q}_\mathrm{E}) \arrow[r, "\mathrm{J}_s^*"] \arrow[dr, "E_s"'] & L^2(\mathrm{Q}) \arrow[d, "\mathrm{J}_s"] \arrow[dl, phantom, bend left, "\circlearrowright"] \\
& L^2(\mathrm{Q}_\mathrm{E})
\end{tikzcd}
$$

Figure 10: Projection E_s

$$
\begin{tikzcd}
L^2(\mathrm{Q}_\mathrm{E}) \arrow[r, "\mathrm{J}_0^*"] \arrow[d, "U_{-s}^*"'] \arrow[dr, phantom, "\circlearrowright"] & L^2(\mathrm{Q}) \arrow[d, "\mathrm{J}_s"] \\
L^2(\mathrm{Q}_\mathrm{E}) \arrow[r, "E_s"'] & L^2(\mathrm{Q}_\mathrm{E})
\end{tikzcd}
$$

Figure 11: E_s and U_s

Theorem 3.8 *Suppose Assumption 3.2. Then* $\{\mathrm{K}_t : t \geq 0\}$ *is a symmetric* C_0*-semigroup.*

Proof: It was shown that $\{\mathrm{K}_t : t \geq 0\}$ is a C_0-semigroup. Hence it is enough to show that K_t is symmetric for each $t \geq 0$. Recall that $\mathrm{R} = \Gamma(r)$ is the second quantization of the reflection r, where $r : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)$ such that $rf(x_0, x) = f(-x_0, x)$ for $(x_0, x) \in \mathbb{R} \times \mathbb{R}^3$. We have

$$(F, \mathrm{K}_t G) = (U_t \mathrm{R} F, U_t \mathrm{R} \mathrm{K}_t G) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} \left(\mathrm{J}_t F(B_0), e^{-\mathrm{i}\mathbb{A}(u_t r K_t)} \mathrm{J}_0 G(B_t) \right) \right] \mathrm{d}x.$$

Notice that $u_t r K_t = \int_0^t \mathrm{j}_{t-s}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu$. Note that $\dot{B}_s = B_{t-s} - B_t$, $0 \le s \le t$ is also Brownian motion. Exchanging integrals $\int_{\mathscr{X}} \mathrm{d}\mathcal{W}^0$ and $\int_{\mathbb{R}^3} \mathrm{d}x$ and changing the variable x to $y + B_t$, we can see

$$(F, \mathrm{K}_t G) = \mathbb{E}^0 \left[\int_{\mathbb{R}^3} e^{-\int_0^t V(B_{t-s}+y)\mathrm{d}s} \Big(\mathrm{J}_t F(B_t + y), e^{+\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_0 G(y) \Big) \right] \mathrm{d}y.$$

Then

$$(F, \mathrm{K}_t G) = \int_{\mathbb{R}^3} \mathbb{E}^y \left[e^{-\int_0^t V(B_s)\mathrm{d}s} \Big(\mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t F(y + B_t), G(y) \Big) \right] \mathrm{d}y = (\mathrm{K}_t F, G)$$

and K_t is symmetric. Then the proof is complete.

QED

By Theorem 3.8 and the Stone's theorem for semigroups there exists a self-adjoint operator K such that

$$\mathrm{K}_t = e^{-t\mathrm{K}}, \quad t \ge 0.$$

3.8 Positivity improving

It is known that second quantizations and positivity improving are deeply related. We refer to see [129, 125, 43, 48, 124, 36, 100, 101, 108, 109] for positivity improving, and this section is due to [62]. Let T be a contraction operator on $L^2(\mathbb{R}^3)$. Then it is known that $\Gamma(T)$ is positivity preserving in $L^2(\mathrm{Q})$. I.e.,

$$(\Phi, \Gamma(T)\Psi) \ge 0, \quad \Phi \ge 0, \ \Psi \ge 0.$$

Furthermore let $f : \mathbb{R}^3 \to \mathbb{R}$ be a non-negative function such that the Lebesgue measure of $\{k \in \mathbb{R}^3 \mid f(k) = 0\}$ is zero. Then $\Gamma(e^{-f(-\mathrm{i}\nabla)})$ is positivity improving on $L^2(\mathrm{Q})$. I.e.,

$$(\Phi, \Gamma(e^{-f(-\mathrm{i}\nabla)})\Psi) \gneq 0, \quad \Phi \ge 0, \ \Psi \ge 0.$$

In particular $\mathrm{J}_0^* \mathrm{J}_t = e^{-tH_\mathrm{f}} = \Gamma(e^{-t\hat{\omega}})$ is positivity improving on $L^2(\mathrm{Q})$. See Figure 12. We are interested in asking if $\mathrm{J}_0^* X \mathrm{J}_t$ is positivity improving or not for some X.

Let $\mathfrak{S}_\# = \exp(-\mathrm{i}\frac{\pi}{2} N_\#)$, where N denotes the Number operator on $L^2(\mathrm{Q})$ and N_E that on $L^2(\mathrm{Q}_\mathrm{E})$. $\mathfrak{S}$ (resp. $\mathfrak{S}_\mathrm{E}$) is a unitary operator on $L^2(\mathrm{Q})$ (resp. $L^2(\mathrm{Q}_\mathrm{E})$). Then we can show that $\mathfrak{S}_\mathrm{E}^{-1} e^{-\mathrm{i}\mathbb{A}(f)} \mathfrak{S}_\mathrm{E}$ is a shift operator on $L^2(\mathrm{Q}_\mathrm{E})$. In particular, it is positivity preserving on $L^2(\mathrm{Q}_\mathrm{E})$. Since J_t and J_0^* are positivity preserving, we see that $\mathrm{J}_0^* \mathfrak{S}_\mathrm{E}^{-1} e^{-\mathrm{i}\mathbb{A}(f)} \mathfrak{S}_\mathrm{E} \mathrm{J}_t$ is positivity preserving on $L^2(\mathrm{Q})$. We can show a stronger statement below. See Figure 13.

Proposition 3.9 ([62]) $\mathrm{J}_0^* \mathfrak{S}_\mathrm{E}^{-1} e^{-\mathrm{i}\mathbb{A}(f)} \mathfrak{S}_\mathrm{E} \mathrm{J}_t$ *is positivity improving in* $L^2(\mathrm{Q}_\mathrm{E})$.

Corollary 3.10 ([62]) $\mathfrak{S}^{-1} e^{-tH} \mathfrak{S}$ *is positivity improving.*

Proof: Let $F, G \in \mathcal{H}$ be non-negative but not identically zero. We have

$$\begin{aligned}(F, \mathfrak{S}^{-1} e^{-tH} \mathfrak{S} G) &= \int_{\mathbb{R}^3} \mathbb{E}^x [e^{-\int_0^t V(B_s)\mathrm{d}s} (F(B_0), \mathrm{J}_0^* \mathfrak{S}_\mathrm{E}^{-1} e^{-\mathrm{i}\mathbb{A}(K_t)} \mathfrak{S}_\mathrm{E} \mathrm{J}_t G(B_t))] \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \mathbb{E}^0 [e^{-\int_0^t V(B_s + x)\mathrm{d}s} (F(x), \mathrm{J}_0^* \mathfrak{S}_\mathrm{E}^{-1} e^{-\mathrm{i}\mathbb{A}(K_t(x))} \mathfrak{S}_\mathrm{E} \mathrm{J}_t G(B_t + x))] \mathrm{d}x,\end{aligned}$$

$$
\begin{tikzcd}
L^2(\mathrm{Q}) \arrow[r, "\mathrm{J}_t"] \arrow[d, "e^{-tH_{\mathrm{f}}}=\mathrm{PI}"'] & L^2(\mathrm{Q}_{\mathrm{E}}) \arrow[dl, "\mathrm{J}_0^*"] \\
L^2(\mathrm{Q}) \arrow[ur, phantom, "\circlearrowright"] &
\end{tikzcd}
$$

Figure 12: Positivity improving I

$$
\begin{tikzcd}
L^2(\mathrm{Q}) \arrow[r, "\mathrm{J}_t"] \arrow[d, "\mathrm{PI}"'] \arrow[dr, phantom, "\circlearrowright"] & L^2(\mathrm{Q}_{\mathrm{E}}) \arrow[d, "\mathfrak{S}^{-1}e^{-\mathrm{i}\mathbb{A}(f)}\mathfrak{S}"] \\
L^2(\mathrm{Q}) & L^2(\mathrm{Q}_{\mathrm{E}}) \arrow[l, "\mathrm{J}_0^*"]
\end{tikzcd}
$$

Figure 13: Positivity improving II

where $K_t(x) = \int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s - x)\mathrm{d}B_s$.

$$f(x,w) = e^{-\int_0^t V(B_s)\mathrm{d}s}(F(x), \mathrm{J}_0^*\mathfrak{S}_{\mathrm{E}}^{-1}e^{-\mathrm{i}\mathbb{A}(K_t(x))}\mathfrak{S}_{\mathrm{E}}\mathrm{J}_t G(B_t+x))$$

is a function with respect to $(x,w) \in \mathbb{R}^3 \times \mathscr{X}$. We can see that there exists a measurable set $M \times B \subset \mathbb{R}^3 \times \mathscr{X}$ such that $\int_{M\times B} \mathrm{d}\mathcal{W}\mathrm{d}x > 0$ and $f(x,w) > 0$ for $(x,w) \in M \times B$ from Proposition 3.9. Hence $(F, \mathfrak{S}^{-1}e^{-tH}\mathfrak{S}G) = \int_{\mathbb{R}^3} \mathbb{E}^0[f]\mathrm{d}x \geq \int_{M\times B} f\mathrm{d}\mathcal{W}\mathrm{d}x > 0$ and the corollary is proven.

QED

Corollary 3.11 ([62]) *If H has a ground state Ψ_{g}. Then $\mathfrak{S}^{-1}\Psi_{\mathrm{g}}$ is strictly positive.*

Proof: This follows from Corollary 3.10 and the Perron Frobenius theorem.

QED

3.9 Baker-Campbell-Hausdorff formula and Fock representation

In Sections 3.6 and 3.7 we obtain the functional integral representation of the semigroup generated by H in the Schrödinger representation. In this section we show a functional integral representation of e^{-tH} in the Fock representation by applying Baker-Campbell-Hausdorff formula. Let $a_{\mathrm{E}}^\sharp$ be the annihilation operator and the creation operator acting in $\mathcal{F}(L^2(\mathbb{R}^4) \oplus L^2(\mathbb{R}^4))$. Let

$$\tilde{\varphi}_j = (\tilde{\varphi}_{1j}, \tilde{\varphi}_{2j}, \tilde{\varphi}_{3j}), \quad j = \pm 1,$$

where $\tilde{\varphi}_{\mu j}$ denotes the inverse Fourier transform of $e_\mu^j \frac{\hat{\varphi}}{\sqrt{\omega}}$. Let

$$\hat{\mathrm{j}}_s = F^{-1}\mathrm{j}_s F, \tag{3.24}$$

where F denotes the Fourier transform on $L^2(\mathbb{R}^3)$. We note that following identification:

$$\hat{A}(K_t) \cong \frac{1}{\sqrt{2}}\left(a_{\mathrm{E}}^*(M_t) + a_{\mathrm{E}}(\tilde{M}_t)\right),$$

where

$$M_t = \bigoplus_{j=\pm1} \int_0^t \hat{\mathrm{j}}_s \frac{\hat{\varphi}}{\sqrt{\omega}} e^{-\mathrm{i}kB_s} e^j \cdot \mathrm{d}B_s,$$

$$\tilde{M}_t = \bigoplus_{j=\pm1} \int_0^t \hat{\mathrm{j}}_s \frac{\tilde{\hat{\varphi}}}{\sqrt{\omega}} e^{+\mathrm{i}kB_s} e^j \cdot \mathrm{d}B_s.$$

We have the theorem below.

Theorem 3.12 *(3.10) can be represented as*

$$(F, e^{-tH}G) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} e^{-\mathfrak{K}_t} (F(B_0), e^{a^*(L_t)} e^{-tH_\mathrm{f}} e^{a(\tilde{L}_t)} G(B_t))_{L^2(\mathrm{Q})} \right] \mathrm{d}x. \tag{3.25}$$

Here

$$L_t = -\frac{\mathrm{i}}{\sqrt{2}} \bigoplus_{j=\pm 1} \int_0^t e^{-|t-s|\omega(k)} \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} e^{-\mathrm{i}kB_s} e^j(k) \cdot \mathrm{d}B_s, \tag{3.26}$$

$$\tilde{L}_t = -\frac{\mathrm{i}}{\sqrt{2}} \bigoplus_{j=\pm 1} \int_0^t e^{-|s|\omega(k)} \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} e^{+\mathrm{i}kB_s} e^j(k) \cdot \mathrm{d}B_s \tag{3.27}$$

and

$$\mathfrak{K}_t = \frac{1}{4} \sum_{j=\pm 1} \left\| \int_0^t \mathrm{j}_s \tilde{\varphi}_j(\cdot - B_s) \cdot \mathrm{d}B_s \right\|^2_{L^2(\mathbb{R}^4)} = \frac{1}{2} \mathrm{q}_\mathrm{E}(K_t, K_t). \tag{3.28}$$

Proof: By Baker-Campbell-Hausdorff formula we can see that

$$\mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t = \mathrm{J}_0^* e^{-\mathfrak{K}_t} e^{-\mathrm{i}a_\mathrm{E}^*(M_t)} e^{-\mathrm{i}a_\mathrm{E}(\tilde{M}_t)} \mathrm{J}_t = e^{-\mathfrak{K}_t} e^{-\mathrm{i}a^*(\mathrm{j}_0^* M_t)} e^{-tH_\mathrm{f}} e^{-\mathrm{i}a(\mathrm{j}_t^* \tilde{M}_t)}.$$

It can be seen that $-\mathrm{i}\mathrm{j}_t^* M_t = L_t$ and $-\mathrm{i}\mathrm{j}_0^* \tilde{M}_t = \tilde{L}_t$. Together with them we obtain the desired results.

QED

The exponent $\mathfrak{K}_t$ is formally written as

$$\mathfrak{K}_t \stackrel{formal}{=} \frac{1}{4} \sum_{1 \le \mu,\nu \le 3} \int_0^t \mathrm{d}B_s^\mu \int_0^t \mathrm{d}B_r^\nu \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} e^{-|s-r|\omega(k)} e^{-\mathrm{i}k(B_s - B_r)} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2} \right) \mathrm{d}k.$$

This expression is formal. The double stochastic integral $\int_0^t \int_0^t \ldots \mathrm{d}B_s^\mu \mathrm{d}B_r^\nu$ is delicate. We discuss this in [15]. In fact it is derived in [15, Proposition 3.1] that

$$\mathfrak{K}_t = \frac{1}{4} \sum_{1 \le \mu,\nu \le 3} \int_{\mathbb{R}^3} \mathrm{d}k \frac{|\hat{\varphi}(k)|^2}{\omega(k)} \int_0^t e^{-\mathrm{i}kB_s} \mathrm{d}B_s^\mu \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2} \right) \int_0^s e^{-(s-r)\omega(k)} e^{\mathrm{i}kB_r} \mathrm{d}B_r^\nu.$$

3.10 Translation invariant Pauli-Fierz Hamiltonian

We consider the translation invariant Pauli-Fierz Hamiltonian, which is obtained by setting the external potential V identically zero, resulting in the fact that H commutes with the total momentum operator. This section is due to [64].

We begin with defining a fiber Hamiltonian. As said, a standing assumption throughout this section is Assumption 3.2 but $V = 0$. Put

$$\mathrm{P}_{\mathrm{f}\mu} = \mathrm{d}\Gamma(k_\mu), \quad \mu = 1, 2, 3,$$

which describes the field momentum. The total momentum operator P^T on $\mathcal{H}$ is defined by the sum of the momentum operator for the particle and that of field:

$$\mathrm{P}_\mu^\mathrm{T} = -\mathrm{i}\nabla_\mu + \mathrm{P}_{\mathrm{f}\mu}, \quad \mu = 1, 2, 3.$$

It follows that

$$[H, \mathrm{P}_\mu^\mathrm{T}] = 0, \quad \mu = 1, 2, 3. \tag{3.29}$$

Remark 3.1 *We give a comment on (3.29). Both H and P^{T} are unbounded, and then (3.29) holds on $D(H\mathrm{P}^{\mathrm{T}}) \cap D(H\mathrm{P}^{\mathrm{T}})$. Precisely we can show that $[e^{-tH}, e^{-\mathrm{i}s\mathrm{P}^{\mathrm{T}}}] = 0$ for $t \geq 0$ and $s \in \mathbb{R}$ on the whole Hilbert space by a functional integral representation.*

This leads to a decomposition of H with respect to the spectrum of the total momentum operator $\sigma(\mathrm{P}^{\mathrm{T}}_\mu) = \mathbb{R}$. The Pauli-Fierz Hamiltonian with total momentum $p \in \mathbb{R}^3$ is defined by

$$H(p) = \frac{1}{2}(p - \mathrm{P}_{\mathrm{f}} - A(0))^2 + H_{\mathrm{f}}, \quad p \in \mathbb{R}^3,$$

with domain $D(H(p)) = D(H_{\mathrm{f}}) \cap D(\mathrm{P}_{\mathrm{f}}^2)$, where $A_\mu(0) = A_\mu(x = 0)$. We give a relationship between H and $H(p)$. Define the unitary operator

$$\mathscr{T} : L^2(\mathbb{R}_x) \otimes \mathcal{F} \to L^2(\mathbb{R}_p) \otimes \mathcal{F}$$

by

$$\mathscr{T} = (\hat{F} \otimes 1\!\!1) \int_{\mathbb{R}^3}^{\oplus} \exp(ix\mathrm{P}_{\mathrm{f}}) \mathrm{d}x, \tag{3.30}$$

with $\hat{F}$ denoting the Fourier transformation from $L^2(\mathbb{R}_x)$ to $L^2(\mathbb{R}_p)$. For $\Psi \in \mathcal{H}$,

$$(\mathscr{T}\Psi)(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}xp} e^{\mathrm{i}x\mathrm{P}_{\mathrm{f}}} \Psi(x) \mathrm{d}x.$$

Theorem 3.13 *For each $p \in \mathbb{R}^3$, $H(p)$ is a non-negative self-adjoint operator, and*

$$\int_{\mathbb{R}^3}^{\oplus} \mathcal{F} \mathrm{d}p \cong \mathcal{H}, \quad \int_{\mathbb{R}^3}^{\oplus} H(p) \mathrm{d}p \cong H.$$

Here the unitary equivalence is implemented by $\mathscr{T}$.

The self-adjointness of $H(p)$ ensures that $\{e^{-tH(p)} : t \geq 0\}$ is a symmetric C_0-semigroup. As in the previous section, we transform $H(p)$ from the Fock representation to the Schrödinger representation in order to construct a functional integral representation. Then $H(p)$ becomes

$$H(p) = \frac{1}{2}(p - \mathrm{P}_{\mathrm{f}} - \hat{A}(0))^2 + H_{\mathrm{f}}$$

on $L^2(\mathrm{Q})$, where $\hat{A}(0) = \hat{A}(x = 0)$. We use the same notations $H(p)$, H_{f} and P_{f} in both the Fock representation and the Schrödinger representation. Recall that $\mathrm{P}_{\mathrm{f}} = \mathrm{d}\Gamma(-\mathrm{i}\nabla)$ and $H_{\mathrm{f}} = \mathrm{d}\Gamma(\omega(-\mathrm{i}\nabla))$. The functional integral representation of $e^{-tH(p)}$ can be also constructed as an application of that of e^{-tH}.

Theorem 3.14 ([64]) *Let $\Psi, \Phi \in L^2(\mathrm{Q})$. Then*

$$(\Psi, e^{-tH(p)}\Phi) = \mathbb{E}^0[(\mathrm{J}_0\Psi, e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t e^{+\mathrm{i}(p-\mathrm{P}_{\mathrm{f}})B_t} \Phi)_{L^2(\mathrm{Q_E})}]. \tag{3.31}$$

In particular

$$e^{-tH(p)}\Phi = \mathbb{E}^0[\mathrm{J}_0^* e^{-\mathrm{i}\mathbb{A}(K_t)} \mathrm{J}_t e^{+\mathrm{i}(p-\mathrm{P}_{\mathrm{f}})B_t} \Phi]. \tag{3.32}$$

Proof: Write $F_s = \Pi_s \otimes \Psi$ and $G_r = \Pi_r \otimes \Phi$, where Π_s is the 3D-heat kernel:

$$\Pi_s(x) = \frac{1}{(2\pi s)^{3/2}} \exp(-|x|^2/2s).$$

Due to the fact that $H = \mathscr{T}(\int_{\mathbb{R}^3}^{\oplus} H(p)\mathrm{d}p)\mathscr{T}^{-1}$, we have the key identity:

$$(F_s, e^{-tH} e^{-\mathrm{i}\xi \mathrm{P}^{\mathrm{T}}} G_r) = \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi p}((\mathscr{T}F_s)(p), e^{-tH(p)}(\mathscr{T}G_r)(p))\mathrm{d}p$$

for any $\xi \in \mathbb{R}^3$. Note that $\lim_{s\to 0}(\mathscr{T}F_s)(p) = \frac{1}{(2\pi)^{3/2}}\Psi$ strongly in $L^2(\mathrm{Q})$ for each $p \in \mathbb{R}^3$. Hence

$$\lim_{s\to 0}(F_s, e^{-tH} e^{-\mathrm{i}\xi \mathrm{P}^{\mathrm{T}}} G_r) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\Psi, e^{-tH(p)} e^{-\mathrm{i}\xi p}(\mathscr{T}G_r)(p))\mathrm{d}p. \tag{3.33}$$

By using the fact $e^{\mathrm{i}\xi \mathrm{P}^{\mathrm{T}}} G_r(x) = \Pi_r(x-\xi)e^{-\mathrm{i}\xi \mathrm{P}_{\mathrm{f}}}\Phi$, we obtain by the functional integral representation in Theorem 3.3 that

$$(F_s, e^{-tH} e^{-\mathrm{i}\xi \mathrm{P}^{\mathrm{T}}} G_r) = \int_{\mathbb{R}^3} \mathbb{E}^x[\Pi_s(x)\Pi_r(B_t - \xi)(\mathrm{J}_0\Psi, e^{-\mathrm{i}\mathbb{A}(K_t)}\mathrm{J}_t e^{-\mathrm{i}\xi \mathrm{P}_{\mathrm{f}}}\Phi)]\mathrm{d}x.$$

Then it follows from $\Pi_s(x) \to \delta(x)$ as $s \to 0$ that

$$\lim_{s\to 0}(F_s, e^{-tH} e^{-\mathrm{i}\xi \mathrm{P}^{\mathrm{T}}} G_r) = \mathbb{E}^0[\Pi_r(B_t - \xi)(\mathrm{J}_0\Psi, e^{-\mathrm{i}\mathbb{A}(K_t)}\mathrm{J}_t e^{-\mathrm{i}\xi \mathrm{P}_{\mathrm{f}}}\Phi)]. \tag{3.34}$$

Combining (3.33) and (3.34) leads to

$$\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi p}(\Psi, e^{-tH(p)}(\mathscr{T}G_r)(p))\mathrm{d}p = \mathbb{E}^0[\Pi_r(B_t - \xi)(\mathrm{J}_0\Psi, e^{-\mathrm{i}\mathbb{A}(K_t)}\mathrm{J}_t e^{-\mathrm{i}\xi \mathrm{P}_{\mathrm{f}}}\Phi)]. \tag{3.35}$$

Taking the inverse Fourier transform on both sides of (3.35) with respect to p gives

$$(\Psi, e^{-tH(p)}(\mathscr{T}G_r)(p)) = \frac{1}{(2\pi)^{3/2}} \mathbb{E}^0\left[\int_{\mathbb{R}^3} e^{\mathrm{i}\xi p}\Pi_r(B_t - \xi)(\mathrm{J}_0\Psi, e^{-\mathrm{i}\mathbb{A}(K_t)}\mathrm{J}_t e^{-\mathrm{i}\xi \mathrm{P}_{\mathrm{f}}}\Phi)d\xi\right] \tag{3.36}$$

for almost every $p \in \mathbb{R}^3$. Since both sides of (3.36) are continuous in p, the equality stays valid for all $p \in \mathbb{R}^3$. After taking $r \to 0$ on both sides we arrive at (3.31).

QED

By Baker-Campbell-Hausdorff formula we can also have a functional integral representation in the Fock representation. Then the proof is similar to that of Theorem 3.12

Corollary 3.15 *Let $\Psi, \Phi \in L^2(\mathrm{Q})$. Then*

$$(\Psi, e^{-tH(p)}\Phi) = \mathbb{E}^0[e^{-\mathfrak{K}_t}(\Psi, e^{a^*(L_t)} e^{-tH_{\mathrm{f}}} e^{a(\tilde{L}_t)} e^{+\mathrm{i}(p-\mathrm{P}_{\mathrm{f}})B_t}\Phi)]. \tag{3.37}$$

In particular

$$e^{-tH(p)}\Phi = \mathbb{E}^0[e^{-\mathfrak{K}_t} e^{a^*(L_t)} e^{-tH_{\mathrm{f}}} e^{a(\tilde{L}_t)} e^{+\mathrm{i}(p-\mathrm{P}_{\mathrm{f}})B_t}\Phi]. \tag{3.38}$$

Here L_t, $\tilde{L}_t$ and $\mathfrak{K}_t$ are given by (3.26), (3.27) and (3.28), respectively.

3.11 Pauli-Fierz Hamiltonian with the dipole approximation

The Pauli-Fierz Hamiltonian with the dipole approximation is defined by H with $A = \int_{\mathbb{R}^3}^{\oplus} A(x)\mathrm{d}x$ replaced by $\mathbb{1}\otimes A(0)$. This implies that collisions between an electron and photons are ignored. Hence it is of the form

$$H_{\mathrm{dip}} = \frac{1}{2}(-\mathrm{i}\nabla\otimes\mathbb{1} - \mathbb{1}\otimes A(0)) + V\otimes\mathbb{1} + \mathbb{1}\otimes H_{\mathrm{f}}.$$

By introducing the dipole approximation the model can be solvable. More precisely it can be diagonalized when $V = 0$.

3.11.1 Bogoliubov transformation

In this section we suppose that

(1) $\hat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^3)$, $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$, and $\hat{\varphi}(k) = \hat{\varphi}(|k|)$.

(2) $\hat{\varphi} \in C_0^\infty(\mathbb{R}^3)$.

Note that we can weaken these conditions, but for simplicity we assume (1) and (2). A weaker conditions are given in [76, Assumption 3.4].

Let K be a Hilbert-Schmidt operator on $L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ such that

$$Kf = \sum_{n=0}^{\infty} \lambda_n(\psi_n, f)\phi_n.$$

Here $\{\psi_n\}$ and $\{\phi_n\}$ are orthogonal vectors in $L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$, and $\sum_{n=1}^\infty \lambda_n^2 < \infty$. Note that $\{\psi_n\}$ and $\{\phi_n\}$ are not necessarily orthonormal systems in $L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$. Then we define

$$\Delta_K^* = \lim_{M\to\infty}\sum_{n=1}^{M} \lambda_n a^*(\bar{\psi}_n)a^*(\phi_n),$$
$$\Delta_K = \lim_{M\to\infty}\sum_{n=1}^{M} \lambda_n a(\bar{\psi}_n)a(\phi_n).$$

It is established in [6] that Δ_K^* and Δ_K are densely defined closed operators in $\mathcal{F}$. Moreover let $\{e_n\}$ be a orthonormal system in $L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$, and S a bounded operator on $L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$. We define

$$N_S = \lim_{M\to\infty}\sum_{n=1}^{M} a^*(e_n)a(\overline{S^* e_n}).$$

This is also a densely defined closed operator. In particular choosing $S = \mathbb{1}$, we see that $N_{\mathbb{1}}$ is the number operator. These are studied in [6, 118, 119]. Let

$$W_\pm = (W_{\pm ij})_{1\le i,j\le 2}$$

be the operators defined by

$$W_{+ij} = \frac{1}{2} \sum_{1\le\mu,\nu\le 3} e_\mu^i \left(\frac{1}{\sqrt{\omega}} T_{\mu\nu}^* \sqrt{\omega} + \sqrt{\omega} T_{\mu\nu}^* \frac{1}{\sqrt{\omega}} \right) e_\nu^j,$$

$$W_{-ij} = \frac{1}{2} \sum_{1\le\mu,\nu\le 3} e_\mu^i \left(\frac{1}{\sqrt{\omega}} T_{\mu\nu}^* \sqrt{\omega} - \sqrt{\omega} T_{\mu\nu}^* \frac{1}{\sqrt{\omega}} \right) T e_\nu^j.$$

Hete T is defined by $Tf(k) = f(-k)$ and $T_{\mu\nu}$ by

$$T_{\mu\nu} f = \delta_{\mu\nu} f + \frac{\hat{\varphi}}{D_+(\omega^2)} \sqrt{\omega} G \sqrt{\omega} \delta_{\mu\nu}^\perp \hat{\varphi} f$$

with

$$Gf(k) = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^3} \frac{f(k')}{(\omega(k)^2 - \omega(k')^2 + \mathrm{i}\varepsilon)\sqrt{\omega(k)\omega(k')}} \mathrm{d}k'$$

and

$$D_+(s) = 1 - \frac{4\pi}{3} \left(\lim_{\varepsilon\downarrow 0} \int_{|s-x|>\varepsilon} \frac{|\hat{\varphi}(\sqrt{x})|^2 \sqrt{|x|}}{s-x} \mathrm{d}x - \pi\mathrm{i} |\hat{\varphi}(\sqrt{s})|^2 \sqrt{s} \right).$$

Let

$$S_p = \exp(-\mathrm{i}\Pi_p),$$

where

$$\Pi_p = \frac{\mathrm{i}}{\sqrt{2}} \frac{1}{m_{\mathrm{eff}}} \sum_{j=\pm 1} \left(a^* \left(p \cdot e^j \frac{\hat{\varphi}}{\omega^{3/2}}, j \right) - a \left(p \cdot e^j \frac{\hat{\varphi}}{\omega^{3/2}}, j \right) \right)$$

with

$$m_{\mathrm{eff}} = 1 + \frac{2}{3} \left\| \frac{\hat{\varphi}}{\omega} \right\|^2.$$

Let

$$U = C \exp(-\frac{1}{2} \Delta^*_{W_- W_+^{-1}}) : \exp(-N_{\mathbb{1} - \overline{(W_+^{-1})^*}}) : \exp(-\frac{1}{2} \Delta_{-W_+^{-1} \overline{W_-}}),$$

where

$$C = \det(\mathbb{1} - (W_- W_+^{-1})^* (W_- W_+^{-1}))^{1/4}.$$

Note that $\|U\mathbb{1}\| = 1$. Define

$$\mathcal{U} = \int_{\mathbb{R}^3}^{\oplus} S_p U e^{\mathrm{i}\frac{\pi}{2} N} \mathrm{d}p.$$

Thus we can see that $\mathcal{U}$ is the unitary operator on $\mathcal{H}$. See [5] and [76, Section 3.4].

Proposition 3.16 ([5, 77]) *It follows that*

$$\mathcal{U}^{-1} H_{\mathrm{dip}} \mathcal{U} = -\frac{1}{2m_{\mathrm{eff}}} \Delta \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + g. \tag{3.39}$$

Here

$$g = \frac{3}{2\pi} \int_{-\infty}^{\infty} \frac{\frac{2}{3} \left\| \frac{t}{\sqrt{t^2+\omega^2}} \frac{\hat{\varphi}}{\sqrt{t^2+\omega^2}} \right\|^2}{1 + \frac{2}{3} \left\| \frac{\hat{\varphi}}{\sqrt{t^2+\omega^2}} \right\|^2} \mathrm{d}t.$$

A functional integral representation of $e^{-tH_{\rm dip}}$ is a minor modification of the full Hamiltonian. The stochastic integral $\int_0^t {\rm j}_s\tilde\varphi(\cdot - B_s){\rm dB}_{\rm s}^\mu$ appearing in the functional integral representation of e^{-tH} is replaced by $\int_0^t {\rm j}_s\tilde\varphi{\rm dB}_{\rm s}^\mu$.

Proposition 3.17 *It follows that*

$$(F, e^{-tH_{\rm dip}}G) = \mathbb{E}^x[e^{-\int_0^t V(B_s){\rm d}s}(F(B_0), {\rm J}_0^* e^{-{\rm i}\mathbb{A}(K_t^{\rm dip})}{\rm J}_t G(B_t)] \tag{3.40}$$

where $K_t^{\rm dip} = \bigoplus_{\mu=1}^3 \int_0^t {\rm j}_s\tilde\varphi{\rm d}B_s^\mu$.

Let $V = 0$. Then together with Propositions 3.16 and 3.17 we have the identity:

$$(F, e^{-tH_{\rm dip}}G) = e^{-tg}\int_{\mathbb{R}^3}\mathbb{E}^x[(\mathcal{U}^{-1}F(B_0), e^{-tH_{\rm f}}\mathcal{U}^{-1}G(B_{t/m_{\rm eff}})]{\rm d}x. \tag{3.41}$$

3.11.2 Translation invariant Pauli-Fierz Hamiltonian with the dipole approximation

Let $V = 0$. Then

$$[H_{\rm dip}, p_\mu] = 0 \quad \mu = 1, 2, 3.$$

We have

$$H_{\rm dip} = \int_{\mathbb{R}^3}^\oplus H_{\rm dip}(p){\rm d}p,$$

where

$$H_{\rm dip}(p) = \frac{1}{2}(p - A(0)) + H_{\rm f}.$$

Thus we have the proposition below.

Proposition 3.18 *It follows that*

$$(\Psi, e^{-tH_{\rm dip}(p)}\Phi) = \mathbb{E}^0[(\Psi, {\rm J}_0^* e^{-{\rm i}\mathbb{A}(K_t^{\rm dip})}{\rm J}_t e^{+{\rm i}(p-{\rm P}_{\rm f})B_t}\Phi)]. \tag{3.42}$$

Proof: The proof is a minor modification of that of Theorem 3.14.

QED

Let $p = 0$ and $\Psi = \Phi = 1\!\!1$. Thus we have

$$(1\!\!1, e^{-tH_{\rm dip}(p)}1\!\!1) = \mathbb{E}^0[e^{-\frac{1}{2}{\rm q}_{\rm E}(K_t^{\rm dip}, K_t^{\rm dip})}]. \tag{3.43}$$

Here we can see that

$$\frac{1}{2}{\rm q}_{\rm E}(K_t^{\rm dip}, K_t^{\rm dip}) \overset{formal}{=} \frac{1}{4}\sum_{1\le\mu,\nu\le 3}\int_0^t {\rm d}B_s^\mu\int_0^t {\rm d}B_r^\nu\int_{\mathbb{R}^3}\frac{|\hat\varphi(k)|^2}{\omega(k)}e^{-|s-r|\omega(k)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right){\rm d}k.$$

This expression is formal. In a similar manner to [15, Proposition 3.1] it is derived that

$$\frac{1}{2}{\rm q}_{\rm E}(K_t^{\rm dip}, K_t^{\rm dip}) = \frac{1}{4}\sum_{1\le\mu,\nu\le 3}\int_{\mathbb{R}^3}{\rm d}k\frac{|\hat\varphi(k)|^2}{\omega(k)}\int_0^t {\rm d}B_s^\mu\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right)\int_0^s e^{-(s-r)\omega(k)}{\rm d}B_r^\nu.$$

Corollary 3.19 *It follows that*

$$\mathbb{E}^0[e^{-\frac{1}{2}\mathrm{q}_{\mathrm{E}}(K_t^{\mathrm{dip}},K_t^{\mathrm{dip}})}] = e^{-tg}\det(\mathbb{1}-(W_-W_+^{-1})^*(W_-W_+^{-1}))^{1/2}(e^{-\frac{1}{2}\Delta^*_{K_0}}\mathbb{1}, e^{-\frac{1}{2}\Delta^*_{K_t}}\mathbb{1}),$$

where $K_t = -\mathrm{j}_t^*(W_+^{-1}\overline{W_-})^*\mathrm{j}_t$ *and* $K_0 = -\mathrm{j}_0^*(W_+^{-1}\overline{W_-})^*\mathrm{j}_0$.

Proof: We have

$$U(p)^{-1}H_{\mathrm{dip}}(p)U(p) = \frac{1}{2m_{\mathrm{eff}}}p^2 + H_{\mathrm{f}} + g$$

for every $p \in \mathbb{R}^3$. Here

$$U(p) = S_pUe^{\mathrm{i}\frac{\pi}{2}N}.$$

Then

$$(\mathbb{1}, e^{-tH_{\mathrm{dip}}(p)}\mathbb{1}) = e^{-tg}\mathbb{E}^0[(U(0)^{-1}\mathbb{1}, e^{-tH_{\mathrm{f}}}U(0)^{-1}\mathbb{1})] = e^{-tg}\mathbb{E}^0[(\mathrm{J}_0U(0)^{-1}\mathbb{1}, \mathrm{J}_tU(0)^{-1}\mathbb{1})].$$

Note that

$$U(0)^{-1}\mathbb{1} = \det(\mathbb{1}-(W_-W_+^{-1})^*(W_-W_+^{-1}))^{1/4}e^{-\frac{1}{2}\Delta^*_{-(W_+^{-1}\overline{W_-})^*}}\mathbb{1}.$$

Since $\mathrm{j}_\# e^{-\frac{1}{2}\Delta^*_{-(W_+^{-1}\overline{W_-})^*}}\mathbb{1} = e^{-\frac{1}{2}\Delta^*_{K_\#}}\mathbb{1}$ for $\# = 0, t$. Then the corollary follows.

QED

4 Relativistic Pauli-Fierz model

4.1 Relativistic Pauli-Fierz Hamiltonian

In quantum mechanics a relativistic Schrödinger operator with a vector potential a is defined by $H_{\mathrm{R}}(a) = ((-\mathrm{i}\nabla - a)^2 + m^2)^{1/2} - m + V$, and a functional integral representation of $e^{-tH_{\mathrm{R}}(a)}$ is shown in Section 2.2. A key element in the construction of the functional integral representation of $e^{-tH_{\mathrm{R}}(a)}$ is to use the subordinator $(T_t)_{t\geq 0}$. In this section the analogue version of the Pauli-Fierz model is defined and its functional integral representation is given. This section is due to [65, 55]. We also refer to see [107, 89, 90, 91, 92, 56, 57]. We say that $V \in \mathcal{R}_{\mathrm{RKato}}$ iff $D(\sqrt{-\Delta}) \subset D(V)$ and there exist $0 \leq a < 1$ and $0 \leq b$ such that $\|Vf\| \leq a\|\sqrt{-\Delta}f\| + b\|f\|$ for $f \in D(\sqrt{-\Delta})$ with $0 \leq a < 1$ and $0 \leq b$. Instead of Assumption 3.1 throughout this section we suppose the assumption below unless otherwise stated.

Assumption 4.1 $\varphi \in \mathcal{U}$, $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$, *and* $V \in \mathcal{R}_{\mathrm{RKato}}$.

In Assumptions 4.1 we add the extra condition $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$ to Assumption 3.1, and instead of $V \in \mathcal{R}_{\mathrm{Kato}}$, we suppose that $V \in \mathcal{R}_{\mathrm{RKato}}$. Let $\mathscr{D} = D(\Delta) \cap C^\infty(N)$ and

$$H(\hat{A}) = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2\lceil_{\mathscr{D}}.$$

To define the relativistic Pauli-Fierz Hamiltonian $(2H(\hat{A}) + m^2)^{1/2} - m + H_{\mathrm{f}} + V$, we have to define $(2H(\hat{A})+m^2)^{1/2}$ as a self-adjoint operator. It is however not trivial to choose a self-adjoint extension of $H(\hat{A})$. We have the lemma below.

Proposition 4.1 ([61]) *Suppose that* $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3), \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$ *and* $\hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$. *Then* $H(\hat{A})$ *is essentially self-adjoint.*

We keep denoting the unique self-adjoint extension of $H(\hat{A})$ by the same symbol $H(\hat{A})$ for simplicity, and we define $(2H(\hat{A})+m^2)^{1/2}$ by the spectral resolution of $H(\hat{A})$. The relativistic Pauli-Fierz Hamiltonian is defined by

$$H_{\mathrm{R}} = (2H(\hat{A})+m^2)^{1/2} - m \dotplus H_{\mathrm{f}} \dotplus V.$$

Here $m \geq 0$ is a parameter, but it describes the mass of an electron in physics.

4.2 Functional integral representations for the relativistic Pauli-Fierz Hamiltonian

We will construct a functional integral representation of $e^{-tH_{\mathrm{R}}}$ through the Trotter product formula. We set

$$T_{\mathrm{k}} = (2H(\hat{A})+m^2)^{1/2} - m. \tag{4.1}$$

By the Trotter product formula for quadratic form sums we see that

$$(F, e^{-tH_{\mathrm{R}}}G) = \lim_{n\to\infty}\left(F, \left(e^{-\frac{t}{2^n}T_{\mathrm{k}}}e^{-\frac{t}{2^n}H_{\mathrm{f}}}e^{-\frac{t}{2^n}V}\right)^{2^n} G\right).$$

Suppose $V \in C_0^\infty(\mathbb{R}^3)$. Then

$$\begin{aligned}&\left(F, \left(e^{-\frac{t}{2^n}T_{\mathrm{k}}}e^{-\frac{t}{2^n}H_{\mathrm{f}}}e^{-\frac{t}{2^n}V}\right)^{2^n} G\right)\\ &= \int_{\mathbb{R}^3}\mathbb{E}^{x,0}\left[\left(\mathrm{J}_0F(B_{T_0}), e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}}(n))}\mathrm{J}_tG(B_{T_t})\right)e^{-\sum_{j=0}^{2^n}\frac{t}{2^n}V(B_{T_{t_j}})}\right]\mathrm{d}x,\end{aligned}$$

where

$$K_t^{\mathrm{R}}(n) = \bigoplus_{\mu=1}^3\sum_{j=1}^{2^n}\int_{T_{t_{j-1}}}^{T_{t_j}}\mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu$$

with $t_j = \frac{jt}{2^n}$, and $\int_{T_{t_{j-1}}}^{T_{t_j}}\mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu$ denotes an $L^2(\mathbb{R}^4)$-valued stochastic integral

$$\int_T^S \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu$$

evaluated at $T = T_{t_{j-1}}$ and $S = T_{t_j}$. $(K_t^{\mathrm{R}}(n))_{t\geq 0}$ can be regarded as a sequence of $\bigoplus^3 L^2(\mathbb{R}^4)$-valued random processes on the product probability space $(\mathscr{X}\times\mathscr{X}_\nu, \mathcal{B}\times\mathcal{B}_\nu, \mathcal{W}^x\otimes\nu)$. Let $N_\nu \in \mathcal{B}_\nu$ be a null set, i.e., $\nu(N_\nu) = 0$, such that for arbitrary $w \in \mathscr{X}_\nu \setminus N_\nu$, the path $t \mapsto T_t(w)$ is nondecreasing and right-continuous, and has the left-limit. We have the lemma below.

Lemma 4.2 ([65]) *For each* $w \in \mathscr{X}_\nu\setminus N_\nu$ *and each* $t \geq 0$, $K_t^{\mathrm{R}}(n)$ *strongly converges in* $L^2(\mathscr{X})\otimes(\bigoplus^3 L^2(\mathbb{R}^4))$ *as* $n\to\infty$, *i.e,. there exists*

$$K_t^{\mathrm{R}} \in L^2(\mathscr{X})\otimes(\bigoplus^3 L^2(\mathbb{R}^4))$$

such that $\lim_{n\to\infty}\mathbb{E}^x\left[\|K_t^{\mathrm{R}}(n) - K_t^{\mathrm{R}}\|^2\right] = 0$.

By Lemma 4.2 and a limiting argument it follows that

$$(F, e^{-t(T_{\rm k}\dotplus H_{\rm f})}G) = \int_{\mathbb{R}^3} \mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0 F(B_{T_0}), e^{-\mathrm{i}\mathbb{A}(K_t^{\rm R})}\mathrm{J}_t G(B_{T_t})\Big)\Big]\,\mathrm{d}x.$$

The immediate consequence is bounds of $e^{-t(T_{\rm k}\dotplus H_{\rm f})}$. Let $F, G \in \mathcal{H}$. Then it follows that

$$\begin{aligned}
&|(F, e^{-t(T_{\rm k}\dotplus H_{\rm f})}G)| \le (|F|, e^{-t((-\Delta+m^2)^{1/2}-m+H_{\rm f})}|G|)_{\mathcal{H}},\\
&|(F, e^{-t(T_{\rm k}\dotplus H_{\rm f})}G)| \le (\|F\|_{L^2(\mathrm{Q})}, e^{-t((-\Delta+m^2)^{1/2}-m)}\|G\|_{L^2(\mathrm{Q})})_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

From these bounds we can conclude a relative boundedness with respect to V.

Lemma 4.3 *(1) and (2) follow.*

(1) *If V is relatively form bounded with respect to $(-\Delta+m^2)^{1/2}-m$ with a relative bound a, then $|V|$ is also relatively form bounded with respect to $(T_{\rm k}\dotplus H_{\rm f})$ with a relative bound smaller than a.*

(2) *If V is relatively bounded with respect to $(-\Delta+m^2)^{1/2}-m$ with a relative bound a, then V is also relatively bounded with respect to $T_{\rm k}\dotplus H_{\rm f}$ with a relative bound a.*

Suppose Assumption 4.1. Then $(T_{\rm k}\dotplus H_{\rm f})+V$ is self-adjoint on $D(T_{\rm k}\dotplus H_{\rm f})$ by (1) of Lemma 4.3.

Theorem 4.4 *Suppose Assumption 4.1. Then*

$$(F, e^{-tH_{\rm R}}G) = \int_{\mathbb{R}^3} \mathbb{E}^{x,0}\left[e^{-\int_0^t V(B_{T_s})\mathrm{d}s}(\mathrm{J}_0F(B_0), e^{-\mathrm{i}\mathbb{A}(K_t^{\rm R})}\mathrm{J}_tG(B_{T_t}))\right]\mathrm{d}x. \tag{4.2}$$

Proof: When V is bounded and continuous, the theorem can be proven by the Trotter formula. Furthermore, it can be extended to a general V in the same way as that of Theorem 3.5.

QED

We have the corollary. Let $\mathfrak{S}_\# = \exp(-\mathrm{i}\frac{\pi}{2}N_\#)$.

Corollary 4.5 ([65]) *(1) $\mathfrak{S}^{-1}e^{-tH_R}\mathfrak{S}$ is positivity improving. (2) If $H_{\rm R}$ has a ground state $\Psi_{\rm g}$. Then $\mathfrak{S}^{-1}\Psi_{\rm g}$ is strictly positive.*

Proof: Let $F, G \in \mathcal{H}$ be non-negative but not identically zero. We have

$$(F, \mathfrak{S}^{-1}e^{-tH_R}\mathfrak{S}G) = \int_{\mathbb{R}^3} \mathbb{E}^{x,0}\left[e^{-\int_0^t V(B_s)\mathrm{d}s}(F(B_0), \mathrm{J}_0^*\mathfrak{S}_{\rm E}^{-1}e^{-\mathrm{i}\mathbb{A}(K_t^{\rm R})}\mathfrak{S}_{\rm E}\mathrm{J}_tG(B_{T_t}))\right]\mathrm{d}x.$$

Since $\mathrm{J}_0^*\mathfrak{S}_{\rm E}^{-1}e^{-\mathrm{i}\mathbb{A}(K_t^{\rm R})}\mathfrak{S}_{\rm E}\mathrm{J}_t$ is positivity improving, the corollary can be proven in a similar manner to Corollary 3.10.

QED

4.3 Invariant domain and self-adjointness

In this section by using the functional integral representation derived in Theorem 4.4 we show the self-adjointness of H_{R}. The fundamental lemma is as follows.

Lemma 4.6 *Let K be a non-negative self-adjoint operator. Suppose that there exists a dense domain D such that $D \subset D(K)$ and $e^{-tK}D \subset D$ for all $t \geq 0$. Then $K\lceil_D$ is essentially self-adjoint.*

Proof: It suffices to show that for some $\lambda > 0$, $\mathrm{Ran}((\lambda + K)\lceil_D)$ is dense. Suppose the contrary. Then there exists nonzero f such that $(f, (\lambda + K)\psi) = 0$ for all $\psi \in D$. We have

$$\frac{d}{dt}(f, e^{-tK}\psi) = (f, -Ke^{-tK}\psi) = \lambda(f, e^{-tK}\psi).$$

Thus $(f, e^{-tK}\psi) = e^{\lambda t}(f, \psi)$. If $(f, \psi) \neq 0$, then $\lim_{t\to\infty} |(f, e^{-tK}\psi)| = \infty$, contradicting the fact that e^{-tK} is a contraction. Hence $(f, \psi) = 0$ for all $\psi \in D(K)$, but (f, ψ) can not equal zero for all $\psi \in D(K)$, since $D(K)$ is dense. Hence we conclude that $\mathrm{Ran}((\lambda + K)\lceil_D)$ is dense.

QED

To prove the essential self-adjointness of H_{R} we find an invariant domain D so that

$$D \subset D(H_{\mathrm{R}}), \quad e^{-tH_{\mathrm{R}}}D \subset D.$$

Let $\hat{\omega} = \omega(-\mathrm{i}\nabla) \otimes \mathbb{1}$ under $L^2(\mathbb{R}^4) \cong L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R})$. Let $\alpha > 0$. We can estimate the forth moment $\mathbb{E}^{x,0}\Big[\|\hat{\omega}^{\alpha/2}K_t^{\mathrm{R}}\|^4_{\bigoplus^3 L^2(\mathbb{R}^4)}\Big]$. Suppose that $\hat{\varphi}/\sqrt{\omega}, \omega^{(\alpha-1)/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$. Then the Burkholder-Davis-Gundy (BDG) type inequality [61, 65] holds:

$$\mathbb{E}^{x,0}\Big[\|\hat{\omega}^{\alpha/2}K_t^{\mathrm{R}}\|^4_{\bigoplus^3 L^2(\mathbb{R}^4)}\Big] \leq C\|\omega^{(\alpha-1)/2}\hat{\varphi}\|^4_{L^2(\mathbb{R}^3)},$$

where C is a constant.

Lemma 4.7 *Let $V = 0$. Then (1) and (2) hold true.*
(1) For $F \in D(-\mathrm{i}\nabla_\mu)$ and $G \in D(-\mathrm{i}\nabla_\mu) \cap D(H_{\mathrm{f}}^{1/2})$ it follows that

$$(-\mathrm{i}\nabla_\mu F, e^{-tH_{\mathrm{R}}}G) \leq C\left((\|\sqrt{\omega}\hat{\varphi}\| + \|\hat{\varphi}\|)\|(H_{\mathrm{f}} + \mathbb{1})^{1/2}G\| + \|-\mathrm{i}\nabla_\mu G\|\right)\|F\|.$$

In particular $e^{-tH_{\mathrm{R}}}D(\sqrt{-\Delta}) \cap D(H_{\mathrm{f}}^{1/2}) \subset D(\sqrt{-\Delta})$ for $t \geq 0$.
(2) For $F, G \in D(H_{\mathrm{f}})$ it follows that

$$(H_{\mathrm{f}}F, e^{-tH_{\mathrm{R}}}G) \leq \left(\|H_{\mathrm{f}}G\| + (\|\sqrt{\omega}\hat{\varphi}\| + \|\hat{\varphi}\|)\|(H_{\mathrm{f}} + \mathbb{1})^{1/2}G\| + \|\hat{\varphi}/\sqrt{\omega}\|^2\|G\|\right)\|F\|.$$

In particular $e^{-tH_{\mathrm{R}}}D(H_{\mathrm{f}}) \subset D(H_{\mathrm{f}})$ for $t \geq 0$.

Proof: We show the outline of a proof. Notice that

$$(e^{\mathrm{i}s(-\mathrm{i}\nabla_\mu)}F, e^{-tH_{\mathrm{R}}}G) = \int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}), e^{+\mathrm{i}s\mathrm{P}_{\mathrm{f}\mu}}e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}e^{-\mathrm{i}s\mathrm{P}_{\mathrm{f}\mu}}\mathrm{J}_te^{-\mathrm{i}s(-\mathrm{i}\nabla_\mu)}G(B_{T_t})\Big)\Big]\,\mathrm{d}x. \tag{4.3}$$

Here we used

$$e^{\mathrm{i}s(-\mathrm{i}\nabla_\mu+\mathrm{P}_{\mathrm{f}\mu})}e^{-tH_{\mathrm{R}}}=e^{-tH_{\mathrm{R}}}e^{\mathrm{i}s(-\mathrm{i}\nabla_\mu+\mathrm{P}_{\mathrm{f}\mu})}.$$

We see that

$$e^{+\mathrm{i}s\mathrm{P}_{\mathrm{f}\mu}}e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}e^{-\mathrm{i}s\mathrm{P}_{\mathrm{f}\mu}}=e^{-\mathrm{i}\mathbb{A}(e^{\mathrm{i}s(-\mathrm{i}\nabla_\mu\otimes 1)}K_t^{\mathrm{R}})}.$$

Take the derivative at $s=0$ on both sides of (4.3). We have

$$\begin{aligned}(\nabla_\mu F, e^{-tH_{\mathrm{R}}}G)=&\int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}),-\mathrm{i}\mathbb{A}_\mu(\nabla_\mu K_t^{\mathrm{R}})e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}\mathrm{J}_tG(B_{T_t})\Big)\Big]\,\mathrm{d}x\\&+\int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}),e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}\mathrm{J}_t(-\nabla_\mu G)(B_{T_t})\Big)\Big]\,\mathrm{d}x.\end{aligned}\tag{4.4}$$

It is trivial to see that

$$\left|\int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}),e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}\mathrm{J}_t(-\nabla_\mu G)(B_{T_t})\Big)\Big]\,\mathrm{d}x\right|\leq\|F\|\|\nabla_\mu G\|.$$

We can estimate the first term on the right-hand side of (4.4) by BDG inequality as

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}),\mathbb{A}_\mu(\nabla_\mu K_t^{\mathrm{R}})e^{-\mathrm{i}\mathbb{A}_\mu(K_t^{\mathrm{R}})}\mathrm{J}_tG(B_{T_t})\Big)\Big]\,\mathrm{d}x\right|\\&\leq C(\|\omega^{1/2}\hat{\varphi}\|+\|\hat{\varphi}\|)\|(H_{\mathrm{f}}+\mathbb{1})^{1/2}F\|\|G\|.\end{aligned}$$

Then (1) follows. We have

$$(H_{\mathrm{f}}F, e^{-tH_{\mathrm{R}}}G)=\int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}),e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}S\mathrm{J}_tG(B_{T_t})\Big)\Big]\,\mathrm{d}x,$$

where $S=e^{\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}H_{\mathrm{f}}e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}=H_{\mathrm{f}}-i[H_{\mathrm{f}},\mathbb{A}(K_t^{\mathrm{R}})]+g$ with $g=\mathrm{q}(K_t^{\mathrm{R}},K_t^{\mathrm{R}})$. It is trivial to see that

$$\left|\int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}),e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}H_{\mathrm{f}}\mathrm{J}_tG(B_{T_t})\Big)\Big]\,\mathrm{d}x\right|\leq\|F\|\|H_{\mathrm{f}}G\|.$$

In the same way as the estimate of the first term of the right-hand side of (4.4) we can see that

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}),e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}[H_{\mathrm{f}},\mathbb{A}(K_t^{\mathrm{R}})]\mathrm{J}_tG(B_{T_t})\Big)\Big]\,\mathrm{d}x\right|\\&\leq C(\|\sqrt{\omega}\hat{\varphi}\|+\|\hat{\varphi}\|)\|F\|\|(H_{\mathrm{f}}+\mathbb{1})^{1/2}G\|\end{aligned}$$

with some constant $C>0$. Here we used BDG inequality. Finally we see that $g\leq C\|K_t^{\mathrm{R}}\|^2$ and by BDG inequality again,

$$\left|\int_{\mathbb{R}^3}\mathbb{E}^{x,0}\Big[\Big(\mathrm{J}_0F(B_{T_0}),e^{-\mathrm{i}\mathbb{A}(K_t^{\mathrm{R}})}g\mathrm{J}_tG(B_{T_t})\Big)\Big]\,\mathrm{d}x\right|\leq C\|\hat{\varphi}/\sqrt{\omega}\|^2\|F\|\|G\|.$$

Then (2) follows.

QED

Theorem 4.8 ([65]) *Suppose Assumption 4.1 and $m>0$. Then H_{R} is essentially self-adjoint on $D(\sqrt{-\Delta})\cap D(H_{\mathrm{f}})$.*

Proof: Suppose $V = 0$. Let $D_\infty = C_0^\infty(\mathbb{R}^3) \otimes L^2_{\rm fin}(\mathrm{Q})$. Let $F \in D_\infty$. We see that

$$\|(T_{\rm k} \dot{+} H_{\rm f})F\|^2 \leq C_1\|\sqrt{-\Delta}F\|^2 + C_2\|H_{\rm f}F\|^2 + C_3\|F\|^2 \tag{4.5}$$

with some constants C_1, C_2 and C_3. Since D_∞ is a core of $\sqrt{-\Delta} + H_{\rm f}$,

$$D(\sqrt{-\Delta}) \cap D(H_{\rm f}) \subset D(T_{\rm k} \dot{+} H_{\rm f}) \tag{4.6}$$

follows from a limiting argument. By Lemmas 4.7, we also see that

$$e^{-t(T_{\rm k} \dot{+} H_{\rm f})}\left(D(\sqrt{-\Delta}) \cap D(H_{\rm f})\right) \subset \left(D(\sqrt{-\Delta}) \cap D(H_{\rm f})\right). \tag{4.7}$$

(4.6) and (4.7) imply that $T_{\rm k} \dot{+} H_{\rm f}$ is essentially self-adjoint on $D(\sqrt{-\Delta}) \cap D(H_{\rm f})$ by Lemma 4.6. Next we suppose that V satisfies Assumption 4.1. Then V is also relatively bounded with respect to $T_{\rm k} \dot{+} H_{\rm f}$ with a relative bound strictly smaller than one. Then the theorem follows by the Kato-Rellich theorem.

QED

Furthermore we can establish the self-adjointness of $H_{\rm R}$. The key inequality to show the self-adjointness of $H_{\rm R}$ on $D(\sqrt{-\Delta}) \cap D(H_{\rm f})$ is the following inequality.

Lemma 4.9 ([55]) *Suppose Assumption 4.1 and that $m \geq 0$ and $V = 0$. Then there exists a constant C such that*

$$\|\sqrt{-\Delta}F\|^2 + \|H_{\rm f}F\|^2 \leq C\|(T_{\rm k} \dot{+} H_{\rm f} + \mathbb{1})F\|^2 \tag{4.8}$$

for all $F \in D(\sqrt{-\Delta}) \cap D(H_{\rm f})$.

Theorem 4.10 ([55]) *Suppose Assumption 4.1. Then $H_{\rm R}$ is self-adjoint on $D(\sqrt{-\Delta}) \cap D(H_{\rm f})$.*

Proof: Suppose that $V = 0$. We write $H^{(m)}$ for $H_{\rm R}$ to emphasize m-dependence. Let $m > 0$. Then $H^{(m)}$ is essentially self-adjoint on $D(\sqrt{-\Delta}) \cap D(H_{\rm f})$. While by (4.8), $H^{(m)}\lceil_{D(\sqrt{-\Delta})\cap D(H_{\rm f})}$ is closed. Then $H^{(m)}$ is self-adjoint on $D(\sqrt{-\Delta}) \cap D(H_{\rm f})$. Note that

$$H^{(0)} = H^{(m)} + (H^{(0)} - H^{(m)})$$

and $H^{(0)} - H^{(m)}$ is bounded. Then $H^{(0)}$ is also self-adjoint on $D(\sqrt{-\Delta}) \cap D(H_{\rm f})$ for $V = 0$. Finally let V be potential satisfying Assumption 4.1. Then V is also relatively bounded with respect to $H^{(m)}$ with a relative bound strictly smaller than one. Then the theorem follows from the Kato-Rellich theorem.

QED

A spinless hydrogen like atom is defined by introducing the Coulomb potential

$$V(x) = -\frac{g}{|x|}, \quad g > 0,$$

which is relatively form bounded with respect to $(-\Delta + m^2)^{1/2}$ with a relative bound strictly smaller than one if $g \leq 2/\pi$ by [53]. Furthermore if $g < 1/2$, V is relatively bounded with respect to $(-\Delta + m^2)^{1/2}$ with a relative bound strictly smaller than one. Let $\hat{A}_\Lambda$ be the quantized radiation field with

$$\hat{\varphi}(k) = \frac{1}{(2\pi)^{3/2}}\mathbb{1}_{|k|\leq\Lambda}(k).$$

By Theorem 4.10 when $g < 1/2$, $H_{\rm R}$ is self-adjoint on $D(\sqrt{-\Delta}) \cap D(H_{\rm f})$.

4.4 Non-relativistic limit

Schrödinger operators do not contain the velocity of the light in the definition. On the other hand Dirac operators

$$H(c) = c\alpha \cdot \left(-\mathrm{i}\nabla - \frac{1}{c}a \right) + \beta mc^2 + V$$

can be regarded as relativistic versions of Schrödinger operators. Here c denotes the velocity of the light, and α and β are 4×4 Dirac matrices given by

$$\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, i = 1, 2, 3, \quad \beta = \begin{pmatrix} 1\!\!1_{2\times 2} & 0 \\ 0 & -1\!\!1_{2\times 2} \end{pmatrix}$$

with 2×2 Pauli matrices σ_1, σ_2 and σ_3. This was introduced by P. A. M. Dirac to construct a relativistically covariant quantum theory. Hence non-relativistic limits of Dirac operators have been studied so far. We refer to see [132, Section 6] for non-relativistic limits of Dirac operators and [102, 103] for those of non-linear Dirac and Klein-Gordon equations. Formally it can be shown that

$$\lim_{c\to\infty} (H(c) - mc^2 - z)^{-1} = \begin{pmatrix} (\frac{1}{2m}(-\mathrm{i}\nabla - a)^2 + V - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand we are concerned with semi-relativistic Schrödinger operators in this section. Since we have

$$\sqrt{c^2|p|^2 + m^2c^4} - mc^2 = \frac{1}{2m}|p|^2 + \mathcal{O}(\frac{|p|^4}{m^3c^2}),$$

intuitively we have

$$\exp\left(-t\big(\sqrt{c^2(-\Delta) + m^2c^4} - mc^2\big) \right) \to \exp\left(\frac{t}{2m}\Delta \right)$$

as $c \to \infty$. This intuition becomes substantial by means of the so-called non-relativistic limit of semi-relativistic Schrödinger operators. Define

$$H_c = \sqrt{-c^2\Delta + m^2c^4} - mc^2 + V.$$

By using a functional integral representation we can show that $H_c \to H_\infty$ as $c \to \infty$ in a specific sense, and the limit operator is the Schrödinger operator

$$H_\infty = -\frac{1}{2m}\Delta + V.$$

For every $c > 0$ consider the subordinator $(T_t^c)_{t\geq 0}$ with parameter c such that

$$\mathbb{E}[e^{-uT_t^c}] = e^{-t(\sqrt{2c^2u + m^2c^4} - mc^2)}, \quad u \geq 0.$$

Proposition 4.11 *Let f be a bounded continuous function on $\mathbb{R}$. Then*

$$\lim_{c\to\infty} \mathbb{E}_P[f(T_t^c)] = f(\frac{t}{m}).$$

Proof: Let $f \in \mathscr{S}(\mathbb{R})$. We have

$$\mathbb{E}[f(T_t^c)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \check{f}(k) \mathbb{E}[e^{\mathrm{i}kT_t^c}] \mathrm{d}k.$$

We see that

$$\mathbb{E}[e^{\mathrm{i}kT_t^c}] = \int_0^\infty \frac{ct}{\sqrt{2\pi}} e^{mc^2 t} s^{-3/2} \exp\left(-\frac{1}{2}\left(\frac{c^2t^2}{s} + (2k\mathrm{i} + m^2c^2)s\right)\right) \mathrm{d}s$$

and then

$$\mathbb{E}_\mu[e^{\mathrm{i}kT_t^c}] = e^{-(\sqrt{2k\mathrm{i}+m^2c^2}-mc)ct}.$$

Furthermore, we have

$$c(\sqrt{2k\mathrm{i} + m^2c^2} - mc) = c((4k^2 + m^4c^4)^{1/4} - mc)e^{\mathrm{i}\theta/2} + mc^2(e^{\mathrm{i}\theta/2} - 1),$$

where $\tan\theta = 2k/m^2c^2$ with $|\theta| < \pi/2$. The first term converges to zero as $c \to \infty$. For the second term it can be seen that $mc^2(e^{\mathrm{i}\theta/2} - 1) = mc^2(\cos(\theta/2) - 1 + \mathrm{i}\sin(\theta/2))$. Since

$$mc^2(\cos(\theta/2) - 1) \sim mc^2(\theta/2)^2/2 \sim mc^2(k/m^2c^2)^2/2 \sim 0$$

and

$$\mathrm{i}mc^2\sin(\theta/2) \sim \mathrm{i}mc^2\theta/2 \sim \mathrm{i}k/m$$

as $c \to \infty$, we get

$$\lim_{c\to\infty} \mathbb{E}_P[e^{\mathrm{i}kT_t^c}] = e^{\mathrm{i}\frac{tk}{m}}$$

as $c \to \infty$, and the proposition follows. When f is chosen to be a bounded continuous function, it can be uniformly approximated by functions in $\mathscr{S}(\mathbb{R})$ and the proof is completed by a simple limiting argument.

QED

We derive the non-relativistic limit of e^{-tH_c}.

Corollary 4.12 *Let V be a bounded continuous function. Then*

$$s - \lim_{c\to\infty} e^{-tH_c} = e^{-tH_\infty}.$$

Proof: We suppose that V is non-negative without loss of generality. It is enough to show the weak limit

$$\lim_{c\to\infty} (f, e^{-tH_c} g) = (f, e^{-tH_\infty} g). \tag{4.9}$$

Since $H_c \geq 0$ for every $c > 0$, $\|e^{-tH_c}\| \leq 1$ uniformly with respect to $c > 0$. It is also sufficient to show (4.9) for arbitrary $f, g \in \mathscr{S}(\mathbb{R})$ by a simple limiting argument. Note that by Proposition 4.11 it can be seen that

$$\begin{aligned}(f, e^{-t(\sqrt{-\Delta+m^2c^4}-mc^2+V)}g) &= \int_{\mathbb{R}^3} \mathbb{E}^{x,0}\left[\bar{f}(x)g(B_{T_t^c})e^{-\int_0^t V(B_{T_s^c})\mathrm{d}s}\right]\mathrm{d}x \\ \to \int_{\mathbb{R}^3} \mathbb{E}^x\left[\bar{f}(x)g(B_{\frac{t}{m}})e^{-\int_0^t V(B_{\frac{s}{m}})\mathrm{d}s}\right]\mathrm{d}x &= (f, e^{-t(-\frac{1}{2m}\Delta+V)}g)\end{aligned}$$

as $c \to \infty$.

QED

$K_t^{\mathrm{R}}(c)$ is defined by K_t^{R} with T_t replaced by T_t^c.

Lemma 4.13 *It follows that* $\lim_{c\to\infty} K_t^{\mathrm{R}}(c) = K_t$ *strongly in* $L^2(\mathscr{X}\times\mathscr{X}_\nu)\otimes(\bigoplus^3 L^2(\mathbb{R}^4))$.

Proof: Let

$$\mathrm{I}_n^c = \bigoplus_{\mu=1}^{3}\sum_{j=1}^{2^n}\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu,$$
$$\mathrm{I}_n = \bigoplus_{\mu=1}^{3}\sum_{j=1}^{2^n}\int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu.$$

We have seen that $\mathrm{I}_n^c \to K_t^{\mathrm{R}}(c)$ and $\mathrm{I}_n \to K_t$ as $n\to\infty$ strongly in $L^2(\mathscr{X}\times\mathscr{X}_\nu)\otimes(\bigoplus^3 L^2(\mathbb{R}^4))$. We have

$$\|K_t^{\mathrm{R}}(c) - K_t\| \le \|K_t^{\mathrm{R}}(c) - \mathrm{I}_n^c\| + \|\mathrm{I}_n^c - \mathrm{I}_n\| + \|\mathrm{I}_n - K_t\|$$

and

$$\mathbb{E}^x[\|\mathrm{I}_n^c - \mathrm{I}_k^c\|^2] \le 3T_t^c\|\hat{\varphi}/\sqrt{\omega}\|^2\left(\sum_{j=n+1}^{k} 2^{-j/2}\right)^2.$$

From this we have

$$\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - K_t^{\mathrm{R}}(c)\|^2] \le 3\mathbb{E}^0[T_t^c]\|\hat{\varphi}/\sqrt{\omega}\|^2\left(\sum_{j=n+1}^{\infty} 2^{-j/2}\right)^2.$$

Since $\mathbb{E}^0[T_t^c] = \frac{t}{m}$ which is independent of $c>0$, we obtain that

$$\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - K_t^{\mathrm{R}}(c)\|^2] \le \frac{3t}{m}\|\hat{\varphi}/\sqrt{\omega}\|^2\left(\sum_{j=n+1}^{\infty} 2^{-j/2}\right)^2$$

and we conclude that

$$\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - K_t^{\mathrm{R}}(c)\|^2] \to 0 \tag{4.10}$$

as $n\to\infty$ uniformly in c. Let $\varepsilon>0$ be arbitrary. There exists n_0 such that for all $n>n_0$ $\mathbb{E}^{x,0}[\|K_t^{\mathrm{R}}(c) - \mathrm{I}_n^c\|^2] < \varepsilon^2$ and $\mathbb{E}^{x,0}[\|\mathrm{I}_n - K_t\|^2] < \varepsilon^2$ uniformly in c. Now we estimate $\|\mathrm{I}_n^c - \mathrm{I}_n\|$. We have

$$\mathrm{I}_n^c - \mathrm{I}_n = \bigoplus_{\mu=1}^{3}\sum_{j=1}^{2^n}\left(\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu - \int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu\right).$$

We note that $s\to\int_a^s \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu$ and $s\to\int_s^b \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu$ are almost surely continuous. Hene

$$(S,T)\to\mathbb{E}^x\left[\left(\int_S^T \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu, \int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu\right)\right]$$

is continuous. This implies that for every j,

$$\begin{aligned}
&\mathbb{E}^{x,0}\left[\left(\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s, \int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s\right)\right] \\
&\to \mathbb{E}^{x}\left[\left(\int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s, \int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s\right)\right] = \frac{(t_j - t_{j-1})}{m}\|\hat{\varphi}/\sqrt{\omega}\|^2 \quad (4.11)
\end{aligned}$$

as $c \to \infty$. We have

$$\mathbb{E}^{x,0}[\|\mathrm{I}^c_n - \mathrm{I}_n\|^2] = 3\sum_{j=1}^{2^n}\mathbb{E}^{x,0}\left[\left\|\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\varphi(\cdot - B_s)\mathrm{d}B^\mu_s - \int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s\right\|^2\right],$$

and

$$\begin{aligned}
&\mathbb{E}^{x,0}\left[\left\|\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s - \int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s\right\|^2\right] \\
&= \mathbb{E}^{x,0}\left[\left\|\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s\right\|^2\right] + \mathbb{E}^{x,0}\left[\left\|\int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s\right\|^2\right] \\
&\quad - 2\mathbb{E}^{x,0}\left[\left(\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s, \int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s\right)\right] \\
&= \frac{1}{m}\|\hat{\varphi}/\sqrt{\omega}\|^2(\mathbb{E}^0[T^c_{t_j} - T^c_{t_{j-1}}] + t_j - t_{j-1}) \\
&\quad - 2\mathbb{E}^{x,0}\left[\left(\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s, \int_{\frac{t_{j-1}}{m}}^{\frac{t_j}{m}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B^\mu_s\right)\right].
\end{aligned}$$

Note that $\mathbb{E}^0[T^c_{t_j} - T^c_{t_{j-1}}] = t_j - t_{j-1}$ and (4.11). We can see that $\mathbb{E}^{x,0}[\|\mathrm{I}^c_n - \mathrm{I}_n\|^2] \to 0$ as $c \to \infty$. We have

$$\lim_{c\to\infty}(\mathbb{E}^{x,0}[\|K^{\mathrm{R}}_t(c) - K_t\|]^2)^{1/2} \le 2\varepsilon + \lim_{c\to\infty}(\mathbb{E}^{x,0}[\|\mathrm{I}^c_n - \mathrm{I}_n\|])^{1/2} = 2\varepsilon.$$

Thus the lemma is proven.

QED

Now we show a non-relativistic limit of the relativistic Pauli-Fierz Hamiltonian.

Theorem 4.14 ([67]) *Suppose that V is bounded and continuous. Then for every $t \ge 0$ it follows that*

$$\mathrm{s} - \lim_{c\to\infty} e^{-tH_{\mathrm{R}}} = e^{-tH}.$$

Proof: Suppose that $F, G \in C^\infty_0(\mathbb{R}^3) \otimes L^2(\mathrm{Q})$. We have

$$(F, e^{-tH_{\mathrm{R}}}G) = \int_{\mathbb{R}^3}\mathbb{E}^{x,0}\left[e^{-\int_0^t V(B_{T^c_s})\mathrm{d}s}(\mathrm{J}_0 F(x), e^{-\mathrm{i}\mathbb{A}(K^{\mathrm{R}}_t(c))}\mathrm{J}_t G(B_{T^c_t}))\right]\mathrm{d}x.$$

It follows that

$$\lim_{c\to\infty}(F, e^{-tH_{\rm R}}G) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V(B_{\frac{s}{m}}){\rm d}s} ({\rm J}_0 F(x), e^{-{\rm i}\mathbb{A}(K_t)} {\rm J}_t G(B_{\frac{t}{m}})) \right] {\rm d}x = (F, e^{-tH}G).$$

Since $H_{\rm R} \geq \inf_{x\in\mathbb{R}^3} V(x) = g > -\infty$, $e^{-tH_{\rm R}} \leq e^{-tg}$. Let $F, G \in \mathcal{H}$. There exists $F_n, G_n \in C_0^\infty(\mathbb{R}^3) \otimes L^2({\rm Q})$ such that $F_n \to F$ and $G_n \to G$ strongly as $n \to \infty$. By the uniform bound $e^{-tH_{\rm R}} \leq e^{-tg}$, we can show $\lim_{c\to\infty}(F, e^{-tH_{\rm R}}G) = (F, e^{-tH}G)$. Finally since the weak convergence of $e^{-tH_{\rm R}}$ implies the strong convergence, the theorem follows.

QED

4.5 Translation invariant relativistic Pauli-Fierz Hamiltonian

For the relativistic Pauli-Fierz Hamiltonian with $V = 0$, as well as H, it can be seen that $[H_{\rm R}, {\rm P}_\mu^{\rm T}] = 0$. This allows that there exists a self-adjoint operator $H_{\rm R}(p)$ in $\mathcal{F}$ such that

$$H_{\rm R} = \int_{\mathbb{R}^3}^{\oplus} H_{\rm R}(p){\rm d}p.$$

The self-adjoint operator $H_{\rm R}(p)$, $p \in \mathbb{R}^3$, is called the relativistic Pauli-Fierz Hamiltonian with a total momentum p. We can show the similar results to those of $H(p)$ by using the functional integral representation of $e^{-tH_{\rm R}}$. The theorem below can be proven in a similar manner to that of Theorem 3.14.

Theorem 4.15 ([65]) *Suppose Assumption 4.1 with $V = 0$. Let $\Psi, \Phi \in L^2({\rm Q})$. Then*

$$(\Psi, e^{-tH_{\rm R}(p)}\Phi) = \mathbb{E}^{0,0}\left[\left({\rm J}_0\Psi, e^{-{\rm i}\mathbb{A}(K_t^{\rm R})}{\rm J}_t e^{+{\rm i}(p-{\rm P}_{\rm f})B_{T_t}}\Phi\right)\right].$$

We can see the explicit form of the fiber Hamiltonian $H_{\rm R}(p)$. Let

$$K(p) = (p - {\rm P}_{\rm f} - \hat{A}(0))^2 + m^2.$$

Then we have

$$(\Psi, e^{-tK(p)}\Phi) = e^{-tm^2}\mathbb{E}^{0,0}\left[e^{{\rm i}pB_t}\left(\Psi, e^{-{\rm i}\hat{A}(L_t)}e^{-{\rm i}{\rm P}_{\rm f}B_t}\Phi\right)\right].$$

Let $D = D({\rm P}_{\rm f}^2) \cap D(H_{\rm f})$. Set $\bar{K}(p) = \overline{K(p)\lceil_D}$. We define $L_{\rm R}(p)$ by

$$L_{\rm R}(p) = \bar{K}(p)^{1/2} \dotplus H_{\rm f}, \quad p \in \mathbb{R}^3.$$

Theorem 4.16 ([65]) *Suppose Assumption 4.1 with $V = 0$. Then*

$$H_{\rm R} \cong \int_{\mathbb{R}^3}^{\oplus} L_{\rm R}(p){\rm d}p.$$

In particular $H_{\rm R}(p) = L_{\rm R}(p)$.

5 Pauli-Fierz model with spin $1/2$

5.1 Pauli-Fierz Hamiltonian with spin $1/2$

In this section we are concerned with the Pauli-Fierz Hamiltonian with spin 1/2. This section is due to [72], and in this section we assume Assumption 3.1 unless otherwise stated. The Hilbert space consisting of state vectors of the Pauli-Fierz Hamiltonian with spin 1/2 is

$$\mathcal{H}_S = L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes L^2(\mathrm{Q}).$$

The Pauli-Fierz Hamiltonian with spin 1/2 is formally given by

$$\frac{1}{2}\left(\sigma \cdot (-\mathrm{i}\nabla - \hat{A})\right)^2 + V + H_\mathrm{f} = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + V + H_\mathrm{f} - \frac{1}{2}\sigma \cdot \hat{B}. \tag{5.1}$$

Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the 2×2 Pauli matrices, and the quantized magnetic field $\hat{B}$ is defined by the curl of $\hat{A}$ as usual:

$$\hat{B}(x) = (\hat{B}_1(x), \hat{B}_2(x), \hat{B}_3(x)) = \mathrm{curl}_x \hat{A}(x).$$

Both sides of (5.1) are formally identical. It is straightforward to see that

$$\hat{B}_\mu(x) = \sum_{\lambda,\alpha,\nu=1}^{3} \hat{A}_\lambda(\delta_{\lambda\nu}\varepsilon^{\mu\alpha\nu}\nabla_{x_\alpha}\tilde{\varphi}(\cdot - x)), \tag{5.2}$$

where $\varepsilon^{\alpha\beta\gamma}$ denotes the antisymmetric tensor defined by

$$\varepsilon^{\alpha\beta\gamma} = \begin{cases} 1 & \alpha\beta\gamma \text{ is an even permutation of 123,} \\ -1 & \alpha\beta\gamma \text{ is an odd permutation of 123,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\hat{B}_\mu = \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \left\{ a^* \left(-\mathrm{i}(k \times e^j)_\mu \frac{\hat{\varphi}}{\sqrt{\omega}} e^{-\mathrm{i}kx}, j \right) + a^* \left(\mathrm{i}(k \times e^j)_\mu \frac{\tilde{\hat{\varphi}}}{\sqrt{\omega}} e^{\mathrm{i}kx}, j \right) \right\}.$$

In this paper the right-hand side of (5.1) is adopted as the definition of the Pauli-Fierz Hamiltonian with spin 1/2:

$$H_\mathrm{S} = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + V + H_\mathrm{f} - \frac{1}{2}\sigma \cdot \hat{B}. \tag{5.3}$$

H_S is self-adjoint on $D(-\Delta) \cap D(H_\mathrm{f})$ and bounded from below. Moreover, it is essentially self-adjoint on any core of $H_0 = H_\mathrm{p} + H_\mathrm{f}$.

5.2 Scalar representations

As in the classical case in order to construct a functional integral representation of $(F, e^{-tH_\mathrm{S}}G)$ with a scalar integrand we introduce a two-valued spin variable $\theta \in \mathbb{Z}_2$ and redefine the Pauli-Fierz Hamiltonian with spin 1/2. We identify $\mathcal{H}_\mathrm{S}$ with $L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathrm{Q})$ by

$$\mathcal{H}_\mathrm{S} \ni F = \begin{pmatrix} F(\cdot, +1) \\ F(\cdot, -1) \end{pmatrix} \cong F(\cdot, \theta) \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathrm{Q}).$$

Since

$$H_{\mathrm{S}} = \frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + V + H_{\mathrm{f}} - \frac{1}{2}\begin{pmatrix} \hat{B}_3 & \hat{B}_1 - \mathrm{i}\hat{B}_2 \\ \hat{B}_1 + \mathrm{i}\hat{B}_2 & -\hat{B}_3 \end{pmatrix},$$

H_{S} can be regarded as an operator $H_{\mathbb{Z}_2}$ acting on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathrm{Q})$:

$$(H_{\mathbb{Z}_2}F)(\theta) = \left(\frac{1}{2}(-\mathrm{i}\nabla - \hat{A})^2 + V + H_{\mathrm{f}} + \hat{H}_{\mathrm{d}}(\theta)\right) F(\theta) + \hat{H}_{\mathrm{od}}(-\theta)F(-\theta).$$

See Table 5 below.

spin	$\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	$\sigma_2 = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$	$\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
$\mathbb{Z}_2$-action	$f(\theta) \to f(-\theta)$	$f(\theta) \to -\mathrm{i}\theta f(-\theta)$	$f(\theta) \to \theta f(\theta)$

Table 5: Correspondence between spin 1/2 and $\mathbb{Z}_2$-actions

Here $\hat{H}_{\mathrm{d}}$ and $\hat{H}_{\mathrm{od}}$ denote the diagonal part and the off-diagonal part, respectively, which are explicitly given by

$$\hat{H}_{\mathrm{d}}(x,\theta) = -\frac{1}{2}\theta\hat{B}_3(x), \quad \hat{H}_{\mathrm{od}}(x,-\theta) = -\frac{1}{2}\left(\hat{B}_1(x) - i\theta\hat{B}_2(x)\right), \quad \theta \in \mathbb{Z}_2.$$

Write

$$\mathcal{K} = L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathrm{Q}).$$

We define the Pauli-Fierz Hamiltonian with spin 1/2 on $\mathcal{K}$ by $H_{\mathbb{Z}_2}$, and consider a functional integral representation of $(F, e^{-tH_{\mathbb{Z}_2}}G)$ for $(F, e^{-tH_{\mathrm{S}}}G)$.

5.3 Functional integral representations for the Pauli-Fierz Hamiltonian with spin 1/2

The idea of constructing a functional integral representation of $e^{-tH_{\mathbb{Z}_2}}$ is to use the identification:

$$\mathcal{K} \cong \int_{\mathrm{Q}}^{\oplus} L^2(\mathbb{R}^3 \times \mathbb{Z}_2)\mathrm{d}\mu(\phi). \tag{5.4}$$

In other words, we regard $\mathcal{K}$ as the set of $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$-valued L^2-functions on Q. We make the decomposition

$$H_{\mathbb{Z}_2} = \int_{\mathrm{Q}}^{\oplus} K(\phi)\mathrm{d}\mu(\phi) + H_{\mathrm{f}},$$

where $K(\phi)$ is a self-adjoint operator on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ for each $\phi \in \mathrm{Q}$. For each $\phi \in \mathrm{Q}$, we define the Hamiltonian $K(\phi)$ on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by

$$(K(\phi)F)(x,\theta) = \left(\frac{1}{2}(-\mathrm{i}\nabla - \hat{A}(\phi))^2 + V + \hat{H}_{\mathrm{d}}(\theta,\phi)\right) F(x,\theta) + \hat{H}_{\mathrm{od}}(-\theta,\phi)F(x,-\theta). \tag{5.5}$$

We construct the functional integral representation of $e^{-tH_{\mathbb{Z}_2}}$ through functional integral representations of both $e^{-tK(\phi)}$ and $e^{-tH_{\rm f}}$, and the Trotter product formula. In order to do that we will use the identity

$$(F, e^{-t\int_{\rm Q}^{\oplus} K(\phi){\rm d}\mu} G)_{\mathcal{K}} = \int_{\rm Q} \Big(F(\phi), e^{-tK(\phi)}G(\phi)\Big)_{L^2(\mathbb{R}^3\times\mathbb{Z}_2)} {\rm d}\mu(\phi),$$

while we have already done that of $(F(\phi), e^{-tK(\phi)}G(\phi))$, $\phi \in {\rm Q}$, in the classical case $(f, e^{-th_{\rm S}(a,b)}g)$ introduced in Section 2.1. To prevent the off-diagonal part $\hat{H}_{\rm od}$ vanishes we introduce a regularization $H_{\mathbb{Z}_2,\varepsilon}$ of $H_{\mathbb{Z}_2}$ by

$$H_{\mathbb{Z}_2,\varepsilon}F(\theta) = \left(\frac{1}{2}(-{\rm i}\nabla - \hat{A})^2 + V + H_{\rm f} + \hat{H}_{\rm d}(\theta)\right)F(\theta) + \Psi_\varepsilon(\hat{H}_{\rm od}(-\theta))F(-\theta),$$

where $\Psi_\varepsilon(X) = X + \varepsilon\psi_\varepsilon(|X|)$ and $\psi_\varepsilon \in C_0^\infty(\mathbb{R})$ is given by

$$\psi_\varepsilon(z) = \psi_\varepsilon(|z|) = \begin{cases} 1, & |z| < \varepsilon/2, \\ \leq 1, & \varepsilon/2 \leq |z| \leq \varepsilon, \\ 0, & |z| > \varepsilon. \end{cases}$$

Also, let $K_\varepsilon(\phi)$ be the counterpart of $K(\phi)$ with $\hat{H}_{\rm od}(\phi)$ replaced by $\Psi_\varepsilon(\hat{H}_{\rm od}(\phi))$, i.e.,

$$(K_\varepsilon(\phi)F)(x,\theta) = \left(\frac{1}{2}(-{\rm i}\nabla - \hat{A}(\phi))^2 + V + \hat{H}_{\rm d}(\theta,\phi)\right)F(x,\theta) + \Psi_\varepsilon(\hat{H}_{\rm od}(-\theta,\phi))F(x,-\theta).$$

Recall that $\theta_t = (-1)^{N_t}$ and $(X_t)_{t\geq 0} = (B_t, \theta_t)_{t\geq 0}$ is the $(\mathbb{R}^3\times\mathbb{Z}_2)$-valued random process on $\mathscr{X}\times\mathscr{X}_\mu$. If $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$, then for every $\phi \in {\rm Q}$, $K_\varepsilon(\phi)$ is self-adjoint on $D(-\Delta)\otimes\mathbb{Z}_2$ and for $g \in L^2(\mathbb{R}^3\times\mathbb{Z}_2)$,

$$(e^{-tK_\varepsilon(\phi)}g)(x,\alpha) = e^t\mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s){\rm d}s}e^{Z_t(\phi,\varepsilon)}g(X_t)], \quad (x,\alpha) \in \mathbb{R}^3\times\mathbb{Z}_2,$$

where

$$\begin{aligned} Z_t(\phi,\varepsilon) = &-{\rm i}\int_0^t \hat{A}(\tilde{\varphi}(\cdot - B_s),\phi){\rm d}B_s \\ &- \int_0^t \hat{H}_{\rm d}(B_s, \theta_{N_s}, \phi){\rm d}s + \int_0^{t+} \log\Big(-\Psi_\varepsilon(\hat{H}_{\rm od}(B_s, -\theta_{N_{s-}}, \phi))\Big){\rm d}N_s. \end{aligned}$$

Next we define an operator $K_\varepsilon(\hat{A})$ on $\mathcal{K}$ through $K_\varepsilon(\phi)$ and the constant fiber direct integral representation (5.4) of $\mathcal{K}$. Take $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$ and define the self-adjoint operator $K_\varepsilon(\hat{A})$ on $\mathcal{K}$ by

$$K_\varepsilon(\hat{A}) = \int_{\rm Q}^{\oplus} K_\varepsilon(\phi){\rm d}\mu(\phi).$$

Then we can define the self-adjoint operator K_ε by

$$K_\varepsilon = K_\varepsilon(\hat{A}) \dot{+} H_{\rm f}.$$

In what follows we construct a functional integral representation of e^{-tK_ε} and show that $e^{-tK_\varepsilon} = e^{-tH_{\mathbb{Z}_2,\varepsilon}}$. Let us define a dense subspace by $\mathcal{K}_\infty = C_0^\infty(\mathbb{R}^3\times\mathbb{Z}_2)\hat{\otimes}L^2_{\rm fin}({\rm Q})$.

Lemma 5.1 *Under Assumption 3.1 it follows that*

$$s - \lim_{\varepsilon \downarrow 0} e^{-tH_{\mathbb{Z}_2,\varepsilon}} = e^{-tH_{\mathbb{Z}_2}}. \tag{5.6}$$

Suppose that $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$. *Then* $H_{\mathbb{Z}_2,\varepsilon} = K_\varepsilon$ *and in particular it follows that*

$$(F, e^{-tH_{\mathbb{Z}_2}} G) = \lim_{\varepsilon \downarrow 0} (F, e^{-tK_\varepsilon} G). \tag{5.7}$$

Proof: It is seen that $K_\varepsilon = H_{\mathbb{Z}_2,\varepsilon}$ on $\mathcal{K}_\infty$, implying that $K_\varepsilon = H_{\mathbb{Z}_2,\varepsilon}$ as a self-adjoint operator since $\mathcal{K}_\infty$ is a core of $H_{\mathbb{Z}_2,\varepsilon}$. Moreover, $H_{\mathbb{Z}_2,\varepsilon} \to H_{\mathbb{Z}_2}$ on $\mathcal{K}_\infty$ as $\varepsilon \to 0$ and $\mathcal{K}_\infty$ is a common core of the sequence $\{H_{\mathbb{Z}_2,\varepsilon}\}_{\varepsilon \geq 0}$. Thus (5.6) and (5.7) follow.

QED

By Lemma 5.1 to give the functional integral representation of $(F, e^{-tH_{\mathbb{Z}_2}} G)$ for any $\hat{\varphi}$ given in Assumption 3.1 it suffices to construct a functional integral representation of the right-hand side of (5.7) and to take an approximation argument on $\hat{\varphi}$. To obtain the functional integral representation of e^{-tK_ϵ}, we apply the Trotter product formula as usual, i.e.,

$$e^{-tK_\varepsilon} = s - \lim_{n \to \infty} (e^{-\frac{t}{n} K_\epsilon(\hat{A})} e^{-\frac{t}{n} H_{\mathrm{f}}})^n = s - \lim_{n \to \infty} \mathrm{J}_0^* \Big(\prod_{i=0}^{n-1} \mathrm{J}_{\frac{it}{n}} e^{-\frac{t}{n} K_\varepsilon(\hat{A})} \mathrm{J}_{\frac{it}{n}}^* \Big) \mathrm{J}_t.$$

The Euclidean version of $\hat{B}_\mu(g)$ with test function $g \in L^2(\mathbb{R}^4)$ is defined by

$$\mathbb{B}_\mu(g) = \sum_{\lambda,\alpha,\nu=1}^{3} \mathbb{A}_\lambda(\delta_{\lambda\nu} \varepsilon^{\mu\alpha\nu} \nabla_{x_\alpha} g). \tag{5.8}$$

Here $\nabla_{x_\alpha} g = \nabla_{x_\alpha} g(\cdot - x)\lceil_{x=0}$, hence $\widehat{\nabla_{x_\alpha} g} = -\mathrm{i} k_\alpha \hat{g}(k)$. Define the Euclidean versions of $\hat{H}_{\mathrm{d}}(x, \theta)$ and $\hat{H}_{\mathrm{od}}(x, -\theta)$ by

$$\mathbb{H}_{\mathrm{d}}(x, \theta, s) = -\frac{1}{2} \theta \mathbb{B}_3 \left(\mathrm{j}_s \tilde{\varphi}(\cdot - x) \right), \tag{5.9}$$

$$\mathbb{H}_{\mathrm{od}}(x, -\theta, s) = -\frac{1}{2} (\mathbb{B}_1(\mathrm{j}_s \tilde{\varphi}(\cdot - x)) - \mathrm{i}\theta \mathbb{B}_2(\mathrm{j}_s \tilde{\varphi}(\cdot - x))), \tag{5.10}$$

respectively.

5.4 Technical estimates

This section is a brief version of [73, Sections 3.8.7 and 3.8.8]. To avoid complicated computations we introduce self-adjoint operators L and L_R for $R > 0$ which satisfy that

$$(|F|, e^{-tL_R} |G|)| \uparrow (|F|, e^{-tL} |G|) \quad \text{as } R \uparrow \infty$$

and diamagnetic type inequality:

$$|(F, e^{-tH_{\mathbb{Z}_2}} G)| \leq (|F|, e^{-tL} |G|).$$

See Table 6. Applying this inequality we can avoid technical difficulties to construct the functional integral representation of $(F, e^{-tH_{\mathbb{Z}_2}} G)$. We introduce two cutoff functions for each $R > 0$. Let $\chi_R^- \in C^\infty(\mathbb{R})$ be defined by

(1) $\chi_R^-(x) = x$ for $x > -R+1$,

(2) $-R \le \chi_R^-(x) \le -R+1$ for $-R \le x \le -R+1$,

(3) $\chi_R^-(x) = -R$ for $x < -R$.

We also define $\chi_R^+ \in C^\infty(\mathbb{R})$ by

(1) $\chi_R^+(x) = R$ for $x > R$,

(2) $R-1 \le \chi_R^+(x) \le R$ for $R-1 \le x \le R$,

(3) $\chi_R^+(x) = x$ for $x < R-1$.

Furthermore note that $|\chi_R^\pm(x) - \chi_R^\pm(y)| \le c|x-y|$ with some c for any $x, y \in \mathbb{R}$. We define $\Psi_\varepsilon(|\hat{H}_{\rm od}(-\theta)|)_R$ and $\hat{H}_{\rm d}(\theta)_R$ by

$$\Psi_\varepsilon(|\hat{H}_{\rm od}(-\theta)|)_R = \chi_R^+\left(\Psi_\varepsilon(|\hat{H}_{\rm od}(-\theta)|)\right), \tag{5.11}$$

$$\hat{H}_{\rm d}(\theta)_R = \chi_R^-\left(\hat{H}_{\rm d}(\theta)\right). \tag{5.12}$$

Hence $-R \le \hat{H}_{\rm d}(\theta)_R$ and $\varepsilon \le \Psi_\varepsilon(|\hat{H}_{\rm od}(-\theta)|)_R \le R$. Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$. We set for each $\phi \in {\rm Q}$,

$$(M_R(\phi)F)(\theta) = (H_{\rm p} + \hat{H}_{\rm d}(\phi,\theta)_R)F(\theta) - \Psi_\varepsilon(|\hat{H}_{\rm od}(\phi,-\theta)|)_R F(-\theta)$$

for each $R > 0$, and

$$(M(\phi)F)(\theta) = (H_{\rm p} + \hat{H}_{\rm d}(\phi,\theta))F(\theta) - \Psi_\varepsilon(|\hat{H}_{\rm od}(\phi,-\theta)|)F(-\theta).$$

Then $M_R(\phi)$ and $M(\phi)$ are self-adjoint on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ and we define

$$M_R = \int_{\rm Q}^{\oplus} M_R(\phi) d\mu(\phi), \quad M = \int_{\rm Q}^{\oplus} M(\phi) d\mu(\phi).$$

We also define

$$L_R = M_R \dot{+} H_{\rm f}, \qquad L = M \dot{+} H_{\rm f}.$$

In the Fock representation L_R and L are given by

$$L_R = H_{\rm p} + \hat{H}_{\rm f} - \begin{pmatrix} (\frac{1}{2}\hat{B}_3)_R & \Psi_\varepsilon\left(\frac{1}{2}\sqrt{\hat{B}_1^2 + \hat{B}_2^2}\right)_R \\ \Psi_\varepsilon\left(\frac{1}{2}\sqrt{\hat{B}_1^2 + \hat{B}_2^2}\right)_R & -(\frac{1}{2}\hat{B}_3)_R \end{pmatrix},$$

$$L = H_{\rm p} + \hat{H}_{\rm f} - \begin{pmatrix} \frac{1}{2}\hat{B}_3 & \Psi_\varepsilon\left(\frac{1}{2}\sqrt{\hat{B}_1^2 + \hat{B}_2^2}\right) \\ \Psi_\varepsilon\left(\frac{1}{2}\sqrt{\hat{B}_1^2 + \hat{B}_2^2}\right) & -\frac{1}{2}\hat{B}_3 \end{pmatrix}.$$

In the Fock representation we have $|\hat{H}_{\rm od}(-\theta)| = \frac{1}{2}\sqrt{\hat{B}_1^2 + \hat{B}_2^2}$ which leads that $|\hat{H}_{\rm od}(-\theta)|$ is independent of θ. The family of self-adjoint operators L_R, $R > 0$, have also a common core $\mathcal{K}_\infty$ and $\lim_{R\to\infty} L_R F = LF$ for $F \in \mathcal{K}_\infty$. Hence $e^{-tL_R} \to e^{-tL}$ strongly as $R \to \infty$. A functional integral representation of e^{-tL_R} can be done by the Trotter product formula.

Operator	Exponent
e^{-tM_R}	$L(t,\varepsilon,R) = -\int_0^t \hat{H}_\mathrm{d}(B_s,\theta_{N_s})_R \mathrm{d}s + \int_0^{t+} \log\Psi_\varepsilon(\lvert\hat{H}_\mathrm{od}(B_s,-\theta_{N_s})\rvert)_R \mathrm{d}N_s$
$\mathrm{J}_s e^{-tM_R}\mathrm{J}_s^*$	$L(t,s,\varepsilon,R) = -\int_0^t \mathbb{H}_\mathrm{d}(B_r,\theta_{N_r},s)_R \mathrm{d}r + \int_0^{t+} \log\Psi_\varepsilon(\lvert\mathbb{H}_\mathrm{od}(B_r,-\theta_{N_r},s)\rvert)_R \mathrm{d}N_r$
e^{-tL_R}	$Y_R = -\int_0^t \mathbb{H}_\mathrm{d}(B_s,\theta_{N_s},s)_R \mathrm{d}s + \int_0^{t+} \log\Psi_\varepsilon(\lvert\mathbb{H}_\mathrm{od}(B_s,-\theta_{N_s},s)\rvert)_R \mathrm{d}N_s$
e^{-tL}	$Y = -\int_0^t \mathbb{H}_\mathrm{d}(B_s,\theta_{N_s},s) \mathrm{d}s + \int_0^{t+} \log\Psi_\varepsilon(\lvert\mathbb{H}_\mathrm{od}(B_s,-\theta_{N_s},s)\rvert) \mathrm{d}N_s$
$e^{-tH_{\mathbb{Z}_2,\varepsilon}}$	$Z_t(\varepsilon) = -\mathrm{i}\mathbb{A}(K_t) - \int_0^t \mathbb{H}_\mathrm{d}(B_s,\theta_{N_s},s) \mathrm{d}s + \int_0^{t+} \log\Psi_\varepsilon(-\mathbb{H}_\mathrm{od}(B_s,-\theta_{N_s},s)) \mathrm{d}N_s$
$e^{-tH_{\mathbb{Z}_2}}$	$Z_t = -\mathrm{i}\mathbb{A}(K_t) - \int_0^t \mathbb{H}_\mathrm{d}(B_s,\theta_{N_s},s) \mathrm{d}s + \int_0^{t+} \log\Psi_\varepsilon(-\mathbb{H}_\mathrm{od}(B_s,-\theta_{N_s},s)) \mathrm{d}N_s$

Table 6: List of operators and exponents

5.4.1 Estimate of $\mathrm{J}_s e^{-tM_R}\mathrm{J}_s^*$

Let

$$L(t,s,\varepsilon,R) = -\int_0^t \mathbb{H}_\mathrm{d}(B_r,\theta_{N_r},s)_R \mathrm{d}r + \int_0^{t+} \log\Psi_\varepsilon(\lvert\mathbb{H}_\mathrm{od}(B_r,-\theta_{N_r},s)\rvert)_R \mathrm{d}N_r.$$

Here $\mathbb{H}_\mathrm{d}(x,\theta,s)$ and $\mathbb{H}_\mathrm{od}(x,-\theta,s)$ are given by (5.9) and (5.10), respectively, and truncated functions $\mathbb{H}_\mathrm{d}(B_r,\theta_{N_r},s)_R$ and $\Psi_\varepsilon(\lvert\mathbb{H}_\mathrm{od}(x,-\theta,s)\rvert)_R$ are defined in a similar way to (5.11) and (5.12), respectively. Let $\mathcal{K}_\mathrm{E} = L^2(\mathbb{R}^3)\otimes L^2(\mathrm{Q_E})$.

Lemma 5.2 *Assume that $V\in L^\infty(\mathbb{R}^3)$ and $\tilde{\varphi}\in C_0^\infty(\mathbb{R}^3)$. Let $F,G\in\mathcal{K}_\mathrm{E}$. Then*

$$\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_r)\mathrm{d}r}(F(q_0),\mathrm{E}_s e^{L(t,s,\varepsilon,R)}\mathrm{E}_s G(q_t))\right]\mathrm{d}x$$

is finite and

$$(F,\mathrm{J}_s e^{-tM_R}\mathrm{J}_s^* G) = e^t\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_r)\mathrm{d}r}(F(q_0),\mathrm{E}_s e^{L(t,s,\varepsilon,R)}\mathrm{E}_s G(q_t))\right]\mathrm{d}x. \tag{5.13}$$

Proof: Let us set $V_\infty = \sup_{x\in\mathbb{R}^3}\mathbb{E}^x[e^{-2\int_0^t V(B_s)\mathrm{d}s}] < \infty$. Notice that the right-hand side of (5.13) is finite, since $e^{L(t,s,\varepsilon,R)}$ is a bounded function. For each $(x,\alpha,w,m)\in\mathbb{R}^3\times\mathbb{Z}_2\times\mathscr{X}\times\mathscr{X}_\mu$, we have

$$\lvert(F(q_0),\mathrm{E}_s e^{L(t,s,\varepsilon,R)}\mathrm{E}_s G(q_t))\rvert \le \|F(q_0)\|\|G(q_t)\|\|e^{L(t,s,\varepsilon,R)}\|.$$

Here $\|e^{L(t,s,\varepsilon,R)}\|$ is the operator norm of bounded operator $e^{L(t,s,\varepsilon,R)}$ on $L^2(\mathrm{Q_E})$. Then it follows that

$$\lvert\mathrm{RHS}(5.13)\rvert \le e^t\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_r)\mathrm{d}r}\|F(q_0)\|\|G(q_t)\|\|e^{L(t,s,\varepsilon,R)}\|\right]\mathrm{d}x. \tag{5.14}$$

We will also prove in Lemma 5.3 below that there exists a random variable A_t on $(\mathscr{X}_\mu, \mathcal{B}_\mu, \mu)$ such that

(1) $\|e^{L(t,s,\varepsilon,R)}\| \le A_t$,

(2) A_t is independent of $(x, \alpha, w) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X}$,

(3) $\mathbb{E}_\mu[A_t^2] < \infty$.

By (1), (2) and (3) above and (5.14),

$$|\mathrm{RHS}\ (5.13)| \le \|G\|\|F\| V_\infty^{1/2} (\mathbb{E}_\mu[A_t^2])^{1/2} < \infty. \tag{5.15}$$

Next we prove the equality (5.13). Note that M_R is defined by a direct fiber integral representation we have

$$(\mathrm{J}_s^* F, e^{-tM_R} \mathrm{J}_s^* G) = e^t \int_{\mathbb{R}^3} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}^{x,\alpha} \left[e^{-\int_0^t V(B_r)\mathrm{d}r} \mathbb{E}_{\mu_{\mathrm{E}}} \left[\overline{(\mathrm{J}_s^* F)(\phi, q_0)} e^{L_t(\phi,\varepsilon)} (\mathrm{J}_s^* G)(\phi, q_t) \right] \right] \mathrm{d}x.$$

Here we used Fubini's lemma and

$$L_t(\phi, \varepsilon) = -\int_0^t \hat{H}_{\mathrm{d}}(\phi, B_s, \theta_{N_s})_R \mathrm{d}s + \int_0^{t+} \log \Psi_\varepsilon(|\hat{H}_{\mathrm{od}}(\phi, B_s, -\theta_{N_s})|)_R \mathrm{d}N_s.$$

Let

$$L(t, \varepsilon, R) = -\int_0^t \hat{H}_{\mathrm{d}}(B_s, \theta_{N_s})_R \mathrm{d}s + \int_0^{t+} \log \Psi_\varepsilon(|\hat{H}_{\mathrm{od}}(B_s, -\theta_{N_s})|)_R \mathrm{d}N_s.$$

We rewrite as

$$(\mathrm{J}_s^* F, e^{-tM_R} \mathrm{J}_s^* G) = e^t \int_{\mathbb{R}^3} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}^{x,\alpha} \left[e^{-\int_0^t V(B_r)\mathrm{d}r} (F(q_0), \mathrm{J}_s e^{L(t,\varepsilon,R)} \mathrm{J}_s^* G(q_t)) \right] \mathrm{d}x.$$

We can see that $\mathrm{J}_s e^{L(t,\varepsilon,R)} \mathrm{J}_s^* = \mathrm{E}_s e^{L(t,s,\varepsilon,R)} \mathrm{E}_s$ by a limiting argument, leading to (5.13). Then the proof can be completed.

QED

Now we have to prove (1), (2) and (3) used in the proof of Lemma 5.2.

Lemma 5.3 *For each $t \ge 0$, operator $e^{L(t,s,\varepsilon,R)}$ is bounded, and there exists a random variable A_t on $(\mathscr{X}_\mu, \mathcal{B}_\mu, \mu)$ satisfying* (1)-(3) *in the proof of Lemma 5.2.*

Proof: Since $R < \infty$, we have $|e^{L(t,s,\varepsilon,R)}| \le e^{tR} e^{\int_0^{t+} \log R \mathrm{d}N_s}$. For each $m \in \mathscr{X}_\mu$, the number of jumps of map $s \mapsto N_s(m)$ for $0 \le s \le t$ is denoted by $N(m)$. Hence $\int_0^{t+} \log R \mathrm{d}N_s = \log R^{N(m)}$ and then $|e^{L(t,s,\varepsilon,R)}| \le e^{tR} R^{N(m)}$. Set $A_t = A_t(m) = e^{tR} R^{N(m)}$. Then $\mathbb{E}_\mu[A_t^2] = e^{2tR} \sum_{N=0}^\infty \frac{t^N R^{2N}}{N!} e^{-t} = e^{t(R^2+2R-1)} < \infty$. Then A_t satisfies (1), (2) and (3).

QED

5.4.2 Estimate of e^{-tL_R}

In the next lemma we construct a functional integral representation of e^{-tL_R} for $R>0$. Let

$$Y_R = -\int_0^t \mathbb{H}_{\mathrm{d}}(B_s, \theta_{N_s}, s)_R \mathrm{d}s + \int_0^{t+} \log \Psi_\varepsilon(|\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, s)|)_R \mathrm{d}N_s.$$

Lemma 5.4 *Suppose that $V \in L^\infty(\mathbb{R}^3)$ and $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$. Then for every $t \geq 0$ and all $F, G \in \mathcal{K}$ it follows that $\mathrm{J}_0^* e^{Y_R} \mathrm{J}_t$ is a bounded operator on $L^2(\mathrm{Q})$ for every $(x,\alpha,w,m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$ and*

$$(F, e^{-tL_R}G) = e^t \int_{\mathbb{R}^3} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_s)\mathrm{d}s}(F(q_0), \mathrm{J}_0^* e^{Y_R} \mathrm{J}_t G(q_t))\right] \mathrm{d}x. \tag{5.16}$$

Proof: In a similar manner to the proof of Lemma 5.3 for each $m \in \mathscr{X}_\mu$, the number of jumps of map $s \mapsto N_s(m)$ for $0 \leq s \leq t$ is denoted by $N(m)$. Fix $m \in \mathscr{X}_\mu$. Hence by virtue of cutoff parameter $R>0$ it can be seen that $\|\mathrm{J}_0^* e^{Y_R} \mathrm{J}_t \Phi\| \leq e^{tR} R^{N(m)} \|\Phi\|$, which implies that $\mathrm{J}_0^* e^{Y_R} \mathrm{J}_t$ is bounded for every $(x,\alpha,w,m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$.

By the Trotter product formula, the Markov property of $\mathrm{J}_s \mathrm{J}_s^* = \mathrm{E}_s$ and Lemma 5.2 we can see that

$$\begin{aligned}(F, e^{-tL_R}G) &= \lim_{n\to\infty} (F, \prod_{j=0}^{n} \left(e^{-\frac{t}{n}M_R} e^{-\frac{t}{n}H_{\mathrm{f}}}\right)^n G) = \lim_{n\to\infty} (F, \mathrm{J}_0^* \prod_{j=0}^{n-1} \left(\mathrm{J}_{\frac{jt}{n}} e^{-\frac{t}{n}M_R} \mathrm{J}_{\frac{jt}{n}}^*\right) \mathrm{J}_t G) \\ &= \lim_{n\to\infty} \sum_{\alpha\in\mathbb{Z}_2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_s)\mathrm{d}s}(F(q_0), \mathrm{J}_0^* e^{Y_R(n)} \mathrm{J}_t G(q_t))\right] \mathrm{d}x,\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}Y_R(n) = &-\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \mathbb{H}_{\mathrm{d}}\left(B_s, \theta_{N_s}, \frac{(i-1)t}{n}\right)_R \mathrm{d}s \\ &+ \sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}+} \log \Psi_\varepsilon\left(|\mathbb{H}_{\mathrm{od}}\left(B_s, -\theta_{N_{s-}}, \frac{(i-1)t}{n}\right)|\right)_R \mathrm{d}N_s.\end{aligned}$$

Then

$$\begin{aligned}&\left|\sum_{\alpha\in\mathbb{Z}_2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_s)\mathrm{d}s}(F(q_0), \mathrm{J}_0^* e^{Y_R(n)} \mathrm{J}_t G(q_t))\right] \mathrm{d}x\right| \\ &\leq \sum_{\alpha\in\mathbb{Z}_2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_s)\mathrm{d}s} \|F(q_0)\| \|G(q_t))\| e^{tR} R^{N(\cdot)}\right] \mathrm{d}x \leq \|F\| \|G\| V_\infty^{1/2} e^{\frac{t}{2}(R^2+2R-1)} < \infty.\end{aligned} \tag{5.18}$$

Since the right-hand side above is independent of n, it is enough to prove (5.16) for $F, G \in \mathcal{K}_\infty$, hence we suppose that $F, G \in \mathcal{K}_\infty$ in what follows in this proof. We can see that

$$\left\|\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \mathbb{H}_{\mathrm{d}}\left(B_s, \theta_{N_s}, \frac{(i-1)t}{n}\right)_R \mathrm{d}sF\right\| \leq tC \||k|\hat{\varphi}/\sqrt{\omega}\| \|(N + \mathbb{1})^{1/2} F\|$$

and similarly

$$\left\| \int_0^t \mathbb{H}_{\mathrm{d}} \left(B_s, \theta_{N_s}, s\right)_R \mathrm{d}s F \right\| \leq tC \| |k|\hat{\varphi}/\sqrt{\omega} \| \| (N + \mathbb{1})^{1/2} F \|$$

with some constant C. By $|\chi_R^-(x) - \chi_R^-(y)| \leq c|x-y|$ with some constant c, we can also see that

$$\begin{aligned}
&\left\| \left(\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \mathbb{H}_{\mathrm{d}} \left(B_s, \theta_{N_s}, \frac{(i-1)t}{n}\right)_R \mathrm{d}s - \int_0^t \mathbb{H}_{\mathrm{d}} \left(B_s, \theta_{N_s}, s\right)_R \mathrm{d}s \right) F \right\| \\
&\leq c \sum_{i=1}^n \left(\int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \left\| |\mathbb{H}_{\mathrm{d}} \left(B_s, \theta_{N_s}, \frac{(i-1)t}{n}\right) - \mathbb{H}_{\mathrm{d}}(B_s, \theta_{N_s}, s)| F \right\|^2 \mathrm{d}s \right)^{1/2} \frac{\sqrt{t}}{\sqrt{n}}.
\end{aligned}$$

It can be seen that

$$\left\| \left| \mathbb{H}_{\mathrm{d}} \left(B_s, \theta_{N_s}, \frac{(i-1)t}{n}\right) - \mathbb{H}_{\mathrm{d}}(B_s, \theta_{N_s}, s) \right| F \right\|^2 \leq c' \frac{t}{n} \| |k|\hat{\varphi} \|^2 \| (N + \mathbb{1})^{1/2} F \|^2$$

with some constant c'. Then

$$\begin{aligned}
&\left\| \left(\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \mathbb{H}_{\mathrm{d}} \left(B_s, \theta_{N_s}, \frac{(i-1)t}{n}\right)_R \mathrm{d}s - \int_0^t \mathbb{H}_{\mathrm{d}}(B_s, \theta_{N_s}, s)_R \mathrm{d}s \right) F \right\| \\
&\leq cc' \| |k|\hat{\varphi} \| \frac{t\sqrt{t}}{\sqrt{n}} \| (N + \mathbb{1})^{1/2} F \| \to 0
\end{aligned}$$

as $n \to \infty$. Thus it can be straightforwardly seen that

$$\exp\left(- \sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \mathbb{H}_{\mathrm{d}} \left(B_s, \theta_{N_s}, \frac{(i-1)t}{n}\right)_R \mathrm{d}s \right) F \to \exp\left(- \int_0^t \mathbb{H}_{\mathrm{d}}(B_s, \theta_{N_s}, s)_R \mathrm{d}s \right) F. \tag{5.19}$$

Next we consider

$$\exp\left(\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}+} \log \Psi_\varepsilon (|\mathbb{H}_{\mathrm{od}} \left(B_s, -\theta_{N_{s-}}, \frac{(i-1)t}{n})|\right)_R \mathrm{d}N_s \right).$$

Points of discontinuity of map $r \mapsto N_r(m)$ are denoted by $s_1 = s_1(m), \ldots, s_N = s_N(m) \in (0, \infty)$. For a sufficiently large n the number of discontinuous points in interval $(\frac{(i-1)t}{n}, \frac{it}{n}]$ is at most one. Then by taking n large enough and denoting $(n(s_i), n(s_i) + \frac{t}{n}]$ for the interval containing s_i, we get

$$\exp\left(\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}+} \log \Psi_\varepsilon \left(|\mathbb{H}_{\mathrm{od}} \left(B_s, -\theta_{N_{s-}}, \frac{(i-1)t}{n}\right)| \right)_R \mathrm{d}N_s \right) = \prod_{i=1}^N \chi_R^+(|\phi_i| + \varepsilon\psi_\varepsilon(|\phi_i|)),$$

where $\phi_i = \mathbb{H}_{\mathrm{od}}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i))$. By the Lipschitz continuity of χ_R^+ and ψ_ε we have

$$\begin{aligned}
&\left\| \left(\Psi_\varepsilon(|\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, n(s_i))|)_R - \Psi_\varepsilon(|\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, s_i)|)_R \right) F \right\| \\
&\leq C \| \left(|\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, n(s_i))| - |\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, s_i)| \right) F \|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\| \left(|\mathbb{H}_{\rm od}(B_s, -\theta_{N_{s-}}, n(s_i))| - |\mathbb{H}_{\rm od}(B_s, -\theta_{N_{s-}}, s_i)|\right) F\|^2 \\
&\leq C\||k|\hat{\varphi}\|^2 |n(s_i) - s_i| \|(N + \mathbb{1})^{1/2} F\|^2
\end{aligned}$$

with some $C > 0$. Clearly, $n(s_i) \to s_i$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i)) = \mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, s_i)$$

on $\mathcal{K}_\infty$ and we have

$$\begin{aligned}
&\lim_{n\to\infty} \exp\left(\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}+} \log \Psi_\varepsilon\left(|\mathbb{H}_{\rm od}\left(B_s, -\theta_{N_{s-}}, \frac{(i-1)t}{n}\right)|\right)_R {\rm d}N_s\right) \\
&= \prod_{i=1}^N \Psi_\varepsilon(|\mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, s_i)|)_R = \exp\left(\int_0^{t+} \log \Psi_\varepsilon(|\mathbb{H}_{\rm od}(B_s, -\theta_{N_{s-}}, s|)_R {\rm d}N_s\right). \qquad (5.20)
\end{aligned}$$

Then by (5.19) and (5.20) we can see that for each $(x, \alpha, w, m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$,

$$\left(F(q_0), {\rm J}_0^* e^{Y_R(n)} {\rm J}_t G(q_t)\right) \to \left(F(q_0), {\rm J}_0^* e^{Y_R} {\rm J}_t G(q_t)\right)$$

as $n \to \infty$. Together with (5.18), the Lebesgue dominated convergence theorem yields that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}\left[e^{-\int_0^t V(B_r){\rm d}r} \left(F(q_0), {\rm J}_0^* e^{Y_R(n)} {\rm J}_t G(q_t)\right)\right] {\rm d}x \\
&\to \int_{\mathbb{R}^3} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}\left[e^{-\int_0^t V(B_r){\rm d}r} \left(F(q_0), {\rm J}_0^* e^{Y_R} {\rm J}_t G(q_t)\right)\right] {\rm d}x
\end{aligned}$$

for $F, G \in \mathcal{K}_\infty$. Then the proof is complete.

QED

5.4.3 Estimate of e^{-tL}

In this section we estimate the integral kernel of e^{-tL} by using the Baker-Campbell-Hausdorff formula and Theorem B.6 in Appendix B. Let us define

$$Y = -\int_0^t \mathbb{H}_{\rm d}(B_s, \theta_{N_s}, s){\rm d}s + \int_0^{t+} \log \Psi_\varepsilon(|\mathbb{H}_{\rm od}(B_s, -\theta_{N_{s-}}, s)|){\rm d}N_s. \qquad (5.21)$$

We have already seen that ${\rm J}_0^* e^{Y_R} {\rm J}_t$ is bounded for each $R < \infty$, but it is not trivial to see that ${\rm J}_0^* e^Y {\rm J}_t$ is bounded.

Proposition 5.5 ([73, Lemma 3.93],[72, Lemma 4.9]) *For each $(x, \alpha, w, m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$, operator ${\rm J}_0^* e^Y {\rm J}_t$ is bounded in $L^2({\rm Q})$.*

Proof: Let $Y = Y_1 + Y_2$, where

$$\begin{aligned}
Y_1 &= -\int_0^t \mathbb{H}_{\rm d}(B_r, \theta_{N_r}, s)_R {\rm d}r, \\
Y_2 &= \int_0^{t+} \log(\Psi_\varepsilon(|\mathbb{H}_{\rm od}(B_r, -\theta_{N_{r-}}, s)|)){\rm d}N_r.
\end{aligned}$$

Using Baker-Campbell-Hausdorff formula we expand $\mathrm{J}_0^* e^Y \mathrm{J}_t$. Thus formally we have to estimate

$$\mathrm{J}_0^* e^Y \mathrm{J}_t = \mathrm{J}_0^* e^{\frac{B^\dagger}{\sqrt{2}}+\frac{B}{\sqrt{2}}} \prod_{j=1}^N \left(\left| \frac{C_j^\dagger}{\sqrt{2}} + \frac{C_j}{\sqrt{2}} \right| + \varepsilon\psi_\varepsilon(\left| \frac{C_j^\dagger}{\sqrt{2}} + \frac{C_j}{\sqrt{2}} \right|) \right) \mathrm{J}_t, \tag{5.22}$$

where in the Fock representation $B^\sharp$ and $C_i^\sharp$ are defined by

$$\begin{aligned}
B^\dagger &= -a_{\mathrm{E}}^* \left(\bigoplus_{j=\pm 1} \int_0^t S_j(\theta_{N_r}) \hat{\mathrm{j}}_s \left(\frac{\hat{\varphi}}{\sqrt{\omega}} \right) e^{-\mathrm{i}kB_r} \mathrm{d}r \right), \\
B &= -a_{\mathrm{E}} \left(\bigoplus_{j=\pm 1} \int_0^t S_j(\theta_{N_r}) \hat{\mathrm{j}}_s \left(\frac{\hat{\varphi}}{\sqrt{\omega}} \right) e^{\mathrm{i}kB_r} \mathrm{d}r \right), \\
C_i^\dagger &= a_{\mathrm{E}}^* \left(\bigoplus_{j=\pm 1} T_j(\theta_{N_{s_i}}) \hat{\mathrm{j}}_s \left(\frac{\hat{\varphi}}{\sqrt{\omega}} \right) e^{-\mathrm{i}kB_{s_i}} \right), \\
C_i &= a_{\mathrm{E}} \left(\bigoplus_{j=\pm 1} T_j(\theta_{N_{s_i}}) \hat{\mathrm{j}}_s \left(\frac{\hat{\varphi}}{\sqrt{\omega}} \right) e^{\mathrm{i}kB_{s_i}} \right).
\end{aligned}$$

Here

$$\begin{aligned}
S_j(\theta) &= -\frac{1}{2}\theta\eta_3^j, \\
T_j(\theta) &= -\frac{1}{2}(\eta_1^j - \mathrm{i}\theta\eta_2^j)
\end{aligned}$$

with $\eta^j = -\mathrm{i}k \times e^j(k)$. Note that $\|\psi_\varepsilon(\left|\frac{C_j^\dagger}{\sqrt{2}} + \frac{C_j}{\sqrt{2}}\right|)\| \leq 1$ for any j. Now fix $(w, m) \in \mathscr{X} \times \mathscr{X}_\mu$. In a similar way to the case of $R < \infty$ there exist $N = N(m) \in \mathbb{N}$ and points $0 \leq s_1 < \ldots < s_N \leq t$, $s_j = s_j(m)$, $j = 1, ..., N$, depending on m such that $s \mapsto N_s(m)$ is not continuous. Then by taking n large enough and denoting $(n(s_i), n(s_i) + t/n]$ for the interval containing s_i, we get

$$e^{Y_2} = \prod_{i=1}^N (|\phi_i| + \varepsilon\psi_\varepsilon(|\phi_i|)),$$

where $\phi_i = \mathbb{H}_{\mathrm{od}}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i))$. We have

$$\prod_{i=1}^N (|\phi_i| + \varepsilon\psi_\varepsilon(|\phi_i|)) \leq 1 + \prod_{i=1}^N (\phi_i + \varepsilon\psi_\varepsilon(\phi_i))\overline{(\phi_i + \varepsilon\psi_\varepsilon(\phi_i))}$$

and $(\phi_i + \varepsilon\psi_\varepsilon(\phi_i))\overline{(\phi_i + \varepsilon\psi_\varepsilon(\phi_i))} \leq (1+\varepsilon)\phi_i\overline{\phi_i} + \varepsilon^2$. Then

$$|e^{Y_2}| \leq 1 + \prod_{i=1}^N ((1+\varepsilon)\phi_i\overline{\phi_i} + \varepsilon^2).$$

Hence it follows that

$$|(F, \mathrm{J}_0^* e^Y \mathrm{J}_t G)| \leq \left(|F|, \mathrm{J}_0^* e^{Y_1} \left(1 + \prod_{i=1}^N ((1+\varepsilon)\phi_i\overline{\phi_i} + \varepsilon^2) \right) \mathrm{J}_t |G| \right). \tag{5.23}$$

Since Baker-Campbell-Hausdorff formula yields that

$$e^{Y_1} = e^{\beta} e^{\frac{B^\dagger}{\sqrt{2}}} e^{\frac{B}{\sqrt{2}}},$$

where $\beta = [B, B^\dagger]/4$. We have

$$\mathrm{J}_0^* e^{Y_1} \prod_{i=1}^{k} \phi_i \mathrm{J}_t = \mathrm{J}_0^* e^{\beta} e^{\frac{B^\dagger}{\sqrt{2}}} e^{\frac{B}{\sqrt{2}}} \prod_{i=1}^{k} \left(\frac{C_i^\dagger}{\sqrt{2}} + \frac{C_i}{\sqrt{2}} \right) \mathrm{J}_t. \tag{5.24}$$

We apply Wick's theorem to compute the commutator between $\prod_{i=1}^{k} \left(\frac{C_i^\dagger}{\sqrt{2}} + \frac{C_i}{\sqrt{2}} \right)$ and $e^{\frac{B^\dagger}{\sqrt{2}}} e^{\frac{B}{\sqrt{2}}}$. Set $j_i = i$ for simplicity. Hence

$$\prod_{i=1}^{k} \left(\frac{C_i^\dagger}{\sqrt{2}} + \frac{C_i}{\sqrt{2}} \right) = \sum_{p=0}^{[k/2]} \frac{1}{2^p} \sum_{i_1,\ldots,i_{2p}}^{\text{pair}} \left(\prod_{l=1}^{p} C_{i_{2l-1},i_{2l}} \right) c^k_{i_1,\ldots,i_{2k}}.$$

Here $C_{i,j} = [C_i, C_j^\dagger]$, and $\sum_{i_1,\ldots,i_{2p}}^{\text{pair}}$ denotes the summation over all p-pairs chosen from $\{1, \ldots, k\}$, and

$$c^k_{i_1,\ldots,i_{2p}} = 2^{-(k-2p)/2} \sum_{l=0}^{k-2p} \sum_{\{n_1,\ldots,n_l\} \subset \{i_1,\ldots,i_{2p}\}^c} \prod_{i \in \{n_1,\ldots,n_l\}} C_i^\dagger \prod_{i \in \{n_1,\ldots,n_l\}^c \cap \{i_1,\ldots,i_{2p}\}^c} C_i.$$

Hence

$$e^{\frac{B^\dagger}{\sqrt{2}}} e^{\frac{B}{\sqrt{2}}} \prod_{i=1}^{k} \left(\frac{C_i^\dagger}{\sqrt{2}} + \frac{C_i}{\sqrt{2}} \right) = \sum_{p=0}^{[k/2]} \frac{1}{2^p} \sum_{i_1\ldots,i_{2p}}^{\text{pair}} \left(\prod_{l=1}^{p} C_{i_{2l-1},i_{2l}} \right) \sum_{l=0}^{k-2p} \sum_{\{n_1,\ldots,n_l\} \subset \{i_1,\ldots,i_{2p}\}^c} X e^{\frac{B^\dagger}{\sqrt{2}}} e^{\frac{B}{\sqrt{2}}} Z, \tag{5.25}$$

where $X = \prod_{i \in \{n_1,\ldots,n_l\}} (C_i^* + y_i)$ and $Z = \prod_{i \in \{n_1,\ldots,n_l\}^c \cap \{i_1,\ldots,i_{2p}\}^c} C_i$ with $y_j = [B, C_j^\dagger]$. Let us define operators in $\mathcal{F}$ by

$$B^\dagger(l) = -a^* \left(\bigoplus_{j=\pm 1} \int_0^t e^{-|s-l|\omega(k)} S_j(\theta_{N_r}) \frac{\hat{\varphi}}{\sqrt{\omega}} e^{-\mathrm{i}kB_r(w)} \mathrm{d}r \right),$$

$$B(l) = -a \left(\bigoplus_{j=\pm 1} \int_0^t e^{-|s-l|\omega(k)} S_j(\theta_{N_r}) \frac{\hat{\varphi}}{\sqrt{\omega}} e^{\mathrm{i}kB_r(w)} \mathrm{d}r \right),$$

$$C_i^\dagger(l) = a^* \left(\bigoplus_{j=\pm 1} e^{-|s-l|\omega(k)} T_j(\theta_{N_{s_i}}) \frac{\hat{\varphi}}{\sqrt{\omega}} e^{-\mathrm{i}kB_{s_i}} \right),$$

$$C_i(l) = a \left(\bigoplus_{j=\pm 1} e^{-|s-l|\omega(k)} T_j(\theta_{N_{s_i}}) \frac{\hat{\varphi}}{\sqrt{\omega}} e^{\mathrm{i}kB_{s_i}} \right).$$

By intertwining properties $\mathrm{J}_0^* B^\dagger = B^\dagger(0)\mathrm{J}_0^*$, $\mathrm{J}_0 B(0) = B\mathrm{J}_0$, $\mathrm{J}_0 C_i^\dagger = C_i^\dagger(0)\mathrm{J}_0^*$ and $\mathrm{J}_0 C_i(0) = C_i \mathrm{J}_0$, and factorization formula $\mathrm{J}_0^*\mathrm{J}_t = e^{-tH_\mathrm{f}}$, we see by (5.25) that

$$
\begin{aligned}
W(k) &= \mathrm{J}_0^* e^{\frac{B^\dagger}{\sqrt{2}}} e^{\frac{B}{\sqrt{2}}} \prod_{i=1}^k \left(\frac{C_i^\dagger}{\sqrt{2}} + \frac{C_i}{\sqrt{2}} \right) \mathrm{J}_t \\
&= \sum_{p=0}^{[k/2]} \frac{1}{2^p} \sum_{i_1,\ldots,i_{2p}}^{\mathrm{pair}} \left(\prod_{l=1}^p C_{i_{2l-1},i_{2l}} \right) \sum_{l=0}^{k-2p} \sum_{\{n_1,\ldots,n_l\}\subset\{i_1,\ldots,i_{2p}\}^c} X(0) e^{\frac{B^\dagger(0)}{\sqrt{2}}} e^{-t\hat{H}_\mathrm{f}} e^{\frac{B(t)}{\sqrt{2}}} Z(t),
\end{aligned}
$$

where $X(0)$ and $Z(t)$ are defined by X and Z with C_j replaced by $C_j(0)$ and $C_j(t)$, respectively. Let $\alpha = \dfrac{1}{\sqrt{2}}(\||k|\hat{\varphi}/\sqrt{\omega}\| + \||k|\hat{\varphi}/\omega\|)$ and $\beta = \dfrac{1}{\sqrt{2}}\||k|\hat{\varphi}/\sqrt{\omega}\|$. Then $|[C_i(l), C_j^\dagger(l)]| \le \beta^2$ for any i,j. Let

$$
\xi(l,m) = \sum_{j,j'=0}^\infty \frac{\gamma^{j+j''}\sqrt{(j+l)!}\sqrt{(j'+m)!}}{j!j'!}
$$

and

$$
\xi_r(l,m) = \sum_{j,j'=0}^\infty \frac{\gamma^{j+j''}\sqrt{(j+l)!}\sqrt{(j'+m)!}}{r^{(j+l)/2} r^{(j'+m)/2} j!j'!}.
$$

Here $\gamma = \||k|\hat{\varphi}/\sqrt{\omega}\| + \||k|\hat{\varphi}/\omega\|$. The following estimates (5.26) and (5.27) are proven in Theorem B.6.
Case ($t \ge 1$):

$$
\|W(k)\Psi\| \le w(k) \left\| e^{-\frac{1}{2}(t-1)\hat{H}_\mathrm{f}} \right\|^2 \|\Psi\|, \tag{5.26}
$$

where

$$
w(k) = k! \sum_{p=0}^{[k/2]} \sum_{l+m+2p=k} \frac{\left(\frac{\beta^2}{4}\right)^p (\alpha+y)^l \alpha^m}{p!l!m!} \xi(l,m)
$$

and $y = t\beta^2$.
Case ($t < 1$): For any $0 < r < t$, we have

$$
\|W(k)\Psi\| \le w(r,k) \left\| e^{-\frac{1}{2}(t-r)\hat{H}_\mathrm{f}} \right\|^2 \|\Psi\|, \tag{5.27}
$$

where

$$
w(r,k) = k! \sum_{p=0}^{[k/2]} \sum_{l+m+2p=k} \frac{\left(\frac{\beta^2}{4}\right)^p (\alpha+y)^l \alpha^m}{p!l!m!} \xi_r(l,m).
$$

Hence $W(k)$ is bounded. Since

$$
\begin{aligned}
&\mathrm{J}_0^* e^{Y_1} \left(1 + \prod_{i=1}^N ((1+\varepsilon)\phi_i \overline{\phi_i} + \varepsilon^2) \right) \mathrm{J}_t \\
&= \mathrm{J}_0^* e^{Y_1} \mathrm{J}_t + \sum_{k=0}^N \sum_{\{i_1,\ldots,i_k\}\subset\{1,\ldots,N\}} \varepsilon^{2(N-k)} (1+\varepsilon)^k \mathrm{J}_0^* e^{Y_1} \left(\prod_{j=1}^k \phi_{i_j} \bar{\phi}_{i_j} \right) \mathrm{J}_t,
\end{aligned}
$$

in a similar manner to (5.26) and (5.27) we see the bound:

$$\left\| \mathrm{J}_0^* e^{Y_1} \left(\prod_{j=1}^k \phi_{i_j} \bar{\phi}_{i_j} \right) \mathrm{J}_t \right\| \leq w(r, 2k) \left\| e^{-\frac{1}{2}(t-r)\hat{H}_\mathrm{f}} \right\|^2$$

with $0 < r < t$ for $t < 1$ and $r = 1$ for $t \geq 1$, we have

$$\begin{aligned} &\left\| \mathrm{J}_0^* e^{Y_1} \left(1 + \prod_{i=1}^N ((1+\varepsilon)\phi_i \overline{\phi_i} + \varepsilon^2) \right) \mathrm{J}_t \right\| \\ &\leq \left(w(r,0) + \sum_{k=0}^N \frac{N!}{(N-k)!k!} \varepsilon^{2(N-k)} (1+\varepsilon)^k w(r,2k) \right) \left\| e^{-\frac{1}{2}(t-r)\hat{H}_\mathrm{f}} \right\|^2. \end{aligned}$$

Then the proof is complete.

QED

Lemma 5.6 *Let $V \in L^\infty(\mathbb{R}^3)$ and $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$. Then for every $t \geq 0$ and all $F, G \in \mathcal{K}$ it follows that*

$$(F, e^{-tL}G) = e^t \int_{\mathbb{R}^3} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)\mathrm{d}s} (F(q_0), \mathrm{J}_0^* e^Y \mathrm{J}_t G(q_t))]\mathrm{d}x. \tag{5.28}$$

Proof: Let $F, G \in \mathcal{K}$ such that $F \geq 0$ and $G \geq 0$. By Lemma 5.4 we have

$$(F, e^{-tL_R}G) = e^t \int_{\mathbb{R}^3} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)\mathrm{d}s} (F(q_0), \mathrm{J}_0^* e^{Y_R} \mathrm{J}_t G(q_t))]\mathrm{d}x. \tag{5.29}$$

Here $0 \leq g(R) = (F(q_0), \mathrm{J}_0^* e^{Y_R} \mathrm{J}_t G(q_t))$ and $g(R)$ is a monotonously increasing function in R. Hence $(F, e^{-tL_R}G)$ is also increasing in R and $(F, e^{-tL_R}G) \uparrow (F, e^{-tL}G)$ as $R \uparrow \infty$. For each $\phi \in \mathrm{Q}$, it follows that

$$\mathrm{J}_0 F(q_0) \cdot e^{Y_R} \mathrm{J}_t G(q_t) \to \mathrm{J}_0 F(q_0) \cdot e^Y \mathrm{J}_t G(q_t)$$

as $R \to \infty$. Then the monotone convergence theorem yields that the function on the right-hand side above $\mathrm{J}_0 F(q_0) \cdot e^Y \mathrm{J}_t G(q_t)$ is finite for a.e. $(\phi, x, \alpha, w, m) \in \mathrm{Q_E} \times \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$, and

$$\begin{aligned} (F, e^{-tL}G) &= \lim_{R\to\infty} (F, e^{-tL_R}G) \\ &= e^t \int_{\mathbb{R}^3} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}^{x,\alpha} \left[\lim_{R\to\infty} e^{-\int_0^t V(B_s)\mathrm{d}s} (F(q_0), \mathrm{J}_0^* e^{Y_R} \mathrm{J}_t G(q_t)) \right] \mathrm{d}x \\ &= e^t \int_{\mathbb{R}^3} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}^{x,\alpha} \left[e^{-\int_0^t V(B_s)\mathrm{d}s} \mathbb{E}_{\mu_\mathrm{E}} [\mathrm{J}_0 F(q_0) \cdot e^Y \mathrm{J}_t G(q_t)] \right] \mathrm{d}x. \end{aligned}$$

For general $F, G \in \mathscr{K}$, the lemma is proven by decomposing F and G as a linear sum of positive functions: $F = \Re F_+ - \Re F_- + \mathrm{i}(\Im F_+ - \Im F_-)$ and $G = \Re G_+ - \Re G_- + \mathrm{i}(\Im G_+ - \Im G_-)$. Finally we show that $\mathrm{J}_0^* e^Y \mathrm{J}_t G(q_t) \in L^2(\mathrm{Q})$ for each $(x, \alpha, w, m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$ in Proposition 5.5. Then $\mathbb{E}_{\mu_\mathrm{E}}[\mathrm{J}_0 F(q_0) \cdot e^Y \mathrm{J}_t G(q_t)] = (F(q_0), \mathrm{J}_0^* e^Y \mathrm{J}_t G(q_t))$ follows and the lemma is proven.

QED

5.4.4 Estimate of $e^{-tH_{\mathbb{Z}_2,\varepsilon}}$

Let $Z_t = Y_t(1) + Y_t(2) + Y_t(3)$, where

$$\begin{aligned} Y_t(1) &= -\mathrm{i}\mathbb{A}\left(\bigoplus_{\mu=1}^{3}\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu\right), \\ Y_t(2) &= -\int_0^t \mathbb{H}_{\mathrm{d}}(B_s, \theta_{N_s}, s)\mathrm{d}s, \\ Y_t(3) &= \int_0^{t+} \log(-\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, s))\mathrm{d}N_s. \end{aligned}$$

We shall see that Z_t turns to be the exponent of an integral kernel of $e^{-tH_{\mathbb{Z}_2}}$. We furthermore define $Z_t(\varepsilon)$ by Z_t with $Y_t(3)$ replaced by $Y_t(3,\varepsilon)$, i.e., $Z_t(\varepsilon) = Y_t(1) + Y_t(2) + Y_t(3,\varepsilon)$, where

$$Y_t(3,\varepsilon) = \int_0^{t+} \log(-\Psi_\varepsilon(\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, s)))\mathrm{d}N_s.$$

Lemma 5.7 *For each $(x,\alpha,w,m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$, $\mathrm{J}_0^* e^{Z_t(\varepsilon)}\mathrm{J}_t$ is a bounded operator.*

Proof: Let $\Phi, \Psi \in L^2(\mathrm{Q})$. By Y defined in (5.21) we see that

$$|(\Phi, \mathrm{J}_0^* e^{Z_t(\varepsilon)}\mathrm{J}_t\Psi)| \le (|\Phi|, \mathrm{J}_0^* e^{Y}\mathrm{J}_t|\Psi|) \le \|\Phi\|\|\Psi\|\|\mathrm{J}_0^* e^{Y}\mathrm{J}_t\|$$

follows from Proposition 5.5. Then the lemma is proven.

QED

Theorem 5.8 *Suppose Assumption 3.1. Then for every $t \ge 0$ and all $F, G \in \mathcal{K}$ it follows that $(F, e^{-tH_{\mathbb{Z}_2}}G) = \lim_{\varepsilon\downarrow 0}(F, e^{-tH_{\mathbb{Z}_2,\varepsilon}}G)$ and*

$$(F, e^{-tH_{\mathbb{Z}_2,\varepsilon}}G) = e^t \int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}[e^{-\int_0^t V(B_s)\mathrm{d}s}(F(q_0), \mathrm{J}_0^* e^{Z_t(\varepsilon)}\mathrm{J}_t G(q_t))]\mathrm{d}x. \tag{5.30}$$

Proof: Since $e^{-tH_{\mathbb{Z}_2,\varepsilon}} \to e^{-tH^{\mathbb{Z}_2}}$ strongly as $\varepsilon \to 0$, $(F, e^{-tH_{\mathbb{Z}_2}}G) = \lim_{\varepsilon\downarrow 0}(F, e^{-tH_{\mathbb{Z}_2,\varepsilon}}G)$ follows. Now we turn to proving (5.30). Suppose that $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$ and $V \in L^\infty(\mathbb{R}^3)$. Write

$$\begin{aligned} X_{S,T}(\varepsilon, s) = &-\mathrm{i}\int_S^T \mathbb{A}(\mathrm{j}_s\tilde{\varphi}(\cdot - B_r))\mathrm{d}B_r \\ &-\int_S^T \mathbb{H}_{\mathrm{d}}(B_r, \theta_{N_r}, s)\mathrm{d}r + \int_S^{T+}\log(-\Psi_\varepsilon(\mathbb{H}_{\mathrm{od}}(B_r, -\theta_{N_{r-}}, s)))\mathrm{d}N_r. \end{aligned}$$

Define $S_{t,s}^\varepsilon : \mathcal{K}_{\mathrm{E}}^\infty \to \mathcal{K}_{\mathrm{E}}^\infty$ by

$$(S_{t,s}^\varepsilon G)(x, \theta_\alpha) = e^t\mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_r)\mathrm{d}r}e^{X_{0,t}(\varepsilon,s)}G(q_t)].$$

It can be seen that $S_{t,s}^\varepsilon$ has the property:

$$S_{t,s}^\varepsilon S_{t',s'}^\varepsilon G(x,\theta_\alpha) = e^{t+t'}\mathbb{E}^{x,\alpha}[e^{-\int_0^{t+t'}V(B_r)\mathrm{d}r}e^{X_{0,t}(\varepsilon,s)+X_{t,t+t'}(\varepsilon,s')}G(q_{t+t'})]. \tag{5.31}$$

Note that for $s_1 \le \ldots \le s_n$,

$$e^{X_{0,t_1}(\varepsilon,s_1)+X_{t_1,t_1+t_2}(\varepsilon,s_2)+\cdots+X_{t_1+\cdots+t_{n-1},t_1+\cdots+t_n}(\varepsilon,s_n)} \in \mathscr{E}_{[s_1,s_n]}. \tag{5.32}$$

Since $(F, e^{-tH_{\mathbb{Z}_2,\varepsilon}}G) = (F, e^{-tK_\varepsilon}G)$, by the Trotter product formula, (5.31), (5.32) and the Markov property of E_s, $s \in \mathbb{R}$, we obtain that

$$(F, e^{-tH_{\mathbb{Z}_2,\varepsilon}}G) = e^t \lim_{n\to\infty} \int_{\mathbb{R}^3} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_r)\mathrm{d}r}\left(F(q_0), \mathrm{J}_0^* e^{Z_t^n(\varepsilon)}\mathrm{J}_t G(q_t)\right)\right]\mathrm{d}x, \tag{5.33}$$

where $Z_t^n(\varepsilon) = Y_t^n(1) + Y_t^n(2) + Y_t^n(3,\varepsilon)$ with

$$\begin{aligned}
Y_t^n(1) &= -\mathrm{i}\mathbb{A}\left(\bigoplus_{\mu=1}^3 \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^\mu\right), \\
Y_t^n(2) &= -\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathbb{H}_{\mathrm{d}}(B_s, \theta_{N_s}, t_{j-1})\mathrm{d}s, \\
Y_t^n(3,\varepsilon) &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \log(-\Psi_\varepsilon(\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, t_{j-1})))\mathrm{d}N_s
\end{aligned}$$

and $t_j = jt/n$. Put

$$\begin{aligned}
\langle F, T_n G\rangle &= e^t \int_{\mathbb{R}^3} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_r)\mathrm{d}r}\left(F(q_0), \mathrm{J}_0^* e^{Z_t^n(\varepsilon)}\mathrm{J}_t G(q_t)\right)\right]\mathrm{d}x, \\
\langle F, TG\rangle &= e^t \int_{\mathbb{R}^3} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_r)\mathrm{d}r}\left(F(q_0), \mathrm{J}_0^* e^{Z_t(\varepsilon)}\mathrm{J}_t G(q_t)\right)\right]\mathrm{d}x.
\end{aligned}$$

Notice that by the definition of $\langle F, T_n G\rangle$ we see that $|\langle F, T_n G\rangle| \le \|F\|\|G\|$. Suppose that $F_m \to F$ and $G_m \to G$ as $m \to \infty$. Then by a telescoping we can see that $|\langle F, T_n G\rangle - \langle F_m, T_n G_m\rangle| \le \|F - F_m\|\|G\| + \|G - G_m\|\|F_m\|$. By using facts that $|\mathrm{J}_t F| \le \mathrm{J}_t|F|$, $|e^{\pm Y_t(1)}F| = |F|$ and $|e^{Y_t(2)+Y_t(3,\varepsilon)}F| = e^Y|F|$, we can see that $|\langle F, TG\rangle| \le a\|F\|\|G\|$, where $a = e^{-t\inf \mathrm{Spec}(L)}$. Hence we have

$$|\langle F_m, TG_m\rangle - \langle F, TG\rangle| \le a\|F_m - F\|\|G_m\| + a\|F\|\|G - G_m\|.$$

Suppose that $\|F - F_m\| < \varepsilon$ and $\|G - G_m\| < \varepsilon$. Together with them we have

$$|\langle F, T_n G\rangle - \langle F, TG\rangle| \le \varepsilon(\|G\| + a\|G_m\| + \|F_m\| + a\|F\|) + |\langle F_m, T_n G_m\rangle - \langle F_m, TG_m\rangle|. \tag{5.34}$$

It can be concluded from (5.34) that it is enough to show the lemma for arbitrary F, G included in some dense domain. We shall show the lemma for $F, G \in \mathcal{K}_\infty$. We claim that

(1) For each $(x, \alpha, w, m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$, there exists $C_t(m)$ such that

$$|(F, \mathrm{J}_0^* e^{Z_t^n(\varepsilon)}\mathrm{J}_t G)_{L^2(\mathrm{Q})}| \le C_t(m),$$

where $C_t(m)$ is independent of $(x, \alpha, w) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X}$, $\varepsilon > 0$ and n, and it is satisfied that $\mathbb{E}_\mu[C_t^2] < \infty$.

(2) For each $(x,\alpha,m)\in\mathbb{R}^3\times\mathbb{Z}_2\times\mathscr{X}_\mu$,

$$\lim_{n\to\infty}\mathbb{E}^x_{\mathcal{W}}[(F(q_0),\mathrm{J}_0^*e^{Z_t^n(\varepsilon)}\mathrm{J}_tG(q_t))]=\mathbb{E}^x_{\mathcal{W}}[(F(q_0),\mathrm{J}_0^*e^{Z_t(\varepsilon)}\mathrm{J}_tG(q_t))].$$

The proof of (1) and (2) will be given in Lemma C.1 and Lemma C.2 in Appendix C below, respectively. We set

$$\text{RHS (5.33)}=e^t\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^{x,\alpha}[\xi_n]\mathrm{d}x,\tag{5.35}$$

where $\xi_n=e^{-\int_0^tV(B_s)\mathrm{d}s}\big(F(q_0),\mathrm{J}_0^*e^{Z_t^n(\varepsilon)}\mathrm{J}_tG(q_t)\big)$. Thus we have

$$\mathbb{E}^x_{\mathcal{W}}[|\xi_n|]\le\mathbb{E}^x_{\mathcal{W}}[e^{-\int_0^tV(B_s)\mathrm{d}s}C_t(m)\|F(q_0)\|\|G(q_t)\|]$$

and

$$\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^{x,\alpha}[e^{-\int_0^tV(B_s)\mathrm{d}s}C_t(m)\|F(q_0)\|\|G(q_t)\|]\mathrm{d}x\le V_\infty^{1/2}(\mathbb{E}_\mu[C_t^2])^{1/2}\|F\|\|G\|<\infty.$$

Since $\mathbb{E}^x_{\mathcal{W}}[\xi_n]\to\mathbb{E}^x_{\mathcal{W}}[\xi_\infty]$ as $n\to\infty$ for each $(x,\alpha,m)\in\mathbb{R}^3\times\mathbb{Z}_2\times\mathscr{X}_\mu$, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^\alpha_\mu[\mathbb{E}^x_{\mathcal{W}}[\xi_n]]\mathrm{d}x=\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^\alpha_\mu[\mathbb{E}^x_{\mathcal{W}}[\xi_\infty]]\mathrm{d}x$$

follows. For $V\in\mathcal{R}_{\mathrm{Kato}}$ by a limiting argument we can show the theorem. Finally for $\hat\varphi$ in Assumption 3.1, we can see (5.30) by an approximation, which is also shown in Lemma 5.9 below.

QED

Lemma 5.9 (5.30) *is valid for* $\hat\varphi$ *in Assumption 3.1.*

Proof: It is enough to show (5.30) for $F,G\in\mathcal{K}_\infty$ by an approximation argument. Take a sequence $\hat\varphi_n\in C_0^\infty(\mathbb{R}^3)$ such that $|k|\hat\varphi_n/\sqrt{\omega}\to|k|\hat\varphi/\sqrt{\omega}$ as $n\to\infty$. Then (5.30) is valid for each $\hat\varphi_n$:

$$(F,e^{-tH_{\mathbb{Z}_2,\varepsilon}(n)}G)=\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}[e^{-\int_0^tV(B_s)\mathrm{d}s}(F(q_0),\mathrm{J}_0^*e^{Z_t^n(\varepsilon)}\mathrm{J}_tG(q_t))]\mathrm{d}x.$$

Here $Z_t^n(\varepsilon)$ is defined by $Z_t(\varepsilon)$ with $\hat\varphi$ replaced by $\hat\varphi_n$, and $H_{\mathbb{Z}_2,\varepsilon}(n)$ by $H_{\mathbb{Z}_2,\varepsilon}$ with $\hat\varphi$ replaced by $\hat\varphi_n$. It can be seen that $H_{\mathbb{Z}_2,\varepsilon}(n)\to H_{\mathbb{Z}_2,\varepsilon}$ as $n\to\infty$ on a common core $\mathcal{K}_\infty$. Then $e^{-tH_{\mathbb{Z}_2,\varepsilon}(n)}\to e^{-tH_{\mathbb{Z}_2,\varepsilon}}$ strongly as $n\to\infty$. Let $Y_t^n(1)$, $Y_t^n(2)$ and $Y_t^n(3)$ be $Y_t(1)$, $Y_t(2)$ and $Y_t(3)$ with $\hat\varphi$ replaced by $\hat\varphi_n$, respectively. In a similar approximation argument to the proof of Lemma C.2 we can show that

(1) $\mathbb{E}^x_{\mathcal{W}}[\|(e^{Y_t^n(1)}-e^{Y_t(1)})F]\to0$ as $\|(\hat\varphi_n-\hat\varphi)/\sqrt{\omega}\|\to0$,

(2) $\|(e^{Y_t^n(2)}-e^{Y_t(2)})F\|\to0$ as $\||k|(\hat\varphi_n-\hat\varphi)/\sqrt{\omega}\|\to0$,

(3) $\|(e^{Y_t^n(3,\varepsilon)}-e^{Y_t(3,\varepsilon)})F\| \to 0$ as $\||k|(\hat\varphi_n-\hat\varphi)/\sqrt{\omega}\|\to 0$

for $F\in L^2_{\rm fin}(\mathrm{Q})$. Hence

$$
\begin{aligned}
(F, e^{-tH_{\mathbb{Z}_2,\varepsilon}}G) &= \lim_{n\to\infty}\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}[e^{-\int_0^t V(B_s)\mathrm{d}s}(F(q_0), \mathrm{J}_0^* e^{Z_t^n(\varepsilon)}\mathrm{J}_t G(q_t))]\mathrm{d}x \\
&= \int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}[e^{-\int_0^t V(B_s)\mathrm{d}s}(F(q_0), \mathrm{J}_0^* e^{Z_t(\varepsilon)}\mathrm{J}_t G(q_t))]\mathrm{d}x
\end{aligned}
$$

for $F,G\in\mathcal{K}_\infty$. Then the proof is complete.

QED

5.5 Functional integral representations of $e^{-tH_{\mathbb{Z}_2}}$

As was mentioned above we need the regularization $\Psi_\varepsilon(\hat H_{\rm od})$ of $\hat H_{\rm od}$ to prevent zeros of $\hat H_{\rm od}$. The zeros of $\hat H_{\rm od}$ produces the zeros of $e^{\int_0^{t+}\log(-\mathbb{H}_{\rm od}(B_s,-\theta_{N_{s-}},s))\mathrm{d}N_s}$. In order to avoid $\lim_{\varepsilon\to 0}$ in the functional integral representation, instead of introducing regularization Ψ_ε we introduce a subset W of $\mathbb{R}^3\times\mathscr{X}\times\mathscr{X}_N\times\mathrm{Q_E}$ by

$$
W=\left\{\int_0^{t+}\log\left(\frac{1}{2}\sqrt{|b_1(s,x,B_s)|^2+|b_2(s,x,B_s)|^2}\right)\mathrm{d}N_s > -\infty\right\}, \tag{5.36}
$$

where $b_\alpha(s,x,B_s)=\mathbb{B}_\alpha(\mathrm{j}_s\tilde\varphi(\cdot-B_s-x))$.

Theorem 5.10 *Suppose Assumption 3.1. Then for every $t\ge 0$ and all $F,G\in\mathcal{K}$ it follows that*

$$
(F, e^{-tH_{\mathbb{Z}_2}}G)=e^t\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)\mathrm{d}s}(\mathrm{J}_0F(q_0), e^{Z_t}\mathbb{1}_W\mathrm{J}_tG(q_t))]\mathrm{d}x. \tag{5.37}
$$

Here W is given by (5.36) *and the exponent Z_t is defined by*

$$
\begin{aligned}
Z_t = &-\mathrm{i}\int_0^t \mathbb{A}(\mathrm{j}_s\tilde\varphi(\cdot-B_s))\mathrm{d}B_s \\
&-\int_0^t \mathbb{H}_{\rm d}(B_s,\theta_{N_s},s)\mathrm{d}s+\int_0^{t+}\log(-\mathbb{H}_{\rm od}(B_s,-\theta_{N_{s-}},s))\mathrm{d}N_s
\end{aligned}
$$

and

$$
\int_0^{t+}\log(-\mathbb{H}_{\rm od}(B_s,-\theta_{N_{s-}},s))\mathrm{d}N_s=\sum_{\substack{r\in[0,t]\\ N_{r+}\neq N_{r-}}}\log(-\mathbb{H}_{\rm od}(B_r,-\theta_{N_{r-}},r))
$$

possibly takes infinity.

Proof: By the Lebesgue dominated convergence theorem we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)\mathrm{d}s}(\mathrm{J}_0F(q_0), e^{Z_t(\varepsilon)}\mathrm{J}_tG(q_t))]\mathrm{d}x \\
&= \int_{\mathbb{R}^3}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)\mathrm{d}s}\lim_{\varepsilon\to 0}(\mathrm{J}_0F(q_0), e^{Z_t(\varepsilon)}\mathrm{J}_tG(q_t))]\mathrm{d}x.
\end{aligned}
$$

We can also show that $\lim_{\varepsilon\to 0}(\mathrm{J}_0 F(q_0), \mathbb{1}_{W^c} e^{Z_t(\varepsilon)} \mathrm{J}_t G(q_t)) = 0$ and

$$\lim_{\varepsilon\to 0}(\mathrm{J}_0 F(q_0), e^{Z_t(\varepsilon)} \mathrm{J}_t G(q_t)) = \lim_{\varepsilon\to 0}(\mathrm{J}_0 F(q_0), \mathbb{1}_W e^{Z_t(\varepsilon)} \mathrm{J}_t G(q_t)) = (\mathrm{J}_0 F(q_0), \mathbb{1}_W e^{Z_t} \mathrm{J}_t G(q_t)).$$

Then the theorem follows.

QED

5.6 Translation invariant Pauli-Fierz Hamiltonian with spin 1/2

We study the translation invariant Pauli-Fierz Hamiltonian with spin 1/2, that is, H_{S} with V identically zero. We suppose Assumption 3.1 with $V = 0$ in this section. The Pauli-Fierz Hamiltonian with spin 1/2 and a fixed total momentum p is defined by

$$H_{\mathrm{S}}(p) = \frac{1}{2}(p - \mathrm{P_f} - \hat{A}(0))^2 + H_{\mathrm{f}} - \frac{1}{2}\sigma \cdot \hat{B}(0), \quad p \in \mathbb{R}^3,$$

with domain $D(H_{\mathrm{S}}(p)) = D(H_{\mathrm{f}}) \cap D(\mathrm{P_f^2})$. $H_{\mathrm{S}}(p)$ is self-adjoint and essentially self-adjoint on any core of $H_{\mathrm{f}} + \mathrm{P_f^2}$.

As in the case of $H(p)$ and $H_{\mathrm{R}}(p)$ we have the unitary equivalence below:

$$L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes L^2(\mathrm{Q}) \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2 \otimes L^2(\mathrm{Q}) \mathrm{d}p,$$

$$H_{\mathrm{S}} \cong \int_{\mathbb{R}^3}^{\oplus} H_{\mathrm{S}}(p) \mathrm{d}p,$$

which are implemented by $\mathscr{T}$ defined in (3.30). We now construct a functional integral representation for the translation invariant Pauli-Fierz Hamiltonian with spin 1/2. As before, we transform $H_{\mathrm{S}}(p)$ on $\mathbb{C}^2 \otimes L^2(\mathrm{Q})$ to $H_{\mathbb{Z}_2}(p)$ on $\ell^2(\mathbb{Z}_2) \otimes L^2(\mathrm{Q})$ which is defined by

$$(H_{\mathbb{Z}_2}(p)\Psi)(\theta) = \left(\frac{1}{2}(p - \mathrm{P_f} - \hat{A}(0))^2 + \hat{H}_{\mathrm{d}}(0,\theta) + H_{\mathrm{f}}\right)\Psi(\theta) + \hat{H}_{\mathrm{od}}(0,-\theta)\Psi(-\theta),$$

where

$$\hat{H}_{\mathrm{d}}(0,\theta) = -\frac{1}{2}\theta \hat{B}_3(0),$$
$$\hat{H}_{\mathrm{od}}(0,-\theta) = -\frac{1}{2}(\hat{B}_1(0) - \mathrm{i}\theta \hat{B}_2(0)).$$

The strategy of constructing a functional integral representation of $e^{-tH_{\mathbb{Z}_2}(p)}$ is similar to that of the spinless case.

Theorem 5.11 *Let $\Phi, \Psi \in \ell^2(\mathbb{Z}_2) \otimes L^2(\mathrm{Q})$. Let $W_0 \subset \mathscr{X} \times \mathscr{X}_\mu \times \mathrm{Q_E}$ be defined by*

$$W_0 = \left\{(w, m, \phi) \in \mathscr{X} \times \mathscr{X}_\mu \times \mathrm{Q_E} \mid \int_0^{t+} \log\left(\frac{1}{2}\sqrt{|b_1(s, B_s)|^2 + |b_2(s, B_s)|^2}\right) \mathrm{d}N_s > -\infty\right\}.$$

Then

$$(\Phi, e^{-tH_{\mathrm{S}}(p)}\Psi) = e^t \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}^{0,\alpha}[(\mathrm{J}_0\Phi(X_0), e^{Z_t} \mathbb{1}_{W_0} \mathrm{J}_t e^{\mathrm{i}(p-\mathrm{P_f})B_t} \Psi(X_t))],$$

where the exponent Z_t is given by

$$Z_t = -\mathrm{i}\int_0^t \mathbb{A}(\mathrm{j}_s\tilde{\varphi}(\cdot - B_s))\mathrm{d}B_s \\ - \int_0^t \mathbb{H}_{\mathrm{d}}(B_s, \theta_{N_s}, s)\mathrm{d}s + \int_0^{t+} \log(-\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, s))\mathrm{d}N_s.$$

Proof: In addition to the proof of Theorem 3.14 the theorem can be shown by mimicking the proof of Theorem 5.10.

QED

5.7 Symmetry and non-degeneracy of ground states

For the Pauli-Fierz Hamiltonian H and the relativistic Pauli-Fierz Hamiltonian H_R, we can show that the associated semigroups are positivity improving. Thus the ground state for H and H_{R} is unique if it exists. We are concerned with $e^{-tH_{\mathrm{S}}}$ in this section.

Before concerning with H_{S} we study a toy model $H(\varepsilon)$ defined by

$$H(\varepsilon) = \frac{1}{2}(-\mathrm{i}\nabla - A)^2 + V + H_{\mathrm{f}} - \varepsilon\sigma_1, \quad \varepsilon \in \mathbb{R}.$$

When $\varepsilon = 0$,

$$H(0) = \begin{pmatrix} \frac{1}{2}(-\mathrm{i}\nabla - A)^2 + V + H_{\mathrm{f}} & 0 \\ 0 & \frac{1}{2}(-\mathrm{i}\nabla - A)^2 + V + H_{\mathrm{f}} \end{pmatrix}.$$

Then all the eigenvalues of $H(0)$ are degenerate. Let us consider $\varepsilon \neq 0$. Since $H(\varepsilon) \cong H(-\varepsilon)$, we may assume $\varepsilon > 0$.

Proposition 5.12 *Let $\varepsilon > 0$. Then $\mathfrak{S}^{-1}e^{-H(\varepsilon)}\mathfrak{S}$ is positivity improving. In particular the ground state of $H(\varepsilon)$ is unique, if the ground state exists.*

Proof: Let $F, G \in \mathcal{H}_{\mathrm{S}}$. It follows that

$$(F, e^{-tH(\varepsilon)}G) = e^t \sum_{\alpha\in\mathbb{Z}_2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)\mathrm{d}s}(\mathrm{J}_0F(X_0), e^{Z_t}\mathrm{J}_tG(X_t))]\mathrm{d}x.$$

Here $Z_t = -\mathrm{i}\int_0^t \mathbb{A}(\mathrm{j}_s\tilde{\varphi}(\cdot - B_s))\mathrm{d}B_s + \int_0^{t+}\log\varepsilon\mathrm{d}N_s$. Let $F, G \geq 0$. Then

$$(F, \mathfrak{S}^{-1}e^{-tH(\varepsilon)}\mathfrak{S}G) \\ = e^t \sum_{\alpha\in\mathbb{Z}_2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}\left[\varepsilon^{N_t}e^{-\int_0^t V(B_s)\mathrm{d}s}(\mathrm{J}_0F(X_0), \mathfrak{S}_{\mathrm{E}}^{-1}e^{-\mathrm{i}\int_0^t \mathbb{A}(\mathrm{j}_s\tilde{\varphi}(\cdot - B_s))\mathrm{d}B_s}\mathfrak{S}_{\mathrm{E}}\mathrm{J}_tG(X_t))\right]\mathrm{d}x > 0$$

and $\mathfrak{S}^{-1}e^{-tH(\varepsilon)}\mathfrak{S}$ is positivity improving.

QED

We study H_{S} from now on. Suppose that $B = 0$. Then similar to $H(0)$ mentioned above, all the eigenvalues of H_{S} are degenerate. However $B \neq 0$, unlike $H(\varepsilon)$ the ground state of H_{S} is also degenerate. We shall show this in the following. The idea is to show that

$$H_{\mathrm{S}} = \bigoplus_{w\in\mathbb{Z}_{1/2}} H_{\mathrm{S}}(w)$$

and $H_{\mathrm{S}}(w) \cong H_{\mathrm{S}}(-w)$ by symmetries hidden in H_{S}. This section is taken from [78]. Also refer to see [131] and [73, Sections 3.8.4 and 3.8.9].

The polarization vectors $e^{\pm 1}$ are coherent polarization vectors in direction $n \in \mathrm{S}^2$ whenever there exists $z \in \mathbb{Z}$ such that for any $\phi \in [0, 2\pi)$ and any k with $k/|k| \neq n$,

$$\begin{pmatrix} e_\mu^{+1}(\mathcal{R}k) \\ e_\mu^{-1}(\mathcal{R}k) \end{pmatrix} = \begin{pmatrix} \cos(z\phi) & -\sin(z\phi) \\ \sin(z\phi) & \cos(z\phi) \end{pmatrix} \begin{pmatrix} (\mathcal{R}e^{+1}(k))_\mu \\ (\mathcal{R}e^{-1}(k))_\mu \end{pmatrix}, \quad \mu = 1, 2, 3,$$

where $\mathcal{R} = \mathcal{R}(n, \phi)$ is the matrix denoting the rotation around $n \in S^2$ with angle $\phi \in [0, 2\pi)$. Let $\mathfrak{J} : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ be given by

$$\mathfrak{J} = \mathrm{i} \begin{pmatrix} 0 & -\mathbb{1}_{L^2(\mathbb{R}^3)} \\ \mathbb{1}_{L^2(\mathbb{R}^3)} & 0 \end{pmatrix}.$$

We define

$$S_{\mathrm{f}} = \mathrm{d}\Gamma(z\mathfrak{J}).$$

Let $\ell_k = k \times (-\mathrm{i}\nabla_k)$ be the triplet of angular momentum. We define L_{f} by

$$L_{\mathrm{f}} = (L_{\mathrm{f},1}, L_{\mathrm{f},2}, L_{\mathrm{f},3}) = \mathrm{d}\Gamma(\ell_k). \tag{5.38}$$

S_{f} is called the helicity and L_{f} the angular momentum of the field. For $(n, z) \in \mathrm{S}^2 \times \mathbb{Z}$ define $J_{\mathrm{f}} = J_{\mathrm{f}}(n, z)$ by $J_{\mathrm{f}} = n \cdot L_{\mathrm{f}} + S_{\mathrm{f}}$ and $J_{\mathrm{p}} = J_{\mathrm{p}}(n)$ by $J_{\mathrm{p}} = n \cdot \ell_x - \frac{1}{2} n \cdot \sigma$. Write

$$\mathbb{J} = J_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes J_{\mathrm{f}}.$$

Clearly, $\mathbb{J} = \mathbb{J}(n, z)$ is defined for each $(n, z) \in \mathrm{S}^2 \times \mathbb{Z}$.

Proposition 5.13 *If the polarization vectors are coherent in direction n, and $\hat{\varphi}$ and V are rotation-invariant, then $e^{\mathrm{i}\phi\mathbb{J}} H_{\mathrm{S}} e^{-\mathrm{i}\phi\mathbb{J}} = H_{\mathrm{S}}$ for every $\phi \in \mathbb{R}$.*

Proof: Write $a^\sharp(\binom{f}{g})$ for $a^\sharp(f \oplus g)$. Notice that for a rotation-invariant f,

$$e^{\mathrm{i}\phi J_{\mathrm{f}}} a^* \left(f e^{-\mathrm{i}kx} \begin{pmatrix} e_\mu^+ \\ e_\mu^- \end{pmatrix} \right) e^{-\mathrm{i}\phi J_{\mathrm{f}}} = a^* \left(f e^{\mathrm{i}\phi(z\mathfrak{J} + n\ell_k)} e^{-\mathrm{i}kx} \begin{pmatrix} e_\mu^+ \\ e_\mu^- \end{pmatrix} \right).$$

Since the polarization vectors are coherent, we have

$$e^{\mathrm{i}\phi J_{\mathrm{f}}} a^* \left(f e^{-\mathrm{i}kx} \begin{pmatrix} e_\mu^+ \\ e_\mu^- \end{pmatrix} \right) e^{-\mathrm{i}\phi J_{\mathrm{f}}} = \sum_{\nu=1}^{3} \mathcal{R}_{\mu\nu} a^* \left(f e^{-\mathrm{i}k\mathcal{R}^{-1}x} \begin{pmatrix} e_\nu^+ \\ e_\nu^- \end{pmatrix} \right), \tag{5.39}$$

where $\mathcal{R} = \mathcal{R}(n, \phi) = (\mathcal{R}_{\mu\nu})_{1 \le \mu, \nu \le 3}$. By (5.39), we see that

$$(\mathrm{S}) \begin{cases} e^{\mathrm{i}\phi J_{\mathrm{f}}} H_{\mathrm{f}} e^{-\mathrm{i}\phi J_{\mathrm{f}}} = H_{\mathrm{f}}, \\ e^{\mathrm{i}\phi J_{\mathrm{f}}} A_\mu(x) e^{-\mathrm{i}\phi J_{\mathrm{f}}} = (\mathcal{R}A)_\mu(\mathcal{R}^{-1}x), \\ e^{\mathrm{i}\phi n \cdot \ell_x} x_\mu e^{-\mathrm{i}\phi n \cdot \ell_x} = (\mathcal{R}x)_\mu, \\ e^{\mathrm{i}\phi n \cdot \ell_x} (-\mathrm{i}\nabla_x)_\mu e^{-\mathrm{i}\phi n \cdot \ell_x} = (\mathcal{R}(-\mathrm{i}\nabla_x))_\mu, \\ e^{\mathrm{i}\phi n \cdot (1/2)\sigma} \sigma_\mu e^{-\mathrm{i}\phi n \cdot (1/2)\sigma} = (\mathcal{R}^{-1}\sigma)_\mu. \end{cases}$$

Then

$$e^{\mathrm{i}\phi\mathbb{J}}H_{\mathrm{S}}e^{-\mathrm{i}\phi\mathbb{J}} = \frac{1}{2}\left(\mathcal{R}\sigma\cdot(\mathcal{R}(-\mathrm{i}\nabla) - \mathcal{R}A(x))\right)^2 + H_{\mathrm{f}} + V(\mathcal{R}x) = H_{\mathrm{S}}.$$

QED

Denote the set of half integers by $\mathbb{Z}_{1/2} = \{w/2|w \in \mathbb{Z}\}$. For each $(n, z) \in \mathrm{S}^2 \times \mathbb{Z}$, notice that $\sigma(n\cdot(\ell_x + \frac{1}{2}\sigma)) = \mathbb{Z}_{1/2}$, $\sigma(n\cdot L_{\mathrm{f}}) = \mathbb{Z}$ and $\sigma(S_{\mathrm{f}}) = \begin{cases}\mathbb{Z}, & z \neq 0,\\ 0, & z = 0.\end{cases}$ Thus for each $(n, z) \in \mathrm{S}^2 \times \mathbb{Z}$, it follows that

$$\sigma(\mathbb{J}) = \mathbb{Z}_{1/2}. \tag{5.40}$$

Theorem 5.14 *Suppose that the polarization vectors are coherent in direction n, and $\hat{\varphi}$ and V are rotation-invariant. Then $\mathcal{H}_S$ and H_{S} can be decomposed as*

$$\mathcal{H}_S = \bigoplus_{w\in\mathbb{Z}_{1/2}} \mathcal{H}_S(w), \quad H_{\mathrm{S}} = \bigoplus_{w\in\mathbb{Z}_{1/2}} H_{\mathrm{S}}(w). \tag{5.41}$$

Here $\mathcal{H}_S(w)$ is the subspace spanned by eigenvectors of $\mathbb{J}$ associated with eigenvalue $w \in \mathbb{Z}_{1/2}$ and $H_{\mathrm{S}}(w) = H_{\mathrm{S}}\lceil_{\mathcal{H}_S(w)}$.

Proof: This follows from Proposition 5.13 and (5.40).

QED

The Pauli-Fierz Hamiltonians with different polarization vectors, however, are unitary equivalent. Denote the Pauli-Fierz Hamiltonian with polarization vectors $e^{\pm 1}$ by $H_{\mathrm{S}}(e^{\pm 1})$. Let $\eta^{\pm 1}$ be arbitrary polarization vectors. Then

$$H_{\mathrm{S}}(\eta^{\pm 1}) \cong \bigoplus_{w\in\mathbb{Z}_{1/2}} H_{\mathrm{S}}(e^{\pm 1}, w).$$

By using symmetries of the Pauli-Fierz Hamiltonian with spin 1/2 we can show the degeneracy of ground states of H_{S}. Assume that V is rotation-invariant and the polarization vectors $e^{\pm 1}$ are given by

$$e^{+1}(k) = \frac{(-k_2, k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad e^{-1}(k) = \hat{k}\times e^{+1}(k) = \frac{(-k_3k_1, -k_2k_3, k_1^2 + k_2^2)}{|k|\sqrt{k_1^2 + k_2^2}}. \tag{5.42}$$

These are coherent in direction $(0, 0, 1)$ and their helicity is zero. Let $\Lambda : \mathbb{R}^3 \to \mathbb{R}^3$ be the flip defined by $\Lambda\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}k_1\\-k_2\\k_3\end{pmatrix}$, and $\tilde{u}, u : L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ by

$$\tilde{u} : \begin{pmatrix}f\\g\end{pmatrix} \mapsto \begin{pmatrix}f\circ\Lambda\\g\circ\Lambda\end{pmatrix}, \quad u : \begin{pmatrix}f\\g\end{pmatrix} \mapsto \begin{pmatrix}-f\circ\Lambda\\g\circ\Lambda\end{pmatrix}.$$

A computation gives

$$u^{\sharp\,-1}k_\mu u^\sharp = \begin{cases} k_\mu, & \mu = 1, 3,\\ -k_\mu, & \mu = 2,\end{cases} \qquad u^{\sharp\,-1}\nabla_\mu u^\sharp = \begin{cases} \nabla_\mu, & \mu = 1, 3,\\ -\nabla_\mu, & \mu = 2,\end{cases}$$

where $u^\sharp = u$ or $\tilde{u}$. We have for rotation-invariant f and g,

$$u^{-1}\begin{pmatrix} e_\mu^{+1} f \\ e_\mu^{-1} g \end{pmatrix} = \begin{cases} \binom{e_\mu^{+1} f}{e_\mu^{-1} g}, & \mu = 1, 3, \\ -\binom{e_\mu^{+1} f}{e_\mu^{-1} g}, & \mu = 2. \end{cases}$$

Then the second quantization $\Gamma(u)$ of u induces the unitary operator on $\mathcal{F}$ and for rotation-invariant $\hat{\varphi}$ we obtain

$$\Gamma(u)^{-1} A_\mu(x) \Gamma(u) = \begin{cases} A_\mu(\Lambda x), & \mu = 1, 3, \\ -A_\mu(\Lambda x), & \mu = 2, \end{cases} \tag{5.43}$$

$$\Gamma(u)^{-1} H_{\rm f} \Gamma(u) = H_{\rm f}. \tag{5.44}$$

Next we consider the transformation on $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ given by

$$\sigma_\mu \mapsto \sigma_2 \sigma_\mu \sigma_2 = \begin{cases} -\sigma_\mu, & \mu = 1, 3, \\ \sigma_\mu, & \mu = 2. \end{cases} \tag{5.45}$$

Under the identification $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \cong \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$, we define $\tau = \sigma_2 \otimes \tilde{u}$. This satisfies

$$\tau^{-1}(\sigma_\mu \otimes f)\tau = \begin{cases} -\sigma_\mu \otimes f \circ \Lambda^{-1}, & \mu = 1, 3, \\ \sigma_\mu \otimes f \circ \Lambda^{-1}, & \mu = 2, \end{cases} \tag{5.46}$$

$$\tau^{-1}(\sigma_\mu \otimes \nabla_\mu)\tau = -\sigma_\mu \otimes \nabla_\mu, \quad \mu = 1, 2, 3. \tag{5.47}$$

We finally define the unitary operator $\mathfrak{J} : \mathcal{H}_S \to \mathcal{H}_S$ by

$$\mathfrak{J} = \tau \otimes \Gamma(u).$$

Combining (5.43)-(5.47), for rotation-invariant $\hat{\varphi}$ and V

$$\begin{aligned} &\mathfrak{J}^{-1} \sigma_\mu(-\mathrm{i}\nabla_\mu - A_\mu) \mathfrak{J} = -\sigma_\mu(-\mathrm{i}\nabla_\mu - A_\mu), \\ &\mathfrak{J}^{-1} H_{\rm f} \mathfrak{J} = H_{\rm f}, \\ &\mathfrak{J}^{-1} V \mathfrak{J} = V \end{aligned}$$

are obtained. From these relations we can show the theorem below.

Theorem 5.15 *If $\hat{\varphi}$ and V are rotation-invariant, and the polarization vectors $e^{\pm 1}$ are given by* (5.42)*, then $H_{\rm S}(z)$ and $H_{\rm S}(-z)$ are unitary equivalent.*

Proof: Since $e^{\pm 1}$ is coherent in direction $(0, 0, 1)$ and its helicity is zero, $\mathbb{J}$ is of the form $\mathbb{J} = (\ell_{x,3} - \frac{1}{2}\sigma_3) \otimes \mathbb{1} + \mathbb{1} \otimes L_{{\rm f},3}$. It follows that $\mathfrak{J}^{-1} \mathbb{J} \mathfrak{J} = -\mathbb{J}$. This implies that $\mathfrak{J}$ maps $\mathcal{H}_{\rm S}(w)$ onto $\mathcal{H}_{\rm S}(-w)$. Furthermore, $\mathfrak{J}^{-1} H_{\rm S} \mathfrak{J} = H_{\rm S}$. Thus $\mathfrak{J}^{-1} H_{\rm S}(w) \mathfrak{J} = H_{\rm S}(-w)$ follows.

QED

An application of Theorem 5.15 is to estimate the multiplicity of eigenvalues of $H_{\rm S}$.

Corollary 5.16 *Suppose that V and $\hat{\varphi}$ are rotation-invariant. Then the multiplicity of any eigenvalue is a non-negetaive even number. Moreover $\mathfrak{S}^{-1} e^{-tH_{\rm S}} \mathfrak{S}$ is not positivity improving.*

Proof: We may suppose that the polarization vectors of $H_{\rm S}$ are given by (5.42). Thus $H_{\rm S} = \bigoplus_{w \in \mathbb{Z}_{1/2}} H_{\rm S}(w)$. Theorem 5.15 implies $H_{\rm S}(w) \cong H_{\rm S}(-w)$, and the multiplicity of any eigenvalue is even. The existence of the ground state is established in e.g., [10, 12, 11, 13, 45, 96, 9, 8, 51, 21]. $\mathfrak{S}^{-1} e^{-tH_{\rm S}} \mathfrak{S}$ can not be positivity improving, since the ground state is degenerate.

QED

6 Energy comparison inequalities

We show several inequalities derived from functional integral representations for the Pauli-Fierz type Hamiltonians. These inequalities are useful to study the properties of the Pauli-Fierz type Hamiltonians. See Table 7 below.

	Energy comparison inequality
H	$E(0) \leq E(e^2)$
$H(p)$	$E(e^2, 0) \leq E(e^2, p)$
H_{R}	$E_{\mathrm{R}}(0) \leq E_{\mathrm{R}}(e^2)$
$H_{\mathrm{R}}(p)$	$E_{\mathrm{R}}(e^2, 0) \leq E_{\mathrm{R}}(e^2, p)$
H_{S}	$\displaystyle\max_{(\alpha\beta\gamma)=(123),(231),(312)} E(0, \sqrt{\hat{B}_\alpha^2 + \hat{B}_\beta^2}, 0, \hat{B}_\gamma) \leq E(e^2, \hat{B}_1, \hat{B}_2, \hat{B}_3)$
$H_{\mathrm{S}}(p)$	$\displaystyle\max_{(\alpha\beta\gamma)=(123),(231),(312)} E(0, p, \sqrt{\hat{B}_\alpha^2 + \hat{B}_\beta^2}, 0, \hat{B}_\gamma) \leq E(e^2, p, \hat{B}_1, \hat{B}_2, \hat{B}_3)$

Table 7: Energy comparison inequality

6.1 Pauli-Fierz Hamiltonian H and $H(p)$

An application of the functional integral representation of $(F, e^{-tH}G)$ is a diamagnetic inequality.

Corollary 6.1 *Under the conditions of Theorem* 3.3 *it follows that*

$$|(F, e^{-tH}G)| \leq (|F|, e^{-t(H_{\mathrm{p}}+H_{\mathrm{f}})}|G|). \tag{6.1}$$

Proof: By the functional integral representation in Theorem 3.3 we have

$$|(F, e^{-tH}G)| \leq \int_{\mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s}(\mathrm{J}_0|F(B_0)|, \mathrm{J}_t|G(B_t)|)]\mathrm{d}x = (|F|, e^{-t(H_{\mathrm{p}}+H_{\mathrm{f}})}|G|).$$

Here we used that $|\mathrm{J}_s F| \leq \mathrm{J}_s|F|$, since J_s is positivity preserving.

QED

(6.1) is called the diamagnetic inequality. The diamagnetic inequality shows that coupling a particle to a quantized radiation field by minimal interaction increases the ground state energy of the non-interacting system. The exact statement is as follows. We introduce a coupling constant $e \in \mathbb{R}$ to the Pauli-Fierz Hamiltonian H as

$$H = \frac{1}{2}(-\mathrm{i}\nabla - e\hat{A})^2 + V + H_{\mathrm{f}}.$$

By a symmetry we can see that

$$\frac{1}{2}(-\mathrm{i}\nabla - e\hat{A})^2 + V + H_{\mathrm{f}} \cong \frac{1}{2}(-\mathrm{i}\nabla + e\hat{A})^2 + V + H_{\mathrm{f}}.$$

Then $\inf\sigma(H)$ depends on e^2. Let $E(e^2) = \inf\sigma(H)$.

Corollary 6.2 *We have* $E(0) \le E(e^2)$.

Proof: Corollary 6.1 implies that

$$\sup_{F\neq 0} \frac{(F, e^{-tH}F)}{\|F\|^2} \le \sup_{F\neq 0} \frac{(|F|, e^{-t(H_{\rm p}+H_{\rm f})}|F|)}{\|F\|^2} \le \sup_{F\neq 0} \frac{(F, e^{-t(H_{\rm p}+H_{\rm f})}F)}{\|F\|^2}.$$

Hence $\|e^{-tH}\| \le \|e^{-t(H_{\rm p}+H_{\rm f})}\| = \|e^{-tH_{\rm p}}\|$. It implies that $\inf\sigma(H_{\rm p}) \le \inf\sigma(H)$. Thus the corollary follows.

QED

Another significant property of e^{-tH} is positivity improving. Unitary operator $e^{{\rm i}t(-{\rm i}\nabla_\mu)}$ is a shift operator on $L^2(\mathbb{R}^3)$, hence it is positivity preserving. While $e^{-t(-\Delta)}$ is positivity improving. I.e., $(f, e^{-t(-\Delta)}g) > 0$ for any non identically zero functions $f \ge 0$ and $g \ge 0$. By the integral kernel $e^{-{\rm i}\hat{A}(K_t)}$, in general $(F, e^{-tH}G) \in \mathbb{C}$ for $F \ge 0$ and $G \ge 0$. Let $\Psi_{\rm g}$ be the ground state of H. By Corollary 3.10 we see that $e^{-{\rm i}(\pi/2)N}\Psi_{\rm g}$ is strictly positive. Hence $(f\otimes 1\!\!1, e^{-{\rm i}(\pi/2)N}\Psi_{\rm g}) \neq 0$ for any non identically zero $f \ge 0$. It is easy to see that $\lim_{t\to\infty}(f\otimes 1\!\!1, e^{-t(H-E(e^2))}f\otimes 1\!\!1) = |(f\otimes 1\!\!1, \Psi_{\rm g})|^2 > 0$ and hence

$$\begin{aligned} &-\lim_{t\to\infty}\frac{1}{t}\log(f\otimes 1\!\!1, e^{-tH}f\otimes 1\!\!1) = -\lim_{t\to\infty}\frac{1}{t}\log\left\{e^{-tE(e^2)}(f\otimes 1\!\!1, e^{-t(H-E(e^2))}f\otimes 1\!\!1)\right\} \\ &= E(e^2) - \lim_{t\to\infty}\frac{1}{t}\log(f\otimes 1\!\!1, e^{-t(H-E(e^2))}f\otimes 1\!\!1) = E(e^2). \end{aligned}$$

This allows to derive an expression of the ground state energy $E(e^2)$.

Corollary 6.3 *Suppose either Assumption 3.1 or Assumption 3.2. Then the function*

$$e^2 \mapsto E(e^2)$$

is monotonously increasing, continuous and concave.

Proof: Notice that

$$(1\!\!1, e^{-{\rm i}e\hat{A}(K_t)}1\!\!1) = e^{-(e^2/2){\rm q}(K_t,K_t)}.$$

For $f \ge 0$, we have

$$\begin{aligned} E(e^2) &= -\lim_{t\to\infty}\frac{1}{t}\log(f\otimes 1\!\!1, e^{-tH}f\otimes 1\!\!1) \\ &= -\lim_{t\to\infty}\frac{1}{t}\log\int_{\mathbb{R}^3}\mathbb{E}^x[f(B_0)f(B_t)e^{-\int_0^t V(B_s){\rm d}s}e^{-(e^2/2){\rm q}(K_t,K_t)}]{\rm d}x. \end{aligned}$$

As $e^{-(e^2/2){\rm q}_{\rm E}(K_t,K_t)}$ is log-convex in e^2, $E(\cdot)$ is concave. Thus $E(e^2)$ is continuous on $(0,\infty)$. Since $E(e^2)$ is continuous at $e=0$, $E(e^2)$ can be expressed as

$$E(e^2) = E(0) + \int_0^{e^2}\rho(t){\rm d}t$$

with a suitable non-negative function $\rho(t)$. This implies that $E(e^2)$ is monotonously increasing in e^2.

QED

Remark 6.1 *The formula*

$$-\lim_{t\to\infty}\frac{1}{t}\log(f\otimes 1\!\!1, e^{-tH}f\otimes 1\!\!1)=E(e^2)$$

holds for arbitrary strictly positive function f regardless of the existence of ground states. See [73, Lemma 1.56].

We derive a similar inequality for $e^{-tH(p)}$ to that for e^{-tH}.

Corollary 6.4 *It follows that*

$$|(\Psi, e^{-tH(p)}\Phi)|\leq(|\Psi|, e^{-t(\mathrm{P}_\mathrm{f}^2+H_\mathrm{f})}|\Phi|).$$

Proof: By the functional integral representation of $(\Psi, e^{-tH(p)}\Phi)$ we have

$$|(\Psi, e^{-tH(p)}\Phi)|\leq\mathbb{E}[(\mathrm{J}_0|\Psi|, \mathrm{J}_t e^{-\mathrm{i}B_t\mathrm{P}_\mathrm{f}}|\Phi|]=(|\Psi|, e^{-t(\mathrm{P}_\mathrm{f}^2+H_\mathrm{f})}|\Phi|).$$

Here we used that $e^{-\mathrm{i}B_t\mathrm{P}_\mathrm{f}}$ is positivity preserving and hence $|e^{-\mathrm{i}B_t\mathrm{P}_\mathrm{f}}\Phi|\leq e^{-\mathrm{i}B_t\mathrm{P}_\mathrm{f}}|\Phi|$.

QED

From this diamagnetic inequality, we can only deduce the trivial energy comparison inequality $0=\inf\sigma(\mathrm{P}_\mathrm{f}^2+H_\mathrm{f})\leq\inf\sigma(H(p))$. However, combining the unitary transformation $e^{-\mathrm{i}(\pi/2)N}$ with the functional integral representation, we obtain an interesting result. Denote $E(e^2,p)=\inf\sigma(H(p))$.

Corollary 6.5 *It follows that*

(1) *Let $p=0$. Then $e^{\mathrm{i}(\pi/2)N}e^{-tH(0)}e^{-\mathrm{i}(\pi/2)N}$ is positivity improving.*

(2) *The ground state of $H(0)$ is unique whenever it exists.*

(3) *$E(e^2,0)\leq E(e^2,p)$ holds.*

(4) *Map $p\mapsto E(e^2,p)$ is continuous and $E(e^2,0)=\inf\sigma(H_{V=0})$, where $H_{V=0}$ is H with $V=0$.*

Proof: In the case of $p=0$ we remark that $e^{\mathrm{i}pB_t}=1$. Then we have

$$(\Psi, e^{\mathrm{i}(\pi/2)N}e^{-tH(0)}e^{-\mathrm{i}(\pi/2)N}\Phi)=\mathbb{E}^0[(\mathrm{J}_0\Psi, e^{-\mathrm{i}\pi(K_t)}\mathrm{J}_t e^{-\mathrm{i}\mathrm{P}_\mathrm{f}B_t}\Phi)].$$

Since $\mathrm{J}_0^* e^{-\mathrm{i}\pi(K_t)}\mathrm{J}_t$ is positivity improving and $e^{-\mathrm{i}\mathrm{P}_\mathrm{f}B_t}$ is positivity preserving, (1) follows. (2) is implied by (1) and the Perron-Frobenius theorem [47]. We have

$$\begin{aligned}|(\Psi, e^{\mathrm{i}(\pi/2)N}e^{-tH(p)}e^{-\mathrm{i}(\pi/2)N}\Phi)|&\leq\mathbb{E}^0[(\mathrm{J}_0|\Psi|, e^{-\mathrm{i}\pi(K_t)}\mathrm{J}_t e^{-\mathrm{i}\mathrm{P}_\mathrm{f}B_t}|\Phi|)]\\&=(|\Psi|, e^{\mathrm{i}(\pi/2)N}e^{-tH(0)}e^{-\mathrm{i}(\pi/2)N}|\Phi|).\end{aligned}$$

This yields (3). Finally we show (4). From $|(\Psi,(e^{-tH(p)}-e^{-tH(q)})\Phi)|\leq c|p-q|$ the continuity follows. Let $E=\inf\sigma(H_{V=0})$ and we set $E(e^2,p)=E(p)$. We have

$$(\Psi, H\Psi)=\int_{\mathbb{R}^3}(\Psi(p), H(p)\Psi(p))\mathrm{d}p\geq E(0)\|\Psi\|.$$

Then $E(0) \le E$ follows. Let $\Phi_\delta = \int_{\mathbb{R}^3}^{\oplus} \Phi(q) 1_{\{|p-q|\le\delta\}} \mathrm{d}q$, where we assume that $\Phi(q)$ satisfies

$$E(q) \le \frac{(\Phi(q), H(q)\Phi(q))}{\|\Phi(q)\|^2} < E, \quad |p-q| \le \delta.$$

We have

$$E\|\Phi_\delta\|^2 > \int_{|q-p|\le\delta} (\Phi(q), H(q)\Phi(q)) \mathrm{d}q = (\Phi_\delta, H\Phi_\delta).$$

This contradicts that E is the bottom of the spectrum of $H_{V=0}$. Hence $E(0) \le E \le E(p)$ for all $p \in \mathbb{R}^3$. From the continuity of $E(p)$, it follows that $E(0) = E$.

QED

6.2 Relativistic Pauli-Fierz Hamiltonian H_{R} and $H_{\mathrm{R}}(p)$

For H_{R} by using the functional integral representation we can obtain similar results to those of H. Let $E_{\mathrm{R}}(e^2) = \inf \sigma(H_{\mathrm{R}})$ and $E_{\mathrm{R}}(e^2, p) = \inf \sigma(H_{\mathrm{R}}(p))$.

Corollary 6.6 *Suppose Assumption 4.1.*
(1) It follows that

$$|(F, e^{-tH_{\mathrm{R}}}G)| \le \left(|F|, e^{-t((-\Delta+m^2)^{1/2}-m+V+H_{\mathrm{f}})}|G|\right).$$

(2) The function $e^2 \mapsto E_{\mathrm{R}}(e^2)$ is monotonously increasing, continuous and concave.
(3) It follows that

$$|(\Psi, e^{-tH_{\mathrm{R}}(p)}\Phi)| \le (|\Psi|, e^{-t(\sqrt{\mathrm{P}_{\mathrm{f}}^2+m^2}-m+H_{\mathrm{f}})}|\Phi|).$$

(4) Statements (1)-(4) below are satisfied.

(1) *Let $p = 0$. Then $e^{\mathrm{i}(\pi/2)N} e^{-tH_{\mathrm{R}}(0)} e^{-\mathrm{i}(\pi/2)N}$ is positivity improving.*

(2) *The ground state of $H_{\mathrm{R}}(0)$ is unique whenever it exists.*

(3) *Energy comparison inequality $E_{\mathrm{R}}(0) \le E_{\mathrm{R}}(p)$ holds.*

(4) *Map $p \mapsto E_{\mathrm{R}}(p)$ is continuous and $E_{\mathrm{R}}(0) = \inf \sigma(H_{\mathrm{R}V=0})$, where $H_{\mathrm{R}V=0}$ is H_{R} with $V = 0$.*

6.3 Pauli-Fierz Hamiltonian with spin 1/2 $H_{\mathbb{Z}_2}$ and $H_{\mathbb{Z}_2}(p)$

Let us consider $H_{\mathbb{Z}_2}$. By using the functional integral representation we can estimate the ground state energy of $H_{\mathbb{Z}_2}$. Write

$$E(e^2, \hat{B}_1, \hat{B}_2, \hat{B}_3) = \inf \sigma(H_{\mathbb{Z}_2}).$$

For the spinless Pauli-Fierz Hamiltonian H we have $\inf \sigma(H) = E(e^2, 0, 0, 0)$ and the diamagnetic inequality $E(0, 0, 0, 0) \le E(e^2, 0, 0, 0)$ was already seen. We extend this inequality to $H_{\mathbb{Z}_2}$. Define

$$(H_{\mathbb{Z}_2}(0)F)(\theta) = (H_{\mathrm{p}} + H_{\mathrm{f}} + \hat{H}_{\mathrm{d}}(\theta))F(\theta) - |\hat{H}_{\mathrm{od}}(-\theta)|F(-\theta),$$

where

$$|\hat{H}_{\mathrm{od}}(-\theta)| = \frac{1}{2}\sqrt{\hat{B}_1^2(x) + \hat{B}_2^2(x)}$$

is independent of $\theta \in \mathbb{Z}_2$. $H_{\mathbb{Z}_2}(0)$ corresponds to

$$H_{\rm p} + H_{\rm f} - \frac{1}{2} \begin{pmatrix} \hat{B}_3 & \sqrt{\hat{B}_1^2 + \hat{B}_2^2} \\ \sqrt{\hat{B}_1^2 + \hat{B}_2^2} & -\hat{B}_3 \end{pmatrix}$$

acting in $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes L^2(\mathrm{Q})$. The functional integral representation of $e^{-tH_{\mathbb{Z}_2}(0)}$ is given by

$$(F, e^{-tH_{\mathbb{Z}_2}(0)}G) = \lim_{\varepsilon \downarrow 0} e^t \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)\mathrm{d}s}(\mathrm{J}_0 F(X_0), e^{X_t^\perp(\varepsilon)} \mathrm{J}_t G(X_t))]\mathrm{d}x,$$

where

$$X_t^\perp(\varepsilon) = -\int_0^t \mathbb{H}_{\rm d}(B_s, \theta_{N_s}, s)\mathrm{d}s + \int_0^{t+} \log(\Psi_\varepsilon(|\mathbb{H}_{\rm od}(B_s, s)|))\mathrm{d}N_s. \tag{6.2}$$

Corollary 6.7 *It follows that*

$$|(F, e^{-tH_{\mathbb{Z}_2}}G)| \leq (|F|, e^{-tH_{\mathbb{Z}_2}(0)}|G|) \tag{6.3}$$

and

$$\max_{(\alpha\beta\gamma)=(123),(231),(312)} E(0, \sqrt{\hat{B}_\alpha^2 + \hat{B}_\beta^2}, 0, \hat{B}_\gamma) \leq E(e^2, \hat{B}_1, \hat{B}_2, \hat{B}_3). \tag{6.4}$$

Proof: Since $|\Psi_\varepsilon(\mathbb{H}_{\rm od})| \leq \Psi_\varepsilon(|\mathbb{H}_{\rm od}|)$, $|e^{Z_t(\varepsilon)}| \leq e^{X_t^\perp(\varepsilon)}$ and $|\mathrm{J}_t G| \leq \mathrm{J}_t |G|$, by the functional integral representation of $e^{-tH_{\mathbb{Z}_2}}$ we have (6.3). From this,

$$E(0, \sqrt{\hat{B}_1^2 + \hat{B}_2^2}, 0, \hat{B}_3) \leq E(e^2, \hat{B}_1, \hat{B}_2, \hat{B}_3)$$

is obtained. We will show that

$$E(e^2, \hat{B}_1, \hat{B}_2, \hat{B}_3) = E(e^2, \hat{B}_2, \hat{B}_3, \hat{B}_1) = E(e^2, \hat{B}_3, \hat{B}_1, \hat{B}_2) \tag{6.5}$$

by SU(2)-symmetry. Let $\mathcal{R} \in O(3)$ be such that $\mathcal{R}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$. Then there exists $(n, \phi) \in S^2 \times \mathbb{R}$ such that $\mathcal{R} = \mathcal{R}(n, \phi)$. Here $\mathcal{R}(n, \phi)$ describes the rotation around $n \in \mathbb{R}^3$ by angle $\phi \in [0, 2\pi)$. Hence we see that

$$e^{\mathrm{i}\phi n \cdot (1/2)\sigma} \sigma_\mu e^{-\mathrm{i}\phi n \cdot (1/2)\sigma} = (\mathcal{R}^{-1}\sigma)_\mu.$$

Now we write $H_{\mathbb{Z}_2}$ by $H_{\mathbb{Z}_2}(e, \hat{B}_1, \hat{B}_2, \hat{B}_3)$. Thus we obtain that

$$e^{\mathrm{i}\phi n \cdot (1/2)\sigma} H_{\mathbb{Z}_2}(e, \hat{B}_1, \hat{B}_2, \hat{B}_3) e^{-\mathrm{i}\phi n \cdot (1/2)\sigma} = H_{\mathbb{Z}_2}(e, \hat{B}_2, \hat{B}_3, \hat{B}_1)$$

which implies the first equality in (6.5). The second one is proven in the same way.

QED

Write

$$E(p, \hat{A}, \hat{B}_1, \hat{B}_2, \hat{B}_3) = \inf \sigma(H_{\mathbb{Z}_2}(p)),$$

and define $H_{\mathbb{Z}_2}(p,0)$ by

$$(H_{\mathbb{Z}_2}(p,0)\Psi)(\theta) = \left(\frac{1}{2}(p-\mathrm{P_f})^2 + \hat{H}_{\mathrm{d}}(0,\theta)\right)\Psi(\theta) - |\hat{H}_{\mathrm{od}}(0,-\theta)|\Psi(-\theta),$$

where $|\hat{H}_{\mathrm{od}}(0,-\theta)| = \frac{1}{2}\sqrt{\hat{B}_1(0)^2+\hat{B}_2(0)^2}$. This corresponds to

$$\frac{1}{2}(p-\mathrm{P_f})^2 + H_{\mathrm{f}} - \frac{1}{2}\begin{pmatrix} \hat{B}_3(0) & \sqrt{\hat{B}_1(0)^2+\hat{B}_2(0)^2} \\ \sqrt{\hat{B}_1(0)^2+\hat{B}_2(0)^2} & -\hat{B}_3(0) \end{pmatrix}$$

in $\mathbb{C}^2 \otimes L^2(\mathrm{Q})$. Note that $|\hat{H}_{\mathrm{od}}(0,-\theta)|$ is independent of θ.

Corollary 6.8 *It follows that*

$$|(\Phi, e^{-tH_{\mathbb{Z}_2}(p)}\Psi)| \le (|\Phi|, e^{-tH_{\mathbb{Z}_2}(p,0)}|\Psi|) \tag{6.6}$$

and

$$\max_{(\alpha\beta\gamma)=(123),(231),(312)} E(0,0,\sqrt{\hat{B}_\alpha^2+\hat{B}_\beta^2},0,\hat{B}_\gamma) \le E(p,\hat{A},\hat{B}_1,\hat{B}_2,\hat{B}_3). \tag{6.7}$$

Proof: We have

$$|(\Phi, e^{-tH_{\mathrm{S}}(p)}\Psi)| \le e^t \lim_{\varepsilon\downarrow 0}\sum_{\theta\in\mathbb{Z}_2} \mathbb{E}^{0,\alpha}[(\mathrm{J}_0|\Phi(\theta_\alpha)|, e^{X_t^\perp(\varepsilon)}\mathrm{J}_t e^{-\mathrm{i}\mathrm{P_f}B_t}|\Phi(\theta_{N_t})|)] = \text{RHS (6.6)},$$

where $X_t^\perp(\varepsilon)$ is given by (6.2). (6.7) is immediate from (6.6) and a similar argument to (6.4).
QED

7 Concluding remarks

(1) The positivity improvingness of the semigroups generated by translation invariant Hamiltonians is introduced in this paper only for the case $p = 0$. We can also show the positivity improvingness for $p \ne 0$ by functional integral representation. We refer to see [69]. This is also prove in [109] in the different manners.
(2) To find an invariant domain by using functional integral representations can be extended to more general cases. We refer to see [68].
(3) The Nelson model without cutoff function can be defined through a renormalization. The Nelson model with a cutoff function is defined by

$$H = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + \phi(\tilde{\varphi}(\cdot - x)),$$

where $\phi(f)$ for $f \in L^2(\mathbb{R}^3)$ denotes a scalar field satisfying $\mathbb{E}[e^{z\phi(f)}] = e^{z^2\|f\|^2/2}$ for $z \in \mathbb{C}$, $\mathbb{E}[\phi(f)\phi(g)] = \frac{1}{2}(f,g)$ and $\mathbb{E}[\phi(f)] = 0$. The renormalization was succeeded in by Edward Nelson [111, 110], and gave a proof in terms of a path measure by [49, 106]. The spectral properties of the renormalized Nelson Hamiltonian H_∞ is studied in [59, 74, 75]. In particular in [74] a Gibbs measure [14, 17] associated with the ground state Ψ_{g} of H_∞ is constructed, and $(\Psi_{\mathrm{g}}, \mathscr{O}\Psi_{\mathrm{g}})$ for $\mathscr{O} = e^{\beta N}, e^{\phi(f)^2}$ are expressed in terms of the Gibbs measure.

For the Gibbs measure associated with the ground state of the Pauli-Fierz Hamiltonian is constructed in [65], but there is no useful applications.

A Abstract boson Fock space

A.1 Annihilation operators and creation operators

In this Appendix we introduce general tools concerning with abstract boson Fock spaces and second quantization. Instead of physical space-time dimension we work in general d dimension.

Let $\mathfrak{H}$ be a separable Hilbert space over $\mathbb{C}$. Consider the operation $\otimes_{\mathrm{s}}^n$ of n-fold symmetric tensor product defined through the symmetrization operator

$$S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}, \quad n \geq 1,$$

where $f_1, \ldots, f_n \in \mathfrak{H}$ and $\mathfrak{S}_n$ denotes the permutation group of order n. Define

$$\mathcal{F}^{(n)} = \begin{cases} S_n(\otimes^n \mathfrak{H}), & n \geq 1, \\ \mathbb{C}, & n = 0. \end{cases}$$

The space

$$\mathcal{F} = \mathcal{F}(\mathfrak{H}) = \left\{ \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)} \ni \bigoplus_{n=0}^{\infty} \Psi^{(n)} \,\middle|\, \|\Psi\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\mathcal{F}^{(n)}}^2 < \infty \right\}$$

is called the boson Fock space over $\mathfrak{H}$. The boson Fock space $\mathcal{F}$ can be identified with the space of ℓ_2-sequences $(\Psi^{(n)})_{n\geq 0}$ such that $\Psi^{(n)} \in \mathcal{F}^{(n)}$ and $\sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\mathcal{F}^{(n)}}^2 < \infty$. $\mathcal{F}$ is a Hilbert space endowed with the scalar product

$$(\Psi, \Phi)_{\mathcal{F}} = \sum_{n=0}^{\infty} (\Psi^{(n)}, \Phi^{(n)})_{\mathcal{F}^{(n)}}.$$

The vector $\Omega = (1, 0, 0, \ldots)$ is called Fock vacuum. The subspace $\mathcal{F}^{(n)}$ can be interpreted as consisting of the states of the quantum field having exactly n boson particles, while the permutation symmetry corresponds to the fact that the particles are indistinguishable.

There are two fundamental operators, the creation operator denoted by $a^*(f)$, $f \in \mathfrak{H}$, and the annihilation operator by $a(f)$ defined by

$$(a^*(f)\Psi)^{(n)} = \begin{cases} \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), & n \geq 1, \\ 0, & n = 0 \end{cases}$$

with domain

$$D(a^*(f)) = \left\{ (\Psi^{(n)})_{n\geq 0} \in \mathcal{F} \,\middle|\, \sum_{n=1}^{\infty} n \|S_n(f \otimes \Psi^{(n-1)})\|_{\mathcal{F}^{(n)}}^2 < \infty \right\},$$

and

$$a(f) = (a^*(\bar{f}))^*.$$

Note that $a^\sharp(f)$ is linear in f by the definitions.

As the terminology suggests, the action of $a^*(f)$ increases the number of bosons by one, while $a(f)$ decreases it by one. The relation

$$(\Phi, a(f)\Psi) = (a^*(\bar{f})\Phi, \Psi)$$

holds. Furthermore, since both operators are closable by the dense definition of their adjoints, we use and denote their closed extensions by the same symbols. Let $D \subset \mathfrak{H}$ be a dense subset. It is known that

$$\mathrm{LH}\{a^*(f_1)\cdots a^*(f_n)\Omega, \Omega |\, f_j \in D, j = 1, \ldots, n, n \geq 1\}$$

is dense in $\mathcal{F}(\mathfrak{H})$. The space

$$\mathcal{F}_{\mathrm{fin}} = \{(\Psi^{(n)})_{n\geq 0} \in \mathcal{F} |\ \Psi^{(m)} = 0 \text{ for all } m \geq M \text{ with some } M\}$$

is called the finite particle subspace. The field operators $a(f)$ and $a^*(f)$ leave $\mathcal{F}_{\mathrm{fin}}$ invariant and satisfy the canonical commutation relations

$$[a(f), a^*(g)] = (\bar{f}, g)\mathbb{1}, \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)]$$

on $\mathcal{F}_{\mathrm{fin}}$. We introduce several technical estimates. Let

$$\Phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(\bar{f})).$$

Then $\Phi(f)$ is essentially self-adjoint on $\mathcal{F}_{\mathrm{fin}}$. The conjugate momentum operator is also defined by

$$\Pi(f) = \frac{\mathrm{i}}{\sqrt{2}}(a^*(f) - a(\bar{f})).$$

They satisfy that

$$[\Phi(f), \Pi(g)] = i\Re(f, g), \quad [\Pi(f), \Pi(g)] = i\Im(f, g), \quad [\Phi(f), \Phi(g)] = i\Im(f, g).$$

Products $\prod_{j=1}^m \Phi(f_j)$ can be represented by the sum of Wick ordered operators. We see this in the lemma below. We simply write

$$c^m_{j_1,\ldots,j_k} = {:}\Phi(f_1)\cdots\widehat{\Phi(f_{j_1})}\cdots\cdots\widehat{\Phi(f_{j_k})}\cdots\Phi(f_m){:},$$

where $\widehat{\Phi(f)}$ describes neglecting $\Phi(f)$. We also set

$$c^m = {:}\Phi(f_1)\cdots\Phi(f_m){:}.$$

Lemma A.1 *Let $f_j \in \mathfrak{H}$, $j = 1, ..., n$. Then $\prod_{j=1}^m \Phi(f_j)$ can be represented in terms of Wick ordered operators as follows.*
($m = 2n$)

$$\prod_{j=1}^{2n} \Phi(f_j) = c^{2n} + \sum_{k=1}^{n} \frac{1}{2^k} \sum_{j_1,\ldots,j_{2k}}^{\mathrm{pair}} \left(\prod_{i=1}^{k} (f_{j_{2i-1}}, f_{j_{2i}}) \right) c^{2n}_{j_1,\ldots,j_{2k}}, \tag{A.1}$$

where $\sum_{j_1,\ldots,j_{2k}}^{\mathrm{pair}}$ describes the summation over all k-pairs chosen from $\{1, \ldots, 2n\}$.
($m = 2n + 1$)

$$\prod_{j=1}^{2n+1} \Phi(f_j) = c^{2n+1} + \sum_{k=1}^{n} \frac{1}{2^k} \sum_{j_1,\ldots,j_{2k}}^{\mathrm{pair}} \left(\prod_{i=1}^{k} (f_{j_{2i-1}}, f_{j_{2i}}) \right) c^{2n+1}_{j_1,\ldots,j_{2k}}, \tag{A.2}$$

where $\sum_{j_1,\ldots,j_{2k}}^{\mathrm{pair}}$ also describes the summation over all k-pairs chosen from $\{1, \ldots, 2n + 1\}$.

Proof: From the definition of Wick products the lemma directly follows.

QED

Lemma A.2 *Let* $f \in \mathfrak{H}$*. Then*

$$\Phi(f)^n = n! \sum_{k=0}^{[n/2]} \sum_{l+m+2k=n} \frac{(\frac{\|f\|^2}{4})^k}{k!} \frac{(\frac{1}{\sqrt{2}}a^*(f))^l}{l!} \frac{(\frac{1}{\sqrt{2}}a(\bar{f}))^m}{m!}. \tag{A.3}$$

Here $[m]$ *is the integer part of* m*, i.e.,* $[2n/2] = n$ *and* $[(2n+1)/2] = n$*.*

Proof: We see that by Lemma A.1

$$\Phi(f)^n = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} \left(\frac{\|f\|^2}{4}\right)^k :\Phi(f)^{n-2k}:,$$

$$:\Phi(f)^m: = 2^{-m/2} \sum_{k=0}^{m} \binom{m}{k} a^*(f)^k a(\bar{f})^{m-k}.$$

Together with them (A.3) follows. Then the lemma is proven.

QED

A.2 Second quantization

Given a bounded operator T on $\mathfrak{H}$, the second quantization of T is the operator $\Gamma(T)$ on $\mathcal{F}$ defined by

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} (\otimes^n T).$$

Here it is understood that $\otimes^0 T = \mathbb{1}$. In most cases $\Gamma(T)$ is an unbounded operator.

$$\begin{tikzcd}
\mathfrak{H} \arrow[r, "T"] \arrow[d, dotted] & \mathfrak{H} \arrow[d, dotted] \\
\mathcal{F}(\mathfrak{H}) \arrow[r, "\Gamma(T)"'] & \mathcal{F}(\mathfrak{H})
\end{tikzcd}$$

Figure 14: Fanctor Γ

However, for a contraction operator T, the second quantization $\Gamma(T)$ is also a contraction on $\mathcal{F}$, or equivalently, Γ is a functor

$$\Gamma : \mathscr{C}(\mathfrak{H} \to \mathfrak{H}) \to \mathscr{C}(\mathcal{F} \to \mathcal{F}),$$

of the set $\mathscr{C}(X \to Y)$ of contraction operators from X to Y. The functor Γ has the semi-group property, while $\mathscr{C}(\mathfrak{H} \to \mathfrak{H})$ is a $*$-algebra with respect to operator multiplication and conjugation $*$. The map Γ pulls this structure over to $\mathcal{F}$ so that

$$\Gamma(S)\Gamma(T) = \Gamma(ST), \quad \Gamma(S)^* = \Gamma(S^*), \quad \Gamma(\mathbb{1}) = \mathbb{1}, \tag{A.4}$$

$$
\begin{tikzcd}
\mathcal{F}^{(n)} \arrow[r, "a^*(f)"] \arrow[d, "\Gamma(T)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}^{(n+1)} \arrow[d, "\Gamma(T)"] \\
\mathcal{F}^{(n)} \arrow[r, "a^*(Tf)"'] & \mathcal{F}^{(n+1)}
\end{tikzcd}
$$

Figure 15: $a^*(f)$ and $\Gamma(T)$

$$
\begin{tikzcd}
\mathcal{F}^{(n)} \arrow[r, "a(f)"] & \mathcal{F}^{(n-1)} \\
\mathcal{F}^{(n)} \arrow[r, "a(jT^*jf)"'] \arrow[u, "\Gamma(T)"] \arrow[ur, phantom, "\circlearrowright"] & \mathcal{F}^{(n-1)} \arrow[u, "\Gamma(T)"']
\end{tikzcd}
$$

Figure 16: $a(f)$ and $\Gamma(T)$

for $S, T \in \mathscr{C}(\mathfrak{H} \to \mathfrak{H})$.

We can see relationship between $a^\sharp(f)$ and $\Gamma(T)$. Intertwining properties

$$\Gamma(T)a^*(f) = a^*(Tf)\Gamma(T), \tag{A.5}$$

$$a(f)\Gamma(T) = \Gamma(T)a(jT^*jf) \tag{A.6}$$

can be checked directly and from this we can derive commutation relations

$$[\Gamma(T), a^*(f)] = a^*((T - \mathbb{1})f)\Gamma(T), \tag{A.7}$$

$$[\Gamma(T), a(f)] = \Gamma(T)a(j(\mathbb{1} - T^*)jf), \tag{A.8}$$

where $jf = \bar{f}$ denotes the complex conjugate of f. Suppose in particular that T satisfies that $T^*T = \mathbb{1}$ and $TT^* = E$. Then (A.5) and (A.6) yield that

$$\Gamma(T)a^*(f)\Gamma(T^*) = a^*(Tf)\Gamma(E), \tag{A.9}$$

$$\Gamma(T)a(f)\Gamma(T^*) = \Gamma(E)a(jTjf). \tag{A.10}$$

$$
\begin{tikzcd}
\mathcal{F}^{(n-1)} \arrow[d, "\Gamma(T)"'] & \mathcal{F}^{(n)} \arrow[l, "a(f)"'] \arrow[r, "a^*(f)"] & \mathcal{F}^{(n+1)} \arrow[d, "\Gamma(T)"] \\
\mathcal{F}^{(n-1)} \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{F}^{(n)} \arrow[l, "\Gamma(E)a(jTjf)"] \arrow[r, "a^*(Tf)\Gamma(E)"'] \arrow[u, "\Gamma(T^*)"] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{F}^{(n+1)}
\end{tikzcd}
$$

Figure 17: Creation and annihilation operators, and second quantization

A.3 Differential second quantization

For a self-adjoint operator h on $\mathfrak{H}$ the structure relations (A.4) imply in particular that $\{\Gamma(e^{\mathrm{i}th}) : t \in \mathbb{R}\}$ is a strongly continuous one-parameter unitary group on $\mathcal{F}$. Then by the Stone's theorem there exists a unique self-adjoint operator $\mathrm{d}\Gamma(h)$ on $\mathcal{F}$ such that

$$\Gamma(e^{\mathrm{i}th}) = e^{\mathrm{i}t\mathrm{d}\Gamma(h)}, \quad t \in \mathbb{R}.$$

The operator $\mathrm{d}\Gamma(h)$ is called the differential second quantizationof h or simply second quantization of h. Thus we have

$$e^{\mathrm{i}t\mathrm{d}\Gamma(h)}a^*(f)e^{-\mathrm{i}t\mathrm{d}\Gamma(h)} = a^*(e^{\mathrm{i}th}f), \tag{A.11}$$

$$e^{\mathrm{i}t\mathrm{d}\Gamma(h)}a(f)e^{-\mathrm{i}t\mathrm{d}\Gamma(h)} = a(je^{\mathrm{i}th}jf). \tag{A.12}$$

Since $\mathrm{d}\Gamma(h) = -\mathrm{i}\frac{d}{dt}\Gamma(e^{\mathrm{i}th})\lceil_{t=0}$ on some domain, we have

$$\mathrm{d}\Gamma(h) = 0 \oplus \left[\bigoplus_{n=1}^{\infty}\left(\overline{\sum_{j=1}^{n}\underbrace{\mathbb{1}\otimes\cdots\otimes\overset{j}{h}\otimes\cdots\otimes\mathbb{1}}_{n}}\right)\right], \tag{A.13}$$

where j on top of h indicates its position in the product. Thus the action of $\mathrm{d}\Gamma(h)$ is given by

$$\mathrm{d}\Gamma(h)\Omega = 0,$$
$$\mathrm{d}\Gamma(h)a^*(f_1)\cdots a^*(f_n)\Omega = \sum_{j=1}^{n}a^*(f_1)\cdots a^*(hf_j)\cdots a^*(f_n)\Omega.$$

Then it holds that

$$[\mathrm{d}\Gamma(h), a^*(f)] = a^*(hf), \quad [\mathrm{d}\Gamma(h), a(f)] = -a(hf).$$

It can be also seen by (A.13) that

$$\sigma(\mathrm{d}\Gamma(h)) = \overline{\{\lambda_1+\cdots+\lambda_n|\lambda_j\in\sigma(h), j=1,\ldots,n, n\geq 1\}\cup\{0\}},$$
$$\sigma_{\mathrm{p}}(\mathrm{d}\Gamma(h)) = \{\lambda_1+\cdots+\lambda_n|\lambda_j\in\sigma_{\mathrm{p}}(h), j=1,\ldots,n, n\geq 1\}\cup\{0\}.$$

If $0\notin\sigma_{\mathrm{p}}(h)$ and $h\geq 0$, the multiplicity of 0 in $\sigma_{\mathrm{p}}(\mathrm{d}\Gamma(h))$ is one. A crucial operator in quantum field theory is the number operator defined by the second quantization of the identity operator on $\mathfrak{H}$:

$$N = \mathrm{d}\Gamma(\mathbb{1}).$$

Let h be a positive self-adjoint operator and $f\in D(h)$. The following bound is a fundamental. Let $f\in D(h^{-1/2})$ and $\Psi\in D(\mathrm{d}\Gamma(h)^{1/2})$. Then $\Psi\in D(a^\sharp(f))$ and

$$\|a(f)\Psi\| \leq \|h^{-1/2}f\|\|\mathrm{d}\Gamma(h)^{1/2}\Psi\|, \tag{A.14}$$

$$\|a^*(f)\Psi\| \leq \|h^{-1/2}f\|\|\mathrm{d}\Gamma(h)^{1/2}\Psi\| + \|f\|\|\Psi\|. \tag{A.15}$$

In particular, $D(\mathrm{d}\Gamma(h)^{1/2})\subset D(a^\sharp(f))$, whenever $f\in D(h^{-1/2})$.

B The case of $\mathcal{H} = L^2(\mathbb{R}^3)$

This section is taken from [73, Section 1.2].

B.1 Useful bounds

We set $\mathcal{H} = L^2(\mathbb{R}^3)$ and $H_\mathrm{f} = \mathrm{d}\Gamma(\omega)$. Then $\mathcal{F}^{(n)} \ni \Phi$ is a function $\Phi(k_1 \ldots, k_n)$ which is symmetric with respect to any permutation of $k_1, \ldots, k_n$. Thus the annihilation operator and the creation operator are defined by

$$(a(f)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n+1} \int_{\mathbb{R}^3} f(k)\Phi^{(n+1)}(k, k_1 \ldots, k_n)\mathrm{d}k,$$

$$(a^*(f)\Psi)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n f(k_j)\Phi^{(n-1)}(k_1 \ldots, \hat{k}_j \ldots, k_n).$$

Furthermore

$$H_\mathrm{f}\Psi^{(n)}(k_1, \ldots, k_n) = \left(\sum_{j=1}^n \omega(k_j)\right) \Psi^{(n)}(k_1, \ldots, k_n).$$

We show some useful inequalities on $a^\sharp$ and H_f.

Lemma B.1 *Let $\psi \in D(H_\mathrm{f}^{m/2})$. Suppose that $\|f_j/\sqrt{\omega}\| < \infty$ for $j = 1, \ldots, m$. Then $\psi \in D(\prod_{j=1}^n a(f_j))$ and*

$$\left\| \prod_{j=1}^m a(f_j)\psi \right\| \leq \prod_{j=1}^m \|f_j/\sqrt{\omega}\| \|H_\mathrm{f}^{m/2}\psi\|.$$

Proof: Let $n > m$. Since $\psi^{(n)}$ is symmetric, by the definition of H_f we have

$$(\psi^{(n)}, (H_\mathrm{f}^m\psi)^{(n)}) = n(n-1)\cdots(n-m+1) \int_{\mathbb{R}^{dn}} |\psi^{(n)}(k_1, \ldots, k_n)|^2 \prod_{j=1}^n \omega(k_j)\mathrm{d}k_1 \cdots \mathrm{d}k_n.$$

While by the definition of $a(f)$ we can see that

$$\begin{aligned} &\left\| (\prod_{j=1}^m a(f_j)\psi)^{(n-m)} \right\|^2 \\ &\leq n(n-1)\cdots(n-m+1) \prod_{j=1}^m \|f_j/\sqrt{\omega}\| \int_{\mathbb{R}^{dn}} |\psi^{(n)}(k_1, \ldots, k_n)|^2 \prod_{j=1}^m \omega(k_j)\mathrm{d}k_1 \cdots \mathrm{d}k_n. \end{aligned}$$

Thus we have

$$\left\| (\prod_{j=1}^m a(f_j)\psi)^{(n-m)} \right\|^2 \leq \prod_{j=1}^m \|f_j/\sqrt{\omega}\| (\psi^{(n)}, (H_\mathrm{f}^m\psi)^{(n)})$$

for $m \leq n$. For $m > n$, the above inequality also holds true since the left hand side is zero. Summing over n on both sides, we conclude the desired results.

QED

Let us consider to evaluate the product of creation operators $\| \prod_{j=1}^m a^*(f_j)\Phi\|$. Since we know a bound of the product of annihilation operators $\| \prod_{j=1}^m a(f_j)\Phi\|$, we can also evaluate

$\| \prod_{j=1}^m a^*(f_j)\Phi\|$, but which is rather technically complicated than that of $\| \prod_{j=1}^m a(f_j)\Phi\|$. To see this we use the fact

$$\left\| \prod_{j=1}^m a^*(f_j)\Phi \right\|^2 = \left(\Phi, \prod_{j=1}^m a(\bar{f}_j) \prod_{j=1}^m a^*(f_j)\Phi \right)$$

and compute the commutation relation $\left[\prod_{j=1}^m a(\bar{f}_j), \prod_{j=1}^m a^*(f_j)\right]$. We then conclude that

$$\left\| \prod_{j=1}^m a^*(f_j)\Phi \right\|^2 = \left\| \prod_{j=1}^m a(\bar{f}_j)\Phi \right\|^2 + \left(\Phi, \left[\prod_{j=1}^m a(\bar{f}_j), \prod_{j=1}^m a^*(f_j) \right] \Phi \right).$$

Evaluating the second term of the right-hand side above, we can have a bound of $\left\| \prod_{j=1}^m a^*(f_j)\Phi \right\|$.

Lemma B.2 *Let $f_i, g_j \in D(1/\sqrt{\omega})$ for $i, j = 1, ..., n$ and $\Phi \in D(H_{\mathrm{f}}^{n/2})$. Then*

$$\left| \left(\prod_{j=1}^n a^*(g_j)\Phi, \prod_{j=1}^n a^*(f_j)\Phi \right) \right| \le n! 2^n \left(\prod_{l=1}^n \|f_l\|_\omega \|g_l\|_\omega \right) \sum_{m=0}^n \frac{1}{m!} \|H_{\mathrm{f}}^{m/2}\Phi\|^2,$$

where $\|f\|_\omega = \|f\| + \|f/\sqrt{\omega}\|$.

By Lemma B.2 we have useful bounds for products of annihilation operators and creation operators. We summarize them as follows. Suppose that $f_j \in D(1/\sqrt{\omega})$ for $j = 1, ..., n$. Then for $\Phi \in D(H_{\mathrm{f}}^{n/2})$, we have

$$\left\| \prod_{j=1}^n a(f_j)\Phi \right\| \le \left(\prod_{j=1}^n \|f_j/\sqrt{\omega}\| \right) \|H_{\mathrm{f}}^{n/2}\Phi\|, \tag{B.1}$$

$$\left\| \prod_{j=1}^n a^*(f_j)\Phi \right\| \le \sqrt{n!} 2^{n/2} \left(\prod_{l=1}^n \|f_l\|_\omega \right) \left(\sum_{m=0}^n \frac{1}{m!} \|H_{\mathrm{f}}^{m/2}\Phi\|^2 \right)^{1/2}. \tag{B.2}$$

B.2 Exponent of annihilation operators and creation operators

Although exponential operator $e^{a^*(f)}$ is unbounded, it can be seen in the proposition below that $e^{a^*(f)} e^{-\frac{t}{2}H_{\mathrm{f}}}$ is bounded for any $t > 0$.

Proposition B.3 *Let $t > 0$. (1) Let $f \in D(1/\sqrt{\omega})$. Then both $e^{a^*(f)} e^{-\frac{t}{2}H_{\mathrm{f}}}$ and $\overline{e^{-\frac{t}{2}H_{\mathrm{f}}} e^{a(f)}}$ are bounded. (2) Let $f \in L^2(\mathbb{R}^3)$. Then both $e^{a^*(f)} e^{-\frac{t}{2}N}$ and $\overline{e^{-\frac{t}{2}N} e^{a(f)}}$ are bounded.*

Proof: Suppose that $t < 1$. For any $s < t$ we have for $\Phi \in \mathcal{F}$,

$$\left\| \sum_{n=0}^m \frac{1}{n!} a^*(f)^n e^{-\frac{t}{2}H_{\mathrm{f}}}\Phi \right\| \le \sum_{n=0}^m \frac{1}{\sqrt{n!}} 2^{n/2} s^{-n/2} \|f\|_\omega^n \left(\sum_{k=0}^n \frac{1}{k!} \|(sH_{\mathrm{f}})^{k/2} e^{-\frac{t}{2}H_{\mathrm{f}}}\Phi\|^2 \right)^{1/2}.$$

We can see that sequence $\{\sum_{n=0}^m \frac{1}{n!}a^*(f)^n e^{-\frac{t}{2}H_\mathrm{f}}\Phi\}_{m=0}^\infty$ is a Cauchy sequence in $\mathcal{F}$. Hence $e^{-\frac{t}{2}H_\mathrm{f}}\Phi \in D(e^{a^*(f)})$ and as $m\to\infty$ on both sides above and we have

$$\|e^{a^*(f)}e^{-\frac{t}{2}H_\mathrm{f}}\Phi\| \le A(f,s)\|e^{-\frac{1}{2}(t-s)H_\mathrm{f}}\Phi\|,$$

where

$$A(f,s)=\sum_{n=0}^\infty \frac{1}{\sqrt{n!}}2^{n/2}s^{-n/2}\|f\|_\omega^n. \tag{B.3}$$

Choosing s such that $s<t$, we can see that $\|e^{-\frac{1}{2}(t-s)H_\mathrm{f}}\Phi\| \le \|\Phi\|$ and $e^{a^*(f)}e^{-\frac{t}{2}H_\mathrm{f}}$ for $t<1$ is bounded. Suppose $1\le t$. Choosing $s=1$ in the above discussion, we have

$$\|e^{a^*(f)}e^{-\frac{t}{2}H_\mathrm{f}}\Phi\| \le A(f,1)\|e^{-\frac{1}{2}(t-1)H_\mathrm{f}}\Phi\| \le A(f,1)\|\Phi\|.$$

Thus $e^{a^*(f)}e^{-\frac{t}{2}H_\mathrm{f}}$ for $t\ge 1$ is bounded. Finally since $\left(e^{-\frac{t}{2}H_\mathrm{f}}e^{a(f)}\right)^* \supset e^{a^*(\bar f)}e^{-\frac{t}{2}H_\mathrm{f}}$, the second statement follows. Then (1) follows. (2) is similarly proven by replacing ω with the identity $\mathbb{1}$.
QED

B.3 Technical estimates I

Let $X=e^{a^*(g)}e^{a(\bar g)}$ for $g\in L^2(\mathbb{R}^3)$. We already see that $e^{a^*(g)}e^{-\frac{t}{2}H_\mathrm{f}}e^{a(\bar g)}$ is bounded for any $t>0$. We extend this to more general cases. By (A.3) we know the explicit form of Wick ordering of $\Phi(f)^n$, and see that

$$X\Phi(f)^n\Psi = n!\sum_{k=0}^{[n/2]}\sum_{l+m+2k=n}\frac{(\frac{\|f\|^2}{4})^k}{k!}\frac{(\frac{1}{\sqrt2}a^*(f)+y)^l}{l!}X\frac{(\frac{1}{\sqrt2}a(\bar f))^m}{m!}\Psi. \tag{B.4}$$

Here $y=(g,f)/\sqrt2$. Operator $X\Phi(f)^n$ is not bounded, but we are interested in estimating operator $Z(t,n)$ which is defined by inserting e^{-tH_f} into $X\Phi(f)^n$ as

$$Z(t,n)=n!\sum_{k=0}^{[n/2]}\sum_{l+m+2k=n}\frac{(\frac{\|f\|^2}{4})^k}{k!}\frac{(\frac{1}{\sqrt2}a^*(f)+y)^l}{l!}e^{a^*(\bar g)}e^{-tH_\mathrm{f}}e^{a(\bar g)}\frac{(\frac{1}{\sqrt2}a(\bar f))^m}{m!}. \tag{B.5}$$

Lemma B.4 *Let $f\in D(1/\sqrt{\omega})$. Then $Z(t,n)$ is bounded for $t>0$, and the bound of $\|Z(t,n)\Psi\|$ is given as follows.*
$(t\ge 1)$

$$\|Z(t,n)\Psi\| \le z_t(n)\left\|e^{-\frac{1}{2}(t-1)H_\mathrm{f}}\right\|^2\|\Psi\|,$$

where

$$z_t(n)=n!\sum_{k=0}^{[n/2]}\sum_{l+m+2k=n}\frac{\left(\frac{\|f\|^2}{4}\right)^k(\|f\|_\omega+|y|)^l\|f\|_\omega^m}{k!l!m!}\xi(l,m)$$

with $y=(g,f)/\sqrt2$ and

$$\xi(l,m)=\sum_{j,j'=0}^\infty\frac{(\sqrt2\|g\|_\omega)^{j+j''}\sqrt{(j+l)!}\sqrt{(j'+m)!}}{j!j'!}.$$

$(t<1)$ *For any* $0<s<t$,

$$\|Z(t,n)\Psi\| \le z_t(s,n) \left\| e^{-\frac{1}{2}(t-s)H_{\rm f}} \right\|^2 \|\Psi\|,$$

where

$$z_t(s,n) = n! \sum_{k=0}^{[n/2]} \sum_{l+m+2k=n} \frac{\left(\frac{\|f\|^2}{4}\right)^k (\|f\|_\omega + |y|)^l \|f\|_\omega^m}{k!l!m!} \xi_s(l,m)$$

and

$$\xi_s(l,m) = \sum_{j,j'=0}^{\infty} \frac{(\sqrt{2}\|g\|_\omega)^{j+j''} \sqrt{(j+l)!}\sqrt{(j'+m)!}}{s^{(j+l)/2} s^{(j'+m)/2} j!j'!}.$$

Proof: In this proof we set $s=1$ for $t \ge 1$ and $0<s<t$ for $t<1$. Since

$$\left\| \sum_{n=0}^{N} \frac{a^*(f)^n}{n!} \Phi \right\| \le \sum_{n=0}^{N} \frac{(\sqrt{2}\|f\|_\omega)^n s^{-n/2}}{\sqrt{n!}} \left\| e^{\frac{1}{2}sH_{\rm f}} \Phi \right\|,$$

and

$$\begin{aligned}
\|(a^*(f)+y)^m a^*(g)^n \Psi\| &\le \sum_{k=0}^{m} \binom{m}{k} |y|^k \|a^*(f)^{m-k} a^*(g)^n \Psi\| \\
&\le s^{-(n+m)/2} (\sqrt{2}\|f\| + |y|)^m (\sqrt{2}\|g\|_\omega)^n \sqrt{(n+m)!} \|e^{\frac{1}{2}sH_{\rm f}} \Psi\|,
\end{aligned}$$

we have for any non-negative integers $m \ge 0$

$$\left\| (a^*(g)+y)^m e^{a^*(f)} \Phi \right\| \le \sum_{n=0}^{\infty} \frac{(\sqrt{2}\|f\|_\omega + |y|)^m (\sqrt{2}\|g\|_\omega)^n \sqrt{(n+m)!}}{s^{(n+m)/2} n!} \left\| e^{\frac{1}{2}sH_{\rm f}} \Phi \right\|.$$

We have

$$\|Z(t,n)\Psi\| \le \sum_{k=0}^{[n/2]} \sum_{l+m+2k=n} \frac{n! \left(\frac{\|f\|^2}{4}\right)^k}{k!l!m!} \left\| (\frac{a^*(f)}{\sqrt{2}} + y)^l e^{a^*(g)} e^{-\frac{t}{2}H_{\rm f}} \right\| \left\| e^{-\frac{t}{2}H_{\rm f}} e^{a(\bar{g})} (\frac{a(\bar{f})}{\sqrt{2}})^m \right\| \|\Psi\|. \tag{B.6}$$

Inserting inequalities:

$$\left\| (\frac{a^*(f)}{\sqrt{2}} + |y|)^l e^{a^*(g)} e^{-\frac{t}{2}H_{\rm f}} \right\| \le \sum_{j=0}^{\infty} \frac{(\sqrt{2}\|g\|_\omega)^j (\|f\|_\omega + |y|)^l \sqrt{(j+l)!}}{s^{(j+l)/2} j!} A,$$

$$\left\| e^{-\frac{t}{2}H_{\rm f}} e^{a(\bar{g})} (\frac{a(\bar{f})}{\sqrt{2}})^m \right\| \le \sum_{j=0}^{\infty} \frac{(\sqrt{2}\|g\|_\omega)^j \|f\|_\omega^m \sqrt{(j+m)!}}{s^{(j+m)/2} j!} A$$

with $A = \left\| e^{-\frac{1}{2}(t-s)H_{\rm f}} \right\|$ into (B.6), we prove the lemma.

QED

Next we estimate operator $Z(t)$ defined by

$$Z(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} Z(t,n). \tag{B.7}$$

Lemma B.5 *Let $f, g \in D(1/\sqrt{\omega})$. Then $Z(t)$ for $t > 0$ is bounded with*

$$\|Z(t)\| \leq \frac{e^{\frac{\|f\|^2}{4}\left\|e^{-\frac{t-s}{2}H_{\rm f}}\right\|^2} C(h)^2}{\left(1 - \frac{h}{\sqrt{s}}(\sqrt{2}\|g\|_\omega + \|f\|_\omega + y)\right)\left(1 - \frac{h}{\sqrt{s}}(\sqrt{2}\|g\|_\omega + \|f\|_\omega)\right)}. \tag{B.8}$$

Here $s = 1$ for $t \geq 1$ and $0 < s < t$ for $t < 1$, and $C(h)$ is a constant depending on $h > 0$ such that $1 - \frac{h}{\sqrt{s}}(\sqrt{2}\|g\|_\omega + \|f\|_\omega + y) > 0$ for $y = (g, f)/\sqrt{2}$.

Proof: In this proof we set $s = 1$ for $t \geq 1$ and $0 < s < t$ for $t < 1$. By Lemma B.4 and the fact that $Z(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} Z(t, n)$ we have

$$\|Z(t)\Psi\| \leq z_t \left\|e^{-\frac{1}{2}(t-1)H_{\rm f}}\right\|^2 \|\Psi\|,$$

where

$$z_t = \sum_{n=0}^{\infty} \sum_{l+m+2k=n} \frac{\left(\frac{\|tf\|^2}{4}\right)^k (\|tf\|_\omega + t|y|)^l \|tf\|_\omega^m}{k!l!m!} \xi_s(l, m).$$

Then we replace tf and $t|y|$ by f and $|y|$, respectively in what follows. We have

$$z_t = e^{\|f\|^2/4} \xi_s(y) \xi_s(0), \tag{B.9}$$

where

$$\xi_s(y) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{a^j b^l \sqrt{(j+l)!}}{l!j!}. \tag{B.10}$$

with $a = \sqrt{2}\|g\|_\omega/\sqrt{s}$ and $b = (\|f\|_\omega + y)/\sqrt{s}$. Let $0 < h$ be such that $h < 1/(a + b)$. Then

$$\xi_s(y) = \sum_{j,l=0}^{\infty} \frac{a^j b^l h^{j+l} (j+l)!}{l!j!} \frac{(1/h)^j (1/h)^l}{\sqrt{(j+l)!}} \leq \frac{C}{1 - h(a+b)} < \infty.$$

Here $C = C(h) = \sup_{(j,l)\in\mathbb{N}\times\mathbb{N}} \frac{(1/h)^j(1/h)^l}{\sqrt{(j+l)!}}$. It can be also shown that $\xi(0)$ is finite. Hence we have (B.8). Then z_t is finite, and the lemma is proven.

QED

B.4 Technical estimates II

Operator $Z(t, n)$ is defined through commutation relations: $[X, \Phi(f)^n]$, where $X = e^{a^*(g)} e^{a(\bar{g})}$. We also need to see operator $W(t, n)$ defined through commutation relations:

$$\left[X, \prod_{j=1}^{n} \Phi(f_j)\right]. \tag{B.11}$$

To compute the commutation relation (B.11) explicitly we apply Wick's theorem to product $\prod_{j=1}^n \Phi(f_j)$:

$$\prod_{j=1}^n \Phi(f_j) = \sum_{k=0}^{[n/2]} \frac{1}{2^k} \sum_{j_1,\dots,j_{2k}}^{\mathrm{pair}} \left(\prod_{i=1}^k (f_{j_{2i-1}}, f_{j_{2i}}) \right) c^n_{j_1,\dots,j_{2k}}.$$

Since

$$:\prod_{j=1}^n \Phi(f_j): = 2^{-n/2} \sum_{k=0}^m \sum_{\{j_1,\dots,j_k\}\subset\{1,\dots,n\}} \prod_{j\in\{j_1,\dots,j_k\}} a^*(f_j) \prod_{i\in\{j_1,\dots,j_k\}^c} a(\bar{f}_i),$$

we can compute $c^n_{j_1,\dots,j_{2k}}$ as

$$c^n_{j_1,\dots,j_{2k}} = 2^{-(n-2k)/2} \sum_{l=0}^{n-2k} \sum_{\{i_1,\dots,i_l\}\subset\{j_1,\dots,j_{2k}\}^c} \prod_{j\in\{i_1,\dots,i_l\}} a^*(f_j) \prod_{i\in\{i_1,\dots,i_l\}^c\cap\{j_1,\dots,j_{2k}\}^c} a(\bar{f}_i).$$

Hence

$$X \prod_{j=1}^n \Phi(f_j) = \sum_{k=0}^{[n/2]} \frac{1}{2^k} \sum_{j_1,\dots,j_{2k}}^{\mathrm{pair}} \left(\prod_{i=1}^k (f_{j_{2i-1}}, f_{j_{2i}}) \right) \sum_{l=0}^{n-2k} \sum_{\{i_1,\dots,i_l\}\subset\{j_1,\dots,j_{2k}\}^c} AXB$$

where

$$A = \prod_{j\in\{i_1,\dots,i_l\}} (\frac{a^*(f_j)}{\sqrt{2}} + y_j), \quad B = \prod_{i\in\{i_1,\dots,i_l\}^c\cap\{j_1,\dots,j_{2k}\}^c} \frac{a(\bar{f}_i)}{\sqrt{2}} \tag{B.12}$$

with $y_j = (g, f_j)/\sqrt{2}$. Since $X\prod_{j=1}^m \Phi(f_j)$ is not bounded, we then define $W(t,n)$ by inserting $e^{-tH_{\mathrm{f}}}$ into $X\prod_{j=1}^m \Phi(f_j)$ as

$$W(t,n) = \sum_{k=0}^{[n/2]} \frac{1}{2^k} \sum_{j_1,\dots,j_{2k}}^{\mathrm{pair}} \sum_{l=0}^{n-2k} \sum_{\{i_1,\dots,i_l\}\subset\{j_1,\dots,j_{2k}\}^c} \left(\prod_{i=1}^k (f_{j_{2i-1}}, f_{j_{2i}}) \right) A e^{a^*(g)} e^{-tH_{\mathrm{f}}} e^{a(\bar{g})} B. \tag{B.13}$$

Furthermore we set

$$W(t) = \sum_{n=0}^\infty \frac{t^n}{n!} W(t,n). \tag{B.14}$$

$W(t,n)$ and $W(t)$ are extensions of $Z(t,n)$ and $Z(t)$ discussed in (B.5) and (B.7), respectively.

Formula proved in Theorem B.6 below is used to estimate integral kernels of semigroup e^{-tL} in Section 5.4.3. The integral kernel of e^{-tL} is

$$\mathrm{J}_0^* e^Y \mathrm{J}_t = \mathrm{J}_0^* e^{\frac{B^\dagger}{\sqrt{2}} + \frac{B}{\sqrt{2}}} \prod_{j=1}^N \left(\left| \frac{C_j^\dagger}{\sqrt{2}} + \frac{C_j}{\sqrt{2}} \right| + \varepsilon\psi_\varepsilon(\left| \frac{C_j^\dagger}{\sqrt{2}} + \frac{C_j}{\sqrt{2}} \right|) \right) \mathrm{J}_t. \tag{B.15}$$

See (5.22). (B.15) is of the form of $W(t)$ given by (B.14). Theorem B.6 tells that the integral kernel of e^{-tL} is bounded.

Theorem B.6 *Let $f_j \in D(1/\sqrt{\omega})$, $j \in \mathbb{N}$. Suppose that there exists $\alpha > 0$ such that $\|f_j\|_\omega \le \alpha$ for any j. Then $W(t,n)$ and $W(t)$ are bounded for $t > 0$ with*

$$\|W(t)\| \le \frac{e^{\frac{\alpha^2}{4}} \left\| e^{-\frac{t-s}{2}H_\mathrm{f}} \right\|^2 C(h)^2}{\left(1 - \frac{h}{\sqrt{s}}(\sqrt{2}\|g\|_\omega + \alpha + y)\right)\left(1 - \frac{h}{\sqrt{s}}(\sqrt{2}\|g\|_\omega + \alpha)\right)}.$$

Here $s = 1$ for $t \ge 1$ and $0 < s < t$ for $t < 1$, $y = \alpha\|g\|/\sqrt{2}$, and $C(h)$ is a constant depending on $h > 0$ such that $1 - \frac{h}{\sqrt{s}}(\sqrt{2}\|g\|_\omega + \alpha + y) > 0$.

Proof: In this proof we set $s = 1$ for $t \ge 1$ and $0 < s < t$ for $t < 1$. Let A and B be (B.12). We can estimate $\left\|Ae^{a^*(g)}e^{-tH_\mathrm{f}}e^{a(\bar{g})}B\right\|$ as

$$\left\|Ae^{a^*(g)}e^{-tH_\mathrm{f}}e^{a(\bar{g})}B\right\| \le \left\|Ae^{a^*(g)}e^{-\frac{t}{2}H_\mathrm{f}}\right\| \left\|e^{-\frac{t}{2}H_\mathrm{f}}e^{a(\bar{g})}B\right\|,$$

with

$$\left\|Ae^{a^*(g)}e^{-\frac{t}{2}H_\mathrm{f}}\Phi\right\| \le \sum_{j=0}^\infty \frac{(\alpha+y)^l(\sqrt{2}\|g\|_\omega)^j\sqrt{(l+j)!}}{s^{(j+l)/2}j!} \left\|e^{-\frac{1}{2}(t-s)H_\mathrm{f}}\right\|,$$

$$\left\|e^{-\frac{t}{2}H_\mathrm{f}}e^{a(\bar{g})}B\right\| \le \sum_{j=0}^\infty \frac{\alpha^{n-2k-l}(\sqrt{2}\|g\|_\omega)^j\sqrt{(n-2k-l+j)!}}{s^{(j+n-2k-l)/2}j!} \left\|e^{-\frac{1}{2}(t-s)H_\mathrm{f}}\right\|.$$

Let $y = \alpha\|g\|/\sqrt{2}$. Then $|y_j| \le y$. In a similar manner to the proof of Lemma B.4 we have bounds below:
$(t \ge 1)$

$$\|W(t,n)\Psi\| \le w_t(n)\left\|e^{-\frac{1}{2}(t-1)H_\mathrm{f}}\right\|^2\|\Psi\|,$$

where

$$w_t(n) = n!\sum_{k=0}^{[n/2]}\sum_{l+m+2k=n}\frac{\left(\frac{\alpha^2}{4}\right)^k(\alpha+y)^l\alpha^m}{k!l!m!}\xi(l,m).$$

$(t < 1)$ For any $0 < s < t$,

$$\|W(t,n)\Psi\| \le w_t(s,n)\left\|e^{-\frac{1}{2}(t-s)H_\mathrm{f}}\right\|^2\|\Psi\|,$$

where

$$w_t(s,n) = n!\sum_{k=0}^{[n/2]}\sum_{l+m+2k=n}\frac{\left(\frac{\alpha^2}{4}\right)^k(\alpha+y)^l\alpha^m}{k!l!m!}\xi_s(l,m).$$

Here $\xi(l,m)$ and $\xi_s(l,m)$ defined in Lemma B.4. Hence $W(t,n)$ is bounded. We shall show the boundedness of $W(t)$. The proof is also similar to that of Lemma B.5. We have

$$\|W(t)\Psi\| \le w_t\left\|e^{-\frac{1}{2}(t-1)H_\mathrm{f}}\right\|^2\|\Psi\|,$$

where

$$w_t = \sum_{n=0}^{\infty} \sum_{l+m+2k=n} \frac{\left(\frac{t^2\alpha^2}{4}\right)^k (t\alpha+ty)^l (t\alpha)^m}{k!l!m!} \xi_s(l,m).$$

Then we replace $t\alpha$ and ty by α and y, respectively in what follows. We have

$$w_t = \sum_{n=0}^{\infty} \sum_{l+m+2k=n} \frac{\left(\frac{\alpha^2}{4}\right)^k}{k!} \frac{(\alpha+y)^l}{l!} \frac{\alpha^m}{m!} \xi_s(l,m) = e^{\alpha^2/4}\xi_s(y)\xi_s(0),$$

where

$$\xi_s(y) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\sqrt{2}\|g\|_\omega)^j (\alpha+y)^l \sqrt{(j+l)!}}{s^{(j+l)/2} l! j!}.$$

The right-hand side above is finite which is shown in Lemma B.5. Then the proof is complete.

QED

C Proofs of (1) and (2) in the proof of Theorem 5.8

Lemma C.1 *Statement (1) of the proof of Theorem 5.8 is true.*

Proof: For each $(x,\alpha,w,m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$, it can be directly seen that $F, G \in D(e^{Z_t(\varepsilon)}\mathrm{J}_0) \cap D(e^{Z_t^n(\varepsilon)}\mathrm{J}_0)$ for any $n \in \mathbb{N}$. We see the inequality: $\|Y_t(2)\Phi\| \le \gamma\sqrt{m+1}\|\Phi\|$ for $\Phi \in L^2_m(\mathrm{Q})$, where $\gamma = \sqrt{2}t\||k|\hat{\varphi}/\sqrt{\omega}\|$. Since $L^2_{\mathrm{fin}}(\mathrm{Q})$ is the set of analytic vectors for $e^{Y_t^n(2)}$, for $F \in L^2_{\mathrm{fin}}(\mathrm{Q})$ we have $e^{Y_t^n(2)}F = \sum_{j=0}^{\infty} \frac{Y_t^n(2)^j}{j!}F$. We assume that $F \in L^2_{m'}(\mathrm{Q})$ and $G \in L^2_m(\mathrm{Q})$, hence $NF = m'F$ and $NG = mG$ for the number operator N. Then

$$\left\|e^{Y_t^n(2)}F\right\| \le \sum_{j=0}^{\infty} \frac{\sqrt{(m+j-1)\cdots(m+1)m}}{j!}\gamma^j\|F\|.$$

On the other hand by the definition of $Y_t^n(3,\varepsilon)$ we have

$$e^{Y_t^n(3,\varepsilon)} = \prod_{i=1}^{n} \exp\left(\int_{\frac{(i-1)t}{n}}^{\frac{it}{n}+} \log(-\Psi_\varepsilon(\mathbb{H}_{\mathrm{od}}(B_s, -\theta_{N_{s-}}, \frac{(i-1)t}{n})\mathrm{d}N_s\right). \tag{C.1}$$

For every $m \in \mathscr{X}_\mu$ there exists $N = N(m) \in \mathbb{N}$ such that map $t \mapsto N_t(m)$ is not continuous at points $s_1 = s_1(m), \ldots, s_N = s_N(m)$. For sufficiently large n the number of discontinuous points in $(\frac{(i-1)t}{n}, \frac{it}{n}]$ is at most one. Then by taking n large enough and putting $(n(s_i), n(s_i)+t/n]$ for the interval containing s_i, we get

$$e^{Y_t^n(3,\varepsilon)} = \prod_{i=1}^{N}(-\phi_i - \varepsilon\psi_\varepsilon(\phi_i)),$$

where $\phi_i = \mathbb{H}_{\mathrm{od}}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i))$. Fix $m \in \mathscr{X}_\mu$ above. We have

$$\prod_{i=1}^{N}(-\phi_i - \varepsilon\psi_\varepsilon(\phi_i)) = (-1)^N \sum_{p=0}^{N} \varepsilon^{N-p} \sum_{\{j_1,\ldots,j_p\}\subset[N]} \prod_{i\notin\{j_1,\ldots,j_p\}} \psi_i \cdot \prod_{i=1}^{p} \phi_{j_i},$$

where $\psi_i = \psi_\varepsilon(\phi_{j_i})$ and $[N] = \{1, \dots, N\}$, and

$$\begin{aligned}
&(F, \mathrm{J}_0^* e^{Y_t^n(1)} e^{Y_t^n(2)} e^{Y_t^n(3,\varepsilon)} \mathrm{J}_t G) \\
&= \sum_{p=0}^{N} \varepsilon^{N-p} \sum_{\{j_1,\dots,j_p\}\subset[N]} \left(e^{-Y_t^n(1)} \mathrm{J}_0 F, e^{Y_t^n(2)} \prod_{i\notin\{j_1,\dots,j_p\}} \psi_i \cdot \prod_{i=1}^{p} \phi_{j_i} \mathrm{J}_t G \right).
\end{aligned}$$

Notice again that $\|\phi_{j_i}\Phi\| \le 2\gamma\sqrt{m+1}\|\Phi\|$ for $\Phi \in L^2(\mathrm{Q})$ and $i = 1, ..., p$. Since $\psi_\varepsilon \le 1$, each terms on the right-hand side above can be estimated as

$$\begin{aligned}
&\left| \left(e^{-Y_t^n(1)} \mathrm{J}_0 F, e^{Y_t^n(2)} \prod_{i\notin\{j_1,\dots,j_p\}} \psi_i \cdot \prod_{i=1}^{p} \phi_{j_i} \mathrm{J}_t G \right) \right| \\
&\le \sum_{j=0}^{\infty} \varepsilon^{N-p} \frac{\sqrt{(j+p+m-1)\cdots(m+1)m}}{j!} (2\gamma)^{j+p-m+1} \|F\|\|G\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&|(F, \mathrm{J}_0^* e^{Y_t^n(1)} e^{Y_t^n(2)} e^{Y_t^n(3,\varepsilon)} \mathrm{J}_t G)| \\
&\le \sum_{p=0}^{N} \frac{\varepsilon^{N-p} N!}{p!(N-p)!} \sum_{j=0}^{\infty} \frac{\sqrt{(j+p+m-1)\cdots(m+1)m}}{j!} (2\gamma)^{j+p-m+1} \|F\|\|G\| \\
&\le A_t(N)\|F\|\|G\|,
\end{aligned}$$

where

$$A_t(N) = \sum_{j=0}^{\infty} (1+\varepsilon)^N \frac{\sqrt{(j+N+m-1)\cdots(m+1)m}}{j!} (2\gamma)^{j+N-m+1}.$$

The number N depends on $m \in \mathscr{X}_\mu$, then $A_t(N(m))$ turns to be a random process on $(\mathscr{X}_\mu, \mathcal{B}_\mu, \mu)$. We set $C_t(m) = A_t(N(m))$. For each $(x, \alpha, w, m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X} \times \mathscr{X}_\mu$, $|(F, \mathrm{J}_0^* e^{Z_t^n(\varepsilon)} \mathrm{J}_t G)|$ is finite with the bound $|(F, \mathrm{J}_0^* e^{Z_t^n(\varepsilon)} \mathrm{J}_t G)| \le C_t(m)$ and

$$\begin{aligned}
\mathbb{E}_\mu[C_t^2] &= \sum_{N=0}^{\infty} \frac{t^N A_t(N)^2}{N!} e^{-t} \\
&\le e^{-t} \sum_{N=0}^{\infty} \left(\sum_{j=0}^{\infty} (1+\varepsilon)^N (2\gamma)^{j+N-m+1} \frac{\sqrt{(j+N+m-1)\cdots(m+1)m}}{N!j!} \right)^2 < \infty.
\end{aligned}$$

Thus (1) follows.

QED

Lemma C.2 *Statement (2) of the proof of Theorem 5.8 is true.*

Proof: We show the convergence of $\mathrm{J}_0^* e^{Z_t^n(\varepsilon)} \mathrm{J}_t$ as $n \to \infty$. Let $F \in L^2_{\mathrm{fin}}(\mathrm{Q})$. We show the convergences of $e^{Y_t^n(1)}, e^{Y_t^n(2)}$ and $e^{Y_t^n(3,\varepsilon)}$, separately. Thus

$$\|e^{-Y_t(1)} \mathrm{J}_0 F - e^{-Y_t^n(1)} \mathrm{J}_0 F\| \le \left\| \bigoplus_{\mu=1}^{3} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (\mathrm{j}_{t_{j-1}} - \mathrm{j}_s) \tilde{\varphi}(\cdot - B_s) \mathrm{d}B_s^\mu \right\| \|(N+\mathbb{1})^{1/2} F\|.$$

Then it follows that

$$
\begin{aligned}
&\mathbb{E}_{\mathscr{W}}^{x}[\|e^{-Y_t(1)}\mathrm{J}_0F-e^{-Y_t^n(1)}\mathrm{J}_0F\|^2]\le 3\sum_{j=1}^n\int_{t_{j-1}}^{t_j}\|(\mathrm{j}_{t_{j-1}}-\mathrm{j}_s)\tilde{\varphi}(\cdot-B_s)\|^2\mathrm{d}s\|(N+\mathbb{1})^{1/2}F\|^2\\
&=3\sum_{j=1}^n\int_{t_{j-1}}^{t_j}\left(\frac{\hat{\varphi}}{\sqrt{\omega}},(1-e^{-|s-t_{j-1}|\omega})\frac{\hat{\varphi}}{\sqrt{\omega}}\right)\mathrm{d}s\|(N+\mathbb{1})^{1/2}F\|^2\\
&\le 3\frac{t}{n}\|\sqrt{\omega}\hat{\varphi}\|^2\|(N+\mathbb{1})^{1/2}F\|^2.
\end{aligned}
$$

Hence we have

$$
\lim_{n\to\infty}\mathbb{E}_{\mathscr{W}}^{x}[\|e^{-Y_t(1)}\mathrm{J}_0F-e^{-Y_t^n(1)}\mathrm{J}_0F\|]=0. \tag{C.2}
$$

On the other hand in the Fock representation $Y_t(2)$ and $Y_t^n(2)$ are represented as

$$
\begin{aligned}
Y_t(2)&=-\frac{1}{\sqrt{2}}\sum_{j=\pm1}\left\{a_{\mathrm{E}}^*\left(\int_0^t b_j^-(s)\mathrm{d}s,j\right)-a_{\mathrm{E}}\left(\int_0^t b_j^+(s)\mathrm{d}s,j\right)\right\},\\
Y_t^n(2)&=-\frac{1}{\sqrt{2}}\sum_{j=\pm1}\sum_{i=1}^n\left\{a_{\mathrm{E}}^*\left(\int_{t_{i-1}}^{t_i} b_j^-(t_{i-1})\mathrm{d}s,j\right)-a_{\mathrm{E}}\left(\int_{t_{i-1}}^{t_i} b_j^+(t_{i-1})\mathrm{d}s,j\right)\right\}.
\end{aligned}
$$

Here $b_j^{\pm}(s)=S_j(\theta_{N_s})e^{\pm\mathrm{i}kB_s}\hat{\mathrm{j}}_s\frac{\hat{\varphi}}{\sqrt{\omega}}$ and $b_j^{\pm}(t_{i-1})=S_j(\theta_{N_s})e^{\pm\mathrm{i}kB_s}\hat{\mathrm{j}}_{t_{i-1}}\frac{\hat{\varphi}}{\sqrt{\omega}}$. Then the distance between test functions of $Y_t(2)$ and $Y_t^n(2)$ can be estimated as

$$
\left\|\int_0^t b_j^-(s)\mathrm{d}s-\sum_{i=1}^n\int_{t_{i-1}}^{t_i}b_j^-(t_{i-1})\mathrm{d}s\right\|^2\le t\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left\|S_j(\theta_{N_s})e^{-\mathrm{i}kB_s}(\hat{\mathrm{j}}_s\frac{\hat{\varphi}}{\sqrt{\omega}}-\hat{\mathrm{j}}_{t_{i-1}}\frac{\hat{\varphi}}{\sqrt{\omega}})\right\|^2\mathrm{d}s.
$$

Since $\left\|S_j(\theta_{N_s})e^{-\mathrm{i}kB_s}(\hat{\mathrm{j}}_s\frac{\hat{\varphi}}{\sqrt{\omega}}-\hat{\mathrm{j}}_{t_{i-1}}\frac{\hat{\varphi}}{\sqrt{\omega}})\right\|^2\le 2|s-t_{i-1}|\||k|\hat{\varphi}\|^2$, we see that

$$
\left\|\int_0^t b_j^-(s)\mathrm{d}s-\sum_{i=1}^n\int_{t_{i-1}}^{t_i}b_j^-(t_{i-1})\mathrm{d}s\right\|^2\le\frac{2}{n}t^2\||k|\hat{\varphi}\|^2.
$$

It is concluded that

$$
\lim_{n\to\infty}\left\|\int_0^t b_j^-(s)\mathrm{d}s-\sum_{i=1}^n\int_{t_{i-1}}^{t_i}b_j^-(t_{i-1})\mathrm{d}s\right\|=0. \tag{C.3}
$$

Furthermore by (C.3) it can be straightforwardly seen that

$$
e^{Y_t^n(2)}F\to e^{Y_t(2)}F \tag{C.4}
$$

by the expansion $e^{Y_t^n(2)}F=\sum_{j=0}^\infty\frac{Y_t^n(2)^j}{j!}F$.

In the same argument as in the proof of Lemma C.1 for every $m\in\mathscr{X}_\mu$ there exists $N=N(m)\in\mathbb{N}$ such that map $t\mapsto N_t(m)$ is not continuous at points $s_1=s_1(m),\ldots,s_N=s_N(m)$. We suppose that s_i is in $(n(s_i),n(s_i)+t/n]$ for sufficiently large n. We get

$$
e^{Y_t^n(3,\varepsilon)}=\prod_{i=1}^N(-\phi_i-\varepsilon\psi_\varepsilon(\phi_i)),
$$

where $\phi_i = \mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i))$. In Fock representation we can also have

$$\mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, s_i) \cong \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \left\{ a_{\rm E}^* \left(c_j^-(s_i), j \right) - a_{\rm E} \left(c_j^+(s_i), j \right) \right\},$$
$$\mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i)) \cong \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \left\{ a_{\rm E}^* \left(c_j^-(n(s_i)), j \right) - a_{\rm E} \left(c_j^+(n(s_i)), j \right) \right\},$$

where $c_j^\pm(t) = T_j(-\theta_{N_{s_i-}})(\hat{\rm j}_t \frac{\hat{\varphi}}{\sqrt{\omega}}) e^{\pm {\rm i} k B_{s_i}}$. We then estimate the distance between their test functions as

$$\left\| c_j^-(n(s_i)) - c_j^-(s_i) \right\|^2 \leq 2 \left(\frac{|k|\hat{\varphi}}{\sqrt{\omega}}, (1 - e^{-|n(s_i) - s_i|\omega}) \frac{|k|\hat{\varphi}}{\sqrt{\omega}} \right).$$

Clearly, $n(s_i) \to s_i$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i)) = \mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, s_i),$$
$$\lim_{n\to\infty} \psi_\varepsilon(\mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i))) = \psi_\varepsilon(\mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, s_i))$$

on $\mathcal{K}_\infty$ and we have

$$\lim_{n\to\infty} e^{Y_t^n(3,\varepsilon)} = \prod_{i=1}^N (-\Psi_\varepsilon(\mathbb{H}_{\rm od}(B_{s_i}, -\theta_{N_{s_i-}}, s_i))). \tag{C.5}$$

Then by (C.2), (C.4) and (C.5) we can see that for each $(x, \alpha, m) \in \mathbb{R}^3 \times \mathbb{Z}_2 \times \mathscr{X}_\mu$,

$$\mathbb{E}_{\mathcal{W}}^x \left[\left(F(q_0), {\rm J}_0^* e^{Z_t^n(\varepsilon)} {\rm J}_t G(q_t) \right) \right] \to \mathbb{E}_{\mathcal{W}}^x \left[\left(F(q_0), {\rm J}_0^* e^{Z_t(\varepsilon)} {\rm J}_t G(q_t) \right) \right]$$

as $n \to \infty$ and $|\mathbb{E}_{\mathcal{W}}^x[(F(q_0), {\rm J}_0^* e^{Z_t^n(\varepsilon)} {\rm J}_t G(q_t))]| \leq C_t(m) \mathbb{E}_W^x[\|F(q_0)\| \|G(q_t)\|]$ and the dominated function $C_t(m)\|F(q_0)\|\|G(q_t)\|$ is integrable. The Lebesgue dominated convergence theorem yields that

$$\int_{\mathbb{R}^3} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}^{x,\alpha} \left[e^{-\int_0^t V(B_r){\rm d}r} \left(F(q_0), {\rm J}_0^* e^{Z_t^n(\varepsilon)} {\rm J}_t G(q_t) \right) \right] {\rm d}x$$
$$\to \int_{\mathbb{R}^3} \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}^{x,\alpha} \left[e^{-\int_0^t V(B_r){\rm d}r} \left(F(q_0), {\rm J}_0^* e^{Z_t^n(\varepsilon)} {\rm J}_t G(q_t) \right) \right] {\rm d}x$$

for $F, G \in \mathcal{K}_\infty$ as $n \to \infty$. Hence the proof of (2) is complete.

QED

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