Relativistic Pauli-Fierz model in QED by path measures

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1 Relativistic Pauli-Fierz model

The Pauli-Fierz model is a model in the so-called *nonrelativistic QED*. This model can be extended to a relativistic one. This model is defined on $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F}$, where $\mathscr{F}$ is a boson Fock space. Define

$$H = \sqrt{(-i\nabla \otimes 1 - \alpha A)^2 + m^2} - m + V \otimes 1 + 1 \otimes H_{\rm f},$$

where $\alpha \in \mathbb{R}$ is a coupling constant, V is potential relatively bounded with respect to $\sqrt{-\Delta + m^2} - m$ with a bound < 1, A denotes the quantized radiation field given by $A_\mu = \int^\oplus A_\mu(x)dx$ under the identification $\mathscr{H} = \int^\oplus \mathscr{F} dx$ and $A_\mu(x)$ by

$$A_\mu(x) = \sum_{j=1}^{d-1} \int \frac{\hat{\varphi}(k)}{|k|} e_\mu(k,j) \left(a^\dagger(k,j)e^{-ikx} + a(k,j)e^{+ikx}\right) dk.$$

The creation operator $a^\dagger$ and the annihilation operator a satisfy canonical commutation relations $[a(k,j), a^\dagger(k',j')] = \delta_{jj'}\delta(k-k')$, $\hat{\varphi}(k) = \begin{cases} (2\pi)^{-d/2}, & |k| \le \Lambda \\ 0, & |k| > \Lambda \end{cases}$ is the cutoff function with ultraviolet cutoff parameter Λ and $e(k,1), ..., e(k,d-1), k/|k|$ form an orthogonal base on the tangent space of the $d-1$-dimensional unit sphere at k, $\mathrm{T}_k S_{d-1}$. $H_{\rm f}$ is the free Hamiltonian defined by

$$H_{\rm f} = \sum_{j=1}^{d-1} \int |k| a^\dagger(k,j) a(k,j) dk.$$

Here $|k|$ describes the energy of a photon with momentum $k \in \mathbb{R}^d$. In the case of $\alpha = 0$ the Hamiltonian is

$$(\sqrt{-\Delta + m^2} - m + V) \otimes 1 + 1 \otimes H_{\rm f}$$

and all the eigenvalues of $\sqrt{-\Delta + m^2} - m + V$ are embedded in the continuous spectrum since $\sigma(H_{\rm f}) = [0, \infty)$. Thus to investigate the spectrum of H but with $\alpha \neq 0$ is a difficult issue. The boson Fock space is identified with the probability space $L^2(\mathscr{M}, \mu_0)$ with $\mathscr{M} = \oplus^d \mathscr{S}'(\mathbb{R}^d)$ endowed with a certain Gaussian measure μ_0 such that

$$\mathbb{E}[\mathscr{A}_\mu(f)\mathscr{A}_\nu(g)] = \frac{1}{2} \int \bar{\hat{f}}(k)\hat{g}(k) \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right) dk.$$

We can construct the functional integral representation of $(F, e^{-tH_P}G)$. Let $(T_t)_{t \ge 0}$ be the subordinator such that $\mathbb{E}[e^{-uT_t}] = e^{-t(\sqrt{2u+m^2}-m)}$.

Theorem 1.1

$$(F, e^{-tH_P}G) = \int dx \mathbb{E}^{x,0}\left[e^{-\int_0^t V(B_{T_s})ds}\int_{\mathscr{E}} \overline{F(\mathscr{A}_0, B_{T_0})}G(\mathscr{A}_t, B_{T_t})e^{-iK_t}d\mu\right], \quad F, G \in \mathscr{H}.$$

Here $\mathbb{E}^{x,0} = \mathbb{E}\mathbb{E}^x$ and $\mathbb{E}^x$ denotes the expectation with respect to the wiener measure, $\mathscr{E}$ is the Euclidean version of $\mathscr{M}$ and $\mathscr{A}_t$ is the Euclidean field with time t. The exponent is of the form $K_t = \int_0^t \mathscr{A}_s(\tilde{\varphi}(\cdot - B_s)) \cdot dB_s$, where $\tilde{\varphi}$ is the inverse Fourier transform of $\hat{\varphi}/|\cdot|$.

By means of this functional integral representation we can show that

1 H is ess. self-adjoint on $\cap_{\mu=1}^d D(-\nabla_\mu \otimes 1) \cap D(1 \otimes H_{\rm f})$;

2 $e^{-i(\pi/2)N}e^{-tH}e^{i(\pi/2)N}$ is a positivity improving operator, where N denotes the number operator ;

3 the ground state of H is unique;

4 the ground state of H is spatially exponentially decay for $m > 0$.

These results can be extended to more general models of the form:

$$H_\Psi = \Psi\left(\frac{1}{2}(-i\nabla \otimes 1 - \alpha A)^2\right) + V \otimes 1 + 1 \otimes H_{\rm f}$$

with an arbitrary Bernstein functions.

2 Translation invariant models

Let $V = 0$. Then $[H, -i\nabla_\mu \otimes 1 + 1 \otimes P_\mu] = 0$, where $P_\mu = \sum_{j=1}^{d-1}\int k_\mu a^\dagger(k,j)a(k,j)dk$ denotes the field momentum. $\mathscr{H}$ and H are decomposed as $\mathscr{H} = \int_{\mathbb{R}^d}^{\oplus} \mathscr{F} dx$ and $\int_{\mathbb{R}^d}^{\oplus} H(p)dp$ with the fiber Hamiltonian

$$H(p) = \sqrt{(p - P - \alpha A(0))^2 + m^2} - m + H_{\rm f}, \quad p \in \mathbb{R}^d,$$

on $\mathscr{F}$. Her $A(0)$ is defined by $A(x)$ with $x = 0$.

Theorem 2.1

$$(F, e^{-tH(p)}G) = \mathbb{E}^{0,0}\left[\int_{\mathscr{E}} e^{iP\cdot B_{T_t}}\overline{F(\mathscr{A}_0, B_{T_0})}e^{-iK_t}e^{-iP\cdot B_{T_t}}G(\mathscr{A}_t, B_{T_t})d\mu\right], \quad F, G \in \mathscr{F}.$$

The important fact is that if $p = 0$ then the phase $e^{iP\cdot B_{T_t}}$ disappears. By means of this functional integral representation we can show that

1 $H(p)$ is ess. self-adjoint on $\cap_{\mu=1}^d D(P_\mu) \cap D(H_{\rm f})$;

2 $e^{-i(\pi/2)N}e^{-tH(0)}e^{i(\pi/2)N}$ is a positivity improving operator;

3 the ground state of $H(0)$ is unique.