Critical Behaviour of Stochastic Geometric Models and the Lace Expansion

Takashi HARA (hara@math.kyushu-u.ac.jp)

Stochastic geometric models (self-avoiding walk, percolation, lattice trees and animals) exhibit rich critical behaviour. Our understanding of their critical behaviour, especially of those in high dimensions, has been considerably improved in the past twenty years. In particular, it has become apparent (but still a complete proof is missing) that these models have their own *upper critical dimensions*, above which they exhibit so-called *mean-field type critical behavior*. [The upper critical dimension is believed to be 4 for self-avoiding walk, 6 for percolation, and 8 for lattice trees and animals.]

In this talk, I briefly review the current status of our understanding of critical behaviour of these models. I then focus on the lace expansion, which can be a very powerful tool of analysis for these models in high dimensions. I also hope to give a flavor of why these different critical dimensions arise in these models.

In the following, I briefly describe models I consider in the talk (all on d-dimensional hypercubic lattice, $\mathbb{Z}^d$), which will be the focus of the first half of my talk.

1 Models

1.1. Self-Avoiding Walk (SAW) n-step Self-Avoiding Walk (SAW) is a set of $(n+1)$ ordered points $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$ in $\mathbb{Z}^d$, which satisfy

- $|\omega(j) - \omega(j+1)| = 1$ $(0 \leq j < n)$,
- $\omega(i) \neq \omega(j)$ for $i \neq j$ (self-avoiding constraint)

We denote the number of steps of the walk ω by $|\omega|$. We are interested in:

- number of n-step SAW's from x to y: $c_n(x,y)$
- number of n-step SAW's starting from the origin: $c_n = \sum_x c_n(0,x)$
- mean square displacement: $\ell_n := \langle |\omega(n)|^2 \rangle_n^{1/2}$. Here $\langle \cdots \rangle_n$ means the expectation w.r.t. the uniform measure of n-step SAW's from the origin.

We also consider generating functions for the above quantities:

- two-point function: $G_p(x,y) := \sum_n c_n(x,y)\, p^n = \sum_{\omega : x \to y} p^{|\omega|}$
- susceptibility: $\chi_p := \sum_{n \geq 0} c_n\, p^n = \sum_{\omega : 0 \to \bullet} p^{|\omega|} = \sum_x G_p(0,x)$
- correlation length: $\xi_p := -\lim_{n\to\infty} \dfrac{n}{\log G_p(0, ne_1)}$ (e_1 is a unit vector)

1.2. Percolation Fix $0 < p < 1$. Put independent identically distributed random variable n_b on each bond (= a set of nearest-neighbor sites) b as

$$n_b = \begin{cases} 1 & \text{(with probability } p\text{)} \quad \text{— occupied} \\ 0 & \text{(with probability } 1-p\text{)} \text{ — vacant} \end{cases}$$

Given a configuration $\{n_b\}_b$, two sites x, y are called *connected* (written as $x \longrightarrow y$) if there is a path of occupied bonds which connect x and y. The set of all sites connectet to x is called the *connected cluster* of x, and is denoted by $C(x)$. We define:

- two-point function: $\tau_p(x,y) := \mathbb{P}[\, x \longrightarrow y\,]$
- susceptibility: $\chi_p := \sum_y \tau_p(x,y) = \langle |C(0)| \rangle_p$. ($|C|$ is the number og sites in C).
- correlation length: $\xi_p := -\lim_{n\to\infty} \dfrac{n}{\log \tau_p(0, ne_1)}$
- *percolation density*: $\theta_p := \mathbb{P}[\, |C(0)| = \infty\,]$ (e_1 is a unit vector)

1.3. Lattice Trees and Lattice Animals (LTLA) A lattice animal is a set of connected bonds (without any additional condition). A lattice tree is a lattice animal, without loops. We are interested in (for a fixed parameter $p > 0$)

- number of n-sites lattice trees or animals which contain 0: c_n
- radius of gyration: $\ell_n := \langle |\text{distance between two points}|^2 \rangle_n^{1/2}$. Here $\langle \cdots \rangle_n$ means the expectation w.r.t. the uniform measure of size-n lattice trees (or animals).

We also consider generating functions for the above quantities:

- two-point function: $G_p(x,y) := \sum_{A \ni x,y} p^{|A|}$ where the sum is over all lattice trees or animals containing x, y.
- susceptibility: $\chi_p := \sum_{A \ni 0} p^{|\omega|} = \sum_x G_p(0,x)$
- correlation length: $\xi_p := -\lim_{n\to\infty} \dfrac{n}{\log G_p(0, ne_1)}$ (e_1 is a unit vector)

2 (Expected) Critical Behavior

It has been long conjectured in physics and chemistry literature that these models exhibit critical behaviour. In particular it has been believed (not proved!) that

- There exist *critical exponents* γ, ν, η such that

$$c_n \sim A\mu^n\, n^{\gamma-1}, \qquad (\ell_n)^2 \sim D\, n^{2\nu}, \qquad (n \nearrow \infty) \tag{2.1}$$

$$\chi_p \approx (p_c - p)^{-\gamma}, \qquad \xi_p \approx (p_c - p)^{-\nu}, \qquad (p \nearrow p_c) \tag{2.2}$$

$$G_{p_c}(0,x) \approx |x|^{-(d-2+\eta)} \qquad (|x| \nearrow \infty) \tag{2.3}$$

  and so on.
- Critical exponents are *universal*, while μ, p_c, A, D are not universal.
- Critical exponents satisfy *scaling relations* such as $(2-\eta)\nu = \gamma$.
- There exists an *upper critical dimension* d_c. Critical exponents take on simple (mean-field) values for $d > d_c$. d_c is believed to be 4 for self-avoiding walk, 6 for percolation, and 8 for lattice trees and animals.

To my regret, rigorous analysis of critical behavior has not been able to prove most of the above conjectures. However, there is some rigorous understanding as far as mean-field behaviour above upper critical dimensions are concerned. The method of the analysis is called the lace expansion, and this will be the topic of the latter half of my talk.