Semi-classical limit of the lowest eigenvalue of $P(\phi)_2$ Hamiltonian on a finite interval

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In this talk, we discuss the semi-classical limit of the lowest eigenvalue of a $P(\phi)_2$-Hamiltonian on a finite volume interval. Let $I = [-l/2, l/2]$ $(l > 0)$ and $\Delta = \frac{d^2}{dx^2}$ be the Laplace operator with periodic boundary condition on $L^2(I, dx)$. Let $\tilde{A} = (m^2 - \Delta)^{1/4}$ and define

$$H^s = \left\{ h \in \mathrm{D}(\tilde{A}^{2s}) \mid \|h\|_{H^s} := \|\tilde{A}^{2s} h\|_{L^2(I,dx)} \right\}.$$

H^0 is just $L^2(I, dx)$. We denote $H = H^{1/2}$. Let (W, H, μ) be the associated abstract Wiener space. For example, one may take $W = H^{-s_0}$ for any positive s_0. Note that W is the space of Schwartz distributions. Let $A = \Phi \circ \tilde{A} \circ \Phi^{-1}$, where $\Phi : L^2(I, dx) \to H$ is the natural unitary transformation. A is a self-adjoint operator on H. Let $-L_A$ be the generator of the following Dirichlet form:

$$\mathcal{E}_A(f, f) = \int_W \|ADf(w)\|_H^2 d\mu.$$

Let $P(u) = \sum_{k=0}^{2N} a_k u^k$ be a polynomial function with $a_{2N} > 0$ and $N \geq 2$. Let g be a periodic positive smooth function on $\mathbb{R}$ such that $g(x + l) = g(x)$ for all x. We define the potential function on W by

$$V_\lambda(w) = \lambda : V\left(\frac{w}{\sqrt{\lambda}}\right) :, \tag{1}$$

$$: V\left(\frac{w}{\sqrt{\lambda}}\right) : = \int_I : P\left(\frac{w(x)}{\sqrt{\lambda}}\right) : g(x)dx, \tag{2}$$

where $\lambda > 0$ and $: P(w(x)) :$ is defined by the Wick product with respect to μ. $\lim_{n\to\infty} \int_I : P\left(\frac{w_n(x)}{\sqrt{\lambda}}\right) : g(x)dx$ exists in $L^2(\mu)$ and we denote the limit by $: V\left(\frac{w}{\sqrt{\lambda}}\right) :$. The operator $(-L_A + V_\lambda, \mathfrak{F}C_A^\infty(W))$ ($\mathfrak{F}C_A^\infty(W)$ denotes the set of smooth cylindrical functions) is essentially self-adjoint in $L^2(\mu)$ and we denote the self-adjoint extension by $-L_A + V_\lambda$. $-L_A + V_\lambda$ is called a $P(\phi)_2$ Hamiltonian on a finite volume interval I and is a representation of the quantization of the Hamiltonian whose classical field equation is the non-linear Klein-Gordon equation with space-time dimension 2. Physically λ is the inverse of the Planck constant $\hbar$ and our problem is to determine the semi-classical limt of the lowest eigenvalue $E_0(\lambda)$ of $-L_A + V_\lambda$ as $\lambda \to \infty$ in terms of the potential function U which is given below.

Definition 1 Let $U(h) = \frac{1}{4}\|Ah\|_H^2 + V(h)$ for $h \in \mathrm{D}(A)$ and $U(h) = +\infty$ for $h \notin \mathrm{D}(A)$. Here $V(h) = \int_I P(h(x))g(x)dx$ and $h \in H$.

It is easy to see that V is defined on H and U is a smooth functional on H^1. The following is our main theorem.

Theorem 2 *Assume* (A1) *and* (A2).
(A1) $U(h)$ $(h \in H^1)$ *is a non-negative function and has finitely many zero point set* $N = \{h_1, \ldots, h_n\}$.
(A2) *Suppose* (A1). *The Hessian* $\frac{1}{2}D^2U(h_i) \in L(H^1, H^1)$ *is a strictly positive operator for all* $1 \le i \le n$.

Let $E_0(\lambda) = \inf \sigma(-L_A + V_\lambda)$. *Then*

$$\lim_{\lambda \to \infty} E_0(\lambda) = \min_{1 \le i \le n} E_i, \tag{3}$$

where E_i *is the lowest eigenvalue of* $-L_A + Q_{v_i}(w)$ *and* $Q_{v_i}(w) = \int_I : w(x)^2 : v_i(x)dx$, $v_i(x) = \frac{1}{2}P''(h_i(x))g(x)$. *Explicitly,*

$$\begin{aligned} \inf \sigma(-L_A + Q_{v_i}) &= \frac{1}{2}\mathrm{tr}\left(\tilde{A}_{v_i}^2 - \tilde{A}^2 - 2\tilde{A}^{-1}M_{v_i}\tilde{A}^{-1}\right) & (4) \\ &= -\frac{1}{4}\|\left(\tilde{A}_{v_i}^2 - \tilde{A}^2\right)\tilde{A}^{-1}\|_{L_{(2)}(H^0)}^2, & (5) \end{aligned}$$

where $\tilde{A}_{v_i} = (m^2 - \Delta + 4v_i)^{1/4}$ *on* H^0. tr *denotes the trace in* H^0 *and* $\| \ \|_{L_{(2)}(H^0)}$ *denotes the Hilbert-Schmidt norm.*

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