離散的弱ワイル関係式

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作用素論セミナー

Definition (conjugate operator $[A, B] = AB - BA$)

If a self-adjoint operator A in Hilbert space $\mathcal{H}$ admits an operator B satisfying

$$[A, B] = -i$$

on $D(AB) \cap D(BA) \neq \{0\}$, then B is referred to a conjugate operator of A.

▶Algebraic relations:

$$[A, B^n] = -inB^{n-1} \Longrightarrow [A, f(B)] = -if'(B) \Longrightarrow A = -i\frac{d}{dB}$$

▶$[p, q] = -i$
▶Aharonov-Bohm time operator:

$$T_{AB} = \frac{1}{2}(qp^{-1} + p^{-1}q) \Longrightarrow [\frac{1}{2}p^2, T_{AB}] = -i$$

▶Generalization:

$$T_f = \frac{1}{2}(qf'(p)^{-1} + f'(p)^{-1}q) \Longrightarrow [f(p), T_f] = -i$$

▶ ワイル関係式

$$e^{-isq}e^{-itp}\psi = e^{its}e^{-itp}e^{-isq}\psi$$

(1) [Weyl 1928] **ワイル関係式**. T, H sa:

$$e^{-isT}e^{-itH}\psi = e^{its}e^{-itH}e^{-isT}\psi.$$

(2) [Miyamoto 02] **弱ワイル関係式** and **strong time op** T. H sa, T sym:

$$Te^{-itH}\psi = e^{-itH}(T+t)\psi \quad e^{-itH}\psi \in D(T)\forall t.$$

(3) [Heisenberg 1927] **CCR** and **conjugate op** T. H sa:

$$[H,T]\psi = -i\psi \quad \psi \in D(HT)\cap D(TH).$$

$$\boxed{WR \Longrightarrow WWR \Longrightarrow CCR \text{ but } CCR \not\Longrightarrow WWR \not\Longrightarrow WR}.$$

Theorem (一意性定理 von Neumann 1931)

If $\{H, T\}$ *is WR, then* $H \cong \bigoplus_{j=1}^{N} p_j$ *and* $T \cong \bigoplus_{j=1}^{N} q_j$

▶WWR $\{H, T\}$:

$$Te^{-itH} = e^{-itH}(T + t)$$

Theorem (Miyamoto 02)

(1) *If* $\{H, T\}$ *is WWR, then* $\sigma(H)$ *is* **purely absolutely continuous**.

(2) *If* $\{H, T\}$ *is WWR and* $\inf \sigma(H) > -\infty$, *then* T *has* **no sa extension**.

Examples

▶$\{\frac{1}{2}p^2, T_{AB}\}$ is WWR, where $T_{AB} = \frac{1}{2}(p^{-1}q + qp^{-1})$ ($\leftarrow$ **not** ess sa).
▶$\frac{1}{2}(p^2 + q^2)$ has **no** strong time operator. Pauli の話.

▶Harmonic oscillator on $L^2(\mathbb{R})$:

$$\frac{1}{2}(p^2+q^2-1).$$

▶Momentum with quasi-periodic boundary condition on $L^2([-L,L])$:

$$p_\alpha f = -if' \quad D(p_\alpha)=\left\{f\in L^2([-L,L]) \,\middle|\, f(L)=e^{i\alpha}f(-L)\right\}.$$

▶Mathieu operator on $L^2(S^1)$:

$$-\Delta_\theta + (1-\cos\theta).$$

▶Discrete harmonic oscillator on $\ell_2(\mathbb{Z})$:

$$-\frac{1}{2}\Delta_d + n^2.$$

Conjugate operators of $H = \frac{1}{2}(p^2 + q^2 - 1) = a^* a$

- $a = (q + ip)/\sqrt{2}$ and $a^* = (q - ip)/\sqrt{2}$.
- $[H, T] = -i$:

(A) **Angle operator**:

$$T_A = \frac{1}{2}\left(\arctan p^{-1}q + \arctan qp^{-1}\right) \cong -\frac{1}{2}\left(\arctan q^{-1}p + \arctan pq^{-1}\right)$$

(B) **Phase operator**: $a = e^{i\phi}\sqrt{a^* a}$ by Dirac (1927)

$$\phi = \frac{i}{2}(\log a - \log a^*)$$

(C) **POVM**: $\sigma(H) = \mathbb{N} \cup \{0\}$ and $He_n = ne_n$.

$$T_G = i\sum_{n=0}^{\infty}\sum_{m\neq n}\frac{|e_n\rangle\langle e_m|}{n-m}$$

(A) Angle op. $T_A = -\frac{1}{2}\left(\arctan q^{-1}p + \arctan pq^{-1}\right)$

$$t = q^{-1}p \Longrightarrow \left[\frac{1}{2}(p^2+q^2), t\right] = i(1+t^2) \Longrightarrow \left[\frac{1}{2}(p^2+q^2), f(t)\right] = i(1+t^2)f'(t).$$

$$f'(t) = -\frac{1}{1+t^2} \Longrightarrow f(t) = -\arctan t.$$

- $\arctan z = \frac{i}{2}\log\frac{i+z}{i-z}$ for $z \in \mathbb{C}\setminus\{i\}$.
- $t_0 = q^{-1}p$, $t_1 = pq^{-1}$. Eigenvalue equation $0 < \alpha < 1$:

$$\begin{cases} t_0 e^{-\alpha x^2/2} = i\alpha e^{-\alpha x^2/2} \Longrightarrow (\arctan t_0)\, e^{-\alpha x^2/2} = \left(\frac{i}{2}\log\frac{1+\alpha}{1-\alpha}\right) e^{-\alpha x^2/2} \\ t_1 x e^{-\alpha x^2/2} = i\alpha e^{-\alpha x^2/2} \Longrightarrow (\arctan t_1)\, x e^{-\alpha x^2/2} = \left(\frac{i}{2}\log\frac{1+\alpha}{1-\alpha}\right) x e^{-\alpha x^2/2} \end{cases}$$

Theorem (FH-Teranishi 25)

(1) $\arctan t_0$ *is well defined on* $L^2_{even}(\mathbb{R})$ *and* $\arctan t_1$ *on* $L^2_{odd}(\mathbb{R})$.
I.e., T_A **is not densely defined.**

(2) $-i(0,\infty) \subset \sigma(-\arctan t_{\#})$. *I.e.,* T_A *is* **unbounded**.

(B) Phase operator $\phi = \frac{i}{2}(\log a - \log a^*)$

Let A be a linear operator in $L^2(\mathbb{R})$.

$$\begin{cases} \log A = -\sum_{n=1}^{\infty} \frac{1}{n}(\mathbb{1} - A)^n \\ D(\log A) = \left\{ f \in \bigcap_{n=0}^{\infty} D(A^n) \;\middle|\; \exists s \lim_{K\to\infty} \sum_{n=1}^{K} \frac{1}{n}(\mathbb{1} - A)^n f \right\}. \end{cases}$$

$\blacktriangleright D_{\rm fin} = \mathrm{LH}\left\{ \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!}} {a^*}^n e^{-x^2/2} \,\middle|\, n = 0 \text{ for } n \geq \exists m \right\} \subset L^2(\mathbb{R})$.

Theorem (FH+Teranishi 25)

(1) $D_{\rm fin} \subset D(\log a)$ *and* $D_{\rm fin} \cap D(\log a^*) = \{0\}$. *I.e.,* ϕ *is* **not well defined**:

(2) *It follows that*

$$-\arctan(q^{-1}p) \cong -\frac{i}{2}\log\left({a^*}^{-1}a\right) \quad \text{on } L^2_{even}(\mathbb{R}),$$

$$-\arctan(pq^{-1}) \cong -\frac{i}{2}\log\left(a{a^*}^{-1}\right) \quad \text{on } L^2_{odd}(\mathbb{R}).$$

(C) POVM. $T_G = i\sum_{n=0}^{\infty}\sum_{m\neq n}\frac{|e_n\rangle\langle e_m|}{n-m}$

Theorem (Galapon 02)

(1) T_G *is* **bounded and self-adjoint** *st* $\|T_G\| \leq 2\pi$.

(2) $[H, T_G] = -i + iP$. *Here* $P = \lim_{N\to\infty}\sum_{m=0}^{N}\sum_{n=0}^{N}|e_n\rangle\langle e_m|$.

(3) $[H, T_G] = -i$ *on* $\mathrm{LH}\{e_n - e_m \mid n \neq m\}$.

▶Example:

$$\sigma(p_\alpha) = \{E_n\}_{n\in\mathbb{Z}} = \left\{\frac{2\pi n + \alpha}{2L}\right\}_{n\in\mathbb{Z}}.$$

Let $\alpha = 0$. The eigenvector corresponding to E_n is $|n\rangle = \frac{1}{\sqrt{2L}}e^{iE_n x}$.

$$T_\alpha = i\frac{L}{\pi}\sum_{n\in\mathbb{Z}}\sum_{m\neq n}\frac{|n\rangle\langle m|}{n-m}$$

$$[p_\alpha, T_\alpha] = -i + iL\delta(x)$$

Applying $|n\rangle - |m\rangle$ to both sides yields

$$[p_\alpha, T_\alpha](|n\rangle - |m\rangle) = -i(|n\rangle - |m\rangle).$$

Fock-representation $\ell_2 = \{(c_n) \mid \sum_{n=0}^{\infty} |c_n|^2 < \infty\}$

$$(Lc)_n = \begin{cases} c_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}, \quad (L^* c)_n = c_{n+1}, \quad (Nc)_n = nc_n.$$

$$\begin{array}{ccc} \ell_2 & \xrightarrow{L\sqrt{N}, \sqrt{N}L^*, N} & \ell_2 \\ {\scriptstyle F}\downarrow & & \uparrow{\scriptstyle F^{-1}} \\ L^2(\mathbb{R}) & \xrightarrow[a, a^*, a^*a]{} & L^2(\mathbb{R}) \end{array} \overset{\text{time op}}{\Longrightarrow} \begin{array}{ccc} \ell_2 & \xrightarrow{t_A, t_G} & \ell_2 \\ {\scriptstyle F}\downarrow & & \uparrow{\scriptstyle F^{-1}} \\ L^2(\mathbb{R}) & \xrightarrow[T_A, T_G]{} & L^2(\mathbb{R}) \end{array}$$

Theorem (Fock representations of T_G and T_A, HF+Teranishi 25)

$$T_A \cong t_A = \frac{i}{2}\left\{\log\left(\sqrt{\frac{N+2}{N+1}}L^2\right) + \log\left(\sqrt{\frac{N+1}{N+2}}L^2\right)\right\}$$

$$T_G \cong t_G = i\left\{\log(\mathbb{1} - L) + \log(\mathbb{1} - L^*)\right\}$$

▶Generalizations of t_G and t_A:

$$\left.\begin{matrix} t_G \\ t_A \end{matrix}\right\} \Longrightarrow T_{\omega,m} = \frac{i}{m}\log(\omega\mathbb{1} - L^m), \quad (\omega, m) \in \mathbb{D} \times \mathbb{N}$$

▶Classification:

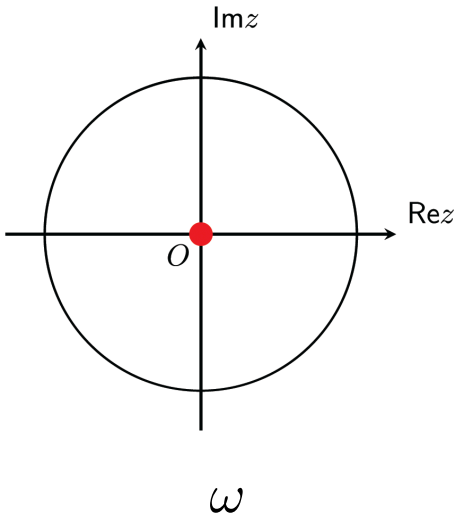


Figure: T_A type

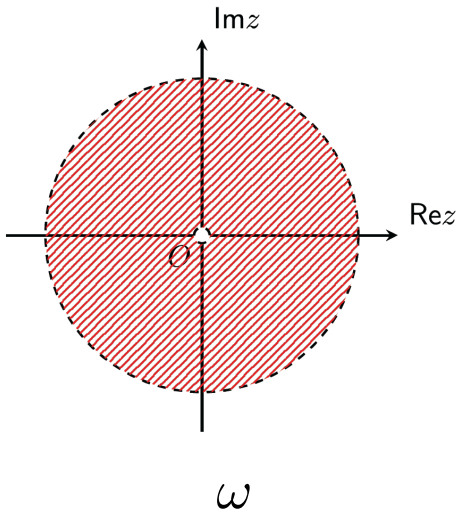


Figure: Interpolation

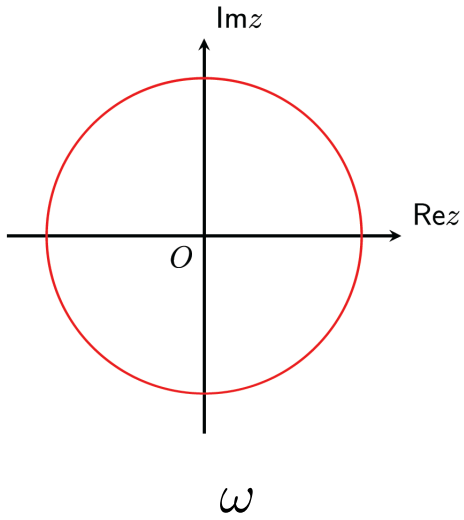


Figure: T_G type

Theorem (離散的弱ワイル関係式 FH+Teranishi25)

$$e^{itH} T_{\omega,m} e^{-itH} = \frac{i}{m} \log(\omega \mathbb{1} - e^{-itm} L^m)$$

and the period is $2\pi/m$.

- 弱ワイル関係式 $Te^{-itH}\psi = e^{-itH}(t+T)\psi \Longrightarrow \sigma(H)$ is abs. cont.

Remark

- *Let* $\omega = 0$. *Formally one might expect*

$$\frac{i}{m} \log(-L^m) e^{-itH} = e^{-itH} \left(t + \frac{i}{m} \log(-L^m) \right)$$

However, this identity is incorrect, since

$$\log(-e^{-itm} L^m) \neq \log\left(e^{-itm}\right) + \log(-L^m).$$

物理実験による存在証明 (Ozawa+Hasegawa+FH+Tanda)

Theorem (Arai-Miyamot 不等式/量子コンピューター (Emori+FH))

Let $H = -\frac{1}{2}p^2$. *Then for* $\psi \in D(T_{AB}^n)$,

$$\left|\langle \psi, e^{-itH}\psi \rangle\right| = O(|t|^{-n}), \qquad |t| \to \infty.$$

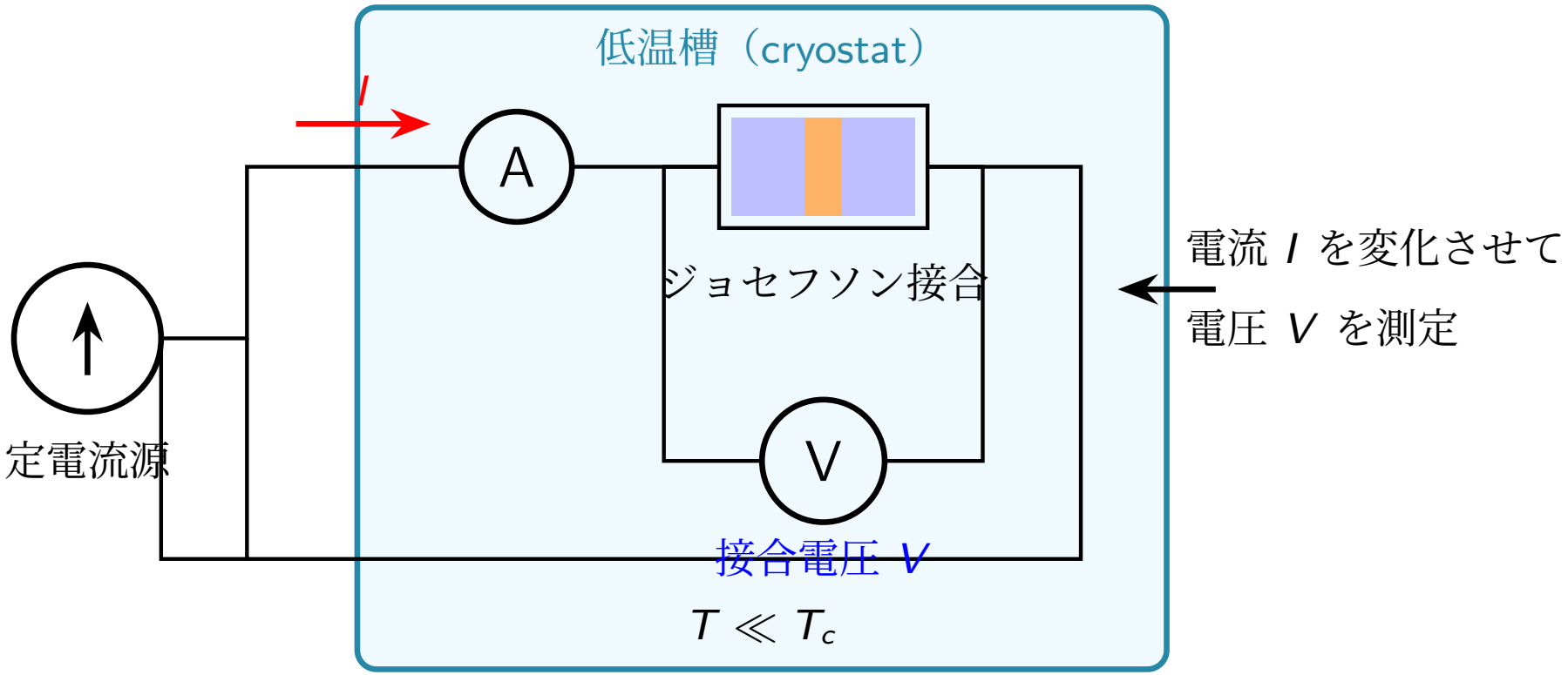
Lemma (CDW と S^1 上のジョセフソン接合での比較実験)

Let $H = -\frac{1}{2}(p^2 + q^2)$. *Then* $\limsup_n \left|\langle \psi, e^{-itH}\psi \rangle\right| = 1$.

Theorem (周期の検出 (たぶん無理))

Time operator	**Period**
T_A	π
T_G	2π

ジョセフソン接合



T. Fujii, FH, and S. Tanda, *Representations of Josephson junction on the unit circle and the derivations of Mathieu operators and Fraunhofer patterns*, J. Math. Phys. **67**, 063502 (2026).

丹田研究室訪問



液体ヘリウム３と４

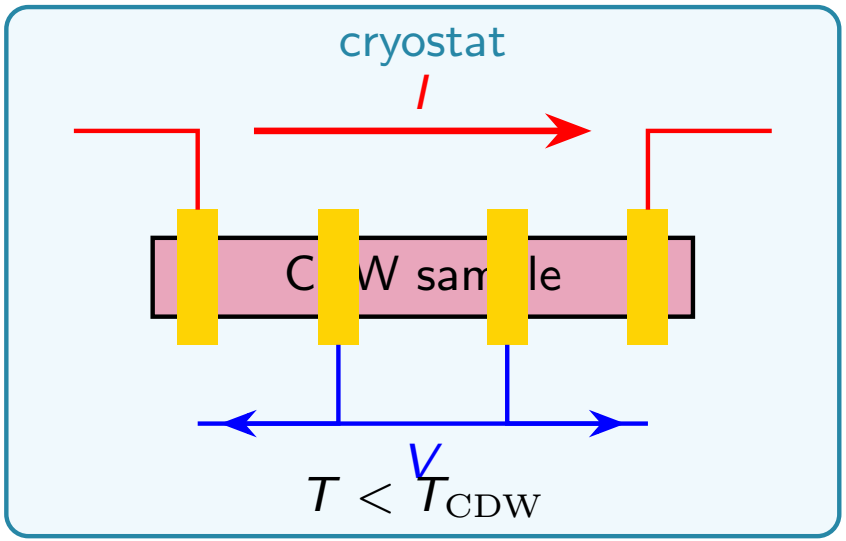


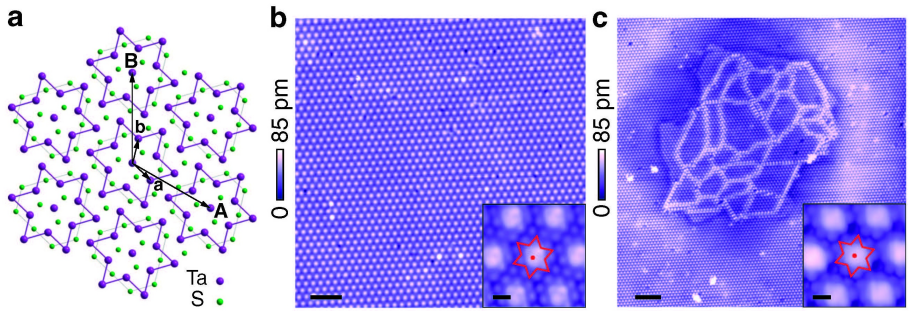
液体ヘリウム３と４



ジョセフソン接合

電荷密度波=Charge-density wave (CDW)





CDW の走査型トンネル顕微鏡写真

L. Ma et al., Nature Communications **7**, 10956 (2016).

周期的な電子密度 $\rho(x) = \rho_0 + \rho_1 \cos(Qx + \phi)$

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