IDS for point interactions supported on the determinantal processes

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Schrödinger operator with point interactions

Let ξ be a locally finite set in $\mathbf{R}^d$ $(d = 1, 2, 3)$, that is,

$$\#(\xi \cap B_0(R)) < \infty$$

for every $R > 0$, where $B_x(R) = \{y \in \mathbf{R}^d\,;\,|y - x| < R\}$ and $\#S$ is the cardinality of a set S. Let $\alpha = (\alpha_y)_{y\in\xi}$ be a sequence of real numbers. We consider the Schrödinger operator $-\Delta_{\alpha,\xi}$, formally written as

$$-\Delta_{\alpha,\xi} = -\Delta + \text{`point interactions on } \xi\text{'},$$

where α_y is the parameter representing the interaction at the point y. Basic facts about $-\Delta_{\alpha,\xi}$ are found in the book 'Solvable models in quantum mechanics' by Albeverio et al.

Definition of Point interactions

A rigorous definition of $-\Delta_{\alpha,\xi}$ is as follows.

$$-\Delta_{\alpha,\xi} u = -\Delta|_{\mathbf{R}^d \setminus \xi} u,$$

$$D(-\Delta_{\alpha,\xi}) = \{u \in H^2_{\mathrm{loc}}(\mathbf{R}^d \setminus \xi) \cap L^2(\mathbf{R}^d)\,;\, -\Delta|_{\mathbf{R}^d\setminus\xi} u \in L^2(\mathbf{R}^d), \; u \text{ satisfies } (BC)_y \text{ for every } y \in \xi\}.$$

Here, $-\Delta|_{\mathbf{R}^d\setminus\xi} u$ is defined as a Schwartz distribution on $\mathbf{R}^d \setminus \xi$. The boundary condition $(BC)_y$ is as follows:

- $d = 1$: $u(y+0) = u(y-0) = u(y)$, $u'(y+) - u'(y-) = \alpha_y u(y)$.
- $d = 2$: $u(x) = u_{y,0} \log|x-y| + u_{y,1} + o(1)$ as $x \to y$, and $2\pi\alpha_y u_{y,0} + u_{y,1} = 0$.
- $d = 3$: $u(x) = u_{y,0}|x-y|^{-1} + u_{y,1} + o(1)$ as $x \to y$, and $-4\pi\alpha_y u_{y,0} + u_{y,1} = 0$.

Point processes

Today we assume that ξ is a locally finite (random) point process, i.e. a random set ξ obeying some probability law, and $\#(\xi \cap B) < \infty$ a.s. for every bounded measurable set B.

We identify a point process ξ on a measurable set Λ in $\mathbf{R}^d$ with a probability measure $\mu = \mu(d\xi)$ on a configuration space Q_Λ, the space of all locally finite subsets of Λ. When $\Lambda = \mathbf{R}^d$, we write $Q = Q_{\mathbf{R}^d}$. We regard Q_Λ as a measure space equipped with the σ-algebra generated by the maps $Q_\Lambda \ni \xi \mapsto \#(\xi \cap E)$ (E: Borel subset in Λ).

Point processes (2)

When Λ is a bounded measurable set in $\mathbf{R}^d$,

$$Q_\Lambda = \sum_{n=0}^{\infty} \Lambda^n / \sim, \quad \Lambda^0 = \{\emptyset\},$$

where the equivalence relation $\sim$ is defined by permutation of coordinates.

For simplicity, we assume that $\alpha = \alpha_y$ is a constant sequence, that is, the value α_y is a real constant independent of y, ξ. We also denote the common value of α_y by α, for simplicity.

Examples of point processes (1)

Today we consider the following point processes.

(1) Poisson point process. Most basic point process, which represents the complete spatial randomness.

(2) Determinantal point process or Fermion point process. Random points have some repulsive interactions. (cf. Macchi 1975, Shirai–Takahashi 2003, Ueki 2019, ..., plenary talk by Shirai 2024 in autumn MSJ meeting, etc.) (Today's main topic)

There are many other point processes, e.g., Gibbs point process (a PP which is absolutely continuous w.r.t. Poisson PP), Cox point process (a Poisson PP with random intensity measure), etc.

Examples of point processes (2)

The book

'Spatial Point Patterns, Methodology and Applications with R'
by Adrian Baddeley, Ege Rubak, and Rolf Turner

contains many examples of point processes, and explains how to simulate point processes by using the R library `spatstat`.

Below we shall show some pictures of point processes created by `spatstat`.

Poisson point process (1)

We say $\mu = \mu(d\xi)$ is the Poisson point process on $\mathbf{R}^d$ with intensity measure ρdx ($\rho > 0$ is a constant) if the following holds.

(1) For any bounded measurable set E of $\mathbf{R}^d$, $\#(\xi \cap E)$ obeys the Poisson distribution with parameter $\rho|E|$, that is,

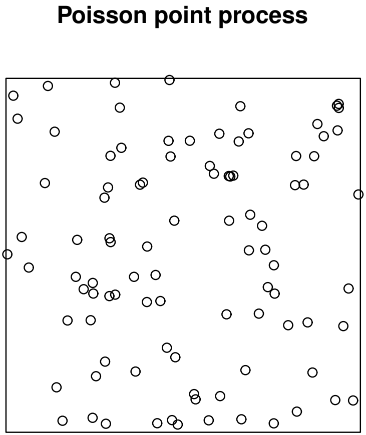
$$\mu\{\#(\xi \cap E) = k\} = \frac{(\rho|E|)^k}{k!} e^{-\rho|E|} \quad (k = 0, 1, 2, \dots),$$

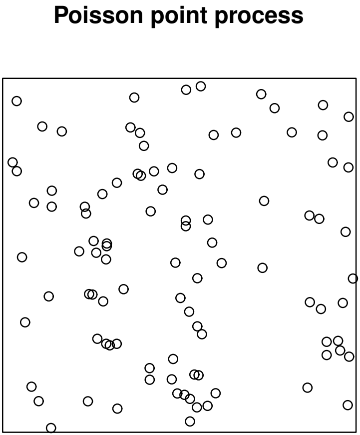
where $|E|$ is the Lebesgue measure of E.

(2) For any disjoint bounded measurable sets E_j ($j = 1, \dots, n$), the random variables $\#(\xi \cap E_j)$ ($j = 1, \dots, n$) are independent.

In the next slide, we show two examples of the Poisson point process on $[0, 1]^2$ with intensity $100dx$.

Poisson point process (2)





Determinantal point process (1)

Let K be a bounded self-adjoint operator on $L^2(\mathbf{R}^d)$ with integral kernel $K(x, y)$. We assume the following.

(K1) $K \geq 0$ (i.e., $(u, Ku) \geq 0$ for every $u \in L^2(\mathbf{R}^d)$) and $\|K\| \leq 1$.

(K2) For every compact set E in $\mathbf{R}^d$, the operator $\chi_E K \chi_E$ is a trace class operator on $L^2(\mathbf{R}^d)$, where χ_E is the characteristic function of the set E.

Determinantal point process (2)

According to Theorem 1.2 of Shirai–Takahashi 2003, the determinantal point process $\mu_{-1,K}$ is defined by its Laplace transform

$$\mathcal{L}_{\mu_{-1,K}}(f) := \int_Q \mu_{-1,K}(d\xi) \exp(-\langle \xi, f \rangle) = \mathrm{Det}(I - K_\varphi),$$
$$K_\varphi(x,y) = \sqrt{\varphi(x)} K(x,y) \sqrt{\varphi(y)},$$
$$\varphi(x) = 1 - \exp(-f(x)), \quad (\mathrm{supp}\,\varphi = \mathrm{supp}\, f)$$

for any non-negative measurable function f on $\mathbf{R}^d$ with compact support. Here, Det is the Fredholm determinant (K_φ is in the trace class), and

$$\langle \xi, f \rangle = \sum_{x \in \xi} f(x), \quad \xi \in Q.$$

n-point correlation function (1)

The n-point correlation function $\rho_n(x_1, \dots, x_n)$ $(n = 1, 2, \dots)$ of a point process μ on $\mathbf{R}^d$ is defined by

$$\int_Q \langle \xi_n, f_n \rangle \mu(d\xi) = \int_{(\mathbf{R}^d)^n} f_n(x_1, \dots, x_n) \rho_n(x_1, \dots, x_n) dx_1 \dots dx_n,$$

$$\langle \xi_n, f_n \rangle = \sum_{x_1, \dots, x_n \in \xi, \text{distinct}} f_n(x_1, \dots, x_n)$$

for any non-negative measurable function f_n on $(\mathbf{R}^d)^n$ with compact support.

n-point correlation function (2)

If we take $f_n = \chi_{E_1}(x_1) \cdots \chi_{E_n}(x_n)$ for disjoint bounded measurable sets $E_1, \ldots, E_n$, we have

$$\mathbf{E}\left[\#(E_1 \cap \xi) \cdots \#(E_n \cap \xi)\right] = \int_{E_1 \times \cdots \times E_n} \rho_n(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

Particularly when $n = 1$, we have

$$\mathbf{E}\left[\#(E \cap \xi)\right] = \int_E \rho_1(x) dx$$

for every bounded measurable set E. From this reason, we call ρ_1 the intensity function of the point process μ.

When μ is the Poisson point process with intensity ρdx, we have

$$\rho_n(x_1, \ldots, x_n) = \rho^n \text{ (constant)}.$$

Determinantal point process (4)

According to Theorem 4.1 of Shirai–Takahashi 2003, the n-point correlation function of the determinantal process $\mu_{-1,K}$ is given by

$$\rho_{n,-1,K}(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{i,j=1}^{n}. \qquad (1)$$

If the kernel K is continuous, (1) means that

$$\rho_{n,-1,K}(x_1, \ldots, x_n) \to 0$$

as $x_i - x_j \to 0$ for some $i \neq j$, which implies that there is some repulsive interaction between points in ξ.

Gaussian kernel (1)

Let $K(x,y)$ be the Gaussian kernel of convolution type, that is

$$K(x,y) = \rho \exp\left(-\frac{|x-y|^2}{\beta^2}\right),$$

where $\rho > 0$ and $\beta > 0$ are constants. If we assume

$$\rho(\beta^2\pi)^{d/2} \le 1,$$

then the assumptions (K1) and (K2) are satisfied. Actually, the Fourier transform of $k(x) = \rho \exp(-|x|^2/\beta^2)$ is

$$\hat{k}(\xi) = \int_{\mathbf{R}^d} k(x) e^{-2\pi i \xi \cdot x} dx = \rho(\beta^2\pi)^{d/2} e^{-\pi^2\beta^2|\xi|^2}.$$

Thus $\sigma(K) = [0, \rho(\beta^2\pi)^{d/2}]$, and (K1) is satisfied. We can prove (K2) by checking that $\chi_E\sqrt{K}$ is a Hilbert–Schmidt operator for any compact set E.

Gaussian kernel (2)

By (1), the n-point correlation function $\rho_{n,-1,K}$ for $\mu_{-1,K}$ is given by

$$\rho_{n,-1,K}(x_1, \dots, x_n) = \rho^n \det \left(e^{-\frac{|x_i - x_j|^2}{\beta^2}} \right)_{i,j=1}^{n}.$$

In particular, the intensity function is given by

$$\rho_{1,-1,K}(x_1, \dots, x_n) = \rho.$$

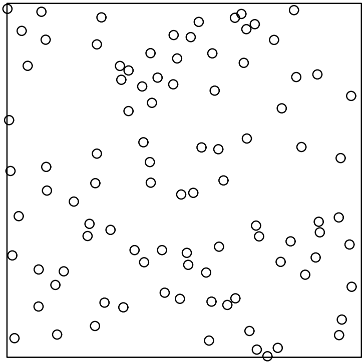
Moreover, we have

$$\rho_{n,-1,K}(x_1, \dots, x_n) \to \rho^n \quad (\beta \to +0),$$

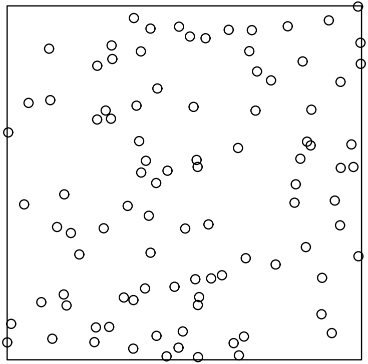
which means that $\mu_{-1,K}$ converges to the Poisson point process with intensity ρdx.

Determinantal point process ($\beta = 0.05$)

Determinantal point process beta 0.05



Determinantal point process beta 0.05



Assumptions

In the sequel, we assume $d = 3$, and assume one of the following.

(PP) The process μ is the Poisson point process with intensity ρdx, where ρ is a positive constant.

(DP) The process μ is the determinantal point process $\mu_{-1,K}$ with the Gaussian kernel

$$K(x, y) = \rho \exp\left(-\frac{|x-y|^2}{\beta^2}\right),$$

where ρ and β are positive constants with

$$\rho(\beta^2\pi)^{3/2} \leq 1.$$

Self-adjointness, Spectrum

Theorem 1

Let $d = 3$. Suppose that either (PP) or (DP) holds.

(i) The operator $-\Delta_{\alpha,\xi}$ is a self-adjoint operator on $L^2(\mathbf{R}^3)$ for any real-valued sequence $\alpha = (\alpha_y)_{y\in\xi}$, almost surely.

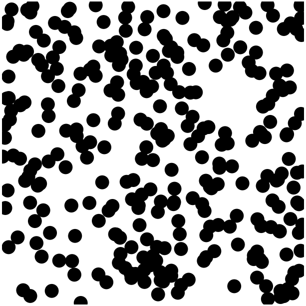
(ii) Assume further that α is a constant sequence. Then, the spectrum of $-\Delta_{\alpha,\xi}$ satisfies $\sigma(-\Delta_{\alpha,\xi}) = \mathbf{R}$, almost surely.

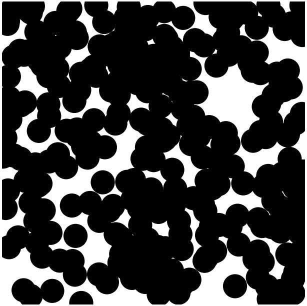
When (PP) holds, Theorem 1 is proved in Kaminaga–M–Nakano 2020. For the proof of (i), we use some special criterion of self-adjointness, called the percolation criterion. For the proof of (ii), we use the method of admissible potentials, a common method for calculating the spectrum of random Schrödinger operators.

Percolation criterion

Theorem 2 (Kaminaga–M–Nakano 2020)

Let $d = 2, 3$ and ξ be a locally finite set in $\mathbf{R}^d$. Assume that there exists $R > 0$ such that every connected component of $(\xi)_R := \{x \in \mathbf{R}^d; \mathrm{dist}(x, \xi) < R\}$ is bounded. Then, $-\Delta_{\alpha,\xi}$ is self-adjoint for any real-valued sequence α.





Admissible potentials

Lemma 3

Let $d = 3$. Suppose that α is a constant sequence, and either (PP) or (DP) holds. Then, we have almost surely

$$\sigma(-\Delta_{\alpha,\xi}) = \overline{\bigcup_{\#\xi' < \infty} \sigma(-\Delta_{\alpha,\xi'})}.$$

This lemma follows from the observation that 'for any finite configuration ξ', we can find an approximation of ξ' in the configuration ξ, almost surely.' Here we use the independence of $\#(\xi \cap E_j)$ for disjoint E_j ($j = 1, \ldots, n$) for (PP), or the mixing property of $\mu_{-1,K}$ with convolution kernel for (DP) (cf. Soshnikov 2000).

Integrated density of states (1)

Let us define the integrated density of states, in the following way.

For $L \in \mathbf{N} := \{1, 2, 3, \ldots\}$, let $C_L = (0, L)^3$. For a locally finite set ξ with $\partial C_L \cap \xi = \emptyset$, let $-\Delta^D_{\alpha,\xi,C_L}$ (resp. $-\Delta^N_{\alpha,\xi,C_L}$) be the operator $-\Delta_{\alpha,\xi}$ restricted to C_L with the Dirichlet boundary conditions $u|_{\partial C_L} = 0$ (resp. Neumann boundary conditions $\frac{\partial u}{\partial n}|_{\partial C_L} = 0$).

For $\lambda \in \mathbf{R}$ and $\sharp = D, N$, let $N^\sharp_{\alpha,\xi,C_L}(\lambda)$ be the number of eigenvalues of $-\Delta^\sharp_{\alpha,\xi,C_L}$ less than or equal to λ, counted with multiplicity.

Integrated density of states (2)

Theorem 4

Let $d = 3$. Assume that α is a constant sequence, and either (PP) or (DP) holds.

(i) For $\sharp = D, N$, we have

$$\lim_{L\to\infty} \frac{N^{\sharp}_{\alpha,\xi,C_L}(\lambda)}{|C_L|} = \lim_{L\to\infty} \frac{\mathbf{E}[N^{\sharp}_{\alpha,\xi,C_L}(\lambda)]}{|C_L|} \tag{2}$$

almost surely. Here $|C_L| = L^3$ is the Lebesgue measure of C_L.

The equality (2) means that the left hand side of (2) is independent of ξ, almost surely. For the proof, we use the translational invariance and the ergodicity of the process μ. Notice that the ergodicity follows from the mixing property.

Integrated density of states (3)

Theorem 4 (continued)

(ii) We denote the right hand side of (2) by $N_\alpha^\sharp(\lambda)$. Then we have

$$N_\alpha^D(\lambda+0) = N_\alpha^N(\lambda+0)$$

for every $\lambda \in \mathbf{R}$, where $f(\lambda+0) := \lim_{\epsilon\to+0} f(\lambda+\epsilon)$.
Moreover, if either N^D or N^N is continuous at λ, then we have

$$N^D(\lambda) = N^N(\lambda) = N^D(\lambda+0) = N^N(\lambda+0).$$

In the sequel, we denote

$$N_\alpha(\lambda) := N_\alpha^\sharp(\lambda+0) \quad (\sharp = D \text{ or } N),$$

and call $N_\alpha(\lambda)$ the integrated density of states (IDS).

Integrated density of states (4)

For $\lambda < 0$, we can use another expression of $N_\alpha(\lambda)$. Let $N_{\alpha,\xi,C_L}(\lambda)$ be the number of eigenvalues of $-\Delta_{\alpha,\xi\cap C_L}$ less than or equal to λ, counted with multiplicity. Notice that $-\Delta_{\alpha,\xi\cap C_L}$ is a self-adjoint operator on $L^2(\mathbf{R}^d)$ (not $L^2(C_L)$).

Proposition 5

Suppose the assumptions of Theorem 4 hold. If $\lambda < 0$ and $N_\alpha^D(\lambda) = N_\alpha^N(\lambda)$, then

$$N_\alpha(\lambda) = \lim_{L\to\infty} \frac{\mathbf{E}[N_{\alpha,\xi,C_L}(\lambda)]}{|C_L|}. \tag{3}$$

The advantage of the expression (3) is that there is an explicit formula for the spectrum of $-\Delta_{\alpha,\xi\cap C_L}$.

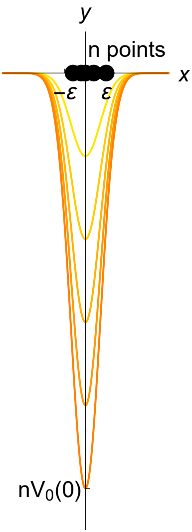
Pastur tail (1)

Let us review the result about the asymptotics of IDS as $\lambda \to -\infty$, for usual Schrödinger operator $H = -\Delta + V$ on $\mathbf{R}^d$ with

$$V(x) = \sum_{y \in \xi} V_0(x-y),$$

where ξ is the Poisson point process with intensity measure ρdx, $V_0 \in C_0^\infty(\mathbf{R}^d)$, $V_0 \le 0$, and the minimum of V_0 is $V_0(0) < 0$.

In this case, we also have $\sigma(H) = \mathbf{R}$, almost surely (cf. Ando–Iwatsuka–Kaminaga–Nakano 2006, Kaminaga–M 2012). If n random points exist near 0, the depth of the potential well is almost multiplied by n.



Pastur tail (2)

The above mechanism also explains very rapid decay of IDS as $\lambda \to -\infty$:

$$\log N(\lambda) = -\frac{|\lambda|}{|V_0(0)|} \log|\lambda| \cdot (1 + o(1)) \quad \text{as } \lambda \to -\infty \tag{4}$$

(Pastur 1974, 1977). So $N(\lambda)$ decays super exponentially $O(|\lambda|^{-C|\lambda|})$ as $\lambda \to -\infty$.

The asymptotics (4) is roughly explained as follows. A negative spectrum λ is created by at least $n = |\lambda|/|V_0(0)|$ random points in a small ball B_ϵ. The probability of this event is

$$p \doteqdot e^{-\rho|B_\epsilon|} \frac{(\rho|B_\epsilon|)^n}{n!}, \quad n! \sim (2\pi n)^{1/2} (n/e)^n .$$

Recently, Nakagawa 2023 gives a remarkable example of the Gibbs process such that $\log N(\lambda) \sim -C\lambda^2$ as $\lambda \to -\infty$.

Auxiliary function

In order to state our main result, we introduce an auxiliary function.

For $s > \max(0, -4\pi\alpha)$, let $R_\alpha(s)$ be the unique solution of the following equation with respect to R:

$$s - \frac{e^{-sR}}{R} = -4\pi\alpha. \tag{5}$$

When $\alpha = 0$, (5) becomes a simple equation $sR = e^{-sR}$. So we have explicitly

$$R_0(s) = \frac{t_0}{s} \quad (s > 0),$$

where t_0 is the unique positive solution of $t = e^{-t}$ ($t_0 \doteqdot 0.567$).
When $\alpha \neq 0$, we have

$$R_\alpha(s) \sim \frac{t_0}{s} \quad (s \to \infty),$$

where, $f \sim g \underset{\text{def}}{\Leftrightarrow} f/g \to 1$.

Asymptotics of IDS for Poisson PI (1)

Theorem 6 (Kaminaga–M–Nakano, submitted)

Let $d = 3$. Suppose that α is a constant sequence, and μ satisfies (PP). Then we have

$$N_\alpha(-s^2) = \frac{2\pi}{3}\rho^2 R_\alpha(s)^3 + O(s^{-6+\epsilon}) \quad (s \to \infty), \tag{6}$$

for every $0 < \epsilon < 3$. In particular, the principal term is

$$N_\alpha(-s^2) \sim \frac{2\pi}{3}\rho^2 t_0^3 s^{-3} \quad (s \to \infty). \tag{7}$$

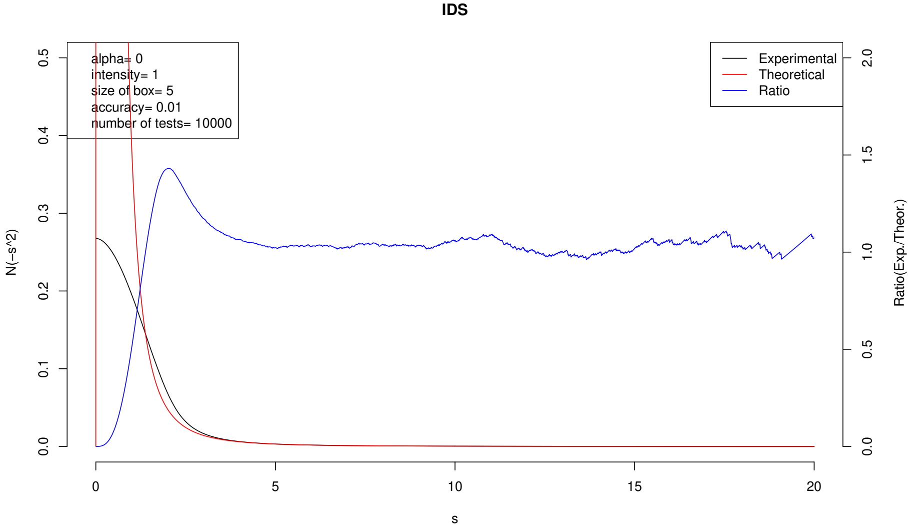
Asymptotics of IDS for Poisson PI (3)

Theorem 6 says $N(\lambda)$ decays polynomially

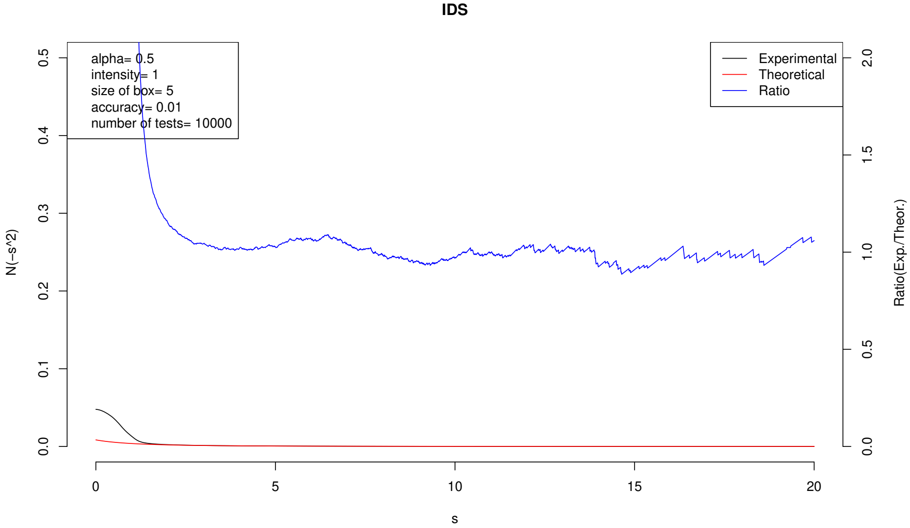
$$N_\alpha(\lambda) = O(|\lambda|^{-3/2}) \quad (\lambda \to -\infty).$$

The principal term given in (7) is independent of α. When we calculate IDS numerically, the first term in RHS of (6) gives more accurate approximation.

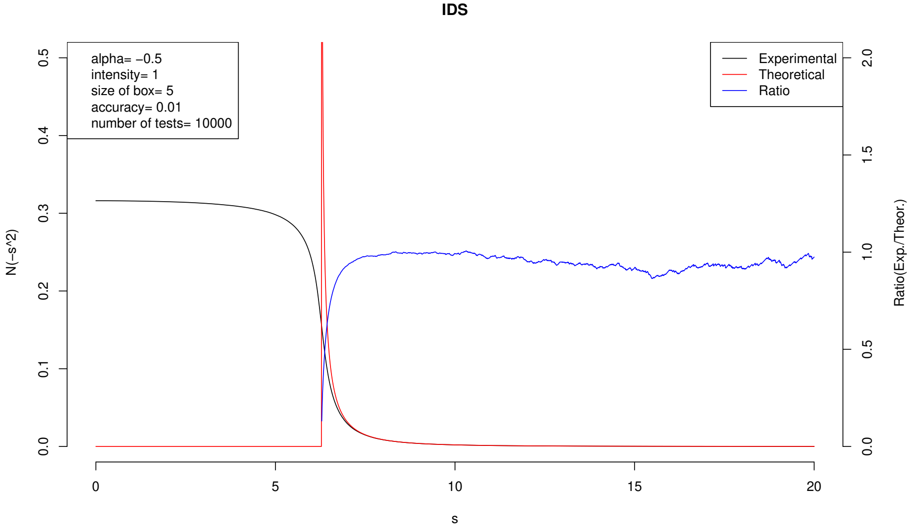
Numerical result ($\alpha = 0.0$, 10000 tests)



Numerical result ($\alpha = 0.5$, 10000 tests)



Numerical result ($\alpha = -0.5$, 10000 tests)



Asymptotics of IDS for Determinantal PI

Theorem 7

Let $d = 3$. Suppose α is a constant sequence and (DP) holds $(K(x, y) = \rho e^{-|x-y|^2/\beta^2})$. Then we have for every $0 < \epsilon < 5$

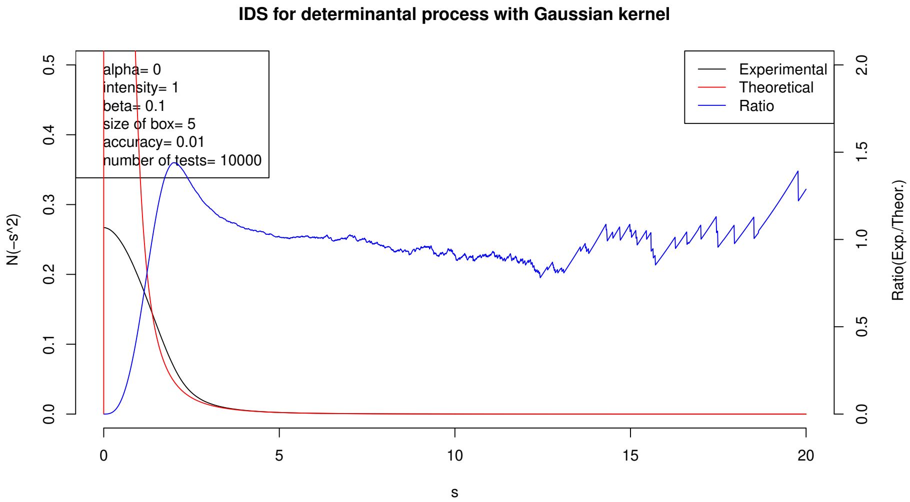
$$N_\alpha(-s^2) = 2\pi\rho^2 \cdot f(R_\alpha(s)) + O(s^{-10+\epsilon}) \quad (s \to \infty),$$
$$f(R) = \int_0^R r^2 \left(1 - e^{-\frac{2r^2}{\beta^2}}\right) dr. \tag{8}$$

The principal term is given by

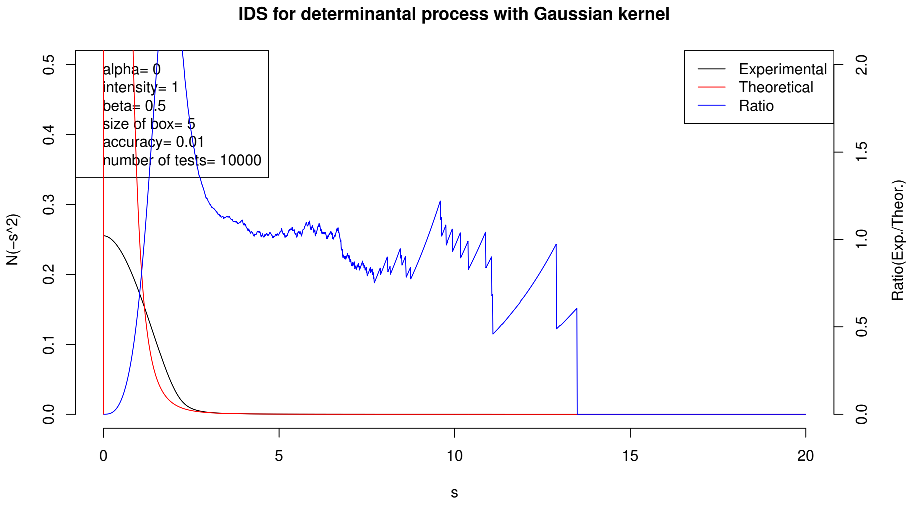
$$N(-s^2) \sim \frac{4\pi\rho^2 t_0^5}{5\beta^2} s^{-5} \quad (s \to \infty). \tag{9}$$

Comparing (9) with (7), we see that the IDS for (DP) decays faster than the IDS for (PP).

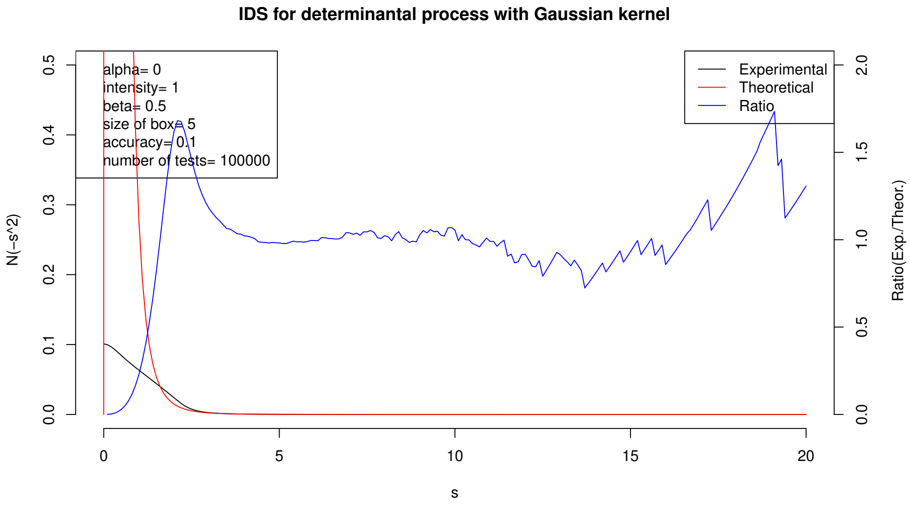
Numerical result ($\rho = 1$, $\beta = 0.1$, 10000 tests)



Numerical result ($\rho = 1$, $\beta = 0.5$, 10000 tests)



Numerical result ($\rho = 1$, $\beta = 0.5$, 100000 tests)



Spectrum of $-\Delta_{\alpha,\xi}$ for finite ξ

The following result is taken from the book of Albeverio et al.

Proposition 8 (Spectrum of $-\Delta_{\alpha,\xi}$ for finite ξ)

Let $d = 3$. Let $\xi = \{y_j\}_{j=1}^N$ be a finite set and $\alpha = (\alpha_j)_{j=1}^N$ ($\alpha_j = \alpha_{y_j}$). Then, for $\lambda = -s^2$ ($s > 0$), λ is an eigenvalue of $-\Delta_{\alpha,\xi}$ if and only if $\det A(s) = 0$, where $A(s) = (a_{jk}(s))$ is the $N \times N$ matrix given by

$$a_{jk}(s) = \begin{cases} \alpha_j + \dfrac{s}{4\pi} & (j = k), \\ -\dfrac{e^{-s|y_j - y_k|}}{4\pi|y_j - y_k|} & (j \neq k). \end{cases}$$

Spectrum of $-\Delta_{\alpha,\xi}$ for $\#\xi = 1$

In the case $\#\xi = 1$ and $\alpha_1 = \alpha$, $\lambda = -s^2$ $(s > 0)$ is an eigenvalue of $-\Delta_{\alpha,\xi}$ if and only if

$$\alpha + \frac{s}{4\pi} = 0.$$

Thus

$$\sigma(-\Delta_{\alpha,\xi}) \cap (-\infty, 0) = \begin{cases} \{-(4\pi\alpha)^2\} & (\alpha < 0), \\ \emptyset & (\alpha \geq 0). \end{cases}$$

Spectrum of $-\Delta_{\alpha,\xi}$ for $\#\xi = 2$

In the case $\#\xi = 2$, $|y_1 - y_2| = R$ and $\alpha_1 = \alpha_2 = \alpha$, $\lambda = -s^2$ $(s > 0)$ is an eigenvalue of $-\Delta_{\alpha,\xi}$ if and only if 0 is an eigenvalue of

$$A(s) = \begin{pmatrix} \alpha + \dfrac{s}{4\pi} & -\dfrac{e^{-sR}}{4\pi R} \\ -\dfrac{e^{-sR}}{4\pi R} & \alpha + \dfrac{s}{4\pi} \end{pmatrix},$$

that is,

$$\alpha + \frac{1}{4\pi}\left(s + \frac{e^{-sR}}{R}\right) = 0 \Leftrightarrow s + \frac{e^{-sR}}{R} = -4\pi\alpha,$$
$$\alpha + \frac{1}{4\pi}\left(s - \frac{e^{-sR}}{R}\right) = 0 \Leftrightarrow s - \frac{e^{-sR}}{R} = -4\pi\alpha.$$

The second equation is the defining equation of $R_\alpha(s)$, and $R_\alpha(s) \to 0$ as $s \to \infty$.

Asymptotics of IDS and close pairs (1)

The calculus in the previous slide roughly suggests that 'if we find a close pair $\{y_1, y_2\} \subset \xi$ with $|y_1 - y_2| < R_\alpha(s)$, then we find an eigenvalue less than $-s^2$ of $-\Delta_{\alpha,\xi}$'. From this consideration, we have

$$
\begin{aligned}
&N_\alpha(-s^2) \\
&\sim \frac{1}{|C_L|} \times \frac{1}{2}\mathbf{E}\left[\#\{y \in \xi \cap C_L\,;\, n_y(R_\alpha(s)) = 2\}\right] \;\; (s \to \infty), \\
&n_y(R) = \#(\xi \cap B_y(R)), \\
&B_y(R) = \{x \in \mathbf{R}^3;\; |x - y| < R\}.
\end{aligned}
$$

Thus the problem is reduced to 'calculate the expectation of the number of close pairs'.

Asymptotics of IDS and close pairs (2)

Lemma 9

Let $d = 3$. Let α is a constant sequence, and μ satisfies (PP). Then, for every δ with $1/2 < \delta < 1$, and for every $m > 0$, there exist constants $R_0 > 0$ and $C > 0$ such that

$$\begin{aligned} &\mathbf{E}[N_{C_L}(-(s_\alpha(R) - R^m))^2] \\ &\geq \frac{1}{2}\mathbf{E}[\#\{y \in \xi \cap C_L\,;\, n_y(R) = 2\}] - CR^{6\delta}|C_L|, \\ &\mathbf{E}[N_{C_L}(-(s_\alpha(R) + R^m))^2] \\ &\leq \frac{1}{2}\mathbf{E}[\#\{y \in \xi \cap C_L\,;\, n_y(R) = 2\}] + CR^{6\delta}|C_L|, \end{aligned}$$

for every $0 < R < R_0$ and every $L > R^{-2}$, where $s = s_\alpha(R)$ is the inverse function of $R = R_\alpha(s)$.

If we assume (DP), then $R^{6\delta}$ becomes $R^{10\delta}$.

Number of close pairs for Poisson PP

Proposition 10

Let $d = 1, 2, 3, \ldots$. Let μ be the Poisson point process in $\mathbf{R}^d$ with intensity measure ρdx, where $\rho > 0$ is a constant. For $L > 0$, let $C_L = (0, L)^d$. Then, we have for $n = 1, 2, 3, \ldots$

$$\frac{\mathbf{E}[\#\{y \in \xi \cap C_L \,;\, n_y(R) = n\}]}{|C_L|} = \frac{1}{(n-1)!} |B_0(R)|^{n-1} \rho^n e^{-\rho |B_0(R)|}.$$

In particular, when $d = 3$, $n = 2$, and $R = R_\alpha(s)$, we have

$$\frac{1}{2} \frac{\mathbf{E}[\#\{y \in \xi \cap C_L \,;\, n_y(R_\alpha(s)) = 2\}]}{|C_L|} \doteqdot \frac{2\pi R_\alpha(s)^3}{3} \rho^2,$$

which is the first term in our result (6).

Proof of Proposition 10 (1)

(Proof) For a point process μ (probability measure on the configuration space Q) and $x \in \mathbf{R}^d$, the reduced Palm measure $\mu_x^!$ is a probability measure on $Q \setminus \{x\}$ (the configuration space on $\mathbf{R}^d \setminus \{x\}$), defined by the formula

$$\int_Q \left(\sum_{x \in \xi} g(x, \xi) \right) d\mu(\xi) = \int_{\mathbf{R}^d} \mathbf{E}_x \left[g(x, \xi \cup \{x\}) \right] d\rho_1(x),$$

$$\mathbf{E}_x \left[g(x, \xi \cup \{x\}) \right] = \int_{Q \setminus \{x\}} g(x, \xi \cup \{x\}) d\mu_x^!(\xi),$$

$$\rho_1(E) = \int_Q \#(E \cap \xi) d\mu(\xi) \quad \text{(intensity measure)},$$

for any non-negative measurable function g on $\mathbf{R}^d \times Q$. The reduced Palm measure $\mu_x^!$ is considered to be the conditional distribution on $Q \setminus \{x\}$ under the condition $x \in \xi$.

Proof of Proposition 10 (2)

If μ is the Poisson PP with intensity ρdx, then it is well-known that $\rho_1(dx) = \rho dx$ and $\mu_x^! = \mu$ (since the configuration on x and the configuration on $\mathbf{R}^d \setminus \{x\}$ are independent). We take

$$g(x,\xi) = \begin{cases} 1 & (x \in C_L,\ n_x(R) = n), \\ 0 & (\text{otherwise}), \end{cases}$$

where $n_x(R) = \#(\xi \cap B_x(R))$. Then we have

$$\begin{aligned} & \mathbf{E}\left[\#\{x \in \xi \cap C_L ; n_x(R) = n\}\right] \\ &= \int_{C_L} \mu\{\#(B_x(R) \cap (\{x\} \cup \xi)) = n\} \rho dx \\ &= \int_{C_L} \mu\{\#(B_x(R) \cap \xi) = n-1\} \rho dx \\ &= \rho |C_L| \cdot \frac{(\rho |B_0(R)|)^{n-1}}{(n-1)!} e^{-\rho |B_0(R)|}. \qquad \square \end{aligned}$$

Palm measure for determinantal PP

Theorem 11 (Shirai–Takahashi 2003, Theorem 1.7)

Let $\mu_{-1,K}$ be the determinantal point process with kernel $K(x,y)$. If $K(x_0,x_0) > 0$, then the reduced Palm measure $\mu^!_{x_0}$ is the determinantal point process $\mu_{-1,K^{x_0}}$ with kernel $K^{x_0}(x,y)$, where

$$K^{x_0}(x,y) = \frac{1}{K(x_0,x_0)} \det \begin{pmatrix} K(x,y) & K(x,x_0) \\ K(x_0,y) & K(x_0,x_0) \end{pmatrix}.$$

If $K(x,y) = \rho e^{-|x-y|^2/\beta^2}$, then we have

$$\begin{aligned} K^{x_0}(x,y) &= \frac{1}{\rho} \begin{vmatrix} \rho e^{-|x-y|^2/\beta^2} & \rho e^{-|x-x_0|^2/\beta^2} \\ \rho e^{-|x_0-y|^2/\beta^2} & \rho \end{vmatrix} \\ &= \rho \left\{ \exp\left(-\frac{|x-y|^2}{\beta^2}\right) - \exp\left(-\frac{|x-x_0|^2 + |x_0-y|^2}{\beta^2}\right) \right\}. \end{aligned}$$

Number of close pairs for determinantal PP

Proposition 12

Let $d = 1, 2, \ldots$. Let $\mu_{-1,K}$ be the determinantal point process with kernel $K(x, y) = \rho e^{-|x-y|^2/\beta^2}$, and ρ, β are positive constants with $\rho(\beta^2\pi)^{d/2} \le 1$. Then, we have

$$\begin{aligned}
&\mathbf{E}\left[\#\{x \in \xi \cap C_L; n_x(R) = 2\}\right] \\
&= \rho^2 |C_L| \left|S^{d-1}\right| f(R)\big(1 + O(R^{d+2})\big) \quad (R \to +0),
\end{aligned}$$

where $|S^{d-1}|$ is the surface volume of S^{d-1}, and

$$\begin{aligned}
f(R) &= \int_0^R r^{d-1}\left(1 - e^{-\frac{2r^2}{\beta^2}}\right) dr \\
&= \frac{2R^{d+2}}{(d+2)\beta^2} + O(R^{d+4}) \quad (R \to +0).
\end{aligned}$$

Proof of Proposition 12 (1)

(Proof) As in the proof of Proposition 10,

$$\mathbf{E}\left[\#\{x \in \xi \cap C_L; n_x(R) = 2\}\right]$$
$$= \int_{C_L} \mu_{-1,K^{x_0}}\{\#(B_{x_0}(R) \cap \xi) = 1\} d\rho_{1,-1,K}(x_0). \tag{7}$$

The intensity measure $d\rho_{1,-1,K}(x_0)$ is given by $K(x_0, x_0)dx_0 = \rho dx_0$. The probability $\mu_{-1,K^{x_0}}\{\cdots\}$ can be calculated by the following.

Probability distribution of $\#(\xi \cap \Lambda)$

Lemma 13

For a bounded measurable set Λ and $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}&\mu_{-1,K}\{\#(\xi \cap \Lambda) = n\}\\ = &\operatorname{Det}(I - K_\Lambda) \operatorname{Tr}\big(\wedge^n((I - K_\Lambda)^{-1} K_\Lambda)\big),\end{aligned}$$

where $K_\Lambda = \chi_\Lambda K \chi_\Lambda$ and $\operatorname{Tr}(\wedge^0 \cdots) = 1$. In particular,

$$\begin{aligned}&\mu_{-1,K}(\{\#(\xi \cap \Lambda) = 1\})\\ = &\operatorname{Det}(I - K_\Lambda) \cdot \operatorname{Tr}\left((I - K_\Lambda)^{-1} K_\Lambda\right).\end{aligned}$$

From this formula, we have the bound

$$\mu_{-1,K}\{\#(\xi \cap \Lambda) = n\} \leq \operatorname{Det}(I - K_\Lambda) \frac{\left\{\operatorname{Tr}\big((I - K_\Lambda)^{-1} K_\Lambda\big)\right\}^n}{n!}.$$

Proof of Proposition 12 (2)

By Lemma 13, we have

$$\mu_{-1,K^{x_0}}(\{\#(\xi \cap B_{x_0}(R)) = 1\})$$
$$= \operatorname{Det}(I - K^{x_0}_{B_{x_0}(R)}) \cdot \operatorname{Tr}((I - K^{x_0}_{B_{x_0}(R)})^{-1} K^{x_0}_{B_{x_0}(R)}),$$

where $K^{x_0}_{B_{x_0}(R)}$ is an integral operator with kernel

$$K^{x_0}_{B_{x_0}(R)}(x, y)$$
$$= \rho \chi_{B_{x_0}(R)}(x) \chi_{B_{x_0}(R)}(y)$$
$$\cdot \left\{ \exp\left(-\frac{|x-y|^2}{\beta^2} \right) - \exp\left(-\frac{|x-x_0|^2 + |x_0 - y|^2}{\beta^2} \right) \right\}.$$

The operator $K^{x_0}_{B_{x_0}(R)}$ is a non-negative, trace-class operator.

Proof of Proposition 12 (3)

Let $\|\cdot\|_1$ be the trace norm. Since $K^{x_0}_{B_{x_0}(R)}$ is non-negative, we have

$$
\begin{aligned}
\|K^{x_0}_{B_{x_0}(R)}\|_1 &= \mathrm{Tr}(K^{x_0}_{B_{x_0}(R)}) \\
&= \int_{B_{x_0}(R)} K^{x_0}_{B_{x_0}(R)}(x,x)dx \\
&= \rho \int_{B_0(R)} \left\{ 1 - \exp\left(-\frac{2|z|^2}{\beta^2} \right) \right\} dz \quad (z = x - x_0) \\
&= \rho \left| S^{d-1} \right| \int_0^R r^{d-1} \left(1 - e^{-\frac{2r^2}{\beta^2}} \right) dr = \rho \left| S^{d-1} \right| f(R), \\
f(R) &= \int_0^R \left(\frac{2r^{d+1}}{\beta^2} + O(r^{d+3}) \right) \\
&= \frac{2R^{d+2}}{(d+2)\beta^2} + O(R^{d+4}) \quad (R \to +0).
\end{aligned}
$$

Proof of Proposition 12 (4)

By Lemma 2.1 of Shirai–Takahashi 2003,

$$\mathrm{Det}(I - K^{x_0}_{B_{x_0}(R)}) = 1 + \sum_{n=1}^{\infty} (-1)^n \,\mathrm{Tr}(\wedge^n K^{x_0}_{B_{x_0}(R)}),$$

$$\| \wedge^n K^{x_0}_{B_{x_0}(R)} \|_1 \leq \frac{1}{n!} \| K^{x_0}_{B_{x_0}(R)} \|_1^n.$$

Thus we have

$$\mathrm{Det}(I - K^{x_0}_{B_{x_0}(R)}) = 1 + O(R^{d+2}) \quad (R \to +0).$$

Moreover

$$(I - K^{x_0}_{B_{x_0}(R)})^{-1} = I + (I - K^{x_0}_{B_{x_0}(R)})^{-1} K^{x_0}_{B_{x_0}(R)},$$

$$\|(I - K^{x_0}_{B_{x_0}(R)})^{-1} K^{x_0}_{B_{x_0}(R)}\| \leq \sum_{n=1}^{\infty} \| K^{x_0}_{B_{x_0}(R)} \|^n = O(R^{d+2}) \quad (R \to +0).$$

Proof of Proposition 12 (5)

By these formulas, we have

$$\begin{aligned} &\mu_{-1,K^{x_0}}(\{\#(\xi \cap B_{x_0}(R)) = 1\}) \\ &= \rho \left| S^{d-1} \right| f(R)\big(1 + O(R^{d+2})\big) \quad (R \to +0). \end{aligned} \tag{8}$$

Substituting (8) into (7), we have

$$\begin{aligned} &\mathbf{E}\left[\#\{x \in \xi \cap C_L; n_x(R) = 2\}\right] \\ &= \rho^2 |C_L| \left| S^{d-1} \right| f(R)\big(1 + O(R^{d+2})\big) \\ &= \rho^2 |C_L| \left| S^{d-1} \right| \left(\frac{2R^{d+2}}{(d+2)\beta^2} + O(R^{d+4}) \right) \quad (R \to +0). \quad \square \end{aligned}$$

We see that the decay rate $O(R^{d+2})$ is faster than that in the Poisson case ($O(R^d)$). This facts also leads us faster decay of IDS $N(-s^2)$ as $s \to \infty$.

Proof of Lemma 13 (1)

(Proof) Assume that we know the Laplace transform

$$\int_Q \mu(d\xi)\exp(-\langle \xi, f\rangle), \quad \langle \xi, f\rangle = \sum_{x\in\xi} f(x).$$

of a point process μ. For a bounded measurable set Λ, take $f = t\chi_\Lambda$. Then the left hand side becomes

$$\int_Q \mu(d\xi)\exp(-t\#(\xi\cap\Lambda)) = \sum_{n=0}^{\infty} \mu\{\#(\xi\cap\Lambda) = n\}e^{-nt}. \quad (9)$$

So we have to calculate the expansion of the above Laplace transform for $\mu = \mu_{-1,K^{x_0}}$ with respect to e^{-t}.

Proof of Lemma 13 (2)

For a general DPP $\mu_{-1,K}$, we have by definition

$$\int_Q \mu_{-1,K}(d\xi) \exp(-\langle \xi, t\chi_\Lambda \rangle) = \mathrm{Det}(I - K_\varphi),$$
$$K_\varphi(x,y) = \sqrt{\varphi(x)} K(x,y) \sqrt{\varphi(y)},$$
$$\varphi(x) = 1 - \exp(-t\chi_\Lambda(x)) = (1 - e^{-t})\chi_\Lambda(x).$$

Thus we have

$$K_\varphi = (1 - e^{-t})K_\Lambda, \quad K_\Lambda = \chi_\Lambda K \chi_\Lambda.$$
$$\begin{aligned} \mathrm{Det}(I - K_\varphi) &= \mathrm{Det}(I - (1 - e^{-t})K_\Lambda) \\ &= \mathrm{Det}(I - K_\Lambda) \, \mathrm{Det}\left(I + e^{-t}(I - K_\Lambda)^{-1} K_\Lambda\right). \end{aligned}$$

Proof of Lemma 13 (3)

We use the expansion of the Fredholm determinant (Proposition 2.1 of Shirai–Takahashi 2003):

$$\mathrm{Det}(I+T) = 1 + \sum_{n=1}^{\infty} \mathrm{Tr}(\wedge^n T),$$

for a trace class operator T on a Hilbert space $\mathcal{H}$, where the operator $\wedge^n T$ on $\wedge^n \mathcal{H}$ is defined by

$$\wedge^n T(e_{i_1} \wedge \cdots \wedge e_{i_n}) = Te_{i_1} \wedge \cdots \wedge Te_{i_n} \quad (i_1 < \cdots < i_n)$$

for an ONB $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{H}$.

Proof of Lemma 13 (4)

Thus we have

$$\begin{aligned}
&\mathrm{Det}(I + e^{-t}(I - K_\Lambda)^{-1}K_\Lambda) \\
= {}& 1 + \sum_{n=1}^{\infty} e^{-nt}\,\mathrm{Tr}\big(\wedge^n((I - K_\Lambda)^{-1}K_\Lambda)\big), \\
&\mathrm{Det}(I - K_\varphi) \\
= {}& \mathrm{Det}(I - K_\Lambda)\left(1 + \sum_{n=1}^{\infty} e^{-nt}\,\mathrm{Tr}\big(\wedge^n((I - K_\Lambda)^{-1}K_\Lambda)\big)\right).
\end{aligned}$$

Comparing with (9), we have

$$\mu_{-1,K}\{\#(\xi \cap \Lambda) = n\} = \mathrm{Det}(I - K_\Lambda)\,\mathrm{Tr}\big(\wedge^n((I - K_\Lambda)^{-1}K_\Lambda)\big)$$

for every $n \in \mathbf{N}$. □

Further problems

There are many remaining problems.

(i) Other kernel K?

(ii) Asymptotics of IDS for the Schrödinger operator with point interactions on α-determinantal process?

(iii) Two-dimensional case?

(iv) IDS for hard core process (Lifshitz tail)?

(v) Anderson localization?

(vi) Level statistics?

(vii) Distribution of resonances?

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