Letters in Mathematical Physics (2025) 115:90
https://doi.org/10.1007/s11005-025-01981-6

Self-adjointness of unbounded time operators

Fumio Hiroshima[1] · **Noriaki Teranishi**[2,3]

Received: 1 March 2025 / Revised: 24 July 2025 / Accepted: 25 July 2025

Abstract
Time operators associated with an abstract semi-bounded self-adjoint operator H possessing a purely discrete spectrum are considered. The existence of a bounded self-adjoint time operator T for such an operator H is known as the Galapon time operator. In this paper, we construct a self-adjoint but *unbounded* time operator T for H with a dense CCR-domain, thereby extending the framework beyond the bounded setting.

Keywords Canonical commutation relation · Self-adjointness · Time operators

Mathematics Subject Classification 81Q10 · 47N50

1 Introduction

We begin by providing the definitions of conjugate operators and time operators as employed in this paper. Let $\mathrm{D}(T)$ denote the domain of the operator T and let $[A, B] = AB - BA$ denote the commutator of A and B on $\mathrm{D}(AB) \cap \mathrm{D}(BA)$ throughout this paper.

Definition 1.1 Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Suppose that T satisfies the canonical commutation relation

$$[H, T] = -i\mathbb{1}$$

on a subset $D_{H,T} \subset \mathrm{D}(HT) \cap \mathrm{D}(TH)$ with $D_{H,T} \neq \{0\}$.

Then, T is called a conjugate operator of H and $D_{H,T}$ is referred to as a CCR-domain. If a conjugate operator T of H is symmetric, then T is called a time operator of H.

✉ Noriaki Teranishi
teranishi@math.sci.hokudai.ac.jp

1 Faculty of Mathematics, Kyushu University, Fukuoka, Japan

2 Faculty of Science, Department of Mathematics, Hokkaido University, Sapporo, Japan

3 Higher Education Support Center, Hokkaido University of Science, Sapporo, Japan

Published online: 20 August 2025

Consequently, the domain of a conjugate operator is not necessarily dense, whereas that of a time operator is dense. Moreover, CCR-domains are not dense in general. It is well known that CCR-domain $D_{H,T}$ does not contain any eigenvector e of H, since $e \notin \mathrm{D}(T)$ or $Te \notin \mathrm{D}(H)$ hold. Therefore, we must pay careful attention to domains. Note that conjugate and time operators associated with a given self-adjoint operator H are, in general, not unique. Time operators have been frequently discussed in the context of physics since early times, as exemplified by historical works [1, 8, 11, 14, 15, 18, 20, 21]. In contrast, time operators of Hamiltonians with purely discrete spectra are studied in [3, 5, 9, 10, 12, 22]. In particular, Galapon [12] and Arai-Matsuzawa [5] have inspired our research on time operators. Motivated by these works, we have recently constructed and studied time and conjugate operators for 1D-harmonic oscillator in [16, 17] and the present paper is a continuation of those studies.

In addition, the definition of the so-called *strong* time operators is given as follows. A symmetric operator T on $\mathcal{H}$ is said to be a strong time operator of a self-adjoint operator H on $\mathcal{H}$ if (1) and (2) are satisfied:

(1) $e^{-itH}\mathrm{D}(T) \subset \mathrm{D}(T)$ for all $t \in \mathbb{R}$;
(2) $Te^{-itH}\psi = e^{-itH}(T+t)\psi$ for all $\psi \in \mathrm{D}(T)$ and all $t \in \mathbb{R}$.

The relation (2) is called the weak Weyl relation. Strong time operators were introduced in [19], their spectral properties were studied in [2, 5–7, 19], and a comprehensive investigation is summarized in [4, Chapter 4]. A strong time operator of $p^2/2$ is given by the so-called Aharonov–Bohm time operator T_{AB} defined by $T_{\mathrm{AB}} = (p^{-1}q + qp^{-1})/2$. We refer the reader to [1]. The weak Weyl relation implies the canonical commutation relation

$$[H, \bar{T}] = -i\mathbb{1} \tag{1.1}$$

on $\mathrm{D}(H\bar{T}) \cap \mathrm{D}(\bar{T}H)$, and $\mathrm{D}(H\bar{T}) \cap \mathrm{D}(\bar{T}H)$ is dense. Here $\bar{T}$ denotes the closure of T. See [4, Proposition 4.5]. Therefore, the closure of every strong time operator is automatically a time operator by (1.1). It is noteworthy that if a self-adjoint operator H admits a strong time operator, then the spectrum of H is purely absolutely continuous. See [19] and [4, Theorem 4.8]. In particular, H has no point spectrum, and hence 1D-harmonic oscillator does not admit a strong time operator, since the spectrum of 1D-harmonic oscillator is purely discrete.

The self-adjointness of time operators constitutes a particularly compelling subject of study, not only from a mathematical standpoint but also from a physical perspective. Interestingly, it is known that if the self-adjoint operator H is bounded from below and has purely absolutely continuous spectrum, then a strong time operator associated with H is neither self-adjoint nor essentially self-adjoint. We refer the reader to [19] and [4, Theorem 4.10]. Nevertheless, it is known that the Galapon time operator defined below for the 1D-harmonic oscillator is a bounded self-adjoint operator, whose CCR-domain is moreover dense in the Hilbert space.

Now we introduce a Galapon time operator for an abstract self-adjoint operator H and establish some fundamental results. Throughout this paper, we adopt the following assumption unless otherwise stated.

Assumption 1.2 Suppose that H is a positive and unbounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$, $\sigma(H)$ consists of only simple eigenvalues and H^{-1} is Hilbert–Schmidt.

Let e_n be an eigenvector of H corresponding to the n-th eigenvalue E_n, for each $n \in \mathbb{N}$. Note that $0 < E_n < E_{n+1}$ and

$$\sum_{n=0}^{\infty} \frac{1}{E_n^2} < \infty. \tag{1.2}$$

Define the Galapon time operator T_{G} associated with an abstract self-adjoint operator H by

$$\begin{aligned} &\mathrm{D}(T_{\mathrm{G}}) = \mathrm{LH}\{e_n \mid n \in \mathbb{N}\}, \\ &T_{\mathrm{G}}\varphi = i \sum_{n=0}^{\infty} \left(\sum_{m \neq n} \frac{(e_m, \varphi)}{E_n - E_m} \right) e_n, \quad \varphi \in \mathrm{D}(T). \end{aligned}$$

Here, for a subset $\mathcal{A}$ of $\mathcal{H}$, LH$\mathcal{A}$ means the linear hull of $\mathcal{A}$. The following proposition is proven in [5, 13]:

Proposition 1.3 *The operator T_{G} is a time operator of H and its CCR-domain is given by* $\mathrm{LH}\{e_n - e_m \mid n, m \in \mathbb{N}\}$.

Let $(p^2 + q^2 - \mathbb{1})/2$ be 1D-harmonic oscillator. The Galapon time operator associated with $(p^2 + q^2 - \mathbb{1})/2$ is given by

$$T_{\mathrm{G}}\varphi = i \sum_{n=0}^{\infty} \left(\sum_{m \neq n} \frac{(e_m, \varphi)}{n - m} \right) e_n, \quad \varphi \in \mathrm{D}(T_{\mathrm{G}}).$$

It can be shown that T_{G} is both bounded and self-adjoint [5]. The proof is straightforward: It is immediate to verify that T_{G} is symmetric, and by virtue of the Hilbert inequality

$$\left| \sum_{n=0}^{\infty} \sum_{m \neq n} \frac{x_n y_m}{n - m} \right| \leq \pi \left(\sum_{n=0}^{\infty} x_n^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} y_n^2 \right)^{1/2},$$

one derives the following key inequality:

$$\|T_{\mathrm{G}}\varphi\| \leq \pi \|\varphi\|. \tag{1.3}$$

Hence, T_{G} is bounded, and therefore self-adjoint. We emphasize that the self-adjointness follows from the boundedness, since it is symmetric. Now let us consider the self-adjoint operator H so that $\sigma(H) = \{n^\lambda \mid n \in \mathbb{N}\}$ for some $1/2 < \lambda < 1$. One might expect that

$$T_{\mathrm{G}}\varphi = i\sum_{n=0}^{\infty}\left(\sum_{m\neq n}\frac{(e_m,\varphi)}{n^{\lambda}-m^{\lambda}}\right)e_n \tag{1.4}$$

remains self-adjoint in this setting, as in the case $\lambda = 1$, but the self-adjointness of (1.4) could not been proven so far. We note that T_{G} defined by (1.4) is unbounded, as established in [5]. Although (1.4) is a candidate for an unbounded self-adjoint time operator with a dense CCR-domain, to the best of our knowledge, no example of an *unbounded* self-adjoint time operator with a dense CCR-domain has been found so far. A central motivation for this paper lies in the fact that there is currently no known proof of the self-adjointness of Galapon-type time operators that does not rely on boundedness.

In our previous paper [16], we constructed three distinct classes $\mathcal{T}_{\{0\}}$, $\mathcal{T}_{\mathbb{D}\setminus\{0\}}$, $\mathcal{T}_{\partial\mathbb{D}}$ of conjugate operators for 1D-harmonic oscillator. Each element of these classes is denoted by $T_{m,\omega}$ with two parameters $\omega \in \mathbb{C}$ and $m \in \mathbb{N}$. $\mathcal{T}_{\{0\}}$ consists of $T_{0,m}$, and $\mathcal{T}_{\mathbb{D}\setminus\{0\}}$ consists of $T_{\omega,m}$ with $0 < |\omega| < 1$. $\mathcal{T}_{\partial\mathbb{D}}$ consists of $T_{\omega,m}$ with $|\omega| = 1$, which are extensions of the Galapon time operator. Here, the subscript $\partial\mathbb{D}$ indicates that parameter ω are included in the unit circle in $\mathbb{C}$. In fact $T_{1,1} + T_{1,1}^{*}$ coincides with T_{G} defined in (1.3). Although the time operators $T_{m,\omega} \in \mathcal{T}_{\partial\mathbb{D}}$ are bounded and self-adjoint, the time and conjugate operators included in $\mathcal{T}_{\{0\}}$ and $\mathcal{T}_{\mathbb{D}\setminus\{0\}}$ are not self-adjoint.

Motivated by the above considerations, the purpose of this paper is to construct a time operator that is both unbounded and self-adjoint with a dense CCR-domain. This paper is organized as follows: In Section 2, we unitarily transform T_{G} into an operator T_f on $\ell^2(\mathbb{N})$. Section 3 is devoted to constructing unbounded self-adjoint time operators with a dense CCR-domain. Section 4 deals with self-adjoint extensions of time operators. The main results are presented in Theorems 3.13.

2 Galapon time operator on $\ell^2(\mathbb{N})$

We denote the set of all square summable functions on $\mathbb{N}$ by $\ell^2(\mathbb{N})$. In this paper, the investigation of time operators is carried out on $\ell^2(\mathbb{N})$ instead of a given separable Hilbert space $\mathcal{H}$. As a first step, we show that T_{G} is unitarily equivalent to an operator T_f on $\ell^2(\mathbb{N})$. Let $\xi_n \in \ell^2(\mathbb{N})$ be the function on $\mathbb{N}$ defined by

$$\xi_n(m) = \delta_{nm}, \quad m \in \mathbb{N},$$

where δ_{nm} denotes the Kronecker delta function. We write $\ell^2_{\mathrm{fin}}(\mathbb{N})$ for the set of $\varphi \in \ell^2(\mathbb{N})$ with a finite support, i.e., there exist $m \in \mathbb{N}$ and $(c_n)_{n=0}^{m} \in \mathbb{C}^{m+1}$ such that φ can be expressed as $\sum_{n=0}^{m} c_n\xi_n$. Note that $\ell^2_{\mathrm{fin}}(\mathbb{N})$ is dense in $\ell^2(\mathbb{N})$. Let L be the left shift operator on $\ell^2(\mathbb{N})$:

$$L\xi_n = \begin{cases}\xi_{n-1} & (n \geq 1),\\ 0 & (n = 0).\end{cases}$$

The adjoint operator L^* of L is given by

$$L^*\xi_n = \xi_{n+1}, \quad n \in \mathbb{N}.$$

Let N be the number operator on $\ell^2(\mathbb{N})$. Then $N\xi_n = n\xi_n$ for $n \in \mathbb{N}$. It is well known that N is a self-adjoint operator, $\ell^2_{\text{fin}}(\mathbb{N})$ is a core for N, and N satisfies commutation relations: $[N, L] = -L$ and $[N, L^*] = L^*$ on $\ell^2_{\text{fin}}(\mathbb{N})$. We introduce sets $\mathcal{K}$ and $\mathcal{K}^-$ as follows.

Definition 2.1 ($\mathcal{K}$ and $\mathcal{K}^-$) We denote by $\mathcal{K}$ the set of all real valued functions on $\mathbb{N}$ which satisfy the following conditions:

(1) $f(0) > 0$,
(2) $f(n) < f(n+1)$ for all $n \in \mathbb{N}$.

Set $\mathcal{K}^- = \left\{ f \in \mathcal{K} \;\middle|\; 1/f \in \ell^2(\mathbb{N}) \right\}$.

To define T_f for $f \in \mathcal{K}$, we set

$$\Delta_k(f, n) = f(n+k) - f(n).$$

Lemma 2.2 *Let $f \in \mathcal{K}$. Then $\ell^2_{\text{fin}}(\mathbb{N}) \subset \mathrm{D}\big(\Delta_k(f, N)^{-1}\big)$ for every natural number $k \geq 1$.*

Proof Since f is strictly increasing, $\Delta_k(f, N)$ is injective. Clearly $\ell^2_{\text{fin}}(\mathbb{N}) \subset \mathrm{D}(\Delta_k(f, N))$ and ξ_n is an eigenvector of $\Delta_k(f, N)$:

$$\Delta_k(f, N)\xi_n = \Delta_k(f, n)\xi_n.$$

This implies that $\ell^2_{\text{fin}}(\mathbb{N}) \subset \mathrm{D}\big(\Delta_k(f, N)^{-1}\big)$. □

Remark 2.3 Note that, for any $f \in \mathcal{K}$, $\inf_{n\in\mathbb{N}} \Delta_k(f, n) > 0$ if and only if $\Delta_k(f, N)^{-1}$ is a bounded operator.

Definition 2.4 Let $f \in \mathcal{K}$. We define operators $T_{f,m}$ and T_f on $\ell^2(\mathbb{N})$ by

$$T_{f,m} = i \sum_{k=1}^{m} \left(L^{*k} \Delta_k(f, N)^{-1} - \Delta_k(f, N)^{-1} L^k \right),$$

$$\mathrm{D}(T_f) = \left\{ \varphi \in \bigcap_{m\geq 1} \mathrm{D}\left(T_{f,m}\right) \;\middle|\; \lim_{m\to\infty} T_{f,m}\varphi \text{ exists in } \ell^2(\mathbb{N}) \right\},$$

$$T_f\varphi = \lim_{m\to\infty} T_{f,m}\varphi, \quad \varphi \in \mathrm{D}(T_f).$$

Lemma 2.5 *Let $f \in \mathcal{K}$. Then $f \in \mathcal{K}^-$ if and only if $\ell^2_{\text{fin}}(\mathbb{N}) \subset \mathrm{D}\left(T_f\right)$.*

Proof Suppose that $f \in \mathcal{K}^-$. It is sufficient to show that $\lim_{m\to\infty} T_{f,m}\xi_n$ exists for all $n \in \mathbb{N}$. Since $k \geq n+1$ implies $L^k\xi_n = 0$, for any $n \leq m_1 \leq m_2$,

$$\begin{aligned}\left\|\left(T_{f,m_2} - T_{f,m_1}\right)\xi_n\right\|^2 &= \left\|\sum_{k=m_1+1}^{m_2}\left(L^{*k}\Delta_k(f,N)^{-1} - \Delta_k(f,N)^{-1}L^k\right)\xi_n\right\|^2 \\ &= \sum_{k=m_1+1}^{m_2}\frac{1}{(f(n+k)-f(n))^2} \\ &\leq \left(1 - \frac{f(n)}{f(n+1)}\right)^{-2}\sum_{k=m_1+1}^{m_2}\frac{1}{f(n+k)^2}.\end{aligned}$$

As $m_1, m_2 \to \infty$, the right-hand side above converges to zero. Hence, $(T_{f,m}\xi_n)_{m\in\mathbb{N}}$ is a Cauchy sequence. Therefore, $\lim_{m\to\infty} T_{f,m}\xi_n$ exists and $\xi_n \in \mathrm{D}(T_f)$.

Conversely, we assume that $\xi_0 \in \mathrm{D}(T_f)$. Then

$$\|T_f\xi_0\|^2 = \sum_{k=1}^{\infty}\frac{1}{(f(k)-f(0))^2} \geq \sum_{k=1}^{\infty}\frac{1}{f(k)^2}.$$

Thus $f \in \mathcal{K}^-$ is concluded. □

From Lemma 2.5, we see that T_f is a densely defined symmetric operator for any $f \in \mathcal{K}^-$. A relationship between T_f and T_G is given by the following theorem.

Theorem 2.6 *Suppose that H satisfies Assumption 1.2. Then, there exists a unitary operator $U : \mathcal{H} \to \ell^2(\mathbb{N})$ and a function $f \in \mathcal{K}^-$ such that $f(N) = UHU^*$ and T_f is unitarily equivalent to T_G on $\ell^2_{\mathrm{fin}}(\mathbb{N})$, i.e.,*

$$UT_\mathrm{G}U^* \subset T_f.$$

Here $A \subset B$ means that B is an extension of A.

Proof Let E_n be the n-th eigenvalue of H and $f : \mathbb{N} \to \mathbb{R}$ be a function such that $f(n) = E_n$. Then, by the condition (1.2), $f \in \mathcal{K}^-$. Let $U : \mathcal{H} \to \ell^2(\mathbb{N})$ be the unitary operator defined by $Ue_n = \xi_n$ for any $n \in \mathbb{N}$. Clearly, $UHU^* = f(N)$. For arbitrary $\varphi \in \mathrm{D}(T_G) = U^*\ell^2_{\mathrm{fin}}(\mathbb{N})$, we see that

$$\begin{aligned}UT_\mathrm{G}\varphi &= i\sum_{n=0}^{\infty}\left(\sum_{m<n}\frac{(\xi_m, U\varphi)}{E_n - E_m} + \sum_{m>n}\frac{(\xi_m, U\varphi)}{E_n - E_m}\right)\xi_n \\ &= i\sum_{n=0}^{\infty}\left(\sum_{m<n}\frac{\left(L^{n-m}\xi_n, U\varphi\right)}{E_n - E_m} + \sum_{m>n}\frac{\left(L^{*m-n}\xi_n, U\varphi\right)}{E_n - E_m}\right)\xi_n \\ &= i\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\frac{\left(L^k\xi_n, U\varphi\right)}{E_n - E_{n-k}} - \sum_{k=1}^{\infty}\frac{\left(L^{*k}\xi_n, U\varphi\right)}{E_{n+k} - E_n}\right)\xi_n.\end{aligned}$$

Since $f(N)\xi_n = E_n\xi_n$, it follows that

$$(E_n - E_{n-k})^{-1}L^k\xi_n = \Delta_k(f,N)^{-1}L^k\xi_n$$

and

$$(E_{n+k} - E_n)^{-1}\xi_n = \Delta_k(f,N)^{-1}\xi_n.$$

From Lemma 2.5, we see that both ξ_n and $U\varphi$ belong to $\mathrm{D}(T_f)$. Thus

$$\begin{aligned} UT_{\mathrm{G}}\varphi &= i\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\Delta_k(f,N)^{-1}L^k - L^{*k}\Delta_k(f,N)^{-1}\right)\xi_n, U\varphi\right)\xi_n \\ &= \sum_{n=0}^{\infty}\left(\xi_n, i\sum_{k=1}^{\infty}\left(L^{*k}\Delta_k(f,N)^{-1} - \Delta_k(f,N)^{-1}L^k\right)U\varphi\right)\xi_n. \end{aligned}$$

This shows that $UT_{\mathrm{G}}\varphi = T_fU\varphi$ for any $\varphi \in \mathrm{D}(T_{\mathrm{G}})$. Then, the theorem is proven. □

Corollary 2.7 *For all $f \in \mathcal{K}^-$, the operator T_f is a time operator of $f(N)$ with the dense CCR-domain $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$.*

Proof By the definition of $\mathcal{K}^-$, the operator $f(N)$ satisfies Assumption 1.2. According to Theorem 2.6, the Galapon time operator T_G of $f(N)$ is equal to the operator $T_f|_{\ell^2_{\mathrm{fin}}(\mathbb{N})}$. Here $A|_{\mathcal{S}}$ denotes the restriction of A to the subspace $\mathcal{S}$. Thus, T_f is a time operator of $f(N)$ with the dense CCR-domain $\mathrm{LH}\{\xi_n - \xi_m \mid n, m \in \mathbb{N}\} = (\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$. □

Remark 2.8 It can be shown that the set $\mathrm{D}(f(N)T_f) \cap \mathrm{D}(T_f f(N)) \subset (\mathbb{1} - L^*)\ell^2(\mathbb{N})$ can be utilized as the CCR-domain of $f(N)$ and T_f. For a detailed discussion, refer to [16].

By Theorem 2.6, the set $\{T_f \mid f \in \mathcal{K}^-\}$ includes the Galapon time operators T_{G}. Therefore, in what follows, we focus on the time operators T_f.

3 Self-adjointness of time operators

3.1 Bounded cases

Let us recall the case where the operator T_f is bounded.

Lemma 3.1 *Let $f \in \mathcal{K}$. Suppose that $0 \notin \sigma(\Delta_k(f,N))$ for all $k \geq 1$ and*

$$\sum_{k\geq 1}\left\|\Delta_k(f,N)^{-1}\right\| < \infty.$$

Then, the operator T_f is bounded. In particular, T_f is a self-adjoint operator.

Proof For any $\varphi \in \ell^2(\mathbb{N})$ and $1 \le m_1 < m_2$,

$$\begin{aligned}\left\| \left(T_{f,m_2} - T_{f,m_1}\right) \varphi \right\| &\le \sum_{k=m_1+1}^{m_2} \left(\left\| L^{*k} \Delta_k(f,N)^{-1} \right\| + \left\| \Delta_k(f,N)^{-1} L^k \right\| \right) \|\varphi\| \\ &\le 2\|\varphi\| \sum_{k=m_1+1}^{m_2} \left\| \Delta_k(f,N)^{-1} \right\|.\end{aligned}$$

This shows that $(T_{f,m}\varphi)_{m\in\mathbb{N}}$ is a Cauchy sequence. Therefore, $\mathrm{D}(T_f) = \ell^2(\mathbb{N})$ and T_f are bounded. □

Example 3.2 Let $\lambda > 1$ and $f(x) = x^\lambda + 1$. Then $f \in \mathcal{K}^-$. Since $\Delta_k(f,n) \ge \Delta_k(f,0) = k^\lambda$, we have

$$\sum_{k\ge 1} \left\| \Delta_k(f,N)^{-1} \right\| \le \sum_{k\ge 1} k^{-\lambda} < \infty.$$

Therefore, T_f is a bounded self-adjoint time operator of $f(N)$.

Remark 3.3 A similar result to Lemma 3.1 is obtained in [5, Theorem 4.5]. If

$$E_n - E_m \ge C(n^\lambda - m^\lambda), \quad n > m > a \tag{3.1}$$

for some constants $a > 0$, $C > 0$ and $\lambda > 1$, then T_{G} is bounded. From the condition (3.1), the assumptions of Lemma 3.1 can be derived. However, the converse does not hold, as demonstrated by the following counterexample:

$$f(n) = \begin{cases} (m+1)^2 & (n = 2m), \\ (m+1)^2 + 1 & (n = 2m+1). \end{cases}$$

3.2 Unbounded cases

In this section, we consider the unbounded cases. As a first step, we provide a sufficient condition for T_f to be unbounded. The proposition below states this condition.

Proposition 3.4 *Suppose that $f \in \mathcal{K}^-$ and $0 \in \sigma(\Delta_1(f,N))$. Then, T_f is unbounded.*

Proof We refer the reader to [5, Theorem 5.1]. □

Example 3.5 Let $\lambda \in (1/2, 1)$ and $f(x) = x^\lambda + 1$. Then $f \in \mathcal{K}^-$. Since $\Delta_1(f,n) \le n^{\lambda-1}$, we have $0 \in \sigma(\Delta_1(f,N))$. Hence T_f is unbounded by Proposition 3.4.

Let $f : \mathbb{N} \to \mathbb{C}$ and $m \in \mathbb{N}$. We denote by f^m the function $f^m : \mathbb{N} \to \mathbb{C}$ such that $f^m(x) = f(x)^m$ for each $x \in \mathbb{N}$. Clearly, if $f \in \mathcal{K}^-$, then $f^2 \in \mathcal{K}^-$ and T_{f^2} can be defined. In what follows we consider operators of the form $f(N)T_{f^2} + T_{f^2} f(N)$.

Lemma 3.6 *Let* $f \in \mathcal{K}^-$. *Then* $\ell^2_{\mathrm{fin}}(\mathbb{N}) \subset \mathrm{D}\left(f(N)T_{f^2}\right)$ *and*

$$\lim_{m\to\infty} f(N)T_{f^2,m}\xi_n = f(N)T_{f^2}\xi_n$$

for all $n \in \mathbb{N}$.

Proof Similarly to the proof of Lemma 2.5, for any $n \le m_1 \le m_2$, we have

$$\begin{aligned}\left\| f(N)\left(T_{f^2,m_2} - T_{f^2,m_1}\right)\xi_n \right\|^2 &= \sum_{k=m_1+1}^{m_2} \frac{f(n+k)^2}{(f(n+k)^2 - f(n)^2)^2} \\ &\le \left(1 - \frac{f(n)^2}{f(n+1)^2}\right)^{-2} \sum_{k=m_1+1}^{m_2} \frac{1}{f(n+k)^2}.\end{aligned}$$

Therefore, a sequence $\left(f(N)T_{f^2,m}\xi_n\right)_{m\in\mathbb{N}}$ is a Cauchy sequence for any $n \in \mathbb{N}$ and then $\lim_{m\to\infty} f(N)T_{f^2,m}\xi_n$ exists. Since $f(N)$ is a closed operator, we obtain the desired conclusion. □

The next lemma shows that T_f is identical to $f(N)T_{f^2} + T_{f^2}f(N)$ on $\ell^2_{\mathrm{fin}}(\mathbb{N})$.

Lemma 3.7 *Let* $f \in \mathcal{K}^-$. *Then*

$$f(N)T_{f^2} + T_{f^2}f(N) = T_f \tag{3.2}$$

on $\ell^2_{\mathrm{fin}}(\mathbb{N})$ *and*

$$[f(N), f(N)T_{f^2} + T_{f^2}f(N)] = -i\mathbb{1} \tag{3.3}$$

on $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$.

Proof From Lemma 3.6, for any $\varphi \in \ell^2_{\mathrm{fin}}(\mathbb{N})$,

$$\left(f(N)T_{f^2} + T_{f^2}f(N)\right)\varphi = \lim_{m\to\infty}\left(f(N)T_{f^2,m} + T_{f^2,m}f(N)\right)\varphi.$$

For each $m \ge 1$, we obtain

$$\begin{aligned}&\left(f(N)T_{f^2,m} + T_{f^2,m}f(N)\right)\varphi \\ &= i\sum_{k=1}^{m}\left(L^{*k}\left(f(N+k) + f(N)\right)\Delta_k\left(f^2,N\right)^{-1}\right. \\ &\quad \left. -\Delta_k\left(f^2,N\right)^{-1}\left(f(N+k)+f(N)\right)L^k\right)\varphi \\ &= i\sum_{k=1}^{m}\left(L^{*k}\Delta_k(f,N)^{-1} - \Delta_k(f,N)^{-1}L^k\right)\varphi \\ &= T_{f,m}\varphi.\end{aligned}$$

Hence, we see that $\varphi \in \mathrm{D}(T_f)$ and $f(N)T_{f^2} + T_{f^2}f(N) = T_f$ on $\ell^2_{\mathrm{fin}}(\mathbb{N})$. Since T_f is a time operator of $f(N)$ with a CCR-domain $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$,

$$\left[f(N), f(N)T_{f^2} + T_{f^2}f(N)\right] = [f(N), T_f] = -i\,\mathbb{1}$$

on $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$ holds true. □

Intuitively it may be difficult to establish the self-adjointness or essential self-adjointness of operators $f(N)T_{f^2} + T_{f^2}f(N)$ or T_f themselves, since both operators $f(N)T_{f^2} + T_{f^2}f(N)$ and T_f are unbounded from above and below, and the CCR-domain $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$ is not a core of $f(N)$. To overcome this difficulty, we introduce an additional term $f(N)^\beta$ into $f(N)T_{f^2} + T_{f^2}f(N)$. Note that $[N, f(N)^\beta] \subset 0$. Thus, we consider the modified operator $f(N)T_{f^2} + T_{f^2}f(N) + rf(N)^\beta$ instead of $f(N)T_{f^2} + T_{f^2}f(N)$ and we shall show that it is a self-adjoint time operator of $f(N)$. This result is based on the fact that $f(N)T_{f^2} + T_{f^2}f(N)$ is relatively small compared to $f(N)^\beta$.

We introduce classes $\mathcal{M}(\beta)$ and $\mathcal{M}_{\mathrm{s}}(\beta)$ of functions on $\mathbb{N}$.

Definition 3.8 ($\mathcal{M}(\beta)$ and $\mathcal{M}_{\mathrm{s}}(\beta)$) Let $\beta \geq 0$. The class $\mathcal{M}(\beta)$ consists of all functions $f \in \mathcal{K}^-$ for which there exist functions $g\colon \mathbb{N} \to (0, \infty)$ and $h \in \ell^1(\mathbb{N}_{\geq 1}, \mathbb{R})$ satisfying the following conditions:

(1) $f^2/g \in \ell^1(\mathbb{N})$,
(2) for any $n \in \mathbb{N}$ and $k \geq 1$,

$$\frac{g(n)^{1/2}}{f(n)^\beta \Delta_k(f^2, n)} \leq h(k). \tag{3.4}$$

The class $\mathcal{M}_{\mathrm{s}}(\beta)$ consists of all functions $f \in \mathcal{M}(\beta)$ such that, for the above function g, there exists a constant $C > 0$ satisfying

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \frac{g(n)}{\left\{f(n-k)^\beta \left(f(n)^2 - f(n-k)^2\right)\right\}^2} \leq C. \tag{3.5}$$

Lemma 3.9 *Let $f \in \mathcal{M}(1)$. Then, T_{f^2} is bounded.*

Proof By (1) of Definition 3.8, $\sup_{n\in\mathbb{N}} f(n)^2/g(n)$ is finite. From (3.4), we have

$$\left\| \Delta_k\left(f^2, N\right)^{-1} \right\| = \sup_{n\in\mathbb{N}} \Delta_k\left(f^2, n\right)^{-1} \leq \sup_{n\in\mathbb{N}} \left(f(n)^2/g(n)\right)^{1/2} h(k).$$

Since $h \in \ell^1(\mathbb{N}_{\geq 1}, \mathbb{R})$, T_{f^2} is bounded by Lemma 3.1. □

Lemma 3.10 *Let $f \in \mathcal{M}_{\mathrm{s}}(\beta)$. Then, the closure $\overline{f(N)T_{f^2}}$ of $f(N)T_{f^2}$ is relatively bounded with respect to $f(N)^\beta$, i.e., there exists a constant $a > 0$ such that for all*

$\varphi \in \mathrm{D}\big(f(N)^{\beta}\big)$

$$\left\| \overline{f(N)T_{f^2}\varphi} \right\| \leq a \left\| f(N)^{\beta}\varphi \right\|.$$

Proof Since $\ell^2_{\mathrm{fin}}(\mathbb{N})$ is a core for $f(N)^{\beta}$, it is sufficient to show that $\overline{f(N)T_{f^2}} f(N)^{-\beta}$ is bounded on $\ell^2_{\mathrm{fin}}(\mathbb{N})$. For any $\varphi \in \ell^2_{\mathrm{fin}}(\mathbb{N})$, it follows that $\varphi \in \mathrm{D}\big(f(N)T_{f^2} f(N)^{-\beta}\big)$ and

$$\begin{aligned}
\left\| f(N)T_{f^2} f(N)^{-\beta}\varphi \right\|^2 &= \sum_{n=0}^{\infty} \left|\left(\xi_n, f(N)T_{f^2} f(N)^{-\beta}\varphi\right)\right|^2 \\
&\leq \|\varphi\|^2 \sum_{n=0}^{\infty} \left\| f(N)^{-\beta} T_{f^2} f(N)\xi_n \right\|^2 \\
&= \|\varphi\|^2 \sum_{n=0}^{\infty} f(n)^2 \left\| f(N)^{-\beta} T_{f^2}\xi_n \right\|^2.
\end{aligned}$$

For all $n \in \mathbb{N}$, we see that, from (3.4) and (3.5),

$$\begin{aligned}
\left\| f(N)^{-\beta} T_{f^2}\xi_n \right\|^2 &= \left\| \sum_{k\geq 1} f(N)^{-\beta} \left(L^{*k}\Delta_k(f^2,N)^{-1} - \Delta_k(f^2,N)^{-1}L^k\right)\xi_n \right\|^2 \\
&= \sum_{k\geq 1} \frac{1}{f(n+k)^{2\beta}\Delta_k\left(f^2,n\right)^2} \\
&\quad + \sum_{k=1}^{n} \frac{1}{f(n-k)^{2\beta}(f(n)^2 - f(n-k)^2)^2} \\
&\leq \frac{1}{g(n)} \left(\sum_{k\geq 1} h(k)^2 + C \right).
\end{aligned}$$

Thus we have

$$\left\| f(N)T_{f^2} f(N)^{-\beta}\varphi \right\|^2 \leq \|\varphi\|^2 \left(\|h\|^2_{\ell^2} + C\right) \sum_{n=0}^{\infty} f(n)^2 g(n)^{-1}.$$

From the condition (1) of $\mathcal{M}(\beta)$, $\overline{f(N)T_{f^2}} f(N)^{-\beta}$ is bounded on $\ell^2_{\mathrm{fin}}(\mathbb{N})$. Therefore, the conclusion follows. □

Remark 3.11 Let $\beta \geq 1$, $f \in \mathcal{M}_{\mathrm{s}}(\beta)$ and T_{f^2} be bounded. Since the operator T_f is equal to $f(N)T_{f^2} + T_{f^2} f(N)$ on $\ell^2_{\mathrm{fin}}(\mathbb{N})$ by Lemma 3.7, it is easy to see that $\overline{T_f}$ is also relatively bounded with respect to $f(N)^{\beta}$, i.e., there exists a constant $a > 0$ such that for all $\varphi \in \mathrm{D}(f(N)^\beta)$

$$\|\overline{T_f}\varphi\| \le a\,\|f(N)^\beta\varphi\|.$$

Proposition 3.12 *Let $\beta \ge 1$ and $f \in \mathcal{M}_{\mathrm{s}}(\beta)$. If the operator T_{f^2} is bounded, then $f(N)T_{f^2} + T_{f^2}f(N)$ is relatively bounded to $f(N)^\beta$, and $f(N)T_{f^2} + T_{f^2}f(N) + rf(N)^\beta$ is a self-adjoint operator for sufficiently large $|r|$.*

Proof Since T_{f^2} is a bounded operator, $f(N)T_{f^2}$ is closed and $T_{f^2}f(N)$ is relatively bounded with respect to $f(N)^\beta$. From Lemma 3.10, it follows that there exists a relative bound a for $f(N)T_{f^2} + T_{f^2}f(N)$ with respect to $f(N)^\beta$. Therefore, by the Kato-Rellich theorem, the operator is self-adjoint for all $|r| > a$. □

We are in the position to state the main theorem in this paper.

Theorem 3.13 *Let $\beta \ge 1$, $f \in \mathcal{M}_{\mathrm{s}}(\beta)$ and $\gamma > \beta$. If T_{f^2} is bounded, then $f(N)T_{f^2} + T_{f^2}f(N) + rf(N)^\gamma$ is a self-adjoint time operator of $f(N)$ with a dense CCR-domain for all $r \in \mathbb{R}\backslash\{0\}$.*

Proof Since $f(N)^\beta$ is infinitesimally small with respect to $f(N)^\gamma$, from Proposition 3.12, it follows that $f(N)T_{f^2} + T_{f^2}f(N)$ is also infinitesimally small with respect to $f(N)^\gamma$. Hence, the operator $f(N)T_{f^2} + T_{f^2}f(N) + rf(N)^\gamma$ is self-adjoint for all $r \in \mathbb{R}\backslash\{0\}$. Since

$$[f(N), f(N)T_{f^2} + T_{f^2}f(N) + rf(N)^\gamma] = -i\mathbb{1}$$

on $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$ by Lemma 3.7, $f(N)T_{f^2} + T_{f^2}f(N) + rf(N)^\gamma$ is a self-adjoint time operator of $f(N)$ with a dense CCR-domain. □

Remark 3.14 (1) From (3.2) and Remark 3.11, we see that $\overline{T_f} + rf(N)^\gamma$ is self-adjoint time operator of $f(N)$ with a dense CCR-domain $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$ for all $r \in \mathbb{R}\backslash\{0\}$ provided that $\beta \ge 1$, $f \in \mathcal{M}_{\mathrm{s}}(\beta)$ and $\gamma > \beta$.

(2) It seems unlikely that the self-adjointness or essential self-adjointness of the operators $T = T_{f^2}f(N) + f(N)T_{f^2}$ and T_f can be established by means of the commutator theorem applied to the auxiliary operator $A = f(N)^\gamma + T$, since the weak commutator $[T, f(N)^\gamma]_{\mathrm{w}}$ fails to be bounded in terms of $f(N)^\gamma$. Although the relation (3.3) might indicate that $[T, f(N)^\gamma]_{\mathrm{w}}$ is controllable, the proof does not go through because the CCR-domain does not form the core of $f(N)^\gamma$ or T is not relative bounded with respect to $f(N)^\gamma$. Likewise, the case of the weak commutator $[T, A]_{\mathrm{w}}$ does not appear to admit a bound in terms of A.

Example 3.15 Let $f(x) = x^\lambda + 1$ for $\lambda \in (3/4, 1)$. We show that $f \in \mathcal{M}_{\mathrm{s}}(1)$. Firstly, it is immediate to see that $f \in \mathcal{K}^-$. Let $\alpha \in (1 + 2\lambda, 6\lambda - 2)$, $g(x) = x^\alpha + 1$ and $\delta = 6\lambda - 2 - \alpha$. Then, the condition (1) of $\mathcal{M}(1)$ is satisfied.

Secondly, by the mean value theorem, we have

$$f(n+k) - f(n) \ge \frac{\lambda k}{(n+k)^{1-\lambda}}.$$

Then we obtain that

$$\frac{g(n)}{f(n)^2\Delta_k\left(f^2,n\right)^2}=\frac{n^\alpha+1}{f(n)^2(f(n+k)^2-f(n)^2)^2}$$
$$\leq\frac{(n^\alpha+1)(n+k)^{2(1-\lambda)}}{\lambda^2(n^\lambda+1)^2(n+k)^{2\lambda}k^2}\leq\frac{2}{\lambda^2k^{2+\delta}}.$$

Thus, the condition (2) of $\mathcal{M}(1)$ is satisfied and $f\in\mathcal{M}(1)$.

Finally, we see that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{g(n)}{f(n-k)^2\left(f(n)^2-f(n-k)^2\right)^2}$$
$$=\lim_{n\to\infty}\left(\sum_{k=1}^{[n/2]}\frac{g(n)}{f(n-k)^2\left(f(n)^2-f(n-k)^2\right)^2}\right.$$
$$\left.+\sum_{k=[n/2]+1}^{n}\frac{g(n)}{f(n-k)^2\left(f(n)^2-f(n-k)^2\right)^2}\right)$$
$$\leq\lim_{n\to\infty}\frac{4^\lambda(n^\alpha+1)n^{2(1-\lambda)}}{\lambda^2(n^\lambda+1)^2n^{2\lambda}}\left(\sum_{k=1}^{[n/2]}\frac{1}{k^2}+\frac{4^{1-\lambda}}{n^{2(1-\lambda)}}\sum_{k=[n/2]+1}^{n}\frac{1}{f(n-k)^2}\right)<\infty,$$

where $[r]$ denotes the greatest integer less than or equal to $r\in\mathbb{R}$. Then, the condition (3.5) is satisfied and $f\in\mathcal{M}_{\mathrm{s}}(1)$.

In Example 3.5, we showed that T_f is unbounded. We see that, from Lemma 3.9 and Theorem 3.13, $f(N)$ has a self-adjoint time operator with a dense CCR-domain.

Example 3.16 Let $f(x)=x^\lambda+1$ for $\lambda\in(1/2,1)$. Then, T_{f^2} is bounded by Lemma 3.1. Similar to Example 3.15, we can see that $f\in\mathcal{M}_{\mathrm{s}}(2)$. Therefore, $f(N)$ has an unbounded self-adjoint time operator with a dense CCR-domain by Theorem 3.13.

4 Self-adjoint extension of time operators

Up to this point, we have considered the case where $f\in\mathcal{K}^-$. From now on, we turn our attention to the case where $f\in\mathcal{K}\setminus\mathcal{K}^-$. In this setting, Lemma 2.5 implies $\ell^2_{\mathrm{fin}}(\mathbb{N})\not\subset \mathrm{D}(T_f)$, and therefore greater care must be taken in analyzing the domain of the time operators. Accordingly we begin by reexamining the domain of $f(N)T_{f^2}+T_{f^2}f(N)$, as well as the CCR-domain for $f(N)T_{f^2}+T_{f^2}f(N)$ and $f(N)$.

Lemma 4.1 *Let $f^2\in\mathcal{K}^-$. Then $(\mathbb{1}-L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})\subset\mathrm{D}\big(f(N)^2T_{f^2}\big)\cap\mathrm{D}(T_{f^2}f(N))$ and the operator $f(N)T_{f^2}+T_{f^2}f(N)$ is symmetric.*

Proof From Corollary 2.7, T_{f^2} satisfies $\left[f(N)^2,T_{f^2}\right]=-i\mathbb{1}$ on $(\mathbb{1}-L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$. This implies that $(\mathbb{1}-L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})\subset\mathrm{D}\big(f(N)^2T_{f^2}\big)\cap\mathrm{D}(T_{f^2}f(N))$. □

We establish the analogs of Lemmas 3.6 and 3.7 in the case where $f^2 \in \mathcal{K}^-$.

Lemma 4.2 *Let $f^2 \in \mathcal{K}^-$. Then $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N}) \subset \mathrm{D}\left(f(N)T_{f^2}\right)$ and*

$$\lim_{m\to\infty} f(N)T_{f^2,m}(\mathbb{1} - L^*)\xi_n = f(N)T_{f^2}(\mathbb{1} - L^*)\xi_n$$

for all $n \in \mathbb{N}$.

Proof Similarly to the proof of Lemma 3.6, it suffices to prove that $(f(N)T_{f^2,m}(\mathbb{1} - L^*)\xi_n)_m$ converges. For any $n + 1 \le m_1 \le m_2$, we have

$$\begin{aligned}
&\left\| f(N)\left(T_{f^2,m_2} - T_{f^2,m_1}\right)(\mathbb{1} - L^*)\xi_n \right\|^2 \\
&= \frac{f(n+m_1+1)^2}{\Delta_{m_1+1}(f^2,n)^2} + \sum_{k=m_1+1}^{m_2-1} f(n+k+1)^2 \left(\frac{1}{\Delta_{k+1}(f^2,n)} - \frac{1}{\Delta_k(f^2,n+1)}\right)^2 \\
&\quad + \frac{f(n+m_2+1)^2}{\Delta_{m_2}(f^2,n+1)^2} \\
&\le \left(1 - \frac{f(n)^2}{f(n+1)^2}\right)^{-2} \frac{1}{f(n+m_1+1)^2} + \sum_{k=m_1+1}^{m_2-1} \frac{f(n+k+1)^2\Delta_1(f^2,n)^2}{\Delta_{k+1}(f^2,n)^2\Delta_k(f^2,n+1)^2} \\
&\quad + \left(1 - \frac{f(n+1)^2}{f(n+2)^2}\right)^{-2} \frac{1}{f(n+m_2+1)^2} \\
&\le \left(1 - \frac{f(n)^2}{f(n+1)^2}\right)^{-2} \frac{1}{f(n+m_1+1)^2} \\
&\quad + \frac{\Delta_1(f^2,n)^2}{f(n+1)^2}\left(1 - \frac{f(n+1)^2}{f(n+2)^2}\right)^{-4} \sum_{k=m_1+1}^{m_2-1} \frac{1}{f(n+k+1)^4} \\
&\quad + \left(1 - \frac{f(n+1)^2}{f(n+2)^2}\right)^{-2} \frac{1}{f(n+m_2+1)^2}.
\end{aligned}$$

Since $f^2 \in \mathcal{K}^-$, $(f(N)T_{f^2,m}(\mathbb{1} - L^*)\xi_n)_{m\in\mathbb{N}}$ is a Cauchy sequence, and then it converges. We have the desired conclusion. □

Lemma 4.3 *Let $f^2 \in \mathcal{K}^-$. Then $(1 - L^*)\ell^2_{\mathrm{fin}} \subset \mathrm{D}(T_f)$ and*

$$f(N)T_{f^2} + T_{f^2}f(N) = T_f.$$

on $(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$.

Proof The assertion can be derived by modifying the proof of Lemma 3.7, using Lemma 4.2 in place of Lemma 3.6. For brevity, the details are omitted. □

Lemma 4.4 *Let $f \in \mathcal{K}$. Then $f(N)(\mathbb{1} - L^*)\Delta_1(f,N)^{-1}(\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N}) \subset (\mathbb{1} - L^*)\ell^2_{\mathrm{fin}}(\mathbb{N})$.*

Proof On $\ell^2_{\rm fin}(\mathbb{N})$, we have

$$\begin{aligned}
&f(N)(\mathbb{1}-L^*)\Delta_1(f,N)^{-1}(\mathbb{1}-L^*)\\
&=\left(f(N)\Delta_1(f,N)^{-1}-L^*f(N+\mathbb{1})\Delta_1(f,N)^{-1}\right)(\mathbb{1}-L^*)\\
&=\left(f(N)\Delta_1(f,N)^{-1}-L^*-L^*f(N)\Delta_1(f,N)^{-1}\right)(\mathbb{1}-L^*)\\
&=(\mathbb{1}-L^*)\left(f(N)\Delta_1(f,N)^{-1}(\mathbb{1}-L^*)-L^*\right).
\end{aligned}$$

Therefore, we obtain the desired result. □

Theorem 4.5 *Let $f^2\in\mathcal{K}^-$. Then, $f(N)T_{f^2}+T_{f^2}f(N)$ and T_f are time operators of $f(N)$ with an infinite dimensional CCR-domain.*

Proof From Lemmas 4.1 and 4.4, we see that

$$\begin{aligned}
&(\mathbb{1}-L^*)\Delta_1(f,N)^{-1}(\mathbb{1}-L^*)\ell^2_{\rm fin}(\mathbb{N})\\
&\quad\subset \mathrm{D}\left(f(N)^2T_{f^2}\right)\cap\mathrm{D}(f(N)T_{f^2}f(N))\cap\mathrm{D}\left(T_{f^2}f(N)^2\right).
\end{aligned}$$

Therefore, the symmetric operator $f(N)T_{f^2}+T_{f^2}f(N)$ satisfies

$$\left[f(N),\,f(N)T_{f^2}+T_{f^2}f(N)\right]=f^2(N)T_{f^2}-T_{f^2}f^2(N)=-i\,\mathbb{1}$$

on $(\mathbb{1}-L^*)\Delta_1(f,N)^{-1}(\mathbb{1}-L^*)\ell^2_{\rm fin}(\mathbb{N})$, since T_{f^2} is a time operator of $f^2(N)$ with the CCR-domain $(\mathbb{1}-L^*)\ell^2_{\rm fin}(\mathbb{N})$. By Lemma 4.3, T_f is also a time operator of $f(N)$ with an infinite dimensional CCR-domain. □

Since the domain of T_f cannot be expected to contain a core of $f(N)$, it is difficult to obtain an estimate similar to Lemma 3.10. Instead, we consider taking a self-adjoint extension of time operators.

Proposition 4.6 *Let $f\in\mathcal{K}$. If $\mathrm{D}(f(N)T_{f^2})\cap\mathrm{D}(f(N)^2)$ is dense and T_{f^2} is bounded, then $f(N)T_{f^2}+T_{f^2}f(N)+rf(N)^2$ has a self-adjoint extension for all $r\geq 1$.*

Proof From

$$f(N)T_{f^2}+T_{f^2}f(N)+rf(N)^2\subset\left(f(N)+T_{f^2}\right)^2+(r-1)f(N)^2-T_{f^2}^2,$$

we see that $f(N)T_{f^2}+T_{f^2}f(N)+rf(N)^2$ is bounded from below. Thus, it has the Friedrichs extension. □

Example 4.7 Let $f(x)=\sqrt{x+1}$. Clearly, $f^2\in\mathcal{K}^-$. From Theorem 4.5, we see that $T=f(N)T_{f^2}+T_{f^2}f(N)+f(N)^2$ is a time operator of $f(N)$. Since T_{f^2} is bounded by [5, Theorem 4.6], T has a self-adjoint extension by Proposition 4.6. Thus, $f(N)$ has a self-adjoint time operator with an infinite dimensional CCR-domain.

We finally discuss the case where T_{f^2} may be unbounded.

Proposition 4.8 *If $f^2 \in \mathcal{K}^-$ and T_{f^4} is bounded, then $f(N)$ has a self-adjoint time operator.*

Proof By Lemma 3.7, we see that

$$\begin{aligned} &f(N)T_{f^2} + T_{f^2}f(N) \\ &\quad = f(N)^3 T_{f^4} + f(N)^2 T_{f^4} f(N) + f(N) T_{f^4} f(N)^2 + T_{f^4} f(N)^3 \end{aligned}$$

on $(\mathbb{1} - L^*)\ell^2_{\text{fin}}(\mathbb{N})$. We consider the operator

$$T = f(N)^3 T_{f^4} + f(N)^2 T_{f^4} f(N) + f(N) T_{f^4} f(N)^2 + T_{f^4} f(N)^3 + r f(N)^6$$

for some real number $r > 1$. From Theorem 4.5, we see that T is a time operator of $f(N)$. Set $r = r_1 + r_2$ such that $r_1 \geq 1$ and $r_2 > 0$. Clearly the following relations hold:

$$\begin{aligned} &f(N)^3 T_{f^4} + T_{f^4} f(N)^3 + r_1 f(N)^6 \subset \left(f(N)^3 + T_{f^4}\right)^2 + (r_1 - 1) f(N)^6 - T_{f^4}^2, \\ &f(N)^2 T_{f^4} f(N) + f(N) T_{f^4} f(N)^2 + r_2 f(N)^6 \\ &\ = f(N)\left(f(N)^2 + f(N) T_{f^4} + T_{f^4} f(N) + \left\| T_{f^4}^2 \right\|\right) f(N) \\ &\quad + r_2 f(N)^6 - f(N)^4 - \left\| T_{f^4}^2 \right\| f(N)^2. \end{aligned}$$

Since $f(N)^2$ and $f(N)^4$ are infinitesimally small compared to $f(N)^6$, the operators on the right-hand side of the above relations are bounded from below. Consequently the operator T admits a self-adjoint extension $\tilde{T}$ which serves as a self-adjoint time operator of $f(N)$. □

Example 4.9 Let $\lambda \in (1/4, 1)$ and $f(x) = x^\lambda + 1$. Then, $f^2 \in \mathcal{K}^-$ and T_{f^4} are bounded. Hence, $f(N)$ admits a self-adjoint time operator by Proposition 4.8.

Acknowledgements FH is financially supported by JSPS KAKENHI 20K20886, 20H01808 and 25H00595.

Data availability This manuscript has no associated data.

Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

References

1. Aharonov, Y., Bohm, D.: Time in the quantum theory and the uncertainty relation for time and energy. Phys. Rev. **122**, 1649–1658 (1961)

2. Arai, A.: Generalized weak Weyl relation and decay of quantum dynamics. Rev. Math. Phys. **17**, 1071–1109 (2005)
3. Arai, A.: Necessary and sufficient conditions for a Hamiltonian with discrete eigenvalues to have time operators. Lett. Math. Phys. **87**, 67–80 (2009)
4. Arai, A.: Inequivalent Representations of Canonical Commutation and Anti-Commutation Relations. Springer, Cham (2020)
5. Arai, A., Matsuzawa, Y.: Time operators of a Hamiltonian with purely discrete spectrum. Rev. Math. Phys. **20**, 951–978 (2008)
6. Arai, A.: Spectrum of time operators. Lett. Math. Phys. **80**(3), 211–221 (2007)
7. Arai, A.: On the uniqueness of weak weyl representations of the canonical commutation relation. Lett. Math. Phys. **85**(1), 15–25 (2008)
8. Bauer, M.: A time operator in quantum mechanics. Ann. Phys. **150**, 1–21 (1983)
9. Cannata, F., Ferrari, L.: Canonical conjugate momentum of discrete label operators in quantum mechanics i:formalizm. Found. Phys. Lett. **4**, 557–568 (1991)
10. Cannata, F., Ferrari, L.: Canonical conjugate momentum of discrete label operators in quantum mechanics ii: formalizm. Found. Phys. Lett. **4**, 569–579 (1991)
11. Carruthers, P., Niet, M.M.: Phase and angle variable in quantum mechanics. Rev. Mod. Phys. **40**, 411–440 (1968)
12. Galapon, E.A.: Pauli's theorem and quantum mechanical pairs: the consistency of bounded, self-adjoint time operator canonically conjugate to a hamiltonian with non-empty point spectrum. Proc. R. Soc. Lond. A **458**, 451–472 (2002)
13. Galapon, E.A.: Self-adjoint time operator is the rule for discrete semi-bounded Hamiltonians. Proc. R. Soc. Lond. A **458**, 2671–2689 (2002)
14. Galapon, E.A., Caballar, R.F., Bahague, R.: Confined quantum time of arrivals. Phys. Rev. Lett. **93**, 180406 (2004)
15. Galapon, E.A., Caballar, R.F., Bahague, R.: Confined quantum time of arrivals for the vanishing potential. Phys. Rev. A **72**, 062107 (2005)
16. Hiroshima, F., Teranishi, N.: Conjugate operators of 1D-harmonic oscillator. arXiv:2404.12286, (2024)
17. Hiroshima, F., Teranishi, N.: Time operators of harmonic oscillators and their representations. J. Math. Phys. **65**(4), 042105 (2024)
18. Mandelstam, L., Tamm, I.G.: The uncertainty relation between energy and time in non-relativistic quantum mechanics. J. Phys. **9**, 249 (1945)
19. Miyamoto, M.: A generalized Weyl relation approach to the time operator and its connection to the survival probability. J. Math. Phys. **42**, 1038–1052 (2001)
20. Olkhovsky, V.S., Recami, E., Gerasimchuk, A.J.: Time operator in quantum mechanics: I: nonrelativistic case. Il Nuovo Cimento A (1965–1970) **22**(2), 263–278 (1974)
21. Susskind, L., Glogower, J.: Quantum mechanical phase and time operator. Physics **1**, 49–61 (1964)
22. Teranishi, N.: A note on time operators. Lett. Math. Phys. **106**, 1259–1263 (2016)