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On the Spectrum of the Discrete Schrödinger Operator of a Rank-Two Perturbation on $\mathbb{Z}$

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Abstract—We consider a family of Schrödinger operators $\widehat{H}_{\lambda\mu k} = -\Delta - \lambda\delta_{k,\cdot} - \mu\delta_{0,\cdot}$ on the one-dimensional lattice $\mathbb{Z}$, where Δ is a standard discrete Laplacian, $\delta_{\cdot,\cdot}$ is a Kronecker delta function, and $\lambda, \mu \in \mathbb{R}$ and $k \in \mathbb{Z}$ are parameters. Eigenvalue behavior of the operators and their dependence on the parameters are explicitly derived. Moreover, we obtain asymptotics for the eigenvalues as the distance between two elements of the potential function support approaches infinity.

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1. INTRODUCTION

Behavior of eigenvalues lying below the essential spectrum of the standard Schrödinger operators of the form $-\Delta - \varepsilon V$ defined on $L^2(\mathbb{R}^n)$, where V is a positive potential and $\varepsilon \geq 0$ is a parameter that varies, has been thoroughly studied [1]. When ε approaches to a critical point $\varepsilon_c \geq 0$, the negative eigenvalues approach to the left edge of the essential spectrum, and consequently they are absorbed into it. A crucial mathematical problem is to specify whether the edge of the essential spectrum is an eigenvalue or a threshold resonance at the critical point ε_c, which is also dependent on the spatial dimension n.

In this work, we study discrete Schrödinger operators, the lattice counterparts of the continuous Schrödinger operators. Lattice Bose–Hubbard models represent a minimal system with highly controllable parameters such as lattice geometry and dimensionality, particle masses, tunneling, two-body potentials, temperature, etc. (see, e.g., [2–4] and the references therein). In the traditional condensed matter systems, stable composite objects are usually formed by the attractive forces, meanwhile the repulsive forces separate particles in the free space. However, the controllability of the collision properties of ultracold atoms has enabled to experimentally observe a stable repulsive bound pair of ultracold atoms in the optical lattice $\mathbb{Z}^3$ [5–7]. In these observations, Bose–Hubbard Hamiltonians became a link between the basic theoretical approaches and experiments.

Discrete Schrödinger operators have also attracted considerable attention for both combinatorial Laplacians and quantum graphs (see Refs. [8–14] for some summaries). Particularly, eigenvalue behavior of such operators were studied in [15–24] and are briefly discussed in [25–27] for the case when the potential is a delta function with a single point mass. In [15], an explicit example for the parameter-dependent operator $\widehat{H}_{\lambda\mu}$ of the form $-\Delta - \widehat{V}_{\lambda\mu}$ was constructed on the three-dimensional lattice $\mathbb{Z}^3$,

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which possesses both a lower threshold resonance and a lower threshold eigenvalue, where Δ stands for the standard discrete Laplacian in $\ell^2(\mathbb{Z}^3)$ and $\widehat{V}_{\lambda\mu}$ is defined as

$$\widehat{V}_{\lambda\mu}(x) = \mu\delta_{x0} + \frac{\lambda}{2}\sum_{|k|=1}\delta_{xk}, \quad \lambda \geq 0, \quad \mu \geq 0, \tag{1}$$

with $\delta_{\cdot,\cdot}$ being the Kronecker delta.

Restriction of this operator to the Hilbert space $\ell^2_{\mathrm{e}}(\mathbb{Z}^3)$ of all even functions in $\ell^2(\mathbb{Z}^3)$ was studied in [27]. They investigated the dependence of the number of eigenvalues of the operator $\widehat{H}_{\lambda\mu}$ on the parameters λ,μ for $\lambda > 0, \mu > 0$. It was shown that all eigenvalues arise either from a lower threshold resonance or from lower threshold eigenvalues under a variation of the interaction energy. Particularly, they proved that the first lower eigenvalue of the Hamiltonian arises only from a lower threshold resonance. In the case of $\lambda = 0$, Hiroshima et. al. [18] studied a similar problem on the d-dimensional lattice $\mathbb{Z}^3$, and showed that a threshold eigenvalue does appear for $n \geq 5$, but does not for $1 \leq n \leq 4$.

The discrete Schrödinger operators with potential of the form (1) posses very interesting spectral properties which are not yet fully studied. In this paper, we consider the problem for an arbitrary $k \in \mathbb{Z}$ in the one-dimensional case. In this case, the discrete Schrödinger operator is of the form $\widehat{H}_{\lambda\mu k} = \widehat{H}_0 - \hat{v}_{\lambda\mu k}$, where $\widehat{H}_0 = -\Delta$ is the discrete Laplacian and $\hat{v}_{\lambda\mu k}$ is a multiplication operator by the function $\hat{v}_{\lambda\mu k}(x) = \lambda\delta_{x0} + \mu\delta_{xk}$, where $\lambda,\mu \in \mathbb{R}$ and $k \in \mathbb{Z}$ are parameters.

For convenience, we study the spectral properties of the family of operators $H_{\lambda\mu k} = \mathcal{F}^{-1}\widehat{H}_{\lambda\mu k}\mathcal{F}$ acting on $L^2(\mathbb{T})$. We show that the operators $H_{\lambda\mu k}$ may have zero, one or two eigenvalues depending on the parameters. In the latter case, one of the eigenvalues lies below the eigenvalues of the operators $H_{\lambda 0k}$ and $H_{0\mu k}$, while the other is above them. We establish asymptotics for these eigenvalues as $k \to \infty$ and show that they converge to the eigenvalues of $H_{\lambda 0k}$ and $H_{0\mu k}$ as k increases. Additionally, we investigate the spectrum of $H_{\lambda\mu k}$, particularly lower threshold eigenvalues and threshold resonances for any $(\lambda,\mu) \in \mathbb{R}^2$.

The paper is organized as follows. Section 1 is introduction. In Sections 2 and 3, discrete Schrödinger operators are described in the coordinate and momentum representations, respectively. In Section 4, we describe the essential spectrum of the operator. Moreover, the Fredholm determinant and its properties are studied. In Section 5, eigenvalues of the operator $H_{\lambda\mu k}$ are investigated (Theorem 2). Section 6 is devoted to the asymptotics of the eigenvalues of the operator $H_{\lambda\mu k}$ as $k \to \infty$.

2. THE DISCRETE SCHRÖDINGER OPERATOR IN THE POSITION REPRESENTATION

In the one-dimensional case, the standard discrete Laplacian Δ is defined as a self-adjoint (bounded) Toeplitz-type operator on the Hilbert space $\ell^2(\mathbb{Z})$ [28]

$$\Delta = \frac{1}{2}\sum_{x\in\mathbb{Z},|x|=1}\big(T(x) - T(0)\big),$$

where $T(y)$, $y \in \mathbb{Z}$ is a shift operator

$$(T(y)\hat{f})(x) = \hat{f}(x+y), \quad \hat{f} \in \ell^2(\mathbb{Z}), \quad x \in \mathbb{Z}.$$

Let the discrete Schrödinger operator be defined in the Hilbert space $\ell^2(\mathbb{Z})$ as $\widehat{H}_{\lambda\mu k} = \widehat{H}_0 - \hat{v}_{\lambda\mu k}$, where $\widehat{H}_0 = -\Delta$ and the potential operator $\hat{v}$ depending on the parameters $\lambda,\mu \in \mathbb{R}$ and $k \in \mathbb{Z}$ is defined by

$$(\hat{v}_{\lambda\mu k}\hat{f})(x) = \begin{cases} \mu\hat{f}(x), & \text{if} \quad x = 0 \\ \lambda\hat{f}(x), & \text{if} \quad x = k \\ 0, & \text{if} \quad x \in \mathbb{Z}\setminus\{k,0\}, \end{cases} \qquad \hat{f} \in \ell^2(\mathbb{Z}), \quad x \in \mathbb{Z}.$$

3. THE DISCRETE SCHRÖDINGER OPERATOR IN THE MOMENTUM REPRESENTATION

Let $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ be the standard Fourier transform defined as

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{T}} f(\theta) e^{ix\theta} d\theta, \quad f \in L^2(\mathbb{T}), \quad x \in \mathbb{Z}$$

with the inverse $\mathcal{F}^{-1} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ acting as

$$(\mathcal{F}^{-1}f)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{x\in\mathbb{Z}} \hat{f}(x) e^{-ix\theta}, \quad \hat{f} \in \ell^2(\mathbb{Z}), \quad \theta \in \mathbb{T}.$$

Let the Hamiltonian $H_{\lambda\mu k}$ be the momentum representation of the Hamiltonian $\widehat{H}_{\lambda\mu k}$, defined as $H_{\lambda\mu k} = \mathcal{F}^{-1}\widehat{H}_{\lambda\mu k}\mathcal{F}$ acting on $L^2(\mathbb{T})$ as $H_{\lambda\mu k} = H_0 - V_{\lambda\mu k}$, where the non-perturbed operator $H_0 = \mathcal{F}^{-1}\widehat{H}_0\mathcal{F}$ is defined on $L^2(\mathbb{T})$ as a multiplication operator

$$(H_0 f)(p) = e(p)f(p), \quad f \in L^2(\mathbb{T}), \quad p \in \mathbb{T},$$

where $e(p) = 1 - \cos p, \quad p \in \mathbb{T}$. In the physical literature, the function $e(\cdot)$, being a real valued-function on $\mathbb{T}$, is called the dispersion relation of the Laplace operator $(-\Delta)$.

The perturbation $V_{\lambda\mu k} = \mathcal{F}^{-1}\hat{v}_{\lambda\mu k}\mathcal{F}$ is the two-dimensional integral operator,

$$(V_{\lambda\mu k}f)(p) = \frac{\mu}{2\pi} \int\limits_{\mathbb{T}} f(q)dq + \frac{\lambda}{2\pi} \int\limits_{\mathbb{T}} e^{ik(p-q)} f(q)dq, \quad f \in L^2(\mathbb{T}), \quad p \in \mathbb{T}.$$

It is not hard to see that $H_{\lambda\mu k}$ is a self-joint operator.

We remark that for $k = 0$, the potential $V_{\lambda\mu k}$ is an operator of rank one, and that case was investigated in [18, 19]. In the present work, we assume that $k \neq 0$.

Hereafter, we use $V_{\lambda\mu}$ and $H_{\lambda\mu}$ instead of $V_{\lambda\mu k}$ and $H_{\lambda\mu k}$, respectively, for simplicity. But we take into consideration their dependence on k.

4. FREDHOLM DETERMINANT OF THE OPERATOR $H_{\lambda\mu}$

The perturbation operator $V_{\lambda\mu}$ is an operator of rank two and in accordance with Weyl's theorem on the stability of the essential spectrum, we have $\sigma_{ess}(H_{\lambda\mu}) = \sigma_{ess}(H_0)$. As H_0 is a multiplication operator by a function, the essential spectrum of the operator $H_{\lambda\mu}$ consists of the segment $\sigma_{\mathsf{ess}}(H_{\lambda\mu}) = [e_{\min}, e_{\max}]$ on the real axis, where $e_{\min} = 0$ and $e_{\max} = 2$.

Let $\mathcal{T}(z), z \in \mathbb{C} \setminus [0.2]$ be a two-dimensional matrix operator defined as

$$\mathcal{T}(z) = \begin{bmatrix} \mu a(z) & \lambda b(z) \\ \mu b(z) & \lambda a(z) \end{bmatrix},$$

where

$$a(z) = \frac{1}{2\pi} \int\limits_{\mathbb{T}} \frac{1}{e(q) - z} dq \quad \text{and} \quad b(z) = \frac{1}{2\pi} \int\limits_{\mathbb{T}} \frac{e^{ikq}}{e(q) - z} dq = \frac{1}{2\pi} \int\limits_{\mathbb{T}} \frac{\cos(kq)}{e(q) - z} dq.$$

For any $\lambda, \mu \in \mathbb{C}$, we define the Fredholm determinant associated with the operator $I - \mathcal{T}(z)$ as an analytic function in $z \in \mathbb{C} \setminus [0.2]$ as

$$D(\lambda, \mu, z) = \Delta_\mu(z)\Delta_\lambda(z) - \lambda\mu b^2(z), \tag{2}$$

where $\Delta_\mu(z) = 1 - \mu a(z)$ and $\Delta_\lambda(z) = 1 - \lambda a(z)$.

Lemma 1 (Birman–Schwinger principle for $z \in \mathbb{C} \setminus [0.2]$**).**

(1) The number $z \in \mathbb{C} \setminus [0.2]$ *is an eigenvalue of* $H_{\lambda\mu}$ *if only if* $1 \in \sigma(\mathcal{T}(z))$.

(2) Let $\det\big(\mathcal{T}(z)-I\big)=0$ *for* $z\in\mathbb{C}\setminus[0.2]$ *and* $(\lambda,\mu)\in\mathbb{R}^2$*, i.e.,* $D(\lambda,\mu,z)=0$*. Then, the vector* $\omega=(\omega_0,\omega_1)^T\in\mathbb{C}^2$ *is an eigenvector of* $\mathcal{T}(z)$ *associated with the eigenvalue* 1 *if and only if*

$$f=\frac{\mu\omega_0+\lambda\omega_1 e^{-ikp}}{e(p)-z}$$

is an eigenfunction of $H_{\lambda\mu}$ *associated with the eigenvalue* z*.*

Proof. (1) Let $H_{\lambda\mu}f=zf$, $z\in\mathbb{C}\setminus[0.2]$, then $\big(e(p)-z\big)f=V_{\lambda\mu}f$, i.e.,

$$f(p)=\frac{\mu c_1}{2\pi}\frac{1}{e(p)-z}+\frac{\lambda c_2}{2\pi}\frac{e^{-ikp}}{e(p)-z},$$

where $c_1=\int\limits_{\mathbb{T}} f(q)dq$ and $c_2=\int\limits_{\mathbb{T}} e^{-ikq}f(q)dq$. Integrating both sides with respect to the variable $p\in\mathbb{T}$ two times: first as it is and then by successively multiplying all terms by e^{ikp}, we obtain a system of linear equations with respect to c_1 and c_2,

$$\begin{cases} c_1=c_1\mu a(z)+c_2\lambda b(z),\\ c_2=c_1\mu b(z)+c_2\lambda a(z),\end{cases}$$

i.e., $\mathcal{T}(z)(c_1,c_2)^T=(c_1,c_2)^T$, that is 1 is an eigenvalue of $\mathcal{T}(z)$.

On the other hand, if 1 is an eigenvalue of $\mathcal{T}(z)$ corresponding to the eigenvector $(c_0,c_1)^T$, i.e.,

$$\begin{pmatrix}\mu a(z) & \lambda b(z)\\ \mu b(z) & \lambda a(z)\end{pmatrix}\begin{pmatrix}c_0\\ c_1\end{pmatrix}=\begin{pmatrix}c_0\\ c_1\end{pmatrix},$$

then, it can be easily verified that for

$$\psi(p)=\frac{\mu c_0}{e(p)-z}+\frac{\lambda c_1 e^{-ikp}}{e(p)-z},$$

we have $H_{\lambda\mu}\psi=z\psi$, i.e., z is an eigenvalue of $H_{\lambda\mu}$. This proves the second statement of the lemma too. □

4.1. Properties of the Fredholm Determinant

As $b(z)$ is even with respect to the parameter k, hereafter, we assume $k\geq 1$. In the following lemma, we study the properties of the functions $a(z)$ and $b(z)$.

Lemma 2. *(a) For the functions* $a(z)$ *and* $b(z)$*, the followings hold*

$$a(z)=\frac{1}{\sqrt{z^2-2z}},\quad z\in(-\infty,0)\cup(2,+\infty)$$

and

$$b(z)=\frac{\Big(1-z-\sqrt{z^2-2z}\Big)^k}{\sqrt{z^2-2z}},\quad z\in(-\infty,0)\cup(2,+\infty).$$

(b) Functions $a(z)$ *and* $b(z)$ *satisfy the relations* $a(z)>b(z)$ *and* $a'(z)>b'(z)$*.*

Proof. (a) According to the residue theorem we have

$$\int\limits_{|z|=1}\frac{f(z)}{g(z)}dz=2\pi i\frac{f(z_0)}{g'(z_0)},$$

where $f(z)$ and $g(z)$ are analytic functions in the unit ball $B=\{z\in\mathbb{C}:|z|<1\}$, and z_0 is the only zero of the function $g(z)$ in B.

We derive a formula for $b(z)$. Let us make the following change of variables in the integral $b(z)$,

$$e^{iq}=\eta,\quad e^{-iq}=\frac{1}{\eta},\quad ie^{iq}dq=d\eta,\quad dq=\frac{d\eta}{i\eta}.$$

Then, we obtain

$$b(z)=\frac{1}{2\pi}\int\limits_{\mathbb{T}}\frac{e^{ikq}}{e(q)-z}dq=\frac{1}{2\pi}\int\limits_{|\eta|=1}\frac{2\eta^k}{\left[2-2z-\left(\eta+\frac{1}{\eta}\right)\right]}\frac{d\eta}{i\eta}=\frac{i}{\pi}\int\limits_{|\eta|=1}\frac{\eta^k d\eta}{\eta^2-(2-2z)\eta+1}.$$

The residue theorem yields that

$$b(z)=\frac{i}{\pi}\,2\pi i\,\frac{\left(1-z-\sqrt{z^2-2z}\right)^k}{2\left(1-z-\sqrt{z^2-2z}\right)-2+2z}=\frac{\left(1-z-\sqrt{z^2-2z}\right)^k}{\sqrt{z^2-2z}},$$

particularly, when $k=0$, we have

$$a(z)=\frac{1}{2\pi}\int\limits_{\mathbb{T}}\frac{dq}{e(q)-z}=\frac{1}{\sqrt{z^2-2z}}.$$

(b) The proof of the second part of the lemma follows from the relations

$$\frac{d^{(n)}a(z)}{dz^n}=(n-1)!\int\limits_{\mathbb{T}}\frac{dq}{\left(e(q)-z\right)^{n+1}}\quad\text{and}\quad\frac{d^{(n)}b(z)}{dz^n}=(n-1)!\int\limits_{\mathbb{T}}\frac{e^{ikq}}{\left(e(q)-z\right)^{n+1}}dq$$

for any $n\in\mathbb{N}$, where $(\cdot)^{(n)}$ stands for the nth order derivative. Then,

$$\frac{d^{(n)}a(z)}{dz^n}>\frac{d^{(n)}b(z)}{dz^n}.$$

□

Let us rewrite the Fredholm determinant (2) as

$$D(\lambda,\mu,z)=\left(a^2(z)-b^2(z)\right)H_z(\lambda,\mu),$$

where

$$H_z(\lambda,\mu)=\left(\lambda-\gamma(z)\right)\left(\mu-\gamma(z)\right)-\xi^2(z)$$

with

$$\gamma(z)=\frac{a(z)}{a^2(z)-b^2(z)}\quad\text{and}\quad\xi(z)=\frac{b(z)}{a^2(z)-b^2(z)},\quad z\in\mathbb{C}\setminus[0.2].$$

Lemma 3. *a) The function $\gamma(z)$ is monotonically decreasing in the interval $(-\infty,0)$.*
b) The following relations are appropriate

$$\lim_{z\to-\infty}\gamma(z)=+\infty,\quad\lim_{z\to0-}\gamma(z)=\frac{1}{2k}.\tag{3}$$

Proof. a) Consider the derivative

$$\gamma'(z)=\frac{a'(z)\left(a^2(z)-b^2(z)\right)-a(z)2\left(a(z)a'(z)-b(z)b'(z)\right)}{\left(a^2(z)-b^2(z)\right)^2}.$$

Here $a(z)>b(z)>0$ and $a'(z)>b'(z)>0$, and $\left(a^2(z)-b^2(z)\right)^2>0$ follows. Furthermore

$$a'(z)\left(a^2(z)-b^2(z)\right)-a(z)2\left(a(z)a'(z)-b(z)b'(z)\right)=2a(z)b(z)b'(z)-a'(z)\left(a^2(z)+b^2(z)\right)$$
$$\leq 2a(z)b(z)a'(z)-a'(z)\left(a^2(z)+b^2(z)\right)=-a'(z)(a(z)-b(z))^2<0.$$

Therefore, $\gamma(z)$ as a function of z is monotonically decreasing in the interval $(-\infty,0)$.

b) Using the relations

$$a(z)=\frac{1}{\sqrt{z^2-2z}} \quad \text{and} \quad b(z)=\frac{\left(1-z-\sqrt{z^2-2z}\right)^k}{\sqrt{z^2-2z}},$$

we obtain the following limits

$$\lim_{z\to-\infty}\gamma(z)=\lim_{z\to-\infty}\frac{a(z)}{a^2(z)-b^2(z)}=\lim_{z\to-\infty}\frac{\sqrt{z^2-2z}}{1-\left(1-z-\sqrt{z^2-2z}\right)^{2k}}$$

$$=\lim_{z\to-\infty}\frac{-z\sqrt{1-\frac{2}{z}}}{\left(1-\left(1-z-z\sqrt{1-\frac{2}{z}}\right)^k\right)\left(1+\left(1-z-z\sqrt{1-\frac{2}{z}}\right)^k\right)}=+\infty,$$

$$\lim_{z\to0-}\gamma(z)=\lim_{z\to0-}\frac{a(z)}{a^2(z)-b^2(z)}=\lim_{z\to0-}\frac{\sqrt{z^2-2z}}{1-\left(1-z-\sqrt{z^2-2z}\right)^{2k}}$$

$$=\lim_{z\to0-}\frac{z-1}{-2k\left(1-z-\sqrt{z^2-2z}\right)}=\frac{1}{2k}.$$

□

Next, we investigate the function $\xi(z)$.

Lemma 4. *a) If $k=1$, then $\xi(z)=const$.*

b) If $k>1$, then $\xi(z)$ is monotonically increasing in the interval $(-\infty,0)$.

c) The following relations hold

$$\lim_{z\to-\infty}\xi(z)=+\infty, \quad \lim_{z\to0-}\xi(z)=\frac{1}{2k}. \tag{4}$$

Proof. a) Let $k=1$. Then,

$$\xi^2(z)=-\frac{z^2-2z}{4z-2z^2+2(1-z)\sqrt{z^2-2z}}+\frac{1}{z^2-2z}\frac{(z^2-2z)^2}{\left(\sqrt{z^2-2z}\big(2(1-z)-2\sqrt{z^2-2z}\big)\right)^2}$$

$$=\frac{2(z^2-2z)-2(1-z)\sqrt{z^2-2z}+1}{4(1-z)^2-8(1-z)\sqrt{z^2-2z}+4(z^2-2z)}=\frac{1}{4}.$$

b) Let $k>1$. On the right hand side of the equalities in Lemma 2, we make a change of variables $\omega=1-z-\sqrt{z^2-2z}$, where ω satisfies the equalities

$$\frac{1}{\omega}+\omega=2(1-z), \quad \frac{1}{\omega}-\omega=2\sqrt{z^2-2z} \quad \text{and} \quad 0<\omega<1 \quad \text{for} \quad z<0.$$

Hence, $\xi(z)$ takes the form

$$\xi_1(\omega):=\xi(z)=\frac{1}{2}\frac{\omega^{k-1}-\omega^{k+1}}{1-\omega^{2k}}, \quad \omega\in(0,1). \tag{5}$$

The last function can be written in a more convenient way as

$$\frac{\omega^{k-1}-\omega^{k+1}}{1-\omega^{2k}}=\frac{\omega^{k-1}(1-w^2)}{1-\omega^{2k}}=\frac{\omega^{k-1}}{1+\omega^2+\omega^4+\cdots+\omega^{2k-2}}.$$

For the derivative of the function $\xi_1(\omega)$, we have

$$\xi_1'(\omega)=\frac{1}{2}\frac{(k-1)\omega^{k-2}\big(1+\omega^2+\omega^4+\cdots+\omega^{2k-2}\big)-\omega^{k-1}\big(2\omega+4\omega^3+\cdots+(2k-2)\omega^{2k-1}\big)}{\big(1+\omega^2+\omega^4+\cdots+\omega^{2k-2}\big)^2}$$

$$= \frac{w^{k-2}}{2} \frac{\Big((k-1)+(k-3)\omega^2+(k-5)\omega^4+\cdots+\big(k-(2k-1)\big)\omega^{2k}\Big)}{\left(1+\omega^2+\omega^4+\cdots+\omega^{2k-2}\right)^2}.$$

If k is odd, then

$$(k-1)+(k-3)\omega^2+(k-5)\omega^4+\cdots+\big(k-(2k-1)\big)\omega^{2k}$$
$$=(k-1)+(k-3)\omega^2+(k-5)\omega^4+\cdots+2\omega^{k-1}-2\omega^{k+1}-\cdots-(k-3)\omega^{2k-2}-(k-1)\omega^{2k}$$
$$=(k-1)(1-\omega^{2k})+(k-3)(\omega^2-\omega^{2k-2})+\cdots+2(\omega^{k-1}-\omega^{k+1}).$$

If k is even, then

$$(k-1)+(k-3)\omega^2+(k-5)\omega^4+\cdots+\big(k-(2k-1)\big)\omega^{2k}$$
$$=(k-1)+(k-3)\omega^2+(k-5)\omega^4+\cdots+\omega^{k-1}-\omega^{k+1}-\cdots-(k-3)\omega^{2k-2}-(k-1)\omega^{2k}$$
$$=(k-1)\left(1-\omega^{2k}\right)+(k-3)\left(\omega^2-\omega^{2k-2}\right)+\cdots+\left(\omega^{k-1}-\omega^{k+1}\right).$$

As $0<\omega<1$, for $n>m$, we have $\omega^n<\omega^m$. Therefore, in both cases all the terms are positive, i.e., $\xi_1(\omega)>0$. This fact, the chain rule

$$\frac{d\xi(z)}{dz}=\frac{d\xi_1(\omega)}{d\omega}\frac{d\left(1-z-\sqrt{z^2-2z}\right)}{dz}$$

and the relation

$$\left(1-z-\sqrt{z^2-2z}\right)'=-1-\frac{z-1}{\sqrt{z^2-2z}}>0, \quad z\in(-\infty,0),$$

imply that the function $\xi(z)$ is monotonically increasing in the interval $(-\infty,0)$.

c) Equation (5) yields the limit

$$\lim_{z\to-\infty}\xi(z)=\lim_{\omega\to 0+}\frac{\omega^{k-1}}{2}\frac{1-\omega^2}{1-\omega^{2k}}=0.$$

Also, we have that

$$\lim_{z\to 0-}\xi(z)=\lim_{w\to 1-}\frac{1}{2}\frac{w^{k-1}}{\sum_{j=0}^{k-1}w^{2j}}=\frac{1}{2k}.$$

□

4.2. Continuation of $H_z(\lambda,\mu)$ to the Point $z=0$

Using the limits (3) in Lemma 3 and (4) in Lemma 4, we can introduce the continuation of the function $H_z(\lambda,\mu)=(\lambda-\gamma(z))(\mu-\gamma(z))-\xi^2(z)$ at the point $z=0$ as

$$H_0(\lambda,\mu)=\left(\lambda-\frac{1}{2k}\right)\left(\mu-\frac{1}{2k}\right)-\frac{1}{4k^2}.$$

Let us study the family of rectangular hyperbolas $\mathcal{H}_z=\{(\lambda,\mu)\in\mathbb{R}^2|H_z(\lambda,\mu)=0\}$ for $z\in(-\infty,0]$. Denote the left (lower) and right (upper) branches of hyperbola $\mathcal{H}_z$ by Γ_z^l and Γ_z^r, respectively.

Lemma 5. *a) Let $k=1$. If z approaches $-\infty$ from 0, then the hyperbola $\mathcal{H}_z$ parallelly shifts to the upper right corner of the $\lambda\mu$-coordinate plane (see Fig. 1).*

b) Let $k>1$. If z approaches $-\infty$ from 0, the hyperbola $\mathcal{H}_z$ shifts to the upper right corner of the $\lambda\mu$-coordinate plane, but the concavity of a rectangular hyperbola increases (see Fig. 2).

c) Additionally, for any $z_1\neq z_2\in(-\infty,0)$, the respective branches do not intersect, i.e., $\Gamma_{z_1}^l\bigcap\Gamma_{z_2}^l=\varnothing$ and $\Gamma_{z_1}^r\bigcap\Gamma_{z_2}^r=\varnothing$.

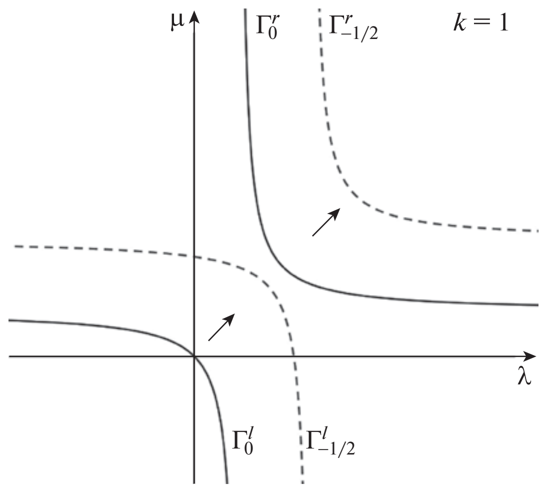

Fig. 1. Motion of hyperbolas for $k = 1$.

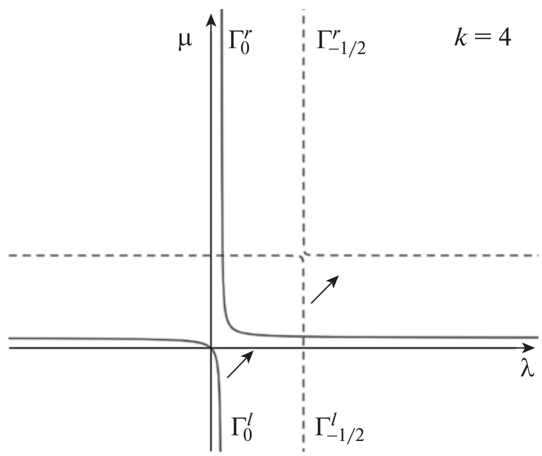

Fig. 2. Motion of hyperbolas for $k > 1$.

Proof. a) According to Lemma 5, $\xi(z) = const$ and hyperbola $\mathcal{H}_z$ is obtained by a parallel movement of $\mathcal{H}_0$ along the vector $\big(-\gamma(0)+\gamma(z), -\gamma(0)+\gamma(z)\big)$,

$$H_z(\lambda,\mu) = H_0(\lambda,\mu) + \big(\lambda-\gamma(z)\big)\big(\mu-\mu(z)\big) - \xi(z)$$
$$= \big(\lambda-\gamma(z)+\gamma(0)-\gamma(0)\big)\big(\mu-\gamma(z)+\gamma(0)-\gamma(0)\big) - \xi(z)$$
$$= H_0\big(\lambda-\gamma(z)+\gamma(0), \mu-\gamma(z)+\gamma(0)\big).$$

From Lemma 3, $\lim_{z\to-\infty}\gamma(z) = +\infty$ and the function $\gamma(z)$ is monotonically decreasing. Therefore, hyperbola $\mathcal{H}_z$ shifts parallelly upwards when z approaches to $-\infty$ from 0.

b) The coordinates of vertices of the hyperbola are

$$(\lambda_1,\mu_1) = \left(\frac{1}{a(z)+b(z)}, \frac{1}{a(z)+b(z)}\right) \quad \text{and} \quad (\lambda_2,\mu_2) = \left(\frac{1}{a(z)-b(z)}, \frac{1}{a(z)-b(z)}\right).$$

As the functions

$$1/\big(a(z)+b(z)\big) \quad \text{and} \quad 1/\big(a(z)-b(z)\big)$$

are monotonically decreasing in the interval $(-\infty, 0)$, the vertices of the hyperbola $\mathcal{H}_z$ move to the right as z approaches to $-\infty$.

Also, from Lemma 3, the function $\gamma(z)=\dfrac{a(z)}{a^2(z)-b^2(z)}$ is monotonically decreasing in the interval $(-\infty,0)$. Hence, the asymptotes of the hyperbola $\mathcal{H}_z$, the functions

$$\lambda(z)=\frac{a(z)}{a^2(z)-b^2(z)} \quad \text{and} \quad \mu(z)=\frac{a(z)}{a^2(z)-b^2(z)}$$

move upwards as z approaches to $-\infty$ from 0, but not parallelly (see Fig. 2).

c) The proof follows the parts a) and b). □

Definition 1. *(Threshold eigenvalue and threshold resonance). Let the function f be a solution of the equation $H_{\lambda\mu}f=0$.*

a) If $f\in L^2(\mathbb{T})$, 0 is called a lower threshold eigenvalue.

b) If $f\in L^1(\mathbb{T})\setminus L^2(\mathbb{T})$, 0 is called a lower threshold resonance.

c) If $f\in L^\varepsilon(\mathbb{T})\setminus L^1(\mathbb{T})$ for any ε $(0<\varepsilon<1)$, 0 is called a lower super threshold resonance.

Theorem 1. *(a) $z\in(-\infty,0)$ is an eigenvalue of $H_{\lambda\mu}$ if and only if $(\lambda,\mu)\in\Gamma_z^l$* or *$(\lambda,\mu)\in\Gamma_z^r$.*

(b) For any $(\lambda,\mu)\in\mathbb{R}^2$, $H_{\lambda\mu}$ has neither threshold resonances nor super-threshold resonances.

Proof. (a) Assume $(\lambda,\mu)\in\Gamma_z^l$ or $(\lambda,\mu)\in\Gamma_z^r$. Then, $H_z(\lambda,\mu)=0$ that is $D(\lambda,\mu,z)=0$. According to Lemma 1, z is an eigenvalue for the operator $H_{\lambda\mu}$.

(b) Let $f\in L^1(\mathbb{T})$. Consider the equation $H_{\lambda\mu}f=0$. Then, from the relation

$$e(p)f(p)-\frac{\mu}{2\pi}\int\limits_{\mathbb{T}}f(q)dq-\frac{\lambda}{2\pi}\int\limits_{\mathbb{T}}e^{ik(p-q)}f(q)dq=0,$$

we have $f=\frac{\varphi(p)}{e(p)}$, where $\varphi(p)=\mu C_0+\lambda C_1e^{-ikp}$.

As $e(0)=0$, from the inclusion $f(p)\in L^1(\mathbb{T})$ it follows that $\varphi(0)=\mu C_0+\lambda C_1=0$. Hence,

$$f(p)=\frac{\mu}{e(p)}\Big(1-e^{-ikp}\Big)C_0.$$

According to the definition of $e(p)$, the real part of the function

$$\frac{1-e^{-ikp}}{e(p)}=\frac{1-\cos(kp)}{e(p)}+\frac{\sin(kp)}{e(p)}i$$

is integrable, but the imaginary part is not. Therefore, the inclusion $f\in L^1(\mathbb{T})$ yields that $\mu C_0=0$, i.e., $f=0$.

Let $f\notin L^1(\mathbb{T})$. Then, $\mu=0$ and $f=\varphi(p)/e(p)$ with $\varphi(p)=\lambda C_1e^{-ikp}$. From these, we have

$$C_1=\frac{\lambda}{2\pi}\int\limits_{\mathbb{T}}\frac{C_1}{e(p)}dp.$$

Then, $C_1=0$, since $\int\limits_{\mathbb{T}}\dfrac{1}{e(p)}dp=\infty$. Therefore, $f=0$. □

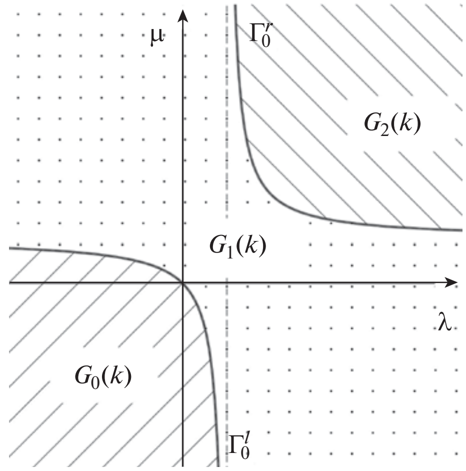

Fig. 3. Branches of the Hyperbola $\mathcal{H}_0$ and the connected components.

5. MAIN RESULTS

The curves Γ_0^l and Γ_0^r split the space $\mathbb{R}^2$ into three open sets (see Fig. 3),

$$G_0(k) = \{(\lambda, \mu) \in \mathbb{R}^2 : H_0(\lambda, \mu) > 0, \lambda < 1/2k\},$$
$$G_1(k) = \{(\lambda, \mu) \in \mathbb{R}^2 : H_0(\lambda, \mu) < 0\},$$
$$G_2(k) = \{(\lambda, \mu) \in \mathbb{R}^2 : H_0(\lambda, \mu) > 0, \lambda > 1/2k\}.$$

The following lemma follows from the definitions of $G_0(k)$ and $G_2(k)$.

Lemma 6. *If $k > m$, then $G_0(k) \subset G_0(m)$ and $G_2(k) \supset G_2(m)$.*

In the next lemma we study zeros of the the operator $H_{\lambda\mu}$ in the interval $(-\infty, 0)$.

Lemma 7. *Let $k \in \mathbb{Z}$ be a fixed positive integer.*

a) If $(\lambda, \mu) \in G_0(k) \cup \Gamma_0^l$, then $H_{\lambda\mu}$ has no eigenvalues in $(-\infty, 0)$;

b) If $(\lambda, \mu) \in G_1(k) \cup \Gamma_0^r$, then $H_{\lambda\mu}$ has a unique simple eigenvalue $z_k \in (-\infty, 0)$, where $z_k = z_{min}$ if $\lambda\mu = 0$, $z_k \in (-\infty, z_{min})$ if $\lambda\mu > 0$, and $z_k \in (z_{min}, 0)$ if $\lambda\mu < 0$;

c) If $(\lambda, \mu) \in G_2(k)$, then $H_{\lambda\mu}$ has two eigenvalues z_k, ζ_k such that $-\infty < z_k < z_{\min} \leq z_{\max} < \zeta_k < 0$.

Proof. According to Lemma 1, $H_{\lambda\mu}$ has an eigenvalue $z \in (-\infty, 0)$, iff $D(\lambda, \mu, z) = 0$, equivalently iff $(\lambda, \mu) \in \Gamma_z^l$ or $(\lambda, \mu) \in \Gamma_z^r$.

a) Let $(\lambda, \mu) \in G_0(k) \cup \Gamma_0^l$. Then, (λ, μ) lies below or on Γ_0^l. From Lemma 5 it follows that Γ_z^l lies in $\Gamma_0^r \cup G_1(k) \cup G_2(k)$ for any $z \in (-\infty, 0)$. Therefore, for any $(\lambda, \mu) \in G_0(k) \cup \Gamma_0^l$, the operator $H_{\lambda\mu}$ has no eigenvalues in the interval $(-\infty, 0)$.

b) Let $(\lambda, \mu) \in G_1(k) \cup \Gamma_0^r$. From Lemma 5, we have $\cup_{\zeta \in (-\infty, 0)} \Gamma_\zeta^l \supset G_1(k)$, $\Gamma_z^l \cap G_1(k) \neq \varnothing$ and $\Gamma_z^r \cap G_1(k) = \varnothing$ for any $z \in (-\infty, 0)$. Also, for $z \neq \zeta \in (-\infty, 0)$, the relation $\Gamma_z^l \cap \Gamma_\zeta^l = \varnothing$ holds. Accordingly, for any $(\lambda, \mu) \in \Gamma_0^l$, the operator $H_{\lambda\mu}$ has a unique eigenvalue in $(-\infty, 0)$.

Let $\lambda\mu = 0$, without loss of generality let $\lambda = 0$, then from the inclusion $(\lambda, \mu) \in G_1(k) \cup \Gamma_0^r$, we have $\mu > 0$. Then, according to the definition of $D(\lambda, \mu, z)$, $D(0, \mu, z_{\min}) = 0$, i.e., $z_k = z_{\min}$.

Let $\lambda\mu > 0\,(< 0)$, then

$$D(\lambda, \mu, z_{\min}) = -\lambda\mu \frac{b^2(z_{\min})}{a^2(z_{\min}) - b^2(z_{\min})} < 0\,(> 0).$$

As $D(\lambda, \mu, z)$ has a unique eigenvalue, from the limit $\lim_{z \to -\infty} D(\lambda, \mu, z) = 1$, we obtain that $z_k \in (-\infty, z_{\min})$ $\big(z_k \in (z_{\min}, 0)\big)$.

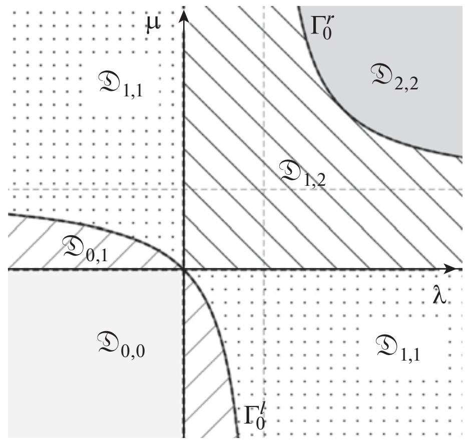

Fig. 4. Connected components.

c) Now let $(\lambda,\mu)\in G_2(k)$. From Lemma 5, we have $\cup_{z\in(-\infty,0)}\Gamma_z^l\supset G_2(k)$ and $\cup_{z\in(-\infty,0)}\Gamma_z^r= G_2(k)$. Moreover, for $z\neq\zeta\in(-\infty,0)$, we have $\Gamma_z^l\cap\Gamma_\zeta^l=\varnothing$, $\Gamma_z^r\cap\Gamma_\zeta^r=\varnothing$ and $\Gamma_z^l\cap\Gamma_z^r=\varnothing$. Therefore, there exist $z\neq\zeta\in(-\infty,0)$ such that $(\lambda,\mu)\in\Gamma_z^l$ and $(\lambda,\mu)\in\Gamma_\zeta^r$. That is, $H_{\lambda\mu}$ has exactly two eigenvalues in $(-\infty,0)$.

We have shown the existence of two eigenvalues $z_k<\zeta_k\in(-\infty,0)$. Next we investigate their location. As $(\lambda,\mu)\in G_2(k)$, $H_0(\lambda,\mu)>0$ and $D(\lambda,\mu,z)=\big(a^2(z)-b^2(z)\big)H_z(\lambda,\mu)$, we obtain the limits

$$\lim_{z\to0-}D(\lambda,\mu,z)=+\infty\quad\text{and}\quad\lim_{z\to-\infty}D(\lambda,\mu,z)=1.$$

Moreover, in $G_2(k)$, we have $\lambda,\mu>0$, therefore,

$$D(\lambda,\mu,z_{\min})=-\lambda\mu\frac{b^2(z_{\min})}{a^2(z_{\min})-b^2(z_{\min})}<0$$

and

$$D(\lambda,\mu,z_{\max})=-\lambda\mu\frac{b^2(z_{\max})}{a^2(z_{\max})-b^2(z_{\max})}<0.$$

Therefore, $D(\lambda,\mu,z)$ has different signs at the edges of the intervals $(-\infty,z_{\min})$ and $(z_{\max},0)$. Consequently, $z_k\in(-\infty,z_{\min})$ and $\zeta_k\in(z_{\max},0)$. □

Let us define the following connected components (see Fig. 4).

$$\mathcal{D}_{0,0}=\big\{(\lambda,\mu)\in\mathbb{R}^2|\lambda\le0,\mu\le0\big\},\quad\mathcal{D}_{0,1}=G_1(1)\cup\Gamma_0^l\setminus\mathcal{D}_{0,0},$$
$$\mathcal{D}_{1,1}=\big\{(\lambda,\mu)\in\mathbb{R}^2|\lambda+\mu>0,\lambda\mu\le0\big\},$$
$$\mathcal{D}_{1,2}=\big\{(\lambda,\mu)\in G_1(k)|\lambda>0,\mu>0\big\}\cup\Gamma_0^r,\quad\mathcal{D}_{2,2}=G_2(k).$$

Theorem 2. *The following statements are true for zeros of the function $D(\lambda,\mu,z)$ in $(-\infty,0)$.*

(a) For any $k\in\mathbb{Z}$ and $(\lambda,\mu)\in\mathcal{D}_{0,0}$, $D(\lambda,\mu,z)$ has no zeros;

(b) If $(\lambda,\mu)\in\mathcal{D}_{0,1}$, then there exists a number $k_0\in\mathbb{N}$ such that $D(\lambda,\mu,z)$ has no zeros for $k<k_0$ and has a unique zero $z_k\in(-\infty,0)$ for $k\ge k_0$;

(c) If $(\lambda,\mu)\in\mathcal{D}_{1,1}$, then for all $k\in\mathbb{Z}$, $D(\lambda,\mu,z)$ has a unique zero $z_k\in(-\infty,0)$;

(d) If $(\lambda,\mu)\in\mathcal{D}_{1,2}$, then there exists a number $k_0\in\mathbb{N}$ such that $D(\lambda,\mu,z)$ has unique zero $z_k\in(-\infty,z_{min})$ for $k<k_0$ and has two zeros $z_k\in(-\infty,z_{min})$ and $\zeta_k\in(z_{max},0)$ for $k\ge k_0$;

(e) If $(\lambda,\mu)\in\mathcal{D}_{2,2}$, then $D(\lambda,\mu,z)$ has two distinct zeros $z_k\in(-\infty,z_{min})$ and $\zeta_k\in(z_{\max},0)$.

Proof. (a) Let $(\lambda,\mu)\in\mathcal{D}_{0,0}$. As $\mathcal{D}_{0,0}=\bigcap_{k=1}^\infty G_0(k)$ (see Fig. 5), we have $(\lambda,\mu)\in\bigcap_{k=1}^\infty G_0(k)$. Therefore, from Lemma 7, $D(\lambda,\mu,z)$ has no zeros in $(-\infty,0)$ for any $k\in\mathbb{N}$.

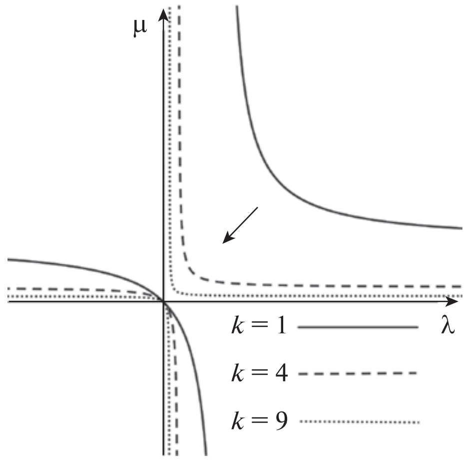

Fig. 5. Motion of hyperbolas with respect to k for $z=0$.

(b) Let $(\lambda,\mu)\in\mathcal{D}_{0,1}$. Then, from Lemma 6, we have the relations $G_0(1)\supset G_0(2)\supset G_0(3)\supset\cdots$. Also, we have $\mathcal{D}_{0,1}=\bigcup_{k=1}^{\infty}G_0(k)\setminus G_0(k+1)$. Therefore, there exists a number $k_0\in\mathbb{N}$ such that $(\lambda,\mu)\in G_0(k_0)\setminus G_0(k_0+1)$, that is

$$(\lambda,\mu)\in\bigcap_{k=1}^{k_0}G_0(k)\quad\text{and}\quad(\lambda,\mu)\in\bigcap_{k=k_0+1}^{\infty}G_1(k).$$

Then, from Lemma 7, $D(\lambda,\mu,z)$ has a unique zero in $(-\infty,z_{\min})$ for all $k>k_0$, and has no zeros if $k\le k_0$.

(c) Let $(\lambda,\mu)\in\mathcal{D}_{1,1}$, then $(\lambda,\mu)\in\bigcap_{k=1}^{\infty}G_1(k)$, therefore, from Lemma 7, $D(\lambda,\mu,z)$ has a unique zero $z_k\in(-\infty,z_{\min})$ for all $k\in\mathbb{N}$.

(d) Let $(\lambda,\mu)\in\mathcal{D}_{1,2}$. Then, from Lemma 6, we have the relations $G_2(1)\subset G_2(2)\subset G_2(3)\subset\cdots$. Also, $\mathcal{D}_{1,2}=\bigcup_{k=1}^{\infty}G_2(k+1)\setminus G_2(k)$. Therefore, there exists a number $k_0\in\mathbb{N}$ such that $(\lambda,\mu)\in G_2(k_0+1)\setminus G_2(k_0)$, that is

$$(\lambda,\mu)\in\bigcap_{k=1}^{k_0}G_1(k)\quad\text{and}\quad(\lambda,\mu)\in\bigcap_{k=k_0+1}^{\infty}G_2(k).$$

Then, from Lemma 7, $D(\lambda,\mu,z)$ two zeros $z_k\in(-\infty,z_{\min})$ and $\zeta_k\in(z_{\max},0)$ for $k>k_0$, and a unique zero $z_k\in(-\infty,z_{\min})$ for $k\le k_0$.

(e) Let $(\lambda,\mu)\in\mathcal{D}_{2,2}$. As $\mathcal{D}_{2,2}=\bigcap_{k=1}^{\infty}G_2(k)$, from Lemma 7, $D(\lambda,\mu,z)$ two zeros $z_k\in(-\infty,z_{\min})$ and $\zeta_k\in(z_{\max},0)$ for all $k\in\mathbb{N}$. $\square$

6. ASYMPTOTICS FOR EIGENVALUES OF THE OPERATOR $H_{\lambda\mu}$

In this section, we study asymptotic behavior of the eigenvalues z_k and ζ_k, as $k\to\infty$, when they exist.

Let z_λ and z_μ be zeros of the function $\Delta_\lambda(z)$ and $\Delta_\mu(z)$, respectively, i.e.,

$$1-\lambda a(z_\lambda)=0\quad\text{and}\quad 1-\mu a(z_\mu)=0.$$

Then, z_μ and z_λ are eigenvalues of the the discrete Schrödinger operators

$$\widehat{H}_{0\mu k}=\widehat{H}_0-\mu\delta(x)\quad\text{and}\quad\widehat{H}_{\lambda 0k}=\widehat{H}_0-\lambda\delta(x-k),$$

respectively. Let

$$z_{\min}=\min\{z_\mu,z_\lambda\}=z_\lambda,\quad z_{\max}=\max\{z_\mu,z_\lambda\}=z_\mu.$$

Let us introduce the numbers

$$w_{\min} = 1 - z_{\min} - \sqrt{z_{\min}^2 - 2z_{\min}} \quad \text{and} \quad w_{\max} = 1 - z_{\max} - \sqrt{z_{\max}^2 - 2z_{\max}}.$$

Then, $0 < w_{\min}, w_{\max} < 1$.

Theorem 3. *(a) The eigenvalue z_k satisfies the asymptotic relation $z_k \approx 1 - \frac{1}{2}\left(w_k + \frac{1}{w_k}\right)$, where*

$$w_k = w_{\min} - \frac{\lambda\mu w_{\min}^{2k+1}}{(w_{\min}+\mu)(\mu-\lambda) + 2\lambda\mu(k+1)w_{\min}^{2k}}.$$

(b) ζ_k satisfies the asymptotics $\zeta_k \approx 1 - \frac{1}{2}\left(w_k + \frac{1}{w_k}\right)$, where

$$w_k = w_{\max} - \frac{\lambda\mu w_{\max}^{2k+1}}{(w_{\max}+\mu)(\lambda-\mu) + 2\lambda\mu(k+1)w_{\max}^{2k}}.$$

Proof. (a) Making the substitution $w = 1 - z - \sqrt{z^2 - 2z}$ on the right hand side of functions $a(z)$ and $b(z)$ in Lemma 2 and using the equalities

$$1 - z = \frac{1}{2}\left(\frac{1}{w} + w\right) \quad \text{and} \quad \sqrt{z^2 - 2z} = \frac{1}{2}\left(\frac{1}{w} - w\right),$$

we rewrite the functions a and b as

$$a(z) = a_1(w) = \frac{2w}{1-w^2}, \quad b(z) = b_1(w) = \frac{2w^{k+1}}{1-w^2}.$$

Then, equation

$$D(\lambda,\mu,z) = \Delta_\lambda(z)\Delta_\mu(z) - \lambda\mu b^2(z) = 0$$

is equivalent to a new equation of the variable w, $0 < w < 1$,

$$(1 - w^2 - 2\mu w)(1 - w^2 - 2\lambda w) = 4\lambda\mu w^{2k+2}.$$

Separating the linear part of the Taylor series of the left hand side of this equation around the point $w_{\min}$, we obtain

$$\left[-4(w_{\min}+\mu)(\mu-\lambda)w_{\min} - 8\lambda\mu(k+1)w_{\min}^{2k+1}\right](w - w_{\min}) = 4\lambda\mu w_{\min}^{2k+2}.$$

From the last equation for $w = w_k = 1 - z_k - \sqrt{z_k^2 - 2z_k}$, i.e., $z_k = 1 - \dfrac{1}{2}\left(w_k + \dfrac{1}{w_k}\right)$, we obtain

$$\begin{aligned} w_k &= w_{\min} - \frac{4\lambda\mu w_{\min}^{2k+2}}{4(w_{\min}+\mu)(\mu-\lambda)w_{\min} + 8\lambda\mu(k+1)w_{\min}^{2k+1}} \\ &= w_{\min} - \frac{\lambda\mu w_{\min}^{2k+1}}{(w_{\min}+\mu)(\mu-\lambda) + 2\lambda\mu(k+1)w_{\min}^{2k}} \end{aligned}$$

as $k \to \infty$.

The part (b) can be proven analogously. □

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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