Where is my book? — Burgers equation in an online bookstore ranking

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We study a hydrodynamic limit approach to move-to-front rules, a stochastic process of particles aligned in a line, each of which jumps randomly to the top. We find that a scaling limit of the joint empirical distribution of jump rate and position satisfies a system of Burgers type partial differential equations, which in turn is globally solved using a standard method of characteristic curves.

Besides known application to LRU-caching in computer sciences, we find a new application in the ranking number of books at an online bookstore Amazon.co.jp . Statistical fit of data to our theory reveals that that the bookstore's dominant source of sales is from a few bestsellers, rather than the majority of seldom sold books, contrary to the expectation of the 'long tail' business model.

1. Move-to-front rules. Let N be a positive integer, and define a Markov process $X^{(N)}(t) = (X_1^{(N)}(t), \cdots, X_N^{(N)}(t))$, $t \geqq 0$, with the state space a set of all permutations of $1, 2, \cdots, N$, as follows. $X^{(N)}(t)$ change values only at discrete random times (jump times) $\{\tau_{i,j}^{(N)} \mid j = 1, 2, \cdots, i = 1, 2, \cdots, N\}$. At a jump time $t = \tau_{i,j}$, we define $X_i^{(N)}(\tau_{i,j}) = 1$ and, for $i' \neq i$,

$$X_{i'}^{(N)}(\tau_{i,j}) = X_{i'}^{(N)}(\tau_{i,j}-0) + \begin{cases} 1, & \text{if } X_{i'}^{(N)}(\tau_{i,j}-0) < X_i^{(N)}(\tau_{i,j}-0), \\ 0, & \text{if } X_{i'}^{(N)}(\tau_{i,j}-0) > X_i^{(N)}(\tau_{i,j}-0). \end{cases}$$

We define the jump times to be independent in i, and for each i, the increments $\{\tau_{i,j+1}^{(N)} - \tau_{i,j}^{(N)},\ j = 0, 1, 2, \cdots\}$ $(\tau_{i,0}^{(N)} := 0)$ are i.i.d., whose distribution is the exponential distribution with parameter $w_i^{(N)} > 0$: $\mathrm{P}[\,\tau_{i,1}^{(N)} > t\,] = \exp(-w_i^{(N)} t)$. Note that as in the Poisson process, with probability 1 the jump times are different for different (i, j). In the following, we regard $X^{(N)}$ as an N particle system aligned on a single line, with $X_i^{(N)}(t)$ denoting the position of the particle i at time t.

2. Hydrodynamic limit. Consider a scaled position: $Y_i^{(N)}(t) := \dfrac{1}{N}(X_i^{(N)}(t) - 1) \in [0, 1)$. Note that $y_C^{(N)}(t) = \dfrac{1}{N}\sharp\{i \mid \tau_{i,1} \leqq t\}$ is the rightmost point in scaled position of particles which have jumped to the top position up to time t. In the following we denote by δ_a the unit distribution concentrated on a, and assume $\lambda^{(N)} := \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \delta_{w_i^{(N)}} \to \lambda$ weakly as $N \to \infty$, for a probability distribution λ.

Proposition 1([1]). $y_C^{(N)}(t) \to y_C(t) := 1 - \displaystyle\int_0^\infty e^{-wt}\,\lambda(dw)$ $(N \to \infty$, in prob.). $\diamond$

Consider a joint empirical distribution $\mu_t^{(N)} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \delta_{(w_i^{(N)}, Y_i^{(N)}(t))}$.

Theorem 2([1]). Assume $\displaystyle\int_0^\infty w\lambda(dw) < \infty$ and $\lambda(\{0\}) = 0$, and assume that the initial distribution $\mu_0^{(N)}$ determined by the initial configuration $Y^{(N)}(0) = y^{(N)}$ converges weakly to a distribution μ_0 as $N \to \infty$. Then for each $t > 0$, there exists a deterministic distribution μ_t such that $\mu_t^{(N)} \to \mu_t$ as $N \to \infty$. μ_t is given by

$$U(dw, y, t) := \mu_t(dw, [y, 1)) = \begin{cases} \lambda(dw)\, e^{-wt_0(y)}\,, & y < y_C(t), \\ U(dw, \hat{y}(y, t), 0)\, e^{-wt}\,, & y > y_C(t), \end{cases}$$

where, $t = t_0(y)$ is the inverse function of $y = y_C(t)$, and $\hat{y}(y,t)$ is the inverse function in y of $y_C(y,t) = 1 - \int_y^1 \int_0^\infty e^{-wt}\mu_0(dw,dz)$. $\diamond$

3. Burgers type equation. We found the explicit limit formula in **Theorem 2** as a solution to a following system of PDEs. For simplicity of notation, consider the case where there are at most countable types of jump rates; $\lambda = \sum_\alpha \rho_\alpha \delta_{f_\alpha}$, where f_α and ρ_α are positive constants, satisfying $\sum_\alpha \rho_\alpha = 1$.

Proposition 3([2]). $U_\alpha(y,t) := U(\{f_\alpha\},y,t) = \mu_t(\{f_\alpha\},[y,1))$ is a unique classical time global solution to the following initial value problem:

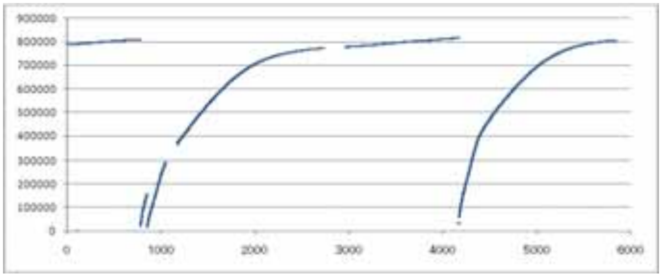
$$\frac{\partial U_\alpha}{\partial t}(y,t) + \sum_\beta f_\beta U_\beta(y,t)\frac{\partial U_\alpha}{\partial y}(y,t) = -f_\alpha U_\alpha(y,t),\ (y,t) \in [0,1) \times [0,\infty),\ \alpha \in \mathbb{N},$$

with boundary conditions $U_\alpha(0,t) = \rho_\alpha$, $t \geqq 0$, $\alpha \in \mathbb{N}$, and 'nice' initial data. $\diamond$

This system is solved by a standard method of characteristic curves, with explicit formula containing inverse functions such as t_0 of the characteristic curve y_C. As in the theory of hydrodynamic limit, PDE explains a structure of the limit formula.

4. Amazon.co.jp. Move-to-front rule first appeared in the literature in 1963, and was extensively studied as a model of least-recently-used caching in the context of data theory in computer sciences, to which our results give some new formulas.

We moreover found a new application in the sales ranking of an online bookstore Amazon.co.jp. For each book which the bookstore sells, there is a web page on which is a number, updated every hour, called sales ranking of the book. The bookstore does not disclose the definition of the number, but from a continuous observation of the number for a less popular book (hmm, such as a book on mathematics ...), we see that we may approximate its behavior by move-to-front rules (see the figure).



Note occasional large discontinuities, which correspond to the points that the book is purchased (as can be proved by ordering a book yourself). By performing a statistical fit of a continuous segment of ranking data to $y_C(t)$, choosing λ as Pareto distribution, we deduced that the total sales of Amazon.co.jp is dominated by those of a few bestsellers (such as Harry Potter's series) rather than a majority of unpopular books, contrary to the notion of 'long tail' business model, of which the Amazon bookstore has been considered as a pioneering company.

References.

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