MEAN-FIELD BEHAVIOUR AND THE LACE EXPANSION

TAKASHI HARA
Department of Applied Physics
Tokyo Institute of Technology
Oh-Okayama, Meguro-ku, Tokyo 152
Japan
e-mail: hara@appana.ap.titech.ac.jp

and

GORDON SLADE
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario
Canada L8S 4K1
e-mail: slade@aurora.math.mcmaster.ca

Abstract. These lectures describe the lace expansion and its role in proving mean-field critical behaviour for self-avoiding walks, lattice trees and animals, and percolation, above their upper critical dimensions. Diagrammatic conditions for mean-field behaviour are also outlined.

Key words: Lace expansion, self-avoiding walk, percolation, trees, lattice animals, mean-field behaviour, critical exponent, bubble diagram, square diagram, triangle condition, upper critical dimension.

1. Introduction

An important problem in statistical mechanics and probability theory is to prove existence of the critical exponents which characterize power-law behaviour in the vicinity of a phase transition. Typically the critical behaviour is dimension-dependent. For example, the susceptibility χ of the Ising model, one of the best-known models of statistical mechanics, is expected to behave as

$$\chi(T) \approx |T - T_c|^{-\gamma}, \tag{1.1}$$

where T_c is the critical temperature and γ is a critical exponent. It is a common feature of many models of statistical mechanics that there exists an *upper critical dimension*, above which the critical behaviour becomes simpler and dimension-independent. For example, above four dimensions the critical behaviour of the Ising model is the same as that of the *mean-field* model known as the Curie–Weiss model, in which spins interact not just with their neighbours but rather with the average of all other spins; in particular $\gamma = 1$. The term 'mean-field' has come also to apply to the behaviour of other statistical mechanical models above their upper critical dimensions, even when there is no explicit field or average. At the upper critical dimension $d = 4$ there are logarithmic corrections to mean-field behaviour and $\chi(T) \approx |T - T_c|^{-1}(\log|T - T_c|)^{1/3}$, while for $d < 4$ there are different power laws and $\gamma > 1$ in (1.1).

Similar behaviour is expected to hold for percolation and for combinatorial systems like the self-avoiding walk and lattice trees and animals. These lectures are concerned with the lace expansion (Brydges and Spencer 1985) and its application to prove existence of critical exponents for these models above their upper critical dimensions of respectively six, four and eight (five for oriented percolation). It remains an open problem to prove existence of critical exponents for these models at or below their upper critical dimensions, although there has been some recent progress in using renormalization group methods to study models related to the 4-dimensional self-avoiding walk in Brydges et al. (1992), Arnaudon et al. (1991), Iagolnitzer and Magnen (1992).

Being an expansion, the lace expansion requires a small parameter to ensure convergence. Here the small parameter will arise in two ways: as the reciprocal of the spatial dimension for nearest-neighbour models, or by considering sufficiently 'spread-out' models having suitable long-range connections. For the self-avoiding walk and for lattice trees and animals, the small parameter could alternatively be introduced through a weak interaction, as in the Domb–Joyce model. The spread-out models are believed to be in the same universality class as the corresponding nearest-neighbour models, and so are believed to have the same critical exponents. Studying sufficiently spread-out models allows for results for all dimensions $d > d_c$, where d_c is the upper critical dimension, whereas current methods give results for the nearest-neighbour models only for d somewhat greater than d_c. An exception is the nearest-neighbour self-avoiding walk, for which a computer-assisted proof has allowed for the treatment of dimensions $d \geq 5$. For lattice trees and animals there are results for the nearest-neighbour model in high dimensions (*how* high has not been computed), or for the spread-out model with $d > 8$. For percolation there are results for the nearest-neighbour model in sufficiently high dimensions (currently $d \geq 19$), and for the spread-out model in more than six dimensions. Oriented percolation has been treated by Nguyen and Yang (1992, 1993) for the nearest-neighbour model in sufficiently high dimensions and for sufficiently spread-out models above 4+1 dimensions. The lace expansion treats each of these models as a perturbation of a simple random walk model.

Although the lace expansion has proved to be sufficiently flexible to treat a variety of models, its use has been limited by its reliance on correlation inequalities to prove convergence. For the self-avoiding walk the correlation inequality is based on the repulsive nature of the self-avoidance interaction, while for percolation it is the BK inequality. The absence of similar correlation inequalities has hindered the application of the lace expansion to other models, such as the 'true' self-avoiding walk (whose interaction combines repulsive and attractive aspects). However there are encouraging indications that a certain attractive walk model can be treated using the lace expansion. In addition difficulties have been encountered in applying lace expansion methods to analyze the upper critical dimension itself, involving problems in reconciliation of the expansion with coupling-constant renormalization. Lower dimensions remain a major challenge for any method.

Historically, mean-field critical behaviour was first proven for Ising and other spin systems using two ingredients. The first is the *infra-red bound*, which is a bound on the behaviour near the origin of the Fourier transform of the two-point function at criticality (Fröhlich et al. 1976). The second ingredient is *correlation inequalities*, which when combined with the infra-red bound have important consequences for critical behaviour. The first manifestation of this combination is Sokal (1979), where it was shown that finiteness of the specific heat in more than four dimensions follows transparently from the infra-red bound. Later more sophisticated correlation inequalities were proven, which together with the infra-red bound led to proof of mean-field behaviour for the susceptibility, magnetization, and so on, in more than four dimensions. (See Aizenman 1982, Fröhlich 1982, Aizenman and Fernández 1986, and Fernández et al. 1992 for details.)

For stochastic geometric models such as the self-avoiding walk, lattice trees and animals, and percolation, there exists no general proof of the infra-red bound. In fact there are claims that the infra-red bound is violated for $d < 8$ for lattice trees and animals, and for some dimensions below six for percolation. Following the successes with spin systems, the correlation inequality methods were soon applied to stochastic geometric systems, resulting in diagrammatic sufficient conditions such as the triangle condition for mean-field critical behaviour. At the time, the diagrammatic conditions could not be verified, due to the absence of an infra-red bound. For self-avoiding walks and lattice trees and animals it is now possible to use the lace expansion to prove mean-field behaviour without appeal to these diagrammatic sufficient conditions, although this is not true for percolation. In this article we shall provide an outline of the diagrammatic conditions, in part because of their essential role in percolation, in part because of their intuitive role in identifying the upper critical dimension and reducing the problem of critical behaviour to two-point functions, and in part because the diagrams play an essential role as small parameters in bounding the lace expansion.

The remainder of this article is organized as follows. In Section 2 we give precise definitions of the models and precise statements of results. Section 3 discusses diagrammatic conditions, and shows in particular the relevance of the triangle condition for mean-field behaviour of the expected cluster size in percolation. Finally in Section 4 we describe the lace expansion in some detail. Convergence issues are discussed only for self-avoiding walks, since the analysis is similar for the other models. A general reference for many of the topics covered below is Madras and Slade (1993); our exposition is based in places on this reference.

2. The Models and Results

In this section we give precise definitions of models and statements of results obtained with the lace expansion. All of the models are set on the hypercubic lattice $\mathbb{Z}^d$. In addition to the usual nearest-neighbour model we will consider also the spread-out model. For the spread-out model, we let Λ denote the set $\{x \in \mathbb{Z}^d : 0 < \|x\|_\infty \le L\}$ for some fixed L which will be taken to be large. The bonds of the spread-out model are then the pairs of sites $\{x, y\}$, with $y - x \in \Lambda$. We at times treat the spread-out and nearest-neighbour models simultaneously, by letting Λ also denote the set of nearest neighbours of the origin.

2.1. The Self-Avoiding Walk

For self-avoiding walks we restrict attention to the usual nearest-neighbour model, and will be interested in the case where the small parameter is the inverse dimension. Results similar to those stated below can be obtained more easily for $d > 4$ for the weakly or spread-out self-avoiding walks, for in these contexts there is a small parameter which can be taken to be arbitrarily small. Results for the nearest-neighbour model for $d \geq 5$ were obtained via a computer-assisted proof, because of difficulties associated with the fact that the small parameter d^{-1} is fixed and cannot be taken to be arbitrarily small.

The fact that the upper critical dimension for the self-avoiding walk is four can be partially understood from the fact that intersection properties of the simple random walk change dramatically at $d = 4$. For example, the probability that two independent n-step simple random walks do not intersect remains bounded away from zero for $d > 4$, but not for $d \leq 4$. (See, e.g., Lawler 1991.) Mean-field behaviour for the self-avoiding walk is behaviour like the simple random walk.

An n-step self-avoiding walk is an ordered set $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$, with each $\omega(i) \in \mathbb{Z}^d$, $|\omega(i+1) - \omega(i)| = 1$ for all i (Euclidean distance), and most importantly $\omega(i) \neq \omega(j)$ when $i \neq j$. If ω is an n-step walk we write $|\omega| = n$ (not to be confused with the Euclidean norm $|\omega(i)|$ of $\omega(i) \in \mathbb{Z}^d$). Let $\Omega_n(x,y)$ denote the set of all n-step self-avoiding walks with $\omega(0) = x$ and $\omega(n) = y$, and let $c_n(x,y)$ be the cardinality of this set. In particular, $c_0(x,y) = \delta_{x,y}$. We also define $\Omega(x,y) = \bigcup_{n=0}^{\infty} \Omega_n(x,y)$ to be the set of all self-avoiding walks, of any length, from x to y. Let c_n be the number of n-step self-avoiding walks which begin at the origin and end anywhere, or in other words $c_n = \sum_y c_n(0,y)$. Hammersley and Morton (1954) observed that the elementary submultiplicativity inequality $c_{n+m} \leq c_n c_m$ implies the existence of the *connective constant* $\mu = \lim_{n\to\infty} c_n^{1/n}$, with $c_n \geq \mu^n$ for all n. The mean-square displacement $\langle |\omega(n)|^2 \rangle_n$ is defined to be the average value of $|\omega(n)|^2$ with respect to the uniform measure on the set of all n-step self-avoiding walks:

$$\langle |\omega(n)|^2 \rangle_n = \frac{1}{c_n} \sum_{\omega \in \Omega_n(0,x)} |\omega(n)|^2. \tag{2.1}$$

The number of n-step self-avoiding walks and the mean-square displacement are believed to behave asymptotically like

$$c_n \sim A\mu^n n^{\gamma-1}, \tag{2.2}$$

$$\langle |\omega(n)|^2 \rangle_n \sim Dn^{2\nu}, \tag{2.3}$$

where the amplitudes A and D and the critical exponents γ and ν are dimension-dependent positive constants. Here we are taking the optimistic viewpoint that the above relations really are asymptotic, in the usual sense of the term that the ratio of the left and right sides has limiting value of unity. The critical exponent γ is believed to be equal to $\frac{43}{32}$ for $d = 2$, about 1.162 for $d = 3$, and 1 for $d \geq 4$, with a logarithmic factor $(\log n)^{1/4}$ multiplying $A\mu^n$ in four dimensions. The exponent ν is believed to be equal to $\frac{3}{4}$ for $d = 2$, about 0.588 for $d = 3$, and $\frac{1}{2}$ for $d \geq 4$, again with a logarithmic factor $(\log n)^{1/4}$ multiplying Dn in four dimensions. In fact for $d \geq 5$ this is a theorem.

Theorem 2.1. *For any $d \geq 5$ there are (dimension-dependent) positive constants A and D such that, as $n \to \infty$,*

$$\begin{aligned} c_n &= A\mu^n[1+O(n^{-\epsilon})] \quad \textit{for any } \epsilon < \tfrac{1}{2}, \\ \langle|\omega(n)|^2\rangle_n &= Dn[1+O(n^{-\epsilon})] \quad \textit{for any } \epsilon < \tfrac{1}{4}. \end{aligned}$$

Rigorous numerical bounds on A and D are available. For example when $d = 5$, $1 \leq A \leq 1.493$ and $1.098 \leq D \leq 1.803$.

A weaker theorem, which is easier to prove, involves corresponding statements about generating functions. To define these generating functions, we let z denote a complex parameter (usually taken here to be non-negative), and first define the *two-point function* as

$$G_z(x,y) = \sum_{n=0}^{\infty} c_n(x,y)z^n = \sum_{\omega \in \Omega(x,y)} z^{|\omega|}. \tag{2.4}$$

Then we define the *susceptibility* $\chi(z)$ as

$$\chi(z) = \sum_x G_z(0,x) = \sum_{n=0}^{\infty} c_n z^n \tag{2.5}$$

and the *correlation length of order two* $\xi_2(z)$ by

$$\xi_2^2(z) = \frac{\sum_x |x|^2 G_z(0,x)}{\sum_x G_z(0,x)}. \tag{2.6}$$

The manner of divergence of the susceptibility and correlation length of order two at the *critical point* $z_c \equiv \mu^{-1}$ reflects the large-n asymptotics of c_n and the mean-square displacement.

Theorem 2.2. *For any $d \geq 5$, as $z \nearrow z_c$ along the positive real axis,*

$$\chi(z) \sim \frac{Az_c}{z_c - z}, \qquad \xi_2(z) \sim \left(\frac{Dz_c}{z_c - z}\right)^{1/2}$$

with the same constants A and D as in Theorem 2.1, and with $f(z) \sim g(z)$ meaning that $\lim_{z \nearrow z_c} f(z)/g(z) = 1$.

Theorem 2.2 is weaker than Theorem 2.1, and needs something like a Tauberian condition to conclude Theorem 2.1. This has been done by combining good error estimates in Theorem 2.2 with contour integration methods using ‘fractional derivatives’.

For $z < z_c$ the two-point function is known to decay exponentially: the *correlation length*

$$\xi(z) = -\left[\lim \frac{1}{n} \log G_z(0,(n,0,\ldots,0))\right]^{-1} \tag{2.7}$$

exists for $0 < z < z_c$, is strictly positive and finite, and $\xi(z) \nearrow \infty$ as $z \nearrow z_c$ (Chayes and Chayes 1986). It is believed that the divergence of $\xi(z)$ at the critical point is via a power law with power ν; for $d \geq 5$ this is a theorem.

Theorem 2.3. *For any $d \geq 5$, as $z \nearrow z_c$ along the positive real axis,*

$$\xi(z) \sim \sqrt{\frac{D}{2d}} \left(\frac{z_c}{z_c - z} \right)^{1/2}.$$

The following theorem shows that the scaling limit of the self-avoiding walk is Brownian motion for $d \geq 5$, and provides a strong statement to the effect that the self-avoiding walk behaves like simple random walk for $d \geq 5$. To state the theorem, we denote by $C_d[0,1]$ the set of continuous $\mathbb{R}^d$-valued functions on $[0,1]$, equipped with the supremum norm. Given an n-step self-avoiding walk ω, we define $X_n \in C_d[0,1]$ by setting $X_n(j/n) = (Dn)^{-1/2}\omega(j)$ for $j = 0, 1, \ldots, n$, and taking $X_n(t)$ to be the linear interpolation of this. Let dW denote the Wiener measure on $C_d[0,1]$, and $\langle \cdot \rangle_n$ denote expectation with respect to the uniform measure on the set of n-step self-avoiding walks beginning at the origin.

Theorem 2.4. *For $d \geq 5$ the scaling limit of self-avoiding walk is Brownian motion. In other words, for any bounded continuous function f on $C_d[0,1]$,*

$$\lim_{n \to \infty} \langle f(X_n) \rangle_n = \int f dW.$$

An important ingredient in the proof of the above theorems is an infra-red bound. This bound reflects the long distance behaviour of the critical two-point function indirectly through the behaviour of the Fourier transform of the two-point function near the origin. In general the Fourier transform of a summable function f on $\mathbb{Z}^d$ is defined by

$$\hat{f}(k) = \sum_x f(x) e^{ik \cdot x}, \qquad k = (k_1, \ldots, k_d) \in [-\pi, \pi]^d, \tag{2.8}$$

where $k \cdot x = \sum_{j=1}^d k_j x_j$. The conjectured behaviour of the critical two-point function is

$$G_{z_c}(x,y) \sim \text{const.} \frac{1}{|x-y|^{d-2+\eta}}, \quad \text{as } |x-y| \to \infty, \tag{2.9}$$

or for the Fourier transform

$$\hat{G}_{z_c}(k) \sim \text{const.} \frac{1}{k^{2-\eta}}, \quad \text{as } k \to 0. \tag{2.10}$$

Scaling theory predicts that the critical exponent η is given in terms of γ and ν by Fisher's scaling relation (Fisher 1969)

$$\gamma = (2 - \eta)\nu. \tag{2.11}$$

According to the conjectured values of γ and ν, η is non-negative in all dimensions. This is a statement of the infra-red bound, which can also be stated in the form $\hat{G}_{z_c}(k) \leq Ck^{-2}$. This is believed to be true in all dimensions, but remains unproven for dimensions 2, 3 and 4. This k^{-2} behaviour is the same as that for simple random walk, for which the analogue of $G_{z_c}(0,x)$ is the Green function $\sum_{n=0}^{\infty} p_n(0,x)$ [where $p_n(0,x)$ is the probability that a simple random walk beginning at 0 is at x after n steps] whose Fourier transform is $(1 - d^{-1}\sum_{j=1}^d \cos k_j)^{-1} \sim (2d)k^{-2}$. The following theorem gives an infra-red bound for self-avoiding walks when $d \geq 5$, with correction term.

Theorem 2.5. *For $d \geq 5$, $\hat{G}_{z_c}(k)^{-1} = (2d)^{-1}k^2[DA^{-1} + O(k^\epsilon)]$ for any $\epsilon < 1/2$.*

Proofs of these theorems, with some further results along these lines, can be found in Hara and Slade (1992a, b). Some of the principal ideas are outlined in Section 4.1 below. As a final application of the lace expansion, we note that it can be used to prove existence of an asymptotic expansion in powers of $1/d$, to all orders, for the connective constant μ, and moreover to compute the coefficients of this expansion. The computation has been done (Hara and Slade 1993) as far as

$$\mu = 2d - 1 - \frac{1}{2d} - \frac{3}{(2d)^2} - \frac{16}{(2d)^3} + O\left(\frac{1}{(2d)^4}\right). \tag{2.12}$$

The coefficient of the term of order $(2d)^{-4}$ was computed by Fisher and Gaunt (1964) and Nemirovsky et al. (1992) to be $-102(2d)^{-4}$, but with no error estimate.

2.2. Lattice Trees and Animals

A *lattice tree* is defined to be a connected set of bonds which contains no closed loops. We consider trees on either the nearest-neighbour or spread-out lattice. Although a tree T is defined as a set of bonds, we will write $x \in T$ if x is an element of a bond in T. The number of bonds in T will be denoted $|T|$. A *lattice animal* is a connected set of bonds which may contain closed loops. We denote a typical lattice animal by A and the number of bonds in A by $|A|$.

Let t_n denote the number of n-bond trees modulo translation, and let a_n denote the number of n-bond animals modulo translation. By a subadditivity argument, $t_n^{1/n}$ and $a_n^{1/n}$ both converge to finite positive limits λ and λ_a as $n \to \infty$. The asymptotic behaviour of both t_n and a_n as $n \to \infty$ is believed to be governed by the same critical exponent θ:

$$t_n \sim \text{const.}\, \lambda^n n^{-\theta}, \quad a_n \sim \text{const.}\, \lambda_a^n n^{-\theta}. \tag{2.13}$$

The typical size of a lattice tree or animal is characterized by the average radius of gyration, which is defined for trees as the average over n-bond trees of their individual radii of gyration:

$$R(n) = \left[\frac{1}{t_n} \sum_{T:|T|=n} \frac{1}{n+1} \sum_{x \in T} |x - \bar{x}_T|^2\right]^{1/2}. \tag{2.14}$$

Here the sum over T is the sum over one tree from each equivalence class modulo translation, and $\bar{x}_T = (n+1)^{-1} \sum_{x \in T} x$ is the centre of mass of T. A similar definition applies for lattice animals. The average radius of gyration is believed to behave asymptotically as

$$R(n) \sim \text{const.}\, n^\nu \tag{2.15}$$

for a critical exponent ν which is the same for both trees and animals. The following theorem, proved in Hara and Slade (1992c), gives results for these critical exponents for trees in high dimensions. Related results have been obtained for lattice animals, but so far at the level of generating functions and not at the level of counts (Hara and Slade 1990b).

Theorem 2.6. *For nearest-neighbour trees with d sufficiently large, or for spread-out trees with $d > 8$ and L sufficiently large, there are positive constants such that for every $\epsilon < \min\{\frac{1}{2}, \frac{1}{4}(d-8)\}$,*

$$\begin{aligned} t_n &= \text{const.}\, \lambda^n n^{-5/2}[1 + O(n^{-\epsilon})], \\ R(n) &= \text{const.}\, n^{1/4}[1 + O(n^{-\epsilon})]. \end{aligned}$$

Some hint can be gleaned from this theorem as to why the upper critical dimension should be eight. The fact that the size of trees typically grows like $n^{1/4}$ is a sign that in some sense trees are 4-dimensional objects, and hence will typically not intersect above eight dimensions. This suggests that for $d > 8$ lattice trees will have similar behaviour to the 'mean-field' model of abstract trees embedded in the lattice with no self-avoidance constraints.

As is the case for the self-avoiding walk, the proof proceeds first by studying generating functions near their closest singularity to the origin, and then uses complex variable methods to extract the large-n asymptotics of t_n and the radius of gyration. Let $z_c = \lambda^{-1}$, and for $|z| \le z_c$ define the two-point function

$$G_z(x, y) = \sum_{T : T \ni x, y} z^{|T|} \tag{2.16}$$

and the susceptibility $\chi(z) = \sum_x G_z(0, x)$. Then in particular, it is shown that under the hypotheses of the theorem the susceptibility $\chi(z)$ obeys

$$\chi(z) = \text{const.}\, (z_c - z)^{-1/2} + O((z_c - z)^{-1/2+\epsilon}) \tag{2.17}$$

for all complex $|z| \le z_c$.

The Fourier transform of the two-point function plays an important role in the proof. In particular, an infra-red bound for $\hat{G}_z(k)$ is obtained. In contrast to the self-avoiding walk, it has been conjectured (Bovier et al. 1986) that the infra-red bound fails for lattice trees and animals below eight dimensions, or in other words that $\hat{G}_{z_c}(k)$ diverges more strongly than k^{-2} for $d < 8$.

2.3. Percolation

In this section we discuss the nearest-neighbour model and the spread-out model simultaneously. We consider independent bond percolation where the bonds are the pairs $\{x, y\}$ of sites with $y - x \in \Lambda$. This means that to each bond $\{x, y\}$ we associate an independent Bernoulli random variable $n_{\{x,y\}}$ which takes the value 1 with probability p and the value 0 with probability $1 - p$, where p is a parameter in the closed interval $[0, 1]$. If $n_{\{x,y\}} = 1$ then we say that the bond $\{x, y\}$ is *occupied*, and otherwise we say that it is *vacant*. A *configuration* is a realization of the random variables for all bonds. Given a configuration and any two sites x and y, we say that x and y are *connected* if there is a self-avoiding walk from x to y consisting of occupied bonds, or if $x = y$. We denote by $C(x)$ the random set of sites connected to x, and denote its cardinality by $|C(x)|$. For $d \ge 2$ we denote by $p_c \in (0, 1)$ the critical value of p such that the probability $\theta(p)$ that the origin is connected to infinitely many sites is zero for $p < p_c$ and strictly positive for $p > p_c$. General references are Grimmett (1989) and Kesten (1982).

We denote the indicator function of an event E by $I[E]$ and expectation with respect to the joint distribution of the Bernoulli random variables $n_{\{x,y\}}$ by $\langle \cdot \rangle_p$. The two-point function $\tau_p(x,y)$ is defined to be the probability that x and y are connected:

$$\tau_p(x,y) = \langle I[x \text{ and } y \text{ are connected}]\rangle_p. \tag{2.18}$$

This is analogous to the functions $G_z(x,y)$ defined previously for self-avoiding walks or for lattice trees and animals. For $p < p_c$ the two-point function is known to decay exponentially as $|x-y| \to \infty$, so that the correlation length

$$\xi(p) = -\left[\lim_{n\to\infty} \frac{1}{n} \log \tau_p(0,(n,0,\ldots,0))\right]^{-1} \tag{2.19}$$

is finite and strictly positive. The susceptibility, or expected cluster size, is defined by

$$\chi(p) = \sum_{x\in\mathbb{Z}^d} \tau_p(0,x) = \langle |C(0)|\rangle_p. \tag{2.20}$$

The susceptibility is known to be finite for $p < p_c$ and to diverge as $p \nearrow p_c$. The magnetization is defined by

$$M(p,h) = 1 - \sum_{1\le n<\infty} e^{-hn} \langle I[|C(0)| = n]\rangle_p. \tag{2.21}$$

The following power laws are believed to hold:

$$\begin{array}{rcll}
\chi(p) & \sim & A_1(p_c-p)^{-\gamma} & \text{as } p \nearrow p_c, \\
\theta(p) & \sim & A_2(p-p_c)^{\beta} & \text{as } p \searrow p_c, \\
M(p_c,h) & \sim & A_3 h^{1/\delta} & \text{as } h \searrow 0, \\
\langle |C(0)|^{m+1}\rangle_p / \langle |C(0)|^m\rangle_p & \sim & A_4(p_c-p)^{-\Delta} & \text{as } p \nearrow p_c, \quad (m=1,2,\ldots), \\
\xi(p) & \sim & A_5(p_c-p)^{-\nu} & \text{as } p \nearrow p_c,
\end{array}$$

for some dimension-dependent amplitudes A_i and critical exponents γ, β, δ, Δ, ν. The next theorem gives existence of these critical exponents under certain conditions. *Asymptotic* behaviour has not been proved, but rather relations of the form $f(x) \simeq g(x)$, meaning that there are positive constants c_1, c_2 such that $c_1 g(x) \le f(x) \le c_2 g(x)$ for x sufficiently close to its limiting value. The methods used to obtain precise asymptotic behaviour for self-avoiding walks and trees made use of the fact that for example the susceptibility of these models is a power series, and do not extend readily to percolation.

Theorem 2.7. *For the nearest-neighbour model with d sufficiently large ($d \ge 19$ is large enough), or for the spread-out model with $d > 6$ and L sufficiently large,*

$$\begin{array}{rcll}
\chi(p) & \simeq & (p_c-p)^{-1} & \textit{as } p \nearrow p_c, \\
\theta(p) & \simeq & (p-p_c)^{1} & \textit{as } p \searrow p_c, \\
M(p_c,h) & \simeq & h^{1/2} & \textit{as } h \searrow 0, \\
\langle |C(0)|^{m+1}\rangle_p / \langle |C(0)|^m\rangle_p & \simeq & (p_c-p)^{-2} & \textit{as } p \nearrow p_c, \quad (m=1,2,\ldots), \\
\xi(p) & \simeq & (p_c-p)^{-1/2} & \textit{as } p \nearrow p_c.
\end{array}$$

The 'mean-field' exponents appearing in this theorem are those for percolation on a tree, where exact calculations can be performed. Theorem 2.7 is a combination of several results which centre on the triangle condition. The triangle condition is the statement that the triangle diagram is finite at the critical point, with the triangle diagram given by

$$\mathsf{T}(p) = \sum_{x,y} \tau_p(0,x)\tau_p(x,y)\tau_p(y,0) = \int_{[-\pi,\pi]^d} \hat{\tau}_p(k)^3 \frac{d^d k}{(2\pi)^d}. \tag{2.22}$$

The triangle condition is discussed in more detail in Section 3.3. Aizenman and Newman (1984) introduced the triangle condition and showed that it implies the mean-field behaviour $\chi(p) \simeq (p_c - p)^{-1}$. Barsky and Aizenman (1991) used differential inequalities to prove that the conclusions of the theorem concerning $\theta(p)$ and $M(p_c, h)$ follow from the triangle condition. Nguyen (1987) showed that the triangle condition implies $\Delta = 2$. Then Hara and Slade (1990a) used the lace expansion to show that the triangle condition holds under the hypotheses of the theorem. No direct implication for the exponent ν is known to follow from the triangle condition, but Hara (1990) has used the lace expansion to obtain the result of the theorem concerning the correlation length $\xi(p)$.

It follows from the behaviour of $\theta(p)$ given in Theorem 2.7 that the percolation probability is zero at the critical point: $\theta(p_c) = 0$. Although this is strongly believed to be true in all dimensions, it has otherwise been proven only for the nearest-neighbour model in two dimensions.

The proof that the triangle condition holds above six dimensions involves proving the infra-red bound

$$0 \le \hat{\tau}_p(k) \le \text{const.}\, k^{-2}, \tag{2.23}$$

with a constant which is uniform in $p < p_c$. The conjectured behaviour here in general dimensions is again

$$\hat{\tau}_{p_c}(k) \sim \text{const.}\, \frac{1}{k^{2-\eta}}, \tag{2.24}$$

with η determined by Fisher's relation (2.11). However for percolation it has been conjectured that the infra-red bound is violated ($\eta < 0$) for some dimensions below six. The triangle condition is expected not to hold for any $d \le 6$.

The lace expansion can also be used to study the high-d behaviour of the critical point of the nearest-neighbour model, with the result (Hara and Slade 1993)

$$p_c = \frac{1}{2d} + \frac{1}{(2d)^2} + \frac{7}{2}\frac{1}{(2d)^3} + O\left(\frac{1}{(2d)^4}\right). \tag{2.25}$$

This expansion was derived to two further terms in Gaunt and Ruskin (1978), but with no control on the error term.

2.4. Oriented Percolation

Consider now the case of bond percolation with bonds oriented in one direction. More precisely, we consider the sites in $\mathbb{Z}^{d+1}$ to be of the form (x, n) with $x \in \mathbb{Z}^d$

and $n \in \mathbb{Z}$, and define an oriented bond to be an ordered pair $((x,n),(y,n+1))$, where $|x-y|=1$ (Euclidean distance in $\mathbb{Z}^d$). This defines the bonds of the nearest-neighbour model. For the simplest version of the spread-out model the bonds consist of all ordered pairs $((x,n),(y,n+1))$ with $\|x-y\|_\infty \leq L$. In oriented percolation we declare that each oriented bond is occupied with probability p and is vacant with probability $1-p$, independently for each bond. Let $\tau_p((x,m),(y,n))$ denote the probability that there is an oriented path consisting of occupied bonds from (x,m) to (y,n). This probability is zero unless $m<n$.

The situation here is in many respects similar to the usual unoriented bond percolation discussed in the previous section. In fact it is somewhat easier, because now there is a Markov property: the event that there is an oriented path consisting of occupied bonds joining (x,m) to (y,n) is independent of the occupation status of bonds lying below the hyperplane $\{(w,m) : w \in \mathbb{Z}^d\}$.

In Nguyen and Yang (1992) it was shown how the lace expansion could be adapted to apply to oriented percolation, either for the spread-out model above $4+1$ dimensions or for the nearest-neighbour model in sufficiently high dimensions. They proved that the triangle condition holds in these two situations, and thus in combination with the results of Aizenman and Newman (1984) and Barsky and Aizenman (1991), obtained (among other results) the following theorem. In the statement of the theorem p_c is of course the critical point for oriented percolation, and $\chi(p)$ and $\theta(p)$ are respectively the expected cluster size and the percolation probability.

Theorem 2.8. *For the nearest-neighbour oriented percolation model in sufficiently high dimensions, or for the spread-out oriented percolation model with $d+1>5$ and L sufficiently large,*

$$\begin{aligned} \chi(p) &\simeq (p_c-p)^{-1} \quad \text{as } p \nearrow p_c, \\ \theta(p) &\simeq (p-p_c)^{1} \quad \text{as } p \searrow p_c. \end{aligned}$$

The proof involves using the lace expansion to show that under the hypotheses of the above theorem oriented percolation can be regarded as a small perturbation of the corresponding model of simple random walk with steps oriented in one direction. Accordingly the infra-red bound must be modified to take the orientation into account. Let

$$\hat{\tau}_p(k,t) = \sum_{(x,n)\in\mathbb{Z}^{d+1}} \tau_p((0,0),(x,n))e^{ik\cdot x}e^{itn}, \qquad k \in [-\pi,\pi]^d, \;\; t \in [-\pi,\pi]. \tag{2.26}$$

Then the infra-red bound used in proving Theorem 2.8, and which holds uniformly in $p<p_c$ under the hypotheses of the theorem, is

$$|\hat{\tau}_p(k,t)| \leq \text{const.} \frac{1}{k^2+|t|}. \tag{2.27}$$

The right side reflects the corresponding behaviour of random walk in the oriented context.

Theorem 2.8 is at the level of generating functions. It is possible to go beyond this by incorporating the 'fractional derivative' methods of Hara and Slade (1990a). In particular, one can study the scaling limit of the hitting distribution on a distant hyperplane $\{(x,n) : x \in \mathbb{Z}^d\}$, with n large. In view of the outlook that in high dimensions at criticality oriented percolation is a small perturbation of a random walk model, a Gaussian scaling limit is to be expected. To state a theorem to this effect at the level of the Fourier transform, recently proved by Nguyen and Yang (1993), we first define

$$Z_p(k;n) = \sum_x \tau_p((0,0),(x,n))e^{ik\cdot x}. \tag{2.28}$$

The theorem is stated for all $p \leq p_c$, but it is at the critical point itself that the result is most interesting.

Theorem 2.9. *For the nearest-neighbour model in sufficiently high dimensions or for the spread-out model in dimensions $d+1>5$ with L sufficiently large, for any $p \leq p_c$ there is a finite positive constant σ_p^2 such that*

$$\lim_{n\to\infty} \frac{Z_p(k/\sqrt{n};n)}{Z_p(0;n)} = e^{-\sigma_p^2 k^2/2}.$$

3. Diagrammatic Conditions and the Upper Critical Dimension

This section describes diagrammatic conditions for the self-avoiding walk, lattice trees and animals, and percolation, namely the finiteness of the bubble diagram, the square diagram, and the triangle diagram respectively. These diagrammatic conditions are sufficient conditions for mean-field behaviour for the susceptibility, and in the case of percolation also for other quantities. For percolation the triangle condition remains a necessary ingredient for proving the results of Sections 2.3 and 2.4. However for self-avoiding walks and lattice trees and animals the bubble and square conditions have been superseded by more powerful lace expansion methods. Nevertheless for each model it is instructive to see how the diagrams appear in the analysis.

The diagrams arise in differential inequalities for the susceptibility. In the case of the self-avoiding walk, the lower bound

$$\chi(z) \geq \frac{z_c}{z_c - z} \tag{3.1}$$

is an immediate consequence of (2.5) and the subadditivity bound $c_n \geq \mu^n = z_c^{-n}$. A complementary upper bound

$$\chi(z) \leq \text{const.}\, \frac{1}{z_c - z} \tag{3.2}$$

is a consequence of the bubble condition, as will be shown below. Together these bounds give the mean-field behaviour

$$\chi(z) \simeq (z_c - z)^{-1}, \tag{3.3}$$

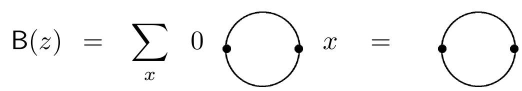
which is only expected to be true above the upper critical dimension. Similar considerations apply for lattice trees and animals and for percolation.

3.1. The Bubble Condition

We restrict attention in this section to the self-avoiding walk. To state the bubble condition we first introduce the *bubble diagram*

$$\mathsf{B}(z) = \sum_{x \in \mathbb{Z}^d} G_z(0,x)^2. \tag{3.4}$$

The name 'bubble diagram' comes from a Feynman diagram notation in which the two-point function or *propagator* evaluated at sites x and y is denoted by a line terminating at x and y. In this notation



where in the diagram on the right it is implicit that one vertex is fixed at the origin and the other is summed over the lattice. The bubble diagram can be rewritten in terms of the Fourier transform of the two-point function, using (3.4) and the Parseval relation, as

$$\mathsf{B}(z) = \|G_z(0,\cdot)\|_2^2 = \|\hat{G}_z\|_2^2 = \int_{[-\pi,\pi]^d} \hat{G}_z(k)^2 \frac{d^d k}{(2\pi)^d}. \tag{3.5}$$

The *bubble condition* is the statement that $\mathsf{B}(z_c) < \infty$.

In view of the definition of η in (2.9) or (2.10), it follows from (3.5) that the bubble condition is satisfied provided $\eta > \frac{1}{2}(4-d)$. Hence the bubble condition for $d > 4$ is implied by the infra-red bound $\eta \geq 0$. If the values for η arising from Fisher's relation and the conjectured values of γ and ν are correct, then the bubble condition will not hold in dimensions $2, 3$ or 4, with the divergence of the bubble diagram being only logarithmic in four dimensions. The next theorem (Bovier et al. 1984) shows that the bubble condition implies (3.2) and hence implies (3.3).

Theorem 3.1. *In all dimensions $\chi(z) \geq z_c(z_c - z)^{-1}$ for $z \in [0, z_c)$. If the bubble condition is satisfied then there is a corresponding upper bound, and for all $z \in [0, z_c)$*

$$\frac{z_c}{z_c - z} \leq \chi(z) \leq \mathsf{B}(z_c)\left(\frac{z_c}{z_c - z} + 1\right).$$

Proof. The lower bound in the statement of the theorem is just (3.1), which holds in all dimensions. So it suffices to prove the upper bound. For this, we begin by obtaining a lower bound on the derivative of the susceptibility. Once this is achieved, integration of this differential inequality will give the upper bound in the statement of the theorem.

The desired lower bound is that for any $z \in [0, z_c)$,

$$\frac{\chi(z)^2}{\mathsf{B}(z)} - \chi(z) \leq z\chi'(z). \tag{3.6}$$

To prove this, we begin by noting that below the critical point the derivative of χ can be obtained by term by term differentiation:

$$z\chi'(z) = \sum_y \sum_{\omega\in\Omega(0,y)} |\omega| z^{|\omega|} = \sum_y \sum_{\omega\in\Omega(0,y)} (|\omega|+1) z^{|\omega|} - \chi(z). \tag{3.7}$$

The summation on the right side can be written

$$\begin{aligned} &\sum_y \sum_{\omega\in\Omega(0,y)} \sum_x I[\omega(j) = x \text{ for some } j] z^{|\omega|} \\ &= \sum_{x,y} \sum_{\substack{\omega^{(1)} \in \Omega(0,x) \\ \omega^{(2)} \in \Omega(x,y)}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)} \cap \omega^{(2)} = \{x\}] \\ &\equiv Q(z), \end{aligned} \tag{3.8}$$

where I denotes the indicator function. Therefore

$$z\chi'(z) = Q(z) - \chi(z). \tag{3.9}$$

The next step is to rewrite $Q(z)$ by using the inclusion-exclusion relation in the form

$$I[\omega^{(1)} \cap \omega^{(2)} = \{x\}] = 1 - I[\omega^{(1)} \cap \omega^{(2)} \neq \{x\}].$$

This gives

$$Q(z) = \chi(z)^2 - \sum_{x,y} \sum_{\substack{\omega^{(1)} \in \Omega(0,x) \\ \omega^{(2)} \in \Omega(x,y)}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)} \cap \omega^{(2)} \neq \{x\}]. \tag{3.10}$$

In the last term on the right side of (3.10), let $w = \omega^{(2)}(l)$ be the site of the last intersection of $\omega^{(2)}$ with $\omega^{(1)}$, where time is measured along $\omega^{(2)}$ beginning at its starting point x. Then the portion of $\omega^{(2)}$ corresponding to times greater than l must avoid all of $\omega^{(1)}$. Relaxing the restrictions that this portion of $\omega^{(2)}$ avoid both the remainder of $\omega^{(2)}$ and the part of $\omega^{(1)}$ linking w to x gives the upper bound

$$\sum_{x,y} \sum_{\substack{\omega^{(1)} \in \Omega(0,x) \\ \omega^{(2)} \in \Omega(x,y)}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)} \cap \omega^{(2)} \neq \{x\}] \leq Q(z)[\mathsf{B}(z) - 1], \tag{3.11}$$

as illustrated in Figure 1. Here the factor $\mathsf{B}(z) - 1$ arises from the two paths joining w and x. The upper bound involves $\mathsf{B}(z) - 1$ rather than $\mathsf{B}(z)$ since there will be no contribution here from the $x = 0$ term in (3.4). Combining (3.10) and (3.11) gives

$$Q(z) \geq \chi(z)^2 - Q(z)[\mathsf{B}(z) - 1]. \tag{3.12}$$

Solving for $Q(z)$ gives

$$Q(z) \geq \frac{\chi(z)^2}{\mathsf{B}(z)}.$$

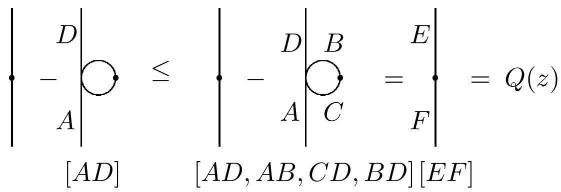


Fig. 1. A diagrammatic representation of the inequality $\chi(z)^2 - Q(z)[\mathsf{B}(z)-1] \le Q(z)$ occurring in the proof of Theorem 3.1. The lists of pairs of lines indicate interactions between the propagators, in the sense that the corresponding walks must avoid each other.

Combining this inequality with (3.9) gives (3.6).

We now integrate (3.6) to obtain the upper bound in the statement of the theorem. Let $z_1 \in [0, z_c)$. By (3.6), for $z \in [z_1, z_c)$ we have

$$z\left(-\frac{d\chi^{-1}}{dz}\right) \ge \frac{1}{\mathsf{B}(z)} - \frac{1}{\chi(z)} \ge \frac{1}{\mathsf{B}(z_c)} - \frac{1}{\chi(z_1)}. \tag{3.13}$$

We bound the factor of z on the left side by z_c and then integrate from z_1 to z_c. Using the fact that $\chi(z_c)^{-1} = 0$ by (3.1), this gives

$$z_c \chi(z_1)^{-1} \ge [\mathsf{B}(z_c)^{-1} - \chi(z_1)^{-1}](z_c - z_1). \tag{3.14}$$

Rewriting gives

$$\chi(z_1) \le \mathsf{B}(z_c)\frac{2z_c - z_1}{z_c - z_1}, \tag{3.15}$$

which is the desired upper bound on the susceptibility. □

3.2. The Square Condition

For lattice trees and animals one can argue similarly, and we just summarize the result for trees. Again for concreteness we consider only the nearest-neighbour case, although there are no difficulties in dealing with greater generality.

We first define the *square diagram* by

$$\mathsf{S}(z) = \sum_{x,y,u} G_z(0,x)G_z(x,y)G_z(y,u)G_z(u,0) = \int_{[-\pi,\pi]^d} \hat{G}_z(k)^4 \frac{d^d k}{(2\pi)^d}, \tag{3.16}$$

where $G_z(x,y)$ is the two-point function defined in (2.16). The *square condition* states that $\mathsf{S}(z_c) < \infty$, and will be satisfied for $d > 8$ if $\hat{G}_{z_c}(k) \le \text{const.}\, k^{-2}$. The following theorem (Bovier et al. 1986, Tasaki 1986, Tasaki and Hara 1987) shows the relevance of the square condition.

Theorem 3.2. *For all d, $\chi(z) \ge \text{const.}\,(z_c - z)^{-1/2}$ for $0 \le z \le z_c$. If the square condition is satisfied then the reverse inequality also holds, and hence*

$$\chi(z) \simeq (z_c - z)^{-1/2}. \tag{3.17}$$

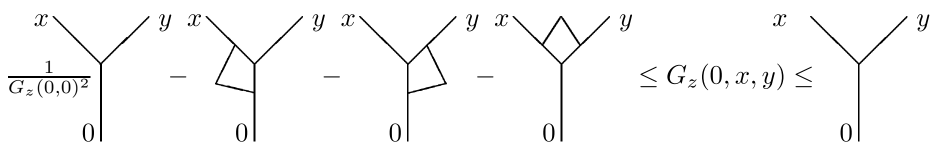


Fig. 2. The skeleton inequality for lattice trees. Each line represents a two-point function and unlabelled vertices are summed over.

Proof. Again, we start with an expression for the derivative of χ. By definition,

$$\begin{aligned} z\frac{d\chi(z)}{dz} &= \sum_x z\frac{dG_z(0,x)}{dz} = \sum_x \sum_{T\ni 0,x} |T| z^{|T|} \\ &= \sum_{x,y} \sum_{T\ni 0,x,y} z^{|T|} - \chi(z) \equiv \sum_{x,y} G_z(0,x,y) - \chi(z), \end{aligned} \tag{3.18}$$

where the last identity defines the three-point function. Using inclusion-exclusion, the three-point function can be bounded according to the 'skeleton inequality' illustrated in Figure 2. This leads to the estimate

$$\chi(z)^3 \left[\frac{1}{G_z(0,0)^2} - 3[\mathsf{S}(z) - 1]\right] - \chi(z) \leq z\frac{d\chi(z)}{dz} \leq \chi(z)^3. \tag{3.19}$$

With some care[1] this differential inequality can be integrated to obtain the lower bound on χ stated in the theorem, and also a corresponding upper bound if $\mathsf{S}(z_c) < 1 + [3G_{z_c}(0,0)]^{-2}$. This argument can be strengthened to prove (3.17) under the weaker hypothesis $\mathsf{S}(z_c) < \infty$ by using an argument of Aizenman (1982); Lemma 3.6 below treats the analogous problem for percolation. □

3.3. The Triangle Condition

It is known from the results of Chayes and Chayes (1987) and Tasaki (1987) that the upper critical dimension of percolation is at least six. In brief, they proved critical exponent inequalities, such as $d\nu \geq 2\Delta - \gamma$, assuming the exponents exist. Inserting the mean-field values $\gamma = 1$, $\Delta = 2$, $\nu = \frac{1}{2}$ (corresponding to percolation on a tree), we see that for $d < 6$ the inequality is not satisfied and hence at least one of these exponents cannot take on its mean-field value.

On the other hand, the triangle condition is a diagrammatic sufficient condition for mean-field behaviour, which is known under some circumstances to hold for $d > 6$ (see Theorem 2.7). The *triangle diagram* is defined by

$$\mathsf{T}(p) = \sum_{x,y} \tau_p(0,x)\tau_p(x,y)\tau_p(y,0) = \int_{[-\pi,\pi]^d} \hat{\tau}_p(k)^3 \frac{d^dk}{(2\pi)^d}, \tag{3.20}$$

and the *triangle condition* is the statement that $\mathsf{T}(p_c) < \infty$. For $d > 6$ the infra-red bound is a sufficient condition for the triangle condition, and for $d \leq 6$ the triangle

[1] Care is required to deal with the possibility that $\chi(z)^{-1}$ has a discontinuity at z_c^-.

condition is believed to be violated. Our present goal is to prove the following theorem due to Aizenman and Newman (1984). Further consequences of the triangle condition are obtained in Barsky and Aizenman (1991) and Nguyen (1987).

Theorem 3.3. *For all $d \geq 2$, $\chi(p) \geq \text{const.}\,(p_c - p)^{-1}$ for $0 \leq p < p_c$. If the triangle condition is satisfied then the corresponding upper bound also holds, and hence*

$$\chi(p) \simeq (p_c - p)^{-1} \quad \text{as } p \nearrow p_c. \tag{3.21}$$

Before beginning, we collect several definitions needed here as well as in Section 4.3. We will make use below of Russo's formula and the BK and FKG inequalities; proofs of these can be found in Grimmett (1989).

Definition 3.4. (a) A *bond* is an unordered pair of distinct sites $\{x, y\}$ with $y - x \in \Lambda$. A *directed bond* is an ordered pair (x, y) of distinct sites with $y - x \in \Lambda$. A *path* from x to y is a self-avoiding walk from x to y, considered to be a set of bonds. Two paths are *disjoint* if they have no bonds in common (they may have common sites). Given a bond configuration, an *occupied path* is a path consisting of occupied bonds.
(b) Given a bond configuration, two sites x and y are *connected* if there is an occupied path from x to y or if $x = y$. We denote by $C(x)$ the random set of sites which are connected to x. Two sites x and y are *doubly-connected* if there are at least two disjoint occupied paths from x to y or if $x = y$. We denote by $D_c(x)$ the random set of sites which are doubly-connected to x. Given a bond $\{u, v\}$ and a bond configuration, we define $C_{\{u,v\}}(x)$ to be the set of sites which remain connected to x in the new configuration obtained by setting the occupation status of $\{u, v\}$ to be vacant.
(c) Given a set of sites $A \subset \mathbb{Z}^d$ and a bond configuration, two sites x and y are *connected in A* if there is an occupied path from x to y having all of its sites in A (so in particular it is required that $x, y \in A$), or if $x = y \in A$. Two sites x and y are *connected through A* if they are connected in such a way that every occupied path from x to y has at least one bond with an endpoint in A, or if $x = y \in A$.
(d) We denote by $\hat{C}^A(x)$ the random set of sites connected to x in $\mathbb{Z}^d \backslash A$. The *restricted two-point function* is defined by

$$\tau_p^A(x, y) = \langle I[x \text{ and } y \text{ are connected in } \mathbb{Z}^d \backslash A] \rangle_p = \langle I[y \in \hat{C}^A(x)] \rangle_p.$$

(e) Given a bond configuration, a bond $\{u, v\}$ (occupied or not) is called *pivotal* for the connection from x to y if (i) either $x \in C(u)$ and $y \in C(v)$, or $x \in C(v)$ and $y \in C(u)$, and (ii) $y \notin C_{\{u,v\}}(x)$. Similarly a *directed* bond (u, v) is pivotal for the connection from x to y if $x \in C_{\{u,v\}}(u)$, $y \in C_{\{u,v\}}(v)$ and $y \notin C_{\{u,v\}}(x)$; this event will be denoted $E_1(x, (u, v), y)$. If x and y are connected then there is a natural order to the set of occupied pivotal bonds for the connection from x to y (assuming there is at least one occupied pivotal bond), and each of these pivotal bonds is directed in a natural way, as follows. The *first pivotal bond from x to y* is the directed occupied pivotal bond (u, v) such that u is doubly-connected to x. If (u, v) is the first pivotal bond for the connection from x to y, then the second pivotal bond is the first pivotal bond for the connection from v to y, and so on.

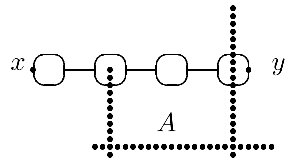


Fig. 3. The event $E_2(x, y; A)$. The line segments represent the pivotal bonds for the connection from x to y, and the circles represent clusters with no such pivotal bonds. The dotted lines represent the sites in A, which need not be connected.

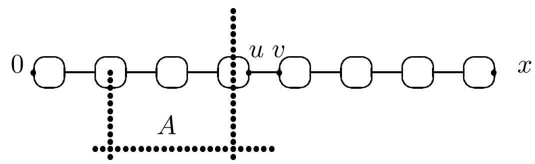


Fig. 4. The event of Lemma 3.5, that $E_2(0, u; A)$ occurs and (u, v) is occupied and pivotal for the connection from 0 to x. There is no restriction on intersections between A and $C_{\{u,v\}}(x)$.

The proof of Theorem 3.3 is based on uniform upper and lower bounds on $\chi'(p)/\chi(p)^2$, much as in the proofs of Theorems 3.1 and 3.2. However for percolation the situation is more complex. For simplicity we consider only the nearest-neighbour model, although there is no difficulty in generalizing the argument. The proof makes use of two lemmas, whose proofs are deferred to the end of this section.

The first lemma will be stated in greater generality than what is needed here, for later use in deriving the lace expansion for percolation. For this greater generality, given sites x, y and a set of sites A we define the event $E_2(x, y; A)$ to be the event that (i) x is connected to y through A and (ii) there is no pivotal bond for the connection from x to y whose first endpoint is connected to x through A; see Figure 3. In particular, $E_2(x, y; A)$ includes the event that x and y are doubly-connected and connected through A. Observe that taking $A = \{y\}$, the event $E_2(x, y; \{y\})$ is simply the event that x is connected to y; this special case serves the needs of this section.In addition, taking $A = \mathbb{Z}^d$, the event $E_2(x, y; \mathbb{Z}^d)$ is precisely the event that $y \in D_c(x)$.

Lemma 3.5. [2] *Given a nonempty set of sites A and a site u, let $E_2 = E_2(0, u; A)$. Let $p < p_c$. Then*

$$\begin{aligned}\langle I[E_2] I[(u, v) \text{ is occupied and pivotal for the connection from 0 to } x]\rangle_p \\ = p\langle I[E_2]\, \tau_p^{C_{\{u,v\}}(0)}(v, x)\rangle_p. \qquad (3.22)\end{aligned}$$

[2] This lemma corresponds to Lemma 2.1 of Hara and Slade (1990a) and corrects an error in that lemma: the class of events in the statement of Lemma 2.1 was too large. However the conclusion of the lemma was correct for the class of events to which it was applied.

In particular, for $E_2 = E_2(0,u;\{u\})$ it follows from the lemma that

$$\langle I[E_1(0,(u,v),x)]\rangle_p = \langle I[u \in C(0)]\, \tau_p^{C_{\{u,v\}}(0)}(v,x)\rangle_p. \tag{3.23}$$

The second lemma will enable us to strengthen a preliminary attempt to prove Theorem 3.3.

Lemma 3.6. *For $|u| = 1$ and $A \equiv \{x \in \mathbb{Z}^d : \|x\|_\infty \le R\} \supset \{0,u\}$, $(R \ge 1)$, there exists $\epsilon_A > 0$ such that*

$$\langle I[0 \in C(x)]\tau_p^{C_{\{0,u\}}(x)}(u,y)\rangle_p \ge \epsilon_A \langle I[0 \in C(x)]\tau_p^{\hat{C}^A(x)}(u,y)\rangle_p. \tag{3.24}$$

Proof of Theorem 3.3.[3] By Russo's formula (a finite volume argument is required here),

$$\begin{aligned} \frac{d\chi}{dp} &= \sum_x \frac{d}{dp}\tau_p(0,x) = \sum_x \sum_{(u,v)} \langle I[E_1(0,(u,v),x)]\rangle_p \\ &= \sum_{x,y}\sum_{|u|=1} \langle I[E_1(x,(0,u),y)]\rangle_p, \end{aligned} \tag{3.25}$$

where in the first line the sum over (u,v) is the sum over directed nearest-neighbour bonds, and in the second line translation invariance was used to shift 0 to the pivotal bond. By (3.25) and (3.23) we have

$$\begin{aligned} \frac{d\chi}{dp} &= \sum_{x,y}\sum_{|u|=1} \langle I[0 \in C(x)]\, \tau_p^{C_{\{0,u\}}(x)}(u,y)\rangle_p \qquad (3.26)\\ &= \sum_{x,y}\sum_{|u|=1} \left[\tau_p(x,0)\tau_p(u,y) - \left\langle I[0 \in C(x)]\, [\tau_p(u,y) - \tau_p^{C_{\{0,u\}}(x)}(u,y)]\right\rangle_p\right] \\ &= 2d\chi(p)^2 - \sum_{x,y}\sum_{|u|=1} \left\langle I[0 \in C(x)]\, [\tau_p(u,y) - \tau_p^{C_{\{0,u\}}(x)}(u,y)]\right\rangle_p . \end{aligned}$$

We seek bounds on the summation on the right side. For this we first note that the difference of two-point functions is exactly the probability that u is connected to y through $C_{\{0,u\}}(x)$, and hence in particular is non-negative. For an upper bound, we note that when u is connected to y through $C_{\{0,u\}}(x)$ there must be a $v \in C_{\{0,u\}}(x)$ which is connected to u and y by disjoint paths. By the BK inequality, the probability of such a configuration is bounded above by $\tau_p(u,v)\tau_p(v,y)$. (In deriving the lace expansion we will require an identity rather than a bound for this probability; see Lemma 4.1 below.) Summing over all possible $v \in C_{\{0,u\}}(x)$ and overcounting gives the bound

$$\begin{aligned} \langle I[0 \in C(x)]\, [\tau_p(u,y) - \tau_p^{C_{\{0,u\}}(x)}(u,y)]\rangle_p &\le \langle I[0 \in C(x)] \sum_{v \in C(x)} \tau_p(u,v)\tau_p(v,y)\rangle_p \\ &= \sum_v \langle I[0, v \in C(x)]\rangle_p\, \tau_p(u,v)\tau_p(v,y). \end{aligned} \tag{3.27}$$

[3] A correct proof involves working first in finite volume and then taking a limit, but we shall sketch only the main ideas and overlook this. Our discussion is deficient in this respect; see Aizenman and Newman (1984) for a more careful treatment.

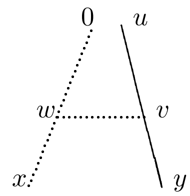


Fig. 5. A schematic representation of the upper bounds (3.27)–(3.28). The dotted lines denote sites in $C_{\{0,u\}}(x)$, while the solid line denotes a connection from u to y through $C_{\{0,u\}}(x)$.

For any configuration in which 0 and v are connected to x, there must be a site w such that there are disjoint connections between 0 and w, v and w, and x and w. By the BK inequality, this implies that the right side of (3.27) is bounded above by

$$\sum_{v,w} \tau_p(0,w)\tau_p(v,w)\tau_p(x,w)\tau_p(u,v)\tau_p(v,y). \tag{3.28}$$

The geometry of the above inequality is depicted in Figure 5. By symmetry, this gives

$$2d\,\chi(p)^2 \left[1 - \sum_{w,v} \tau_p(0,w)\tau_p(w,v)\tau_p(v,e_1)\right] \leq \frac{d\chi}{dp} \leq 2d\,\chi(p)^2 \tag{3.29}$$

where e_1 denotes the unit vector along the first coordinate direction.

With some care integration of the above bound would yield (3.21) if $\mathsf{T}(p_c)$ were less than 1 (and the lower bound $\chi(p) \geq \text{const.}\,(p_c - p)^{-1}$ in any case), since

$$\sum_{w,v} \tau_p(0,w)\tau_p(w,v)\tau_p(v,e_1) = \int \frac{d^dk}{(2\pi)^d} e^{ik_1}\hat{\tau}(k)^3 \leq \mathsf{T}(p) \tag{3.30}$$

(it is known that $\hat{\tau}_p(k) \geq 0$). But in fact $\mathsf{T}(p)$ is always greater than 1, due to the presence of $\tau_p(0,0)^3 = 1$ in the sum. This difficulty is resolved using Lemma 3.6, which instead of Lemma 3.6 gives

$$\frac{d\chi}{dp} \geq \epsilon_A \sum_{x,y} \sum_{|u|=1} \langle I[0 \in C(x)]\, \tau_p^{\hat{C}^A(x)}(u,y)\rangle_p, \tag{3.31}$$

with A any finite set of sites containing 0 and u. Now the expectation on the right is dealt with much as before, but with $\hat{C}^A(x)$ playing the role of $C_{\{0,u\}}(x)$. This gives

$$\frac{d\chi}{dp} \geq 2d\epsilon_A\chi(p)^2 \left[1 - \frac{1}{2d} \sum_{|u|=1} \sum_{w,v \in Z^d \setminus A} \tau_p(0,w)\tau_p(w,v)\tau_p(w,u)\right]. \tag{3.32}$$

When $\mathsf{T}(p_c) < \infty$, the sum on the right can be made arbitrarily small, so in particular less than one, by taking the radius R of A sufficiently large. $\square$

Proof of Lemma 3.5. The event appearing in the left side of (3.22) is depicted in Figure 4. The proof is by conditioning on $C_{\{u,v\}}(0)$, which is the connected cluster of the origin which remains after declaring the bond $\{u,v\}$ to be vacant. This cluster is finite with probability one, since $p < p_c$.

We first observe that the event that E_2 occurs and (u,v) is pivotal (for the connection from 0 to x) is independent of the occupation status of the bond (u,v). Therefore the left side of the identity in the statement of the lemma is equal to

$$p\langle I[E_2]I[(u,v) \text{ is pivotal for the connection from 0 to } x]\rangle_p. \tag{3.33}$$

By conditioning on $C_{\{u,v\}}(0)$, (3.33) is equal to

$$p \sum_{S:S\ni 0} \langle I[E_2 \text{ occurs, } (u,v) \text{ is pivotal, } C_{\{u,v\}}(0) = S]\rangle_p, \tag{3.34}$$

where the sum is over all finite sets of sites S containing 0.

In (3.34), the statement that (u,v) is pivotal can be replaced by the statement that v is connected to x in $\mathbb{Z}^d \backslash S$. This event depends only on the occupation status of the bonds which do not have an endpoint in S. On the other hand, the event E_2 is determined by the occupation status of bonds which have an endpoint in $C_{\{u,v\}}(0)$. Similarly, the event that $C_{\{u,v\}}(0) = S$ depends on the values of n_b only for bonds b which have one or both endpoints in S. Hence the event that both E_2 occurs and $C_{\{u,v\}}(0) = S$ is independent of the event that v is connected to x in $\mathbb{Z}^d \backslash S$, and therefore (3.34) is equal to

$$p \sum_{S:S\ni 0} \langle I[E_2 \text{ occurs and } C_{\{u,v\}}(0) = S]\rangle_p \, \tau_p^S(v,x). \tag{3.35}$$

Bringing the restricted two-point function inside the expectation, replacing the superscript S by $C_{\{u,v\}}(0)$, and performing the sum over S, (3.35) is equal to

$$p\langle I[E_2] \, \tau_p^{C_{\{u,v\}}(0)}(v,x)\rangle_p. \tag{3.36}$$

This completes the proof. □

Proof of Lemma 3.6. The natural inequality here is the reverse of (3.24), since given a bond configuration the fact that $\{0,u\} \subset A$ implies $C_{\{0,u\}}(x) \supset \hat{C}^A(x)$ and hence $\tau^{C_{\{0,u\}}(x)}(u,y) \leq \tau^{\hat{C}^A(x)}(u,y)$. Following Lemma 6.3 of Aizenman and Newman (1984), we show that a reversed inequality (3.24) can be obtained at the cost of a small constant ϵ_A.

We define three events:

$$\begin{aligned} E_1 &= E_1(x,(0,u),y) = \{0 \in C(x) \text{ and } u \text{ is connected to } y \text{ in } \mathbb{Z}^d \backslash C_{\{0,u\}}(x)\}, \\ F &= \{0 \in C(x) \text{ and } u \text{ is connected to } y \text{ in } \mathbb{Z}^d \backslash \hat{C}^A(x)\}, \\ G &= \{C(x) \cap A \neq \phi, \, C(y) \cap A \neq \phi, \text{ and } \hat{C}^A(x) \cap \hat{C}^A(y) = \phi\}. \end{aligned}$$

By definition $G \supset E_1, F$, so

$$\mathrm{Prob}(E_1) = \mathrm{Prob}(G)\,\mathrm{Prob}(E_1 \,|\, G) \geq \mathrm{Prob}(F)\,\mathrm{Prob}(E_1 \,|\, G), \tag{3.37}$$

where $\text{Prob}(\cdot) = \langle I[\cdot] \rangle_p$ and $\text{Prob}(E_1 \,|\, G)$ is a conditional probability.

The event G depends only on bonds having at least one endpoint not in A. And given that G occurs, there is always at least one configuration of bonds having both endpoints within A such that E_1 occurs. Therefore

$$\text{Prob}(E_1 \,|\, G) \geq (\min\{p, 1-p\})^{\#\{\text{bonds within } A\}} \equiv \epsilon_A, \tag{3.38}$$

and hence by (3.37)

$$\text{Prob}(E_1) \geq \epsilon_A \text{Prob}(F). \tag{3.39}$$

By (3.23), $\text{Prob}(E_1) = \langle I[0 \in C(x)] \, \tau_p^{C_{\{0,u\}}(x)}(u, y) \rangle_p$. Hence it suffices to show that[4]

$$\text{Prob}(F) \geq \langle I[0 \in C(x)] \tau_p^{\hat{C}^A(x)}(u, y) \rangle_p. \tag{3.40}$$

Conditioning on $\hat{C}^A(x)$, as in the proof of Lemma 3.5 we have

$$\begin{aligned} \text{Prob}(F) \;=\; & \sum_{S : S \ni x} \text{Prob}(\hat{C}^A(x) = S) \\ & \times \text{Prob}(0 \in C(x), u \text{ is connected to } y \text{ in } \mathbb{Z}^d \backslash S \,|\, \hat{C}^A(x) = S). \end{aligned} \tag{3.41}$$

The events $\{0 \in C(x)\}$ and $\{u \text{ is connected to } y \text{ in } \mathbb{Z}^d \backslash S\}$ are not independent, but they do depend only on those bonds which do not touch S and on those which connect S to A. Restricted to this set of bonds, the two events are increasing. Hence by the FKG inequality

$$\begin{aligned} \text{Prob}(F) \;\geq\; & \sum_{S : S \ni x} \text{Prob}(\hat{C}^A(x) = S) \, \text{Prob}(0 \in C(x) \,|\, \hat{C}^A(x) = S) \\ & \times \text{Prob}(u \text{ is connected to } y \text{ in } \mathbb{Z}^d \backslash S \,|\, \hat{C}^A(x) = S) \\ = \; & \sum_{S : S \ni x} \text{Prob}(\hat{C}^A(x) = S) \, \text{Prob}(0 \in C(x) \,|\, \hat{C}^A(x) = S) \, \tau^S(u, y), \end{aligned} \tag{3.42}$$

where the independence of the events $\{u \text{ is connected to } y \text{ in } \mathbb{Z}^d \backslash S\}$ and $\{\hat{C}^A(x) = S\}$ was used in the last step. This gives (3.40) as in the proof of Lemma 3.5. $\square$

4. The Lace Expansion

In this section the derivation of the lace expansion is described for the various models. Bounds on the expansion and the proof of convergence of the expansion are similar for each model, and these two issues are discussed only for the self-avoiding walk.

4.1. The Self-Avoiding Walk

The lace expansion shows that above four dimensions the self-avoiding walk is a small perturbation of simple random walk. This is exhibited at the level of the Fourier transform of the two-point function, by showing that $\hat{G}_{z_c}(k)$ has the same

[4] This corrects a claim in Aizenman and Newman (1984) that (3.40) is an identity rather than an inequality.

k^{-2} behaviour near the origin as simple random walk. The lace expansion provides an explicit formula for $\hat{G}_z(k)$ which demonstrates this comparison. We first derive the expansion without concern for convergence issues, and then discuss bounds and convergence afterwards. For simplicity we restrict attention here to the nearest-neighbour model, although no real complications arise with more general steps.

4.1.1. Derivation of the Expansion

The lace expansion can be derived in two ways: via a kind of cluster expansion of a type that is well known in statistical mechanics and constructive quantum field theory, or via a repeated application of the inclusion-exclusion relation. These two approaches give the same result. Here we shall follow the inclusion-exclusion approach, which may appeal more to intuition but which has the drawback that it is less explicit in providing precise formulas. The lace expansion produces a linear convolution equation for $G_z(x,y)$, which can then be solved by taking the Fourier transform. This convolution equation is reminiscent of a multi-dimensional renewal equation.

Let $\Omega^{(0)}(x,y)$ denote the set of all simple random walks (with no self-avoidance constraint) of any length, which begin at x and end at y. Then if we define

$$C_z(x,y) = \sum_{\omega \in \Omega^{(0)}(x,y)} z^{|\omega|}, \tag{4.1}$$

for complex z with $|z| \leq (2d)^{-1}$, and set $\hat{D}(k) = d^{-1} \sum_{j=1}^{d} \cos k_j$, we have

$$\hat{C}_z(k) = \frac{1}{1 - 2dz\hat{D}(k)}. \tag{4.2}$$

The corresponding formula for the self-avoiding walk will be

$$\hat{G}_z(k) = \frac{1}{1 - 2dz\hat{D}(k) - \hat{\Pi}_z(k)}, \tag{4.3}$$

where $\hat{\Pi}_z(k)$ will be defined via the lace expansion and in high dimensions will be controlled for all k and for $|z| \leq z_c$.

The first step in deriving the expansion is to extract the term in (2.4) corresponding to a walk which takes no steps:

$$G_z(0,x) = \delta_{0,x} + \sum_{n=1}^{\infty} c_n(0,x) z^n. \tag{4.4}$$

Next, we argue that for $n \geq 1$,

$$c_n(0,x) = \sum_{y:|y|=1} \left[c_1(0,y) c_{n-1}(y,x) - \sum_{\omega \in \Omega_{n-1}(y,x)} I[0 \in \omega] \right]. \tag{4.5}$$

In fact this just follows by inclusion-exclusion: the first term on the right side counts all walks from 0 to x which are self-avoiding *after* the first step, and the second subtracts the contribution overcounted in the first term, due to walks which are self-avoiding apart from a single return to the origin.

Since $c_1(0,y)=1$ for $|y|=1$, substitution of (4.5) into (4.4) gives

$$G_z(0,x)=\delta_{0,x}+z\sum_{y:|y|=1}G_z(y,x)-\sum_{y:|y|=1}\sum_{n=0}^{\infty}z^{n+1}\sum_{\omega\in\Omega_n(y,x)}I[0\in\omega]. \tag{4.6}$$

The second term on the right side is a convolution with G_z, and we wish to write the last term on the right side also as a convolution.

For this purpose, we now apply the inclusion-exclusion relation to the last term on the right side of (4.6), as follows. Let m be the first (and only) time that $\omega(m)=0$. Then

$$\sum_{\omega\in\Omega_n(y,x)}I[0\in\omega]=\sum_{m=1}^{n}\sum_{\substack{\omega^{(1)}\in\Omega_m(y,0)\\ \omega^{(2)}\in\Omega_{n-m}(0,x)}}I[\omega^{(1)}\cap\omega^{(2)}=\{0\}]$$

$$=\sum_{m=1}^{n}\left[c_m(y,0)c_{n-m}(0,x)-\sum_{\substack{\omega^{(1)}\in\Omega_m(y,0)\\ \omega^{(2)}\in\Omega_{n-m}(0,x)}}I[\omega^{(1)}\cap\omega^{(2)}\neq\{0\}]\right].$$

The number $c_m(y,0)$ can be thought of as the number of $(m+1)$-step walks which step from the origin directly to y, then return to the origin in m steps, and which have distinct vertices apart from the fact that they return to their starting point. Let $\mathcal{U}_m$ denote the set of all m-step self-avoiding loops at the origin (m-step walks which begin and end at the origin but which otherwise have distinct vertices), and let u_m be the cardinality of $\mathcal{U}_m$. Then in view of the above equation,

$$\sum_{y:|y|=1}\sum_{n=0}^{\infty}z^{n+1}\sum_{\omega\in\Omega_n(y,x)}I[0\in\omega]$$
$$=\left(\sum_{m=2}^{\infty}u_m z^m\right)G_z(0,x)-\sum_{\substack{m\geq 2\\ n\geq 0}}\sum_{\substack{\omega^{(1)}\in\mathcal{U}_m\\ \omega^{(2)}\in\Omega_n(0,x)}}z^{m+n}I[\omega^{(1)}\cap\omega^{(2)}\neq\{0\}].$$

Thus we have partially achieved our goal of writing the last term on the right side of (4.6) as a convolution equation: the first term on the right side above is a particularly simple convolution of the two-point function with the constant (as a function of x) $\sum_{m=2}^{\infty}u_m z^m$.

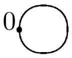
Continuing in this fashion, in the last term on the right side of the above equation let $m_1\geq 1$ be the first time along $\omega^{(2)}$ that $\omega^{(2)}(m_1)\in\omega^{(1)}$, and let $v=\omega^{(2)}(m_1)$. Then the inclusion-exclusion relation can be applied again to remove the avoidance between the portions of $\omega^{(2)}$ before and after m_1, and correct for this removal by the subtraction of a term involving a further intersection. Repetition of this procedure leads to the convolution equation

$$G_z(0,x)=\delta_{0,x}+z\sum_{y\in\Omega}G_z(y,x)+\sum_v\Pi_z(0,v)G_z(v,x), \tag{4.7}$$

where the 'irreducible' two-point function $\Pi_z(0,x)$ is given by

$$\Pi_z(0,v) = \sum_{N=1}^{\infty} (-1)^N \Pi_z^{(N)}(0,v), \tag{4.8}$$

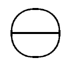
with $\Pi_z^{(N)}(0,v)$ non-negative and defined as follows. The $N=1$ term is given by

$$\Pi_z^{(1)}(0,v) = \delta_{0,v} \sum_{m=2}^{\infty} u_m z^m \equiv \delta_{0,v}\, 0$$  .

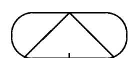
The $N=2$ term is

$$\Pi_z^{(2)}(0,v) = \prod_{i=1}^{3} \left[\sum_{m_i=1}^{\infty} z^{m_i} \sum_{\omega^{(i)} \in \Omega_{m_i}(0,v)} \right] I(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}),$$

where $I(\omega^{(1)}, \omega^{(2)}, \omega^{(3)})$ is equal to 1 if the $\omega^{(i)}$ are pairwise mutually avoiding apart from their common endpoints, and otherwise is equal to 0. Diagrammatically this can be represented by

$$\Pi_z^{(2)}(0,v) = \; 0$$  v

where each line represents a sum over self-avoiding walks between the endpoints of the line, weighted by z^m, with mutual avoidance between the three pairs of lines in the diagram. Similarly

$$\Pi_z^{(3)}(0,v) =$$ 
0 v

where now there is mutual avoidance between some but not all pairs of lines in the diagram. The unlabeled vertex is summed over $\mathbb{Z}^d$. A slashed propagator is used to indicate a walk which may have zero length, i.e., be a single site, whereas propagators without a slash correspond to walks of at least one step. All the higher order terms can be expressed as diagrams in this way. With some care it is possible to discern the pattern of which pairs of lines are mutually avoiding in these diagrammatic expressions, but this is perhaps better understood in the cluster expansion approach, where analytic expressions are used to define the diagrams. For our present purposes it is not important which lines avoid each other, because all such mutual avoidance among lines will be neglected in the upper bounds we will use to estimate the diagrams. Note that neglecting mutual avoidance allows additional configurations, and hence gives an upper bound in x-space for non-negative z.

Using translation invariance, and the fact that the Fourier transform of a convolution is the product of Fourier transforms, taking the Fourier transform of (4.7) and solving for $\hat{G}_z(k)$ gives

$$\hat{G}_z(k) = \frac{1}{1 - 2dz\hat{D}(k) - \hat{\Pi}_z(k)} \tag{4.9}$$

where

$$\hat{\Pi}_z(k) = \sum_{N=1}^{\infty} (-1)^N \hat{\Pi}_z^{(N)}(k). \tag{4.10}$$

Of course, up to now there is no guarantee that this expansion is convergent. Convergence will be discussed in Section 4.1.3, but first we shall indicate how it is possible to estimate the diagrams involved in Π_z in terms of the two-point function itself. Because the goal is to show that (4.9) behaves like k^{-2} for $z = z_c$, we will work in terms of the Fourier transform.

4.1.2. Bounds on $\hat{\Pi}_z(k)$

In this section we show how $\hat{\Pi}_z(k)$ can be bounded in terms of norms of the two-point function, such as the bubble diagram. Let $z \geq 0$. The easiest contribution to estimate is the one-loop diagram $\hat{\Pi}_z^{(1)}(k)$, which is given by

$$\hat{\Pi}_z^{(1)}(k) = \sum_{m=2}^{\infty} \sum_{\omega \in \mathcal{U}_m} z^{|\omega|} = z \sum_{y:|y|=1} G_z(y, 0) \tag{4.11}$$

and hence

$$|\hat{\Pi}_z^{(1)}(k)| \leq 2dz \sup_{x \neq 0} G_z(0, x). \tag{4.12}$$

Writing $\|\cdot\|_\infty$ for the x-space supremum norm

$$\|f\|_\infty = \sup_{x \in \mathbb{Z}^d} |f(x)|, \tag{4.13}$$

and introducing

$$H_z(x, y) = G_z(x, y) - \delta_{x,y} = \begin{cases} G_z(x, y) & x \neq y \\ 0 & x = y, \end{cases} \tag{4.14}$$

(4.12) can be rewritten as

$$|\hat{\Pi}_z^{(1)}(k)| \leq 2dz\, \|H_z\|_\infty. \tag{4.15}$$

(In view of the translation invariance of H_z, in writing norms of H_z we mean norms of the function $H_z(0, \cdot)$ of a single variable.) It is likely that little has been lost in this estimate, as the supremum is probably attained at a neighbour of the origin.

For $\hat{\Pi}_z^{(2)}(k)$, we first use the fact that all lines in the diagram representing $\Pi_z(0, x)$ must take at least one step, and hence there is no contribution from $x = 0$, to obtain

$$|\hat{\Pi}_z^{(2)}(k)| \leq \sum_{x \neq 0} \Pi_z^{(2)}(0, x). \tag{4.16}$$

Let $\|\cdot\|_p$ denote the x-space L^p-norm

$$\|f\|_p = \left[\sum_{x \in \mathbb{Z}^d} |f(x)|^p \right]^{1/p}. \tag{4.17}$$

Neglecting the fact that the three lines in the diagram representing $\Pi_z^{(2)}(0,x)$ mutually avoid, but not that each line is itself self-avoiding, gives the bound

$$|\Pi_z^{(2)}(0,x)| \leq H_z(0,x)^3. \tag{4.18}$$

Therefore by definition of the bubble diagram in (3.4),

$$|\hat{\Pi}_z^{(2)}(k)| \leq \|H_z\|_3^3 \leq \|H_z\|_\infty \|H_z\|_2^2 = \|H_z\|_\infty [\mathsf{B}(z)-1]. \tag{4.19}$$

The role of the bubble diagram in upper bounds now becomes apparent.

Higher order terms can be bounded similarly, using a bit more care than for the $N=2$ case, with the result that

$$|\hat{\Pi}_z^{(N)}(k)| \leq \|H_z\|_\infty [\mathsf{B}(z)-1]^{N/2}[\mathsf{B}(z)]^{N/2-1}. \tag{4.20}$$

It will also be necessary to obtain bounds on

$$\hat{\Pi}_z(0) - \hat{\Pi}_z(k) = \sum_{N=2}^{\infty} (-1)^N [\hat{\Pi}_z^{(N)}(0) - \hat{\Pi}_z^{(N)}(k)] \tag{4.21}$$

(there is no k dependence for $N=1$). This can be bounded in terms of the quantity $\sum_x x_\mu^2 \Pi_z^{(N)}(0,x)$, which is closely related to the second derivative of $\hat{\Pi}_z^{(N)}$ with respect to k_μ. For the two-loop diagram, we have

$$\begin{aligned}\sum_x x_\mu^2 \Pi_z^{(2)}(0,x) &\leq \sum_x x_\mu^2 H_z(0,x)^3 \leq \|x_\mu^2 H_z(0,x)\|_\infty \|H_z\|_2^2 \\ &= \|x_\mu^2 H_z(0,x)\|_\infty [\mathsf{B}(z)-1].\end{aligned} \tag{4.22}$$

This bound provides an indication of the critical nature of $d=4$, in the following way. Assuming that the infra-red bound $\eta \geq 0$ is indeed valid, then $\mathsf{B}(z_c)$ is finite for $d>4$. The infra-red bound is morally the statement that the critical two-point function decays at least as fast as $|x|^{2-d}$, so that for $d>4$ (4.22) will be finite at the critical point, and hence so will $\sum_x x_\mu^2 \Pi_{z_c}^{(2)}(0,x)$. For models with a suitable weak interaction, such as the nearest-neighbour model in sufficiently high dimensions, or a sufficiently spread-out model above four dimensions the quantity $\mathsf{B}(z_c)-1$ will be not only finite but small, and will be the small parameter responsible for convergence of the expansion. Thus the distinction between H_z and G_z is crucial, as $\|G_{z_c}\|_2^2 = \mathsf{B}(z_c)$ is not a small parameter.

Higher order terms can be handled in a similar fashion, and with a careful use of symmetry we obtain the bound

$$\begin{aligned}\hat{\Pi}_z(0) - \hat{\Pi}_z(k) &= \sum_{N=2}^{\infty} (-1)^N \Pi_z^{(N)}(0,x)[1-\cos k\cdot x] \\ &\geq -\sum_{j=1}^{\infty} [\hat{\Pi}_z^{(2j+1)}(0) - \hat{\Pi}_z^{(2j+1)}(k)] \\ &\geq -d[1-\hat{D}(k)]\|x_1^2 H_z\|_\infty \sum_{j=1}^{\infty} (j+1)^2 \|H_z\|_2^{2j+1} \|G_z\|_2^{2j-1}.\end{aligned} \tag{4.23}$$

4.1.3. Convergence of the Expansion

In this section we sketch a proof, based on the method of Slade (1987), that there is a constant K such that for the nearest-neighbour model in sufficiently high dimensions,

$$\|H_{z_c}\|_2^2 \le 2Kd^{-1} \text{ and } \|x_\mu^2 H_{z_c}\|_\infty \le 2Kd^{-1}. \tag{4.24}$$

In particular, $\mathsf{B}(z_c) = 1 + \|H_{z_c}\|_2^2 < \infty$ and the bubble condition is satisfied. The infra-red bound will be obtained in the course of the proof. For simple random walk the analogous behaviour can be proved without difficulty, and in fact the constant K above can be taken to be the smallest constant K such that the bounds

$$\|C_{1/(2d)}\| \;\le\; Kd^{-1}, \tag{4.25}$$

$$\|x_\mu^2 C_{1/(2d)}\|_\infty \;\le\; \left\| \frac{\partial_\mu^2 \hat{D}}{[1-\hat{D}]^2} \right\|_1 + \left\| \frac{(\partial_\mu \hat{D})^2}{[1-\hat{D}]^3} \right\|_1 \le Kd^{-1} \tag{4.26}$$

(here $\partial_\mu \equiv \partial/\partial k_\mu$) hold for all $d \ge 5$. Since $H_z(0,x) \le C_{1/(2d)}(0,x)$ for all $z \in [0, 1/(2d)]$, the bounds (4.24) hold when z_c is replaced by $z \in [0,1/(2d)]$. It is near the critical point that some work is required.

To prove (4.24) it is sufficient to obtain uniform bounds on the norms on the left sides, for all $z < z_c$. So let us fix an activity $z \in [1/(2d), z_c)$. Suppose for the moment that we had bounds on $\|x_\mu^2 H_z\|_\infty$ and $\|H_z\|_2$ that were a bit worse than (4.24), say with the constant factor 2 weakened to a factor 3. Then it would be possible to improve these weak bounds to the stronger bounds with factor 2, by arguing as follows. Defining $\hat{F}_z(k) \equiv \hat{G}_z(k)^{-1}$, the first step is to use (4.9) to write

$$\hat{G}_z(k) = \frac{1}{[\hat{F}_z(k) - \hat{F}_z(0)] + \hat{F}_z(0)} = \frac{1}{\hat{F}_z(0) + 2dz[1-\hat{D}(k)] + \hat{\Pi}_z(0) - \hat{\Pi}_z(k)}. \tag{4.27}$$

The term $\hat{F}_z(0)$ in the denominator is handled by noting that $\hat{F}_z(0) = \chi(z)^{-1} > 0$. For $\hat{\Pi}_z(0) - \hat{\Pi}_z(k)$, if we had the weak bounds, then for large d we could argue that the right side of (4.23) is dominated by the first term, which would be $O(d^{-3/2})[1-\hat{D}(k)]$ and hence a small correction to the simple random walk term $1 - \hat{D}(k)$. The factor $2dz$ in (4.27) is bounded below by 1 for $z \in [1/(2d), z_c)$. This means that given the weak bounds, we have the infra-red bound

$$\hat{G}_z(k) \le [1 + O(d^{-3/2})][1-\hat{D}(k)]^{-1} = [1+O(d^{-3/2})]\hat{C}_{1/(2d)}(k), \tag{4.28}$$

and this can then be used to obtain bounds on the norms appearing in (4.24) which are just slightly worse than the corresponding critical simple random walk bounds (hence the factor 2).

In the above we have assumed weak bounds to obtain stronger bounds. This shows that there is a forbidden region in the graphs of $\|H_z\|_2$ and $\|x_\mu^2 H_z\|_\infty$ versus z, namely the region where the weak bounds hold but the strong ones fail. Since these norms are continuous functions of $z < z_c$, and since the weak bounds hold for $z \le 1/(2d)$, they therefore must also hold for all $z < z_c$, and we are done. In addition, the weak bounds imply the strong bounds, and hence imply the infra-red bound as above.

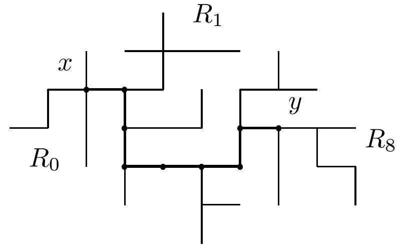


Fig. 6. Decomposition of a tree T containing sites x and y into its backbone and ribs $R_0, ..., R_8$. The vertices of the backbone are indicated by heavy dots.

The above argument can be carried out, with considerable elaboration, for $d \geq 5$. The infra-red bound implies the bubble condition, which gives the conclusion of Theorem 3.1. Sufficient control of $\hat{\Pi}_z(k)$ can be obtained to prove the stronger results of Section 2.1. For lattice trees and animals, and for percolation, the proof of convergence of the expansion has the same structure as that outlined in this section.

Finally we mention that (2.12) can be obtained by using $\chi(z_c)^{-1} = 1 - 2dz_c - \hat{\Pi}_{z_c}(0) = 0$.

4.2. Lattice Trees and Animals

For lattice trees and animals it is not immediately obvious how to adapt the lace expansion, since trees and animals are not one-dimensional structures. There is however a sense in which they are one-dimensional, in high dimensions. Consider the two-point function for trees:

$$G_z(x, y) = \sum_{T:T \ni x,y} z^{|T|}. \tag{4.29}$$

Given two distinct sites x, y and a tree $T \ni x, y$, the *backbone* of T is defined to be the unique path, consisting of bonds of T, which joins x to y. Sites in the backbone are labeled consecutively from x to y. For a tree with an n-bond backbone, removal of the bonds in the backbone disconnects the tree into $n+1$ mutually non-intersecting trees $R_0, ..., R_n$, which we refer to as *ribs*. This decomposition is shown in Figure 6. The rib generating function, or one-point function $G_z(0,0) = \sum_{T:T\ni 0} z^{|T|}$ is equal to $\sum_n (n+1)t_n z^n$. For $\theta = \frac{5}{2}$ this will be finite at the critical point, and in this sense trees can be considered to be one-dimensional structures in high dimensions. The lace expansion allows them to be treated as a perturbation of simple random walk.

The expansion can be performed by repeated use of inclusion-exclusion, as was done for self-avoiding walks. This begins by turning off the interaction between the rib at x and all subsequent ribs, to obtain a simple convolution term. Then there is a correction term in which at least one intersection is required between the rib at x and some subsequent rib. In this term there must be a first rib which intersects the rib at x, and to obtain a convolution the interaction is turned off between all

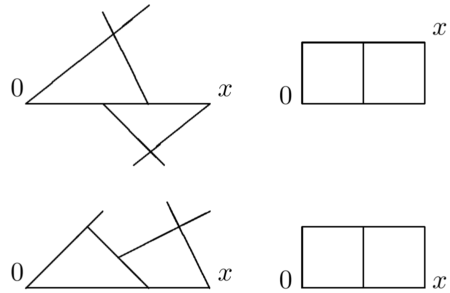


Fig. 7. The two generic types of rib intersection occurring at the two-loop level, and the Feynman diagrams bounding the corresponding contributions to $\Pi_z^{(2)}(0,x)$.

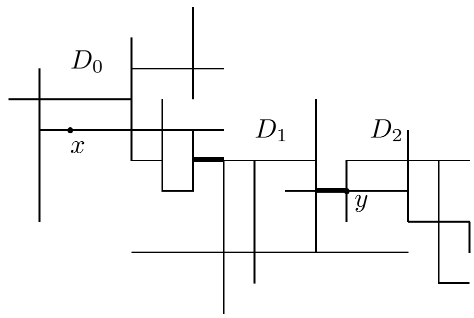


Fig. 8. Decomposition of a lattice animal A containing x and y into backbone and ribs D_0, D_1, D_2. The backbone, consisting of two bonds, is drawn in bold lines.

ribs following and preceding this first rib. This involves a further correction term with further intersections, and so on. A quantity like $\Pi_z(0,x)$ for self-avoiding walks arises, but for trees this is estimated in terms of the square diagram rather than the bubble diagram. It is through controlling this analogue of $\Pi_z(0,x)$ that the critical behaviour is accessed. The generic intersections and resulting Feynman diagrams are illustrated in Figure 7 for the second order contribution to $\Pi_z(0,x)$. Avoidance constraints between distinct diagram lines can be neglected in upper bounds.

For lattice animals there is not a unique backbone as there is for trees. To modify the notion of backbone to suit lattice animals, we first introduce some definitions. A lattice animal A containing x and y is said to have a *double connection* from x to y if there are two *disjoint* (i.e., sharing no common bond) self-avoiding walks in A between x and y or if $x = y$. A bond $\{u,v\}$ in A is called *pivotal* for the connection from x to y if its removal would disconnect the animal into two connected components with x in one connected component and y in the other.

Given two sites x, y and an animal A containing x and y, the *backbone* of A is now defined to be the set of pivotal bonds for the connection from x to y. In general this backbone is not connected. The *ribs* of A are the connected components which remain after the removal of the backbone from A. An example is depicted in Figure 8. One can then produce a lace expansion based on the inclusion-exclusion relation, as was done for trees. Again the square diagram plays a basic role in the estimates. Further details can be found in Hara and Slade (1990b, 1992c).

4.3. Percolation

For percolation the basic idea behind the expansion is similar to that underlying the expansion for lattice animals. Suppose that $p < p_c$, so that the connected cluster of the origin is finite with probability one. Given a configuration in which 0 and x are connected, the connected bond cluster of the origin is a lattice animal containing the sites 0 and x. The occupied pivotal bonds divide the cluster into doubly-connected ribs, as in Figure 8. No two of these pieces can share a common site, so there is a kind of 'repulsive interaction' between these pieces. However the situation is not as simple as it was for lattice animals, because the pieces interact also when they share a common boundary bond, due to the factors of $1 - p$ associated with the unoccupied boundary bonds.

We shall consider the percolation cluster to be like a self-avoiding walk, whose steps correspond to the pivotal bonds and whose sites are the intervening doubly-connected clusters. The first task is to extract the contribution due to the zero-step walk, which in the percolation context corresponds to the event that 0 and x are doubly-connected. Thus we have

$$\tau_p(0,x) = \langle I[x \in D_c(0)]\rangle_p + \langle I[x \in C(0), x \notin D_c(0)]\rangle_p \tag{4.30}$$

(see Definition 3.4 for definitions used in this section; we treat the nearest-neighbour and spread-out models simultaneously). If 0 is connected to x, but not doubly, then there is a pivotal bond for the connection from 0 to x and hence a first pivotal bond, so that

$$\begin{aligned}\tau_p(0,x) = {} & \langle I[x \in D_c(0)]\rangle_p \\ & + \sum_{(u,v)} \langle I[x \in C(0), (u,v) \text{ is the first pivotal bond}]\rangle_p.\end{aligned} \tag{4.31}$$

To proceed further, we need a way of writing the last term on the right side as a convolution with τ_p. This is achieved using Lemma 3.5.

We apply Lemma 3.5 to the second term on the right side of (4.31), with $E_2 = E_2(0,u;\mathbb{Z}^d) = \{u \in D_c(0)\}$. The summand in this term is equal to the probability that 0 is doubly-connected to u and (u,v) is occupied and pivotal for the connection from 0 to x. Hence by the lemma it is equal to

$$p\langle I[u \in D_c(0)]\, \tau_p^{C_{\{u,v\}}(0)}(v,x)\rangle_p. \tag{4.32}$$

To extract a term involving a convolution with τ_p from this quantity, we write

$$\tau_p^{C_{\{u,v\}}(0)}(v,x) = \tau_p(v,x) - [\tau_p(v,x) - \tau_p^{C_{\{u,v\}}(0)}(v,x)]. \tag{4.33}$$

Using (4.32) and (4.33) in (4.31), we obtain

$$\begin{aligned}\tau_p(0,x) &= \langle I[x \in D_c(0)]\rangle_p + p \sum_{(u,v)} \langle I[u \in D_c(0)]\rangle_p \tau_p(v,x) \\ &\quad - p \sum_{(u,v)} \langle I[u \in D_c(0)]\{\tau_p(v,x) - \tau_p^{C_{\{u,v\}}(0)}(v,x)\}\rangle_p. \end{aligned} \tag{4.34}$$

The above equation gives the lowest order expansion with remainder. We abbreviate the notation by writing

$$g_p(0,x) = \langle I[x \in D_c(0)]\rangle_p \tag{4.35}$$

and

$$R_p^{(0)}(0,x) = p \sum_{(u,v)} \langle I[u \in D_c(0)]\{\tau_p(v,x) - \tau_p^{C_{\{u,v\}}(0)}(v,x)\}\rangle_p. \tag{4.36}$$

We denote by D the function on $\mathbb{Z}^d$ which takes the value $|\Lambda|^{-1}$ at sites in Λ and otherwise is zero; here $|\Lambda|$ denotes the cardinality of the set Λ. Then (4.34) can be rewritten as

$$\tau_p(0,x) = g_p(0,x) + g_p * p|\Lambda| D * \tau_p(x) - R_p^{(0)}(0,x), \tag{4.37}$$

where $*$ denotes convolution: $f*g(x) = \sum_y f(x-y)g(y)$. To proceed further, we will use the following lemma to expand the remainder term $R_p^{(0)}(0,x)$. For the statement of the lemma, we write $I_2(v,x;A)$ for the indicator function of the event $E_2(v,x;A)$.

Lemma 4.1. *Given a set of sites A and two sites v and x,*

$$\tau_p(v,x) - \tau_p^A(v,x) = \langle I_2(v,x;A)\rangle_p + p \sum_{(y,y')} \langle I_2(v,y;A)\tau_p^{C_{\{y,y'\}}(v)}(y',x)\rangle_p. \tag{4.38}$$

Proof. The left side is the probability of the event that v and x are connected but are not connected in $\mathbb{Z}^d \backslash A$. By definition, this is the probability that v is connected to x through A. If v is connected to x through A then either (i) there is no pivotal bond for the connection from v to x whose first endpoint is connected to v through A, or (ii) there is such a pivotal bond. Case (i) is exactly the event $E_2(v,x;A)$, and gives the first term on the right side of (4.38). In case (ii), let (y,y') denote the first pivotal bond for the connection from v to x such that y is connected to v through A. The contribution to the left side of (4.38) due to this case is

$$\sum_{(y,y')} \langle I[E_2(v,y;A)] I[(y,y') \text{ is occupied and pivotal for the connection from } v \text{ to } x]\rangle_p. \tag{4.39}$$

Then by Lemma 3.5, with (v,y) playing the role of $(0,u)$, the contribution due to this case gives the second term on the right side of (4.38). □

Using Lemma 4.1 and (4.36), and replacing the summation index (u,v) by (y_1, y_1'), we have

$$\begin{aligned} R_p^{(0)}(0,x) \;=\; & p \sum_{(y_1,y_1')} \langle I[y_1 \in D_c(0)] \langle I_2(y_1', x; C^0_{\{y_1,y_1'\}}(0)) \rangle^{(1)} \rangle^{(0)} \\ & + p^2 \sum_{(y_1,y_1')} \sum_{(y_2,y_2')} \langle I[y_1 \in D_c(0)\,] \langle I_2(y_1', y_2; C^0_{\{y_1,y_1'\}}(0)) \\ & \qquad \times \tau_p^{C^1_{\{y_2,y_2'\}}(y_1')}(y_2', x) \rangle^{(1)} \rangle^{(0)}. \end{aligned}$$

Here and in the following we simplify the notation by dropping the subscript p from the angular brackets denoting expectation. In addition we have introduced a superscript to coordinate random sets with the appropriate expectation, in nested expectations. Thus for example in the second term in the right side of the above equation, the set $C^0_{\{y_1,y_1'\}}(0)$ is random with respect to the outer expectation, but may be treated as deterministic in the evaluation of the inner expectation. Using the analogue of (4.33) to replace the restricted two-point function on the right side by an unrestricted two-point function plus a correction, and defining

$$\Pi_p^{(1)}(0,x) = p \sum_{(y_1,y_1')} \langle I[y_1 \in D_c(0)] \langle I_2(y_1', x; C^0_{\{y_1,y_1'\}}(0)) \rangle^{(1)} \rangle^{(0)}$$

and

$$\begin{aligned} R_p^{(1)}(0,x) \;=\; & p^2 \sum_{(y_1,y_1')} \sum_{(y_2,y_2')} \langle I[y_1 \in D_c(0)] \langle I_2(y_1', y_2; C^0_{\{y_1,y_1'\}}(0)) \\ & \times \{\tau_p(y_2', x) - \tau_p^{C^1_{\{y_2,y_2'\}}(y_1')}(y_2', x)\} \rangle^{(1)} \rangle^{(0)}, \end{aligned}$$

we now have from (4.37) that

$$\tau_p(0,x) = g_p(0,x) - \Pi_p^{(1)}(0,x) + [g_p - \Pi_p^{(1)}] * p|\Lambda| D * \tau_p(x) + R_p^{(1)}(0,x). \tag{4.40}$$

The above procedure can be iterated as many times as desired. The result is the lace expansion for percolation, which is stated in the next theorem. For the statement of the theorem we write $y_0' = 0$ and introduce for $n \geq 1$,

$$C^{n-1} = C^{n-1}_{\{y_n,y_n'\}}(y_{n-1}'), \qquad I^n = I_2(y_n', y_{n+1}; C^{n-1}),$$

$$\begin{aligned} \Pi_p^{(n)}(0,x) \;=\; & p^n \sum_{(y_1,y_1')} \cdots \sum_{(y_n,y_n')} \langle I[y_1 \in D_c(0)] \langle I^1 \langle I^2 \langle I^3 \ldots \langle I^{n-1} \\ & \times \langle I_2(y_n', x; C^{n-1}) \rangle^{(n)} \rangle^{(n-1)} \ldots \rangle^{(3)} \rangle^{(2)} \rangle^{(1)} \rangle^{(0)}, \end{aligned}$$

$$h_p^{(n)}(0,x) = g_p(0,x) + \sum_{j=1}^{n} (-1)^j \Pi_p^{(j)}(0,x)$$

and

$$R_p^{(n)}(0,x) = p^{n+1} \sum_{(y_1,y_1')} \cdots \sum_{(y_{n+1},y_{n+1}')} \langle I[y_1 \in D_c(0)] \langle I^1 \langle I^2 \ldots \langle I^n$$
$$\times \{\tau_p(y_{n+1}',x) - \tau_p^{C^n}(y_{n+1}',x)\}\rangle^{(n)} \ldots \rangle^{(2)} \rangle^{(1)} \rangle^{(0)}.$$

Finally defining $h_p^{(0)}(0,x) = g_p(0,x)$, we have the following theorem.

Theorem 4.2. *For $p < p_c$ and $N \geq 0$,*

$$\tau_p(0,x) = h_p^{(N)}(0,x) + h_p^{(N)} * p\Omega D * \tau_p(x) + (-1)^{N+1} R_p^{(N)}(0,x). \tag{4.41}$$

Now we take the Fourier transform as was done for self-avoiding walks. Bounds on $\hat{\Pi}_p^{(j)}(k)$ and $\hat{R}_p^{(N)}(k)$ involve bounding the nested expectations from the inside out, using the BK inequality. The triangle diagram is important in the bounds. Further details can be found in Hara and Slade (1990a).

4.4. Oriented Percolation

For oriented percolation the lace expansion is closer in some respects to the expansion for lattice animals than for unoriented percolation. The Markov property makes the analysis somewhat simpler. Details can be found in Nguyen and Yang (1992).

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