Generalized Markov numbers
and
generalized cluster algebras

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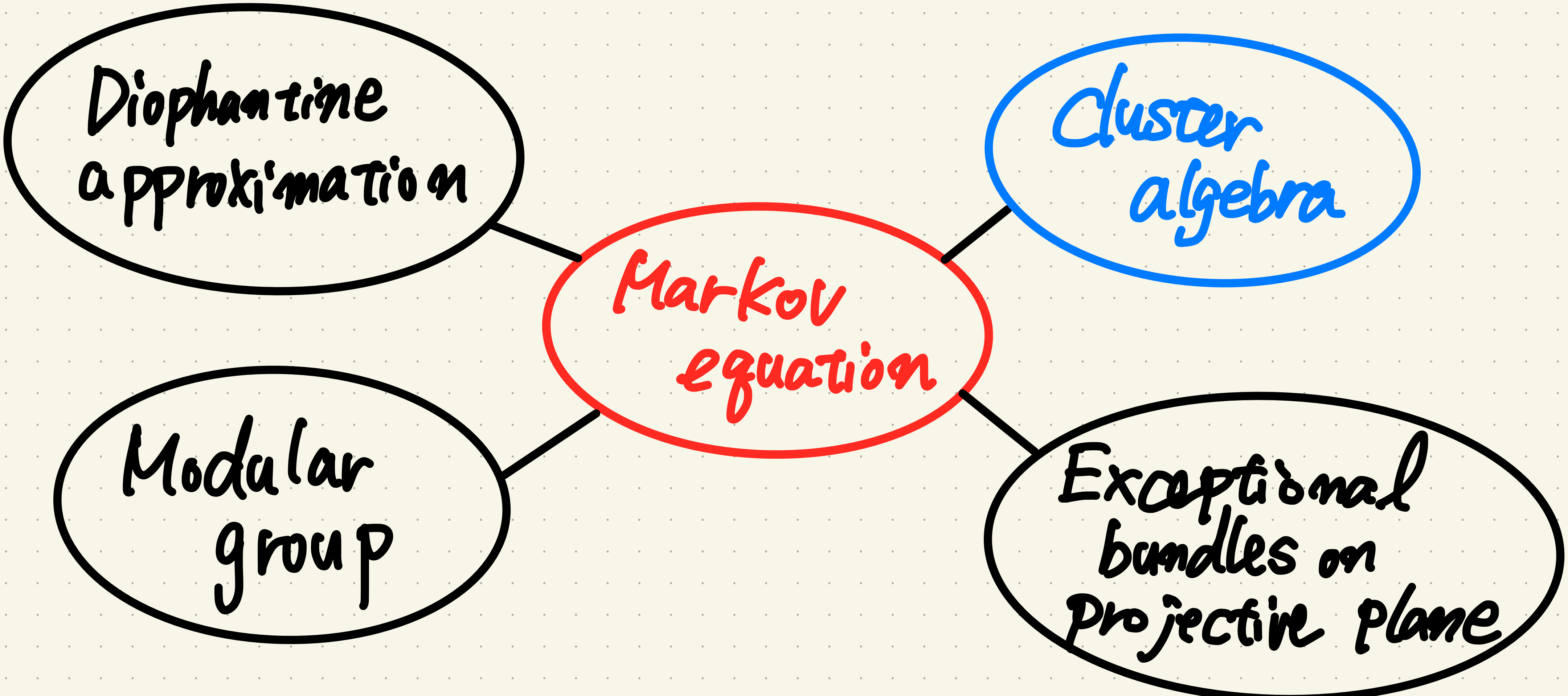
Contents

1. Introduction

Markov equation (ME)

$$x^2+y^2+z^2=3xyz.$$



$$\begin{array}{ccc} \boxed{\text{Cluster algebra}} & \longrightarrow & \boxed{\text{Markov equation}} \\ \cap & & \cap \\ \boxed{\text{generalized Cluster algebra}} & \longrightarrow & \boxed{\text{generalized Markov equation}} \end{array}$$

Generalized Markov Equation (GME)

$$x^2 + y^2 + z^2 + k_1 xy + k_2 yz + k_3 zx = (3 + k_1 + k_2 + k_3) xyz$$

2. Markov equation and its generalization

Markov Equation

$$x^2+y^2+z^2=3xyz$$

positive integer solutions to ME:

$(x,y,z)=(1,1,1),\ (1,1,2),\ (1,2,5),\ (1,5,13),\ \ldots$

The integers appearing in positive integer solutions to ME are called the **Markov numbers**.

Thm (Someone?)

We consider

$$
(a, b, c) \;\longrightarrow\;
\begin{array}{l}
\left(\dfrac{b^2+c^2}{a},\ b,\ c \right) \\[2ex]
\left(a,\ \dfrac{a^2+c^2}{b},\ c \right) \\[2ex]
\left(a,\ b,\ \dfrac{a^2+b^2}{c} \right).
\end{array}
$$

(1) If (a, b, c) is a positive integer solution to ME, then so are the right 3 triplets.

(2) All positive integer solutions to ME are obtained by applying this operation to $(1, 1, 1)$ repeatedly.

Solution tree of $x^2+y^2+z^2=3xyz$

$$
\begin{tikzcd}[column sep=small]
 & & & & (1,34,13)\cdots \\
 & & & (1,5,13) \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & (194,5,13)\cdots \\
(1,1,1) \arrow[r, dash] & (1,1,2) \arrow[r, dash] & (1,5,2) \arrow[uur, dash] \arrow[ddr, dash] & & \\
 & & & & (29,5,433)\cdots \\
 & & & (29,5,2) \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & (29,169,2)\cdots
\end{tikzcd}
.
$$

$$\frac{1^2+5^2}{2}=\frac{26}{2}$$

Generalized Markov Equation (GME(k_1, k_2, k_3))

$$x^2 + y^2 + z^2 + k_1 xy + k_2 yz + k_3 zx = (3 + k_1 + k_2 + k_3) xyz$$

for $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$

$$k_1 = k_2 = k_3 = 0 \Rightarrow x^2 + y^2 + z^2 = 3xyz$$

$$k_1 = k_2 = k_3 = 1 \Rightarrow (x+y)^2 + (y+z)^2 + (z+x)^2 = 12xyz$$

Thm (G.-Matsushita)

We consider

$$(a,b,c) \longrightarrow \begin{cases} \left(\dfrac{b^2+k_2bc+c^2}{a},\ b,\ c\right) \\[2ex] \left(a,\ \dfrac{a^2+k_3ac+c^2}{b},\ c\right) \\[2ex] \left(a,\ b,\ \dfrac{a^2+k_1ab+b^2}{c}\right) \end{cases}$$

(1) If (a,b,c) is a positive integer solution to GME(k_1,k_2,k_3) then so are the right 3 triplets.

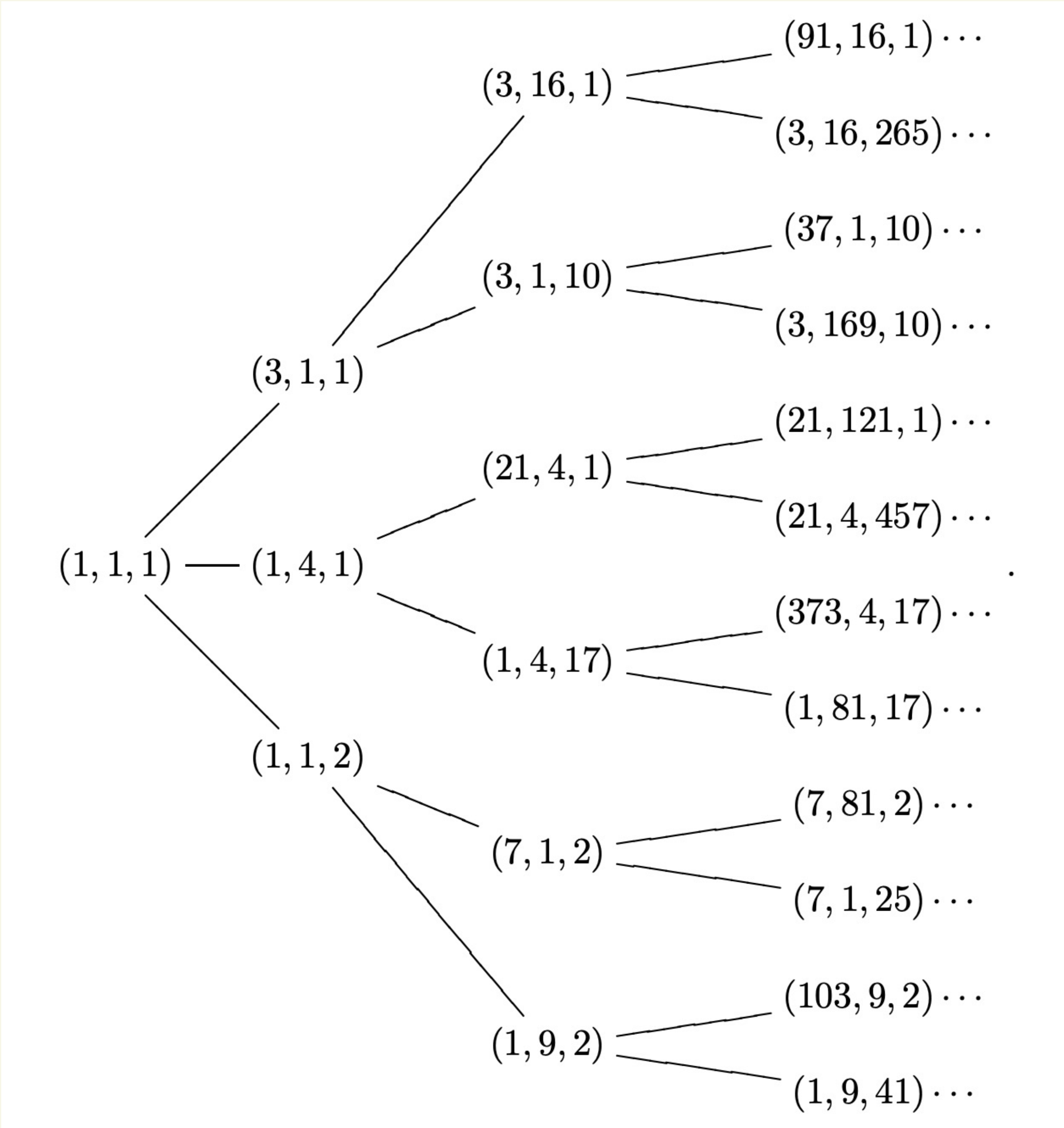
(2) All positive integer solutions to GME(k_1,k_2,k_3) are obtained by applying this operation to $(1,1,1)$ repeatedly.

Solution tree of $x^2+y^2+z^2+xy+yz+zx=6xyz$

$$
\begin{tikzcd}[arrows=dash]
 & & & & (1,291,61)\cdots \\
 & & & (1,13,61) \arrow[ur] \arrow[dr] & \\
 & & & & (4683,13,61)\cdots \\
(1,1,1) \arrow[r] & (1,1,3) \arrow[r] & (1,13,3) \arrow[uur] \arrow[ddr] & & \\
 & & & & (217,13,16693)\cdots \\
 & & & (217,13,3) \arrow[ur] \arrow[dr] & \\
 & & & & (217,3673,3)\cdots
\end{tikzcd}
$$

$$\frac{1^2+3^2+1\cdot1\cdot3}{1} = 13$$

Solution tree of $x^2+y^2+z^2+0xy+1yz+2zx=6xyz$



5. Generalized seed mutation

- $(\mathbf{x}, \mathbf{z}, B)$: seed

$:\Leftrightarrow$

- $\mathbf{x} = (x_1, \ldots, x_n)$: (formal) variables. "cluster"
- $\mathbf{z} := (z_1, \ldots, z_n)$
  n-tuple of palindromic monic polynomials of 1-variable with strictly positive integer coefficients.
- $B = (b_{ij})$: $n \times n$ skew-symmetrizable matrix
  i.e. $\exists S = \mathrm{diag}(s_1, \ldots, s_n), s_i > 0$, SB : skew-symmetric

Generalized seed mutation:

For $k \in [1, n]$ and $(\mathbf{x}, \mathbf{z}, B)$, we obtain a mutated seed $(\mathbf{x}', \mathbf{z}', B') := \mu_k(\mathbf{x}, \mathbf{z}, B)$ at k as follows:

$$
\begin{cases}
\circ\ \mathbf{z}' := \mathbf{z}, \\
\circ\ b_{ij}' = \begin{cases} -b_{ij} & (i \text{ or } j = k) \\ b_{ij} + \deg z_k \cdot ([b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+) & (\text{otherwise}) \end{cases} \\
\circ\ x_i' = \begin{cases} \dfrac{\left(\prod_j x_j^{[b_{jk}]_+}\right)^{\deg z_k} \cdot Z_k\left(\prod_j x_j^{[-b_{jk}]_+}\right)}{x_k} & (i = k) \\ x_i & (i \neq k) \end{cases}
\end{cases}
$$

where $[b]_+ := \max(0, b)$.

Ex.

$$\mathbf{x} = (x_1, x_2, x_3), \quad \mathbb{Z} = \begin{cases} z_1 = 1 + u + u^2 \\ z_2 = 1 + u \\ z_3 = 1 + 2u + u^2, \end{cases} \quad B = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\longrightarrow \quad \mu_i(B) = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix} = -B$$

$$(x_1, x_2, x_3) \xrightarrow{\mu_1} \left(\frac{x_2^2 + x_2 x_3 + x_3^2}{x_1}, x_2, x_3 \right)$$

$$(x_1, x_2, x_3) \xrightarrow{\mu_2} \left(x_1, \frac{x_1^2 + x_3^2}{x_2}, x_3 \right)$$

$$(x_1, x_2, x_3) \xrightarrow{\mu_3} \left(x_1, x_2, \frac{x_1^2 + 2x_1 x_2 + x_2^2}{x_3} \right)$$

Rule of $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + x_2x_3 = 3xyz$!

Thm (G.-Matsushita)

For $(B, \mathbb{Z})$ in the below table, we set $\{\mathbf{x}_t\}_{t\in\mathbb{T}_n}$ all clusters obtained from applying mutations to $(\mathbf{x}, \mathbb{Z}, B)$ repeatedly. Then $\{\mathbf{x}_t\}_{t\in\mathbb{T}_n} \mid x_1=x_2=x_3=1$ gives the set of positive integer solution to the corresponding equation.

	Equation	B	$\mathbf{Z}$
ME	$x^2+y^2+z^2=3xyz$	$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u)=1+u \\ Z_2(u)=1+u \\ Z_3(u)=1+u \end{cases}$
GME$(k_1,0,0)$	$x^2+y^2+z^2+k_1xy=(3+k_1)xyz$	$\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 2 & -2 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u)=1+u \\ Z_2(u)=1+u \\ Z_3(u)=1+k_1u+u^2 \end{cases}$
GME$(k_1,k_2,0)$	$x^2+y^2+z^2+k_1xy+k_2yz$ $=(3+k_1+k_2)xyz$	$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u)=1+k_2u+u^2 \\ Z_2(u)=1+u \\ Z_3(u)=1+k_1u+u^2 \end{cases}$
GME(k_1,k_2,k_3)	$x^2+y^2+z^2+k_1xy+k_2yz+k_3zx$ $=(3+k_1+k_2+k_3)xyz$	$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$	$\begin{cases} Z_1(u)=1+k_2u+u^2 \\ Z_2(u)=1+k_3u+u^2 \\ Z_3(u)=1+k_1u+u^2 \end{cases}$

13. Properties and questions

Markov uniqueness (unicity) conjecture.

For a Markov number c, there is a unique positive integer solution $\{x, y, z\} = \{a, b, c\}$ to ME up to order such that $a, b \leq c$.

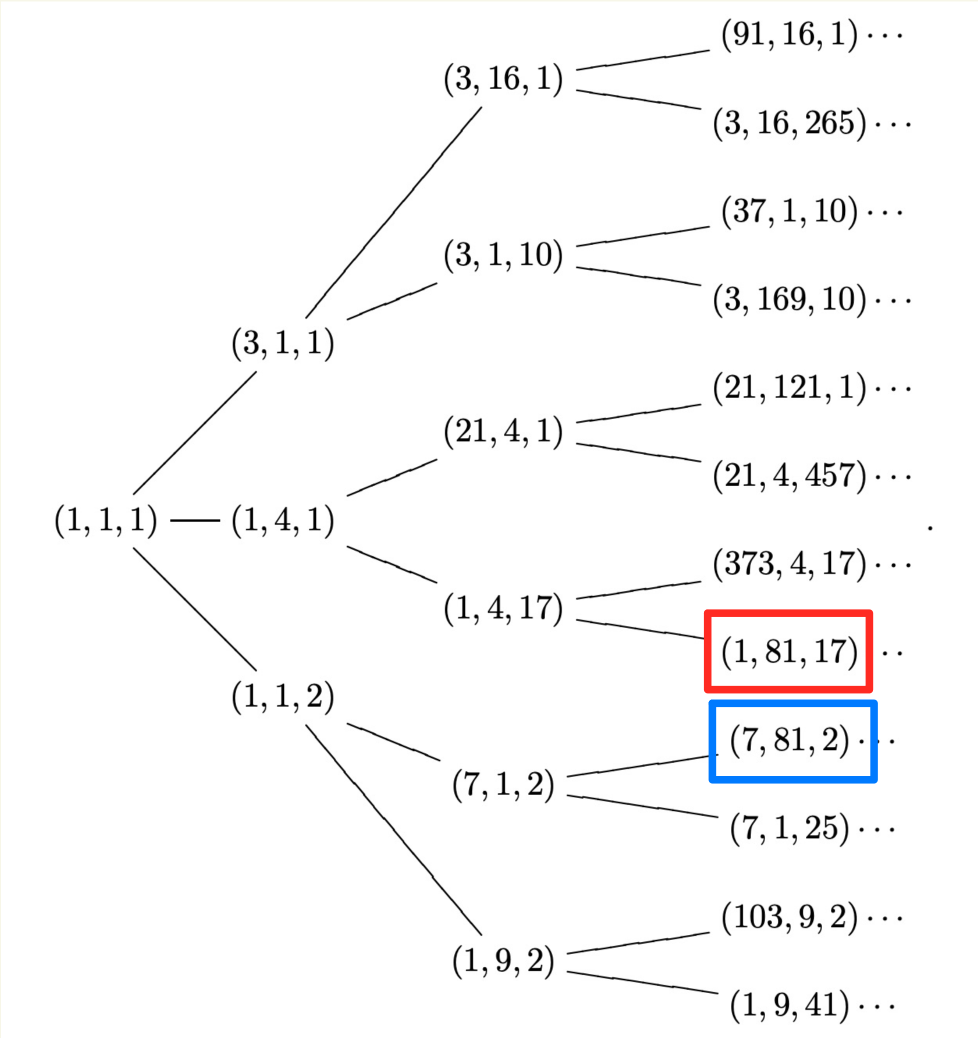
For example, for $c = 5$, positive integer solution $\{a, b, 5\}$ to ME which satisfies $a, b < 5$ is only $\{1, 2, 5\}$.

How about GME version?

Thm (G.-Matsushita)

There is a counterexample of GME version of the conjecture if k_1, k_2, k_3 are not the same number.

When $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $(1.81, 17)$ and $(7, 81, 2)$ are both solutions to GME which have 81 as maximal.



Question (open problem!)

How about the case $k_0 = k_1 = k_2$?

$k_0 = k_1 = k_2 = 2$ case:

Thm (G.-Matsushita)

(1) If $(x, y, z) = (a, b, c)$ is a positive integer solution to ME, then $(x, y, z) = (a^2, b^2, c^2)$ is one to GME $(2,2,2)$.

(2) If $(x, y, z) = (A, B, C)$ is a positive integer solution to GME $(2,2,2)$ then $(x, y, z) = (\sqrt{A}, \sqrt{B}, \sqrt{C})$ is one to ME.

Cor.

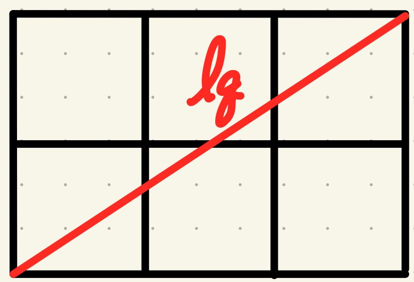
M.U.C. $\Leftrightarrow$ G.M.U.C. $(2,2,2)$

- Irreducible fraction and (generalized) Markov number

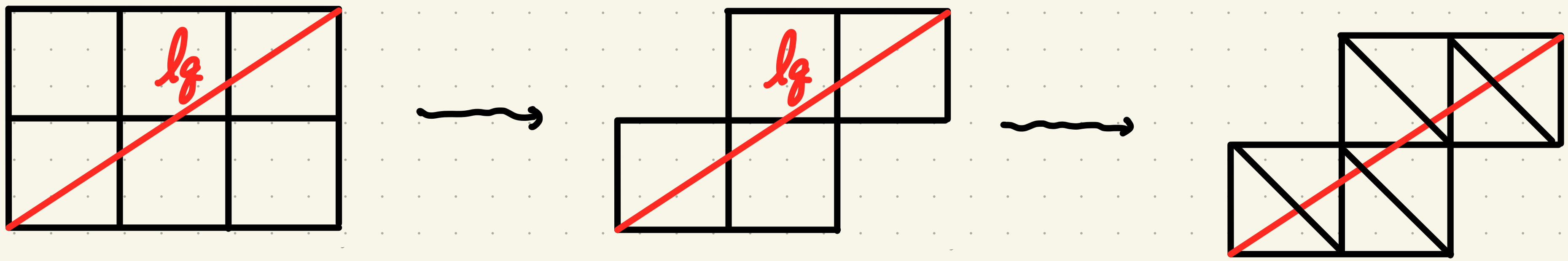
We construct a map

$$\left\{ \begin{matrix} \text{Irreducible fraction} \\ 0 < q \leq 1 \end{matrix} \right\} \longrightarrow \{ \text{Markov number} \}.$$

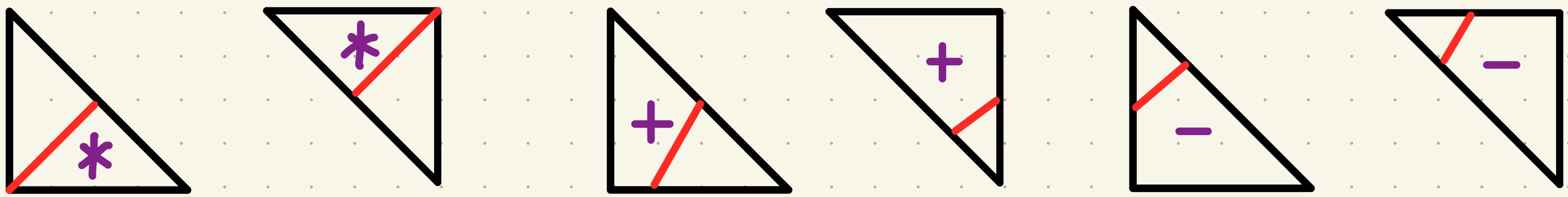
For given $q \in \mathbb{Q}$, $0 < q \leq 1$,
We consider a line segment l_q in $\mathbb{R}^2$ with the slope q from a lattice point to the next one.

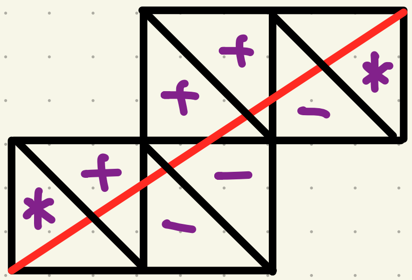
Ex. $q = \frac{2}{3} \Rightarrow$ 

We focus tiles that l_q passes through:



We assign + or − for each triangle of ↑ as follows:



$\rightsquigarrow$ 

$* \in \{+, -\}$

$$\longrightarrow \left[\underbrace{\overset{*}{+}, +}_{2}, \underbrace{-, -}_{2}, \underbrace{+, +}_{2}, \underbrace{-, \overset{*}{-}}_{2}\right]$$

$$\longrightarrow [2, 2, 2, 2] = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{29}{12}$$

m_q

Thm. (Canacki-Schiffler, Banaian).

For $0 < q, q_1, q_2, q_3 \leq 1$,

(1) $q \longmapsto m_q$ is well-defined (not depend on the choice of $*$).

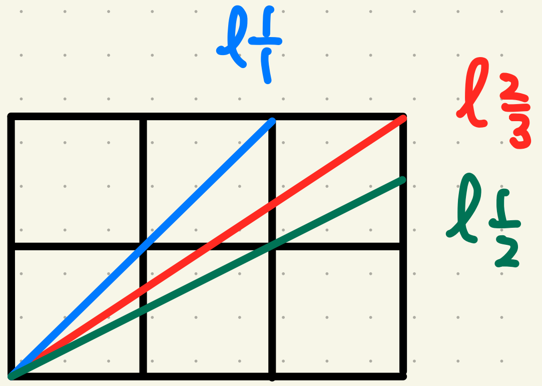
(2) m_q is a Markov number.

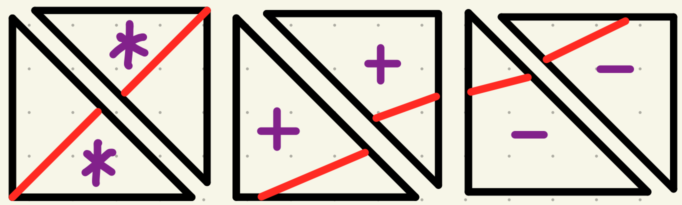
(3) $(m_{q_1}, m_{q_2}, m_{q_3})$ is a solution to ME

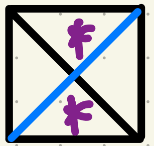
$\Leftrightarrow$ $L_{q_1}, L_{q_2}, L_{q_3}$ cross each other on lattice points only ($L_q := \{$line segments with slope $q\}$).

$\Leftrightarrow$ if $q_1 = \frac{a}{b}, q_2 = \frac{c}{d}, q_3 = \frac{e}{f}$ are reduced expressions, then $\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = \left|\det\begin{pmatrix} c & e \\ d & f \end{pmatrix}\right| = \left|\det\begin{pmatrix} e & a \\ f & b \end{pmatrix}\right| = 1$.

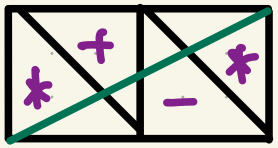
Ex.







$$\rightsquigarrow [\overset{+}{*}, \overset{-}{*}] \longrightarrow [1,1] \rightsquigarrow 1 + \frac{1}{1} = \frac{2}{1} \qquad 2 = m_{\frac{1}{1}}$$



$$\longrightarrow [\underbrace{\overset{+}{*}, +}_{2}, \underbrace{-, \overset{-}{*}}_{2}] \rightsquigarrow [2,2] \longrightarrow 2 + \frac{1}{2} = \frac{5}{2} \qquad 5 - m_{\frac{1}{2}}$$

$$\left(m_{\frac{1}{1}}, m_{\frac{1}{2}}, m_{\frac{2}{3}}\right) = (2, 5, 29)$$

$$2^2 + 5^2 + 29^2 = 870 = 3 \cdot 2 \cdot 5 \cdot 29$$

Thm.

M.U.C. $\Leftrightarrow$ $q \mapsto m_q$ is injective.

Aigner Conjecture (solved, Lee-Li-Rabideau-Schiffler)

For $0 < \frac{a}{b} \leq 1$,

(a) $m_{\frac{a}{b}} < m_{\frac{a}{b+1}}$

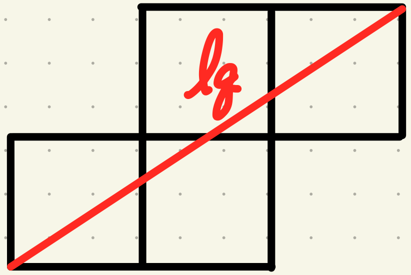
(b) $m_{\frac{a}{b}} < m_{\frac{a+1}{b}}$

(c) $m_{\frac{a}{b}} < m_{\frac{a-1}{b+1}}$

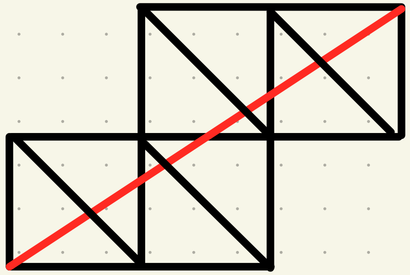
We construct a map

$$\left\{ \begin{array}{r} \text{Irreducible fraction} \\ 0<q\le 1 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Generalized} \\ \text{Markov number} \\ \text{for } k_1=k_2=k_3=1 \end{array}\right\}.$$

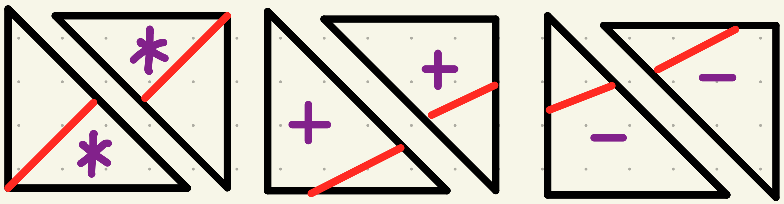
$q = \dfrac{2}{3} \rightsquigarrow$

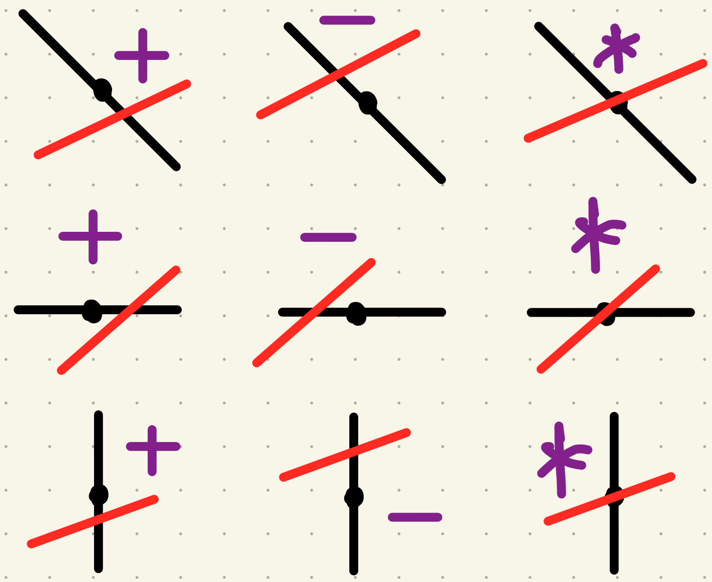


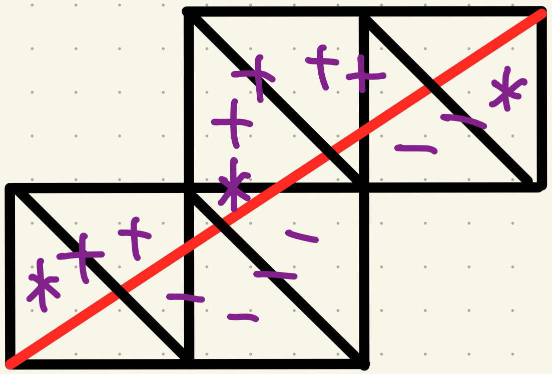
$\rightsquigarrow$



We assign + or − to each triangle and edge that lq pass though:







$$\to \left[\overbrace{\underset{+}{*}, +, +}^{3}, \overbrace{-, -, -, -, \underset{-}{*}}^{5}, \overbrace{+, +, +, +}^{4}, \overbrace{-, -, \underset{-}{*}}^{3}\right]$$

$$\to [3, 5, 4, 3] = 3+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{3}}} = \frac{\overset{Mq}{217}}{68}$$

Thm. (Bananian).

For $0 < q, q_1, q_2, q_3 \leq 1$,

(1) $q \longmapsto M_q$ is well-defined (not depend on the choice of $*$).

(2) M_q is a generalized Markov number for $k_1 = k_2 = k_3 = 1$.

(3) $(M_{q_1}, M_{q_2}, M_{q_3})$ is a solution to $GME(1,1,1)$

$\Leftrightarrow$ $L_{q_1}, L_{q_2}, L_{q_3}$ cross each other on lattice points only,

$\Leftrightarrow$ if $q_1 = \frac{a}{b}, q_2 = \frac{c}{d}, q_3 = \frac{e}{f}$ are reduced expressions,

then $\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = \left|\det\begin{pmatrix} c & e \\ d & f \end{pmatrix}\right| = \left|\det\begin{pmatrix} e & a \\ f & b \end{pmatrix}\right| = 1.$

Thm.

G.M.U.C.(1,1,1) $\Leftrightarrow$ $q \mapsto M_q$ is injective.

Question

- An analogue of Aigner conjecture, that is,

$$\begin{cases} \text{(a) } M_{\frac{a}{b}} < M_{\frac{a}{b+1}}, \\ \text{(b) } M_{\frac{a}{b}} < M_{\frac{a+1}{b}} \\ \text{(c) } M_{\frac{a}{b}} < M_{\frac{a-1}{b+1}} \end{cases}$$ is true or not.

- How about GME(k,k,k) for $k \neq 0, 1$?

Generalized seed mutation $\longrightarrow$ Equation?

There is another "GME-type equation"!

<u>Thm. (G.-Matsushita)</u>

For $(B, \mathbb{Z})$ in the below table, we set $\{x_t\}_{t\in\mathbb{T}_n}$ all clusters obtained from applying mutations to $(x, \mathbb{Z}, B)$ repeatedly. Then $\{x_t\}_{t\in\mathbb{T}_n}|_{x_1=x_2=x_3=1}$ gives the set of positive integer solution to the corresponding equation.

$D = [\deg Z_1 \dots \deg Z_n]$

Equation	B	$\mathbf{Z}$	D
$x^2+y^4+z^4+2xy^2+2z^2x=7xy^2z^2$	$\begin{bmatrix}0&1&-1\\-4&0&2\\4&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$x^2+y^4+z^4+2xy^2+ky^2z^2+2z^2x$ $=(7+k)xy^2z^2$	$\begin{bmatrix}0&1&-1\\-2&0&2\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+ku+u^2\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$

	Equation	B	$\mathbf{Z}$	D
(1)	$x^2+y^2+z^2=3xyz$	$\begin{bmatrix}0&2&-2\\-2&0&2\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
(2)	$x^2+y^2+z^2+k_1xy=(3+k_1)xyz$	$\begin{bmatrix}0&2&-1\\-2&0&1\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+k_1u+u^2\end{cases}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$
(3)	$x^2+y^2+z^2+k_1xy+k_2yz$ $=(3+k_1+k_2)xyz$	$\begin{bmatrix}0&2&-1\\-1&0&1\\1&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+k_2u+u^2\\Z_2(u)=1+u\\Z_3(u)=1+k_1u+u^2\end{cases}$	$\begin{bmatrix}2&0&0\\0&1&0\\0&0&2\end{bmatrix}$
(4)	$x^2+y^2+z^2+k_1xy+k_2yz+k_3zx$ $=(3+k_1+k_2+k_3)xyz$	$\begin{bmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+k_2u+u^2\\Z_2(u)=1+k_3u+u^2\\Z_3(u)=1+k_1u+u^2\end{cases}$	$\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$
(5)	$x^2+y^4+z^4+2xy^2+2z^2x=7xy^2z^2$	$\begin{bmatrix}0&1&-1\\-4&0&2\\4&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
(6)	$x^2+y^4+z^4+2xy^2+ky^2z^2+2z^2x$ $=(7+k)xy^2z^2$	$\begin{bmatrix}0&1&-1\\-2&0&2\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+ku+u^2\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Common Property: $\forall i \in \{1,2,3\}$, $\mu_i(B) = -B$.

BD is $\begin{bmatrix}0&2&-2\\-2&0&2\\2&-2&0\end{bmatrix}$ ((1) ~ (4)) or $\begin{bmatrix}0&1&-1\\-4&0&2\\4&-2&0\end{bmatrix}$ ((5), (6))

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}, \quad \mathbb{Z} = \begin{cases} Z_1 = 1 + k_1 u + k_2 u^2 + k_1 u^3 + u^4 \\ Z_2 = 1 + u \\ Z_3 = 1 + u \end{cases}, \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rightsquigarrow$ $\mu_i(B) = -B$, BD is the same as (5) and (6).

Question

Is there an equation such that it corresponds with the above $(B, \mathbb{Z})$?

General Question

What is the algorithm to construct the corresponding equation from "good" $(B, \mathbb{Z})$?

- Rank 2 observation

By fixing one variable in (x_1, x_2, x_3), we obtain the seed-equation correspondence of rank 2 from that of rank 3.

Ex.

$$\left(x = (x_1, x_2, \underset{\substack{\parallel \\ 1}}{x_3}),\ Z = \begin{cases} z_1 = 1 + u + u^2 \\ z_2 = 1 + u \\ \cancel{z_3 = 1 + 2u + u^2}, \end{cases} \quad B = \begin{bmatrix} 0 & 2 & \cancel{-1} \\ -1 & 0 & \\ \cancel{1} & \cancel{-2} & \cancel{0} \end{bmatrix} \right)$$

$$\longleftrightarrow \quad x^2 + y^2 + \underset{1}{\cancel{z}^2} + 2xy + y\cancel{z} = 3xy\cancel{z}$$

$$\Leftrightarrow \quad x^2 + y^2 + y + 1 = xy$$

Thm. (G.-Matsushita)

For $(B, \mathbb{Z})$ in the below table, we set $\{\mathbf{x}_t\}_{t\in\mathbb{T}_n}$ all clusters obtained from applying mutations to $(\mathbf{x}, \mathbb{Z}, B)$ repeatedly. Then $\{\mathbf{x}_t\}_{t\in\mathbb{T}_n}|_{x_1=x_2=x_3=1}$ gives the set of positive integer solution to the corresponding equation.

Equation	B	$\mathbf{Z}$	
$x^2+y^2+2x+2y+x^2y+xy^2+1=9xy$	$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\ Z_2(u)=1+u\end{cases}$	
$x^2+y^2+k_1x+k_2y+1=(3+k_1+k_2)xy$	$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+k_2u+u^2\\ Z_2(u)=1+k_1u+u^2\end{cases}$	from GME
$x^2+y^2+k_1x+1=(3+k_1)xy$	$\begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\ Z_2(u)=1+k_1u+u^2\end{cases}$	
$x^2+y^4+ky^2+2x+1=(5+k)xy^2$	$\begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+ku+u^2\\ Z_2(u)=1+u\end{cases}$	from "GME-type eq."
$x^2+y^4+2x+1=5xy^2$	$\begin{bmatrix}0 & 1\\ -4 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\ Z_2(u)=1+u\end{cases}$	
$x^2+y^2+1=3xy$	$\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\ Z_2(u)=1+u\end{cases}$	from ME