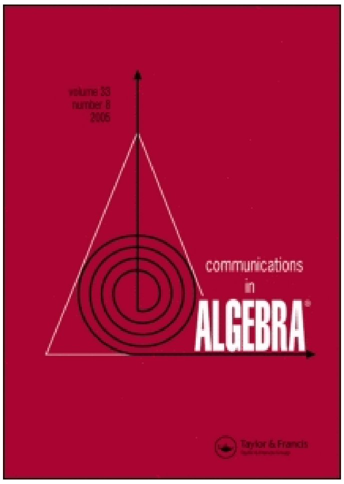

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

Standard modules of level 1 for s_2 in terms of virasoro algebra representations

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To cite this Article Weng, Lin and You, Yucking(1998) 'Standard modules of level 1 for s_2 in terms of virasoro algebra representations', Communications in Algebra, 26: 2, 613 — 625
To link to this Article: DOI: 10.1080/00927879808826151
URL: http://dx.doi.org/10.1080/00927879808826151

COMMUNICATIONS IN ALGEBRA, 26(2), 613–625 (1998)

Standard Modules of Level 1 for $\hat{sl}_2$ In Terms of Virasoro Algebra Representations

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It is well-known that for a level l integrable irreducible highest weight module $L(\Lambda)$ of the affine Lie algebra $\hat{g} = g \otimes \mathbf{C}((t)) + \mathbf{C} \cdot c$ associated to a finite dimensional simple Lie algebra g over the complex number field, we may introduce a set of Virasoro operators $\{L_n\}_{n\in\mathbf{Z}}$ in the sense of Sugawara. The construction is as follows: let $\{J^a\}_{a=1}^{\dim g}$ be an orthonormal basis with respect to the normalized Cartan-Killing form of g, (the inner poduct of the highest long root is two,) then

$$L_n := \frac{1}{2(g^*+l)} \sum_{m\in\mathbf{Z}} \sum_{a=1}^{\dim g} : J^a(m)\, J^a(n-m) : .$$

Here g^* denotes the the dual Coxeter number of g and $: \cdot :$ denotes the normal ordering. The linear span of $\{L_n\}_{n\in\mathbf{Z}}$ together with the one-dimensional central extension forms a Lie algebra, which is usually called the Virasoro algebra (associated to $L(\Lambda)$). Denote this Lie algebra by Vir. Then its central charge is $\frac{l \dim g}{g^*+l}$.

In this paper, we will give the irreducible decomposition of the level one integrable highest weight modules of $\hat{sl}_2$ associated to $sl_2(\mathbf{C})$ for the action of the (associated) Virasoro algebra. In fact, using a realization of the modules, we will explicitly construct the highest weight vectors for the Virasoro algebra. As a by-product, we obtain the norm of these highest weight vectors for the inner product introduced by Garland [G]. We hope that such results can be used in the study of the hermitian theory of vector bundles of conformal blocks in conformal field theory [TUY].

From now on, we consider the affine Lie algebra $\hat{sl}_2$ associated to the simple Lie algebra sl_2 with standard basis

$$E_{12} := E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ E_{21} := F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, the dual Coxeter number g^* is 2. For $X \in sl_2$, let $X(n)$ denote the element $X \otimes t^n$ in $\hat{sl}_2$.

Let $L(\Lambda)$ be an integrable irreducible highest weight module of $\hat{sl}_2$. If l is the level of $L(\Lambda)$, we define

$$L_n(H) := \frac{1}{4l} \sum_{i \in \mathbf{Z}} : H(i)H(n-i) : .$$

One checks that $\{L_n(H)\}_{n \in \mathbf{Z}}$ together with the one-dimensional central extension also form a Virasoro algebra, denoted by $\mathrm{Vir}(H)$, whose central charge is 1. That is, we have

$$[L_n(H), L_m(H)] = (n-m)L_{n+m}(H) + \frac{n^3 - n}{12} \delta_{n,-m}.$$

As a matter of fact, we find that to understand the action of Vir, it is convenient to use $\mathrm{Vir}(H)$.

Remark. The operators $\tilde{L}_n := L_n - L_n(H)$ together with the one-dimensional central extension also form a Virasoro algebra (with central charge $\frac{2(l-1)}{l+1}$), and they are called the *coset Virasoro operators.*

For our purpose, we consider the case $l = 1$, i.e., the level one modules. From the representation theory of affine Lie algebras, there are only two irreducible integrable highest weight $\hat{sl}_2$-modules of level 1, namely $L(\Lambda_0)$ and $L(\Lambda_1)$. More precisely, $L(\Lambda_0)$ is the induced module from the trivial module V_0 of sl_2, while $L(\Lambda_1)$ is the induced module from the irreducible module $V(\lambda_1)$ of sl_2 with the highest weight λ_1, where λ_1 denotes $\alpha/2$, the fundamental dominant weight for sl_2. Moreover, we have very precise realizations for these two level one modules of $\hat{sl}_2$. Indeed, there are several different ways. In the so-called bosonic picture, we have the principal realization and the homogeneous realization. In this paper, we take the homogeneous realization.

Theorem 0. ([LP], [K], [Y]) *For $k = 0, 1$, $L(\Lambda_k) \simeq \mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}[q, q^{-1}]$, and the Lie algebra $\hat{sl}_2$ action is given by*

$$H(n)(f \otimes q^\alpha) = \begin{cases} 2\frac{\partial}{\partial x_n} f \otimes q^\alpha, & \text{if } n > 0, \\ (-n)x_{-n} f \otimes q^\alpha, & \text{if } n < 0, \end{cases}$$

$$H(0)(f \otimes q^\alpha) = \begin{cases} 2\alpha f \otimes q^\alpha, & \text{if } k = 0, \\ (2\alpha - 1) f \otimes q^\alpha, & \text{if } k = 1, \end{cases}$$

$$c(f \otimes q^\alpha) = f \otimes q^\alpha,$$

$$E_{12}(z)(f \otimes q^\alpha) = z^{2\alpha+1-k}(-1)^\alpha \exp[\sum_j z^j x_j] \exp[-\sum_j \frac{2z^{-j}}{j} \frac{\partial}{\partial x_j} f] \otimes q^{\alpha+1},$$

$$E_{21}(z)(f \otimes q^\alpha) = z^{-2\alpha+1+k}(-1)^{\alpha+1} \exp[\sum_j -z^j x_j] \exp[\sum_j \frac{2z^{-j}}{j} \frac{\partial}{\partial x_j} f] \otimes q^{\alpha-1},$$

where $E_{ij}(z) := \sum_{n \in \mathbf{Z}} E_{ij} \otimes z^{-n}$.

First, we consider the module $L(\Lambda_0)$. In this case, one knows that the energy operator $(-d) = L_0$ acts as follows:

$$L_0(f \otimes q^\alpha) = (\deg(f) + \alpha^2) f \otimes q^\alpha$$

if f is principally homogenous, where the gradration is given by $\deg x_i = i$. We also define the principal degree of $f \otimes q^\alpha$ to be $\deg f + \alpha^2$.

Lemma 1. *With the same notation as above, $L_n = L_n(H)$.*

Proof. Recall that, by definition,

$$L_n(H) := \frac{1}{4} \sum_{i \in \mathbf{Z}} : H(m)H(n-m) : .$$

So, for a fixed α, all $L_n(H)$ leave each $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^\alpha$ invariant. Moreover, it is well-known that, for all $m, n \in \mathbf{Z}$,

$$[L_n(H), H(m)] = -mH(n+m). \tag{1}$$

Note that the original Virasoro operator L_n by definition is just

$$L_n := \frac{1}{6} \sum_{i+j=n} \left(: E(i)F(j) : + : F(i)E(j) : + \frac{1}{2} : H(i)H(j) : \right).$$

So it is obvious from the realization theorem that L_n leaves each copy $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^\alpha$

invariant as well. Furthermore,

$$[L_n, X(m)] = -mX(n+m) \tag{2}$$

for all m, $n \in \mathbf{Z}$, and $X \in sl_2$ [TUY].

Define $\tilde{L}_n := L_n - L_n(H)$. Then, by (1) and (2), $\tilde{L}_n$ commutes with the action of $H(m)$ for all $m \in \mathbf{Z}$. Note that these Heisenberg operators $H(m)$ act irreducibly on each copy $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^{\alpha}$. Thus, by the Schur lemma, we see that $\tilde{L}_n$ are scalars on these copies $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^{\alpha}$. Using the Virasoro commutator relations, a simple argument leads that these scalars are all zero, and hence $\tilde{L}_n = 0$. This completes the proof of the lemma.

From Lemma 1, in order to understand the representations of $L(\Lambda_0)$ for Vir, we only need to study the representations of $L(\Lambda_0)$ for $\mathrm{Vir}(H)$. (In general, the $\mathrm{Vir}(H)$-module structure is relatively easy to study.)

To describe the Virasoro representations for $L(\Lambda_0)$, we need a result of Kac's character formula. Let $M(h,l)$ be a highest weight Verma module for the Virasoro algebra with h being its lowest energy and l its central charge. Then we know that for $l = 1$, $M(h,l)$ is irreducible if and only if h is not $m^2/4$ for some integer m; moreover, $M(m^2/4, 1)$ has a unique irreducible quotient $L(m^2/4, 1)$. Since all the L_0-eigenspaces of $L(h,l)$ are finite dimensional, we may define the z-dimension of $L(h,l)$ as

$$\dim_z L(h,l) := z^h \sum_i \dim L(h,l)_i z^i$$

with $L(h,l)_i$ the L_0-eigenspace with eigenvalue i. With this, we state the Kac's character formula as

$$\dim_z L(m^2/4, 1) = \frac{z^h(1 - z^{m+1})}{\phi(z)},$$

where

$$\phi(z) := \prod_{n=1}^{\infty}(1 - z^n).$$

From the fact that the Virasoro algebra Vir leaves each $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^{\alpha}$ invariant, we only need to understand the irreducible decomposition of $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^{\alpha}$ for the representation of Vir. For this purpose, observe from (2) that $[L_n, X] = 0$ for all vectors

$X \in sl_2$. Hence, if v is a highest weight vector of Vir, then $X^k v$ is also a highest weight vector for Vir with all $X \in sl_2$, $k \in \mathbf{Z}_{\geq 0}$. Define

$$L(\Lambda_0)^{\mathrm{Vir}_+} := \{v : L_n v = 0, \forall n > 0\}.$$

Then, the Lie algebra sl_2 also acts on $L(\Lambda_0)^{\mathrm{Vir}_+}$.

We have natural candidates in $L(\Lambda_0)^{\mathrm{Vir}_+}$.

Lemma 2. *For each integer α, $1 \otimes q^\alpha \in L(\Lambda_0)^{\mathrm{Vir}_+}$.*

Proof. $L_n(1 \otimes q^\alpha)$ is a homogenous polynomial with principal degree $\alpha^2 - n$ for all $n \in Z$. Notice that α^2 is the lowest principal degree for homogenous elements in $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^\alpha$. Hence, for n positive, we have $L_n(1 \otimes q^\alpha) = 0$. This completes the proof of the lemma.

According to [TUY], the module $\mathcal{H} = L(\Lambda_0)$ admits a decomposition $\mathcal{H} = \sum_{n \in \mathbf{Z}} \mathcal{H}(n)$, wherer $\mathcal{H}(n) := \{v \in \mathcal{H} : L_0 v = nv\}$ is a finite dimensional vector subspace. In our case, $\mathcal{H}(n)$ is nothing else but the space of homogeneous polynomials of principal degree n (recall $\deg x_i = i$, $\deg q^\alpha = \alpha^2$). Hence the dimension of $\mathcal{H}(n)$ is $\sum_{j=0}^{[\sqrt{n}]} p(n - j^2)$ where $p(n)$ is the partition function. By (1), $\mathcal{H}(n)$ is sl_2-invariant.

Notice that for each nonnegative α, by the realization theorem, a simple calculation shows that E kills $1 \otimes q^\alpha$. Hence, $1 \otimes q^\alpha$ actually is also a highest weight vector for the action of sl_2 on $\mathcal{H}(\alpha^2)$. Therefore it generates a finite dimensional irreducible sl_2-submodule by integrability, with basis

$$1 \otimes q^\alpha,\ F(1 \otimes q^\alpha),\ \frac{F^2}{2}(1 \otimes q^\alpha), \ldots, \frac{F^k}{k!}(1 \otimes q^\alpha), \ldots.$$

Since $H(1 \otimes q^\alpha) = 2\alpha(1 \otimes q^\alpha)$, it follows that the above irreducible sl_2-module has dimension $2\alpha + 1$, which we denote by $V(2\alpha)$. By (1) or (2), all the vectors $v_k^\alpha := \frac{F^k}{k!}(1 \otimes q^\alpha)$, $k = 0, \ldots, 2\alpha$ are in $L(\Lambda_0)^{\mathrm{Vir}_+}$. These vectors can be written down precisely. However, it seems that an explicit expression for $\frac{F^k}{k!}(1 \otimes q^\alpha)$ using the realization theorem directly is rather difficult to obtain. Instead, we use the so-called fermionic realization of $L(\Lambda_0)$ ([DJKM], [Y]).

Let $\{\psi_n^{(1)}, \psi_n^{(1)*}, \psi_n^{(2)}, \psi_n^{(2)*}\}_{n \in \mathbf{Z}}$ be two sets of free fermions, subject to the relations

$$\{\psi_n^{(i)}, \psi_m^{(j)}\}_+ = \{\psi_n^{(i)*}, \psi_m^{(j)*}\}_+ = 0, \quad \{\psi_n^{(i)}, \psi_m^{(j)*}\}_+ = \delta_{ij}\delta_{nm}.$$

Here $\{\cdot,\cdot\}_+$ denotes the anti-commutator. Let Cl be the Clifford algebra subject to the above relations, and I the left ideal of Cl generated by $\psi_n^{(i)}, n \leq 0, i = 1,2$ and $\psi_n^{(i)*}, n > 0, i = 1,2$. We set $\mathcal{F} := Cl/I$. Let $|0\rangle$ be the element $1+I$ in $\mathcal{F}$. It is known that $\mathcal{F}$ is a module of the affine Lie algebra $\underline{\mathbf{a}}_\infty$ of infinite rank, decomposing into a direct sum of irreducible submodules $\mathcal{F}^{(l)}$, each of which corresponds to a fundamental dominant weight ([DJKM]). The subspace $\mathcal{F}^{(l)}$ has a basis $\psi_{i_1}^{(1)}\psi_{i_2}^{(1)}\ldots\psi_{i_u}^{(1)}\psi_{i_{u+1}}^{(2)}\ldots\psi_{i_s}^{(2)}\psi_{j_1}^{(1)*}\ldots\psi_{j_v}^{(1)*}\psi_{j_{v+1}}^{(2)*}\ldots\psi_{j_s}^{(2)*}|0\rangle$ with $s-t = l, i_1 > \ldots > i_u > 0, i_{u+1} > \ldots > i_s > 0, j_1 < \ldots < j_v \leq 0, j_{v+1} < \ldots < j_t \leq 0$.

For our purpose, we consider $l = 0$. (This is the *basic module* for $\underline{\mathbf{a}}_\infty$). Note that

$$\begin{aligned}&\psi_{i_1}^{(1)}\psi_{i_2}^{(1)}\ldots\psi_{i_u}^{(1)}\psi_{i_{u+1}}^{(2)}\ldots\psi_{i_s}^{(2)}\psi_{j_1}^{(1)*}\ldots\psi_{j_v}^{(1)*}\psi_{j_{v+1}}^{(2)*}\ldots\psi_{j_s}^{(2)*}|0\rangle\\ &\quad= \pm\,\psi_{i_1}^{(1)}\psi_{i_2}^{(1)}\ldots\psi_{i_u}^{(1)}\psi_{j_1}^{(1)*}\ldots\psi_{j_v}^{(1)*}\psi_{i_{u+1}}^{(2)}\ldots\psi_{i_s}^{(2)}\psi_{j_{v+1}}^{(2)*}\ldots\psi_{j_s}^{(2)*}|0\rangle,\end{aligned}$$

it follows that

$$\mathcal{F}^{(0)} = \sum_{k\in\mathbf{Z}} \mathcal{F}_{1,k}^{(0)} \otimes \mathcal{F}_{2,-k}^{(0)}$$

where $\mathcal{F}_{i,k}^{(0)}$ is spanned by the monomials $\psi_{i_1}^{(i)}\psi_{i_2}^{(i)}\ldots\psi_{i_u}^{(i)}\psi_{j_1}^{(i)*}\ldots\psi_{j_v}^{(i)*}|0\rangle$, $u-v = k, i = 1,2$. It is well-known from the theory of soliton equations that each $\mathcal{F}_{i,k}^{(0)}$ can be identified with a polynomial algebra $\mathbf{C}[x_1^{(i)}, x_2^{(i)},\ldots]$. For this, we need to recall the Schur polynomials: for a partition $\lambda : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$, $\lambda_i \in \mathbf{Z}_{>0}$, the *Schur polynomials $s_\lambda(x)$ associated to* λ is defined by $s_\lambda := \det(s_{\lambda_i - i+j}(x))$, where $s_j(x)$ are given by the generating function $\sum_{j=0}^\infty s_j(x)t^j := \exp[\sum_{j=1}^\infty x_j t^j]$. With this, our identification map, the so-called *boson-fermion correspondence*,

$$\sigma_k^{(i)} : \mathcal{F}_{i,k}^{(0)} \simeq \mathbf{C}[x_1^{(i)}, x_2^{(i)},\ldots] \otimes u_i^k,$$

is given by $\sigma_n^{(i)}(\Phi_n^{(i)}|0\rangle) = 1 \otimes u_i^n$, where

$$\Phi_n^{(i)}|0\rangle = \begin{cases} \psi_n^{(i)}\ldots\psi_1^{(i)}|0\rangle, & \text{if } n > 0,\\ |0\rangle, & \text{if } n = 0,\\ \psi_{-n+1}^{(i)*}\ldots\psi_0^{(i)*}|0\rangle, & \text{if } n < 0.\end{cases}$$

More generally, for $u - v = n, i_1 > \ldots > i_u > 0, j_1 < \ldots < j_v \leq 0$

$$\sigma_n^{(i)}\left(\psi_{i_1}^{(i)}\psi_{i_2}^{(i)}\ldots\psi_{i_u}^{(i)}\psi_{j_1}^{(i)*}\ldots\psi_{j_v}^{(i)*}|0\rangle\right) = (-1)^{\sum_k (j_k+k-1)} s_\lambda(x^{(i)}) \otimes u_i^n,$$

where the partition λ is obtained by the following process: starting with the positive integers $i_1, i_2, \ldots, i_u$, we add the sequence of consecutive non-positive integers $0, -1, -2, \ldots$;

Call this sequence as Δ. From Δ, by deleting the non-positive integers $j_1, j_2, \ldots, j_v$, we get a new sequence called $k_1, k_2, k_3, \ldots$, which is in decreasing order. Define a sequence λ by $\lambda_j := k_j - (n-j)$ for $j = 1, 2, 3, \ldots$. Then it is obvious that λ is a partition.

Note that using the boson-fermion correspondence $\sigma = \sigma^{(1)} \otimes \sigma^{(2)}$, we have

$$\sigma : \mathcal{F}^{(0)} \simeq B^{(0)} = \sum_{k \in \mathbf{Z}} \mathbf{C}[x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}, \ldots] \otimes q^k.$$

Here $q := \frac{u_2}{u_1}$. This is known as the *two-component construction* of the basic module for $\underline{\mathrm{a}}_\infty$. Then we can show that the $\hat{sl}_2$-module $L(\Lambda_0)$ in the homogeneous picture (the *reduction procedure* in the litterature) is

$$L(\Lambda_0) \simeq \{ f(x^{(1)}, x^{(2)}, q) \in B^{(0)} : \frac{\partial f}{\partial x_j^{(1)}} + \frac{\partial f}{\partial x_j^{(2)}} = 0, \forall j \geq 1 \}$$

and thus it is realized as a polynomial algebra

$$\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}[q, q^{-1}]$$

where $x_i = x_i^{(2)} - x_i^{(1)}$ ([Y]).

We are not interesting in describing this in detail. But we need the action of the operator F on $1 \otimes q^k$. With the above realization, one knows that the action of F corresponds to the left multiplication by the element $\sum_{n \in \mathbf{Z}} \psi_n^{(2)} \psi_n^{(1)*}$. Now we are ready to state our first theorem.

Theorem 1. *With the same notation as above,*

$$\frac{F^k}{k!}(1 \otimes q^\alpha) = (-1)^{k\alpha} s_{k^{(2\alpha-k)}}(x) \otimes q^{\alpha - k}$$

for $k = 0, 1, \ldots, 2\alpha$ where $s_{k^{(2\alpha-k)}}(x)$ is the Schur function corresponding to the rectanglar partition $k^{(2\alpha-k)}$.

Proof. For $n \in \mathbf{Z}_{\geq 0}$, $1 \otimes q^n$ in the fermionic picture is

$$1 \otimes q^n = \psi_n^{(2)} \psi_{n-1}^{(2)} \cdots \psi_1^{(2)} \psi_{-n+1}^{(1)*} \cdots \psi_0^{(1)*} |0\rangle.$$

We have

$$F^k = \sum_{s_1, s_2, \ldots s_k \in \mathbf{Z}} \psi^{(1)}_{s_1}\psi^{(2)*}_{s_1} \ldots \psi^{(1)}_{s_k}\psi^{(2)*}_{s_k} = k! \sum_{s_1 < \ldots < s_k} \psi^{(1)}_{s_1}\psi^{(2)*}_{s_1} \ldots \psi^{(1)}_{s_k}\psi^{(2)*}_{s_k}.$$

It follows that

$$\frac{F^{(k)}(1 \otimes q^n)}{k!} = \sum_{s_1 < \ldots < s_k} \psi^{(1)}_{s_1}\psi^{(2)*}_{s_1} \ldots \psi^{(1)}_{s_k}\psi^{(2)*}_{s_k}\psi^{(2)}_{n}\psi^{(2)}_{n-1} \ldots \psi^{(2)}_{1}\psi^{(1)*}_{-n+1} \ldots \psi^{(1)*}_{0}|0\rangle$$

where the summation is taken on $s_1, \ldots, s_k \in \{n, n-1, \ldots, 1, 0, -1, \ldots, -n+1\}$.

Observe that our polynomial $\frac{F^{(k)}(1\otimes q^n)}{k!}$ depends only on $x_i = x^{(2)}_i - x^{(1)}_i$ (not on $y_i := x^{(1)}_i + x^{(2)}_i$), therefore it would not change if we specialize $x^{(1)}_i$ to 0 and $x^{(2)}_i$ to x.

Case 1. $k \le n$. Then we see that the summand becomes

$$\pm\psi^{(1)}_{s_1}\psi^{(1)}_{s_2} \ldots \psi^{(1)}_{s_k}\psi^{(1)*}_{-n+1} \ldots \psi^{(1)*}_{0}\psi^{(2)*}_{s_1} \ldots \psi^{(2)*}_{s_k}\psi^{(2)}_{n} \ldots \psi^{(2)}_{1}|0\rangle \in \mathcal{F}^{(0)}_{1,k-n} \otimes \mathcal{F}^{(0)}_{2,n-k}.$$

By the explicit boson-fermion correspondence, $\sigma^{(1)}_{k-n}(\psi^{(1)}_{s_1}\psi^{(1)}_{s_2} \ldots \psi^{(1)}_{s_k}\psi^{(1)*}_{-n+1} \ldots \psi^{(1)*}_{0}|0\rangle)$ is a polynomial in $x^{(1)}_i$ with *positive* degree, unless $s_1 = -n+1, s_2 = -n+2, \ldots, s_k = -n+k$. Moreover, when $s_j = -n+j, j = 1, \ldots, k$,

$$\begin{aligned}&\psi^{(1)}_{s_1}\psi^{(2)*}_{s_1} \ldots \psi^{(1)}_{s_k}\psi^{(2)*}_{s_k}\psi^{(2)}_{n}\psi^{(2)}_{n-1} \ldots \psi^{(2)}_{1}\psi^{(1)*}_{-n+1} \ldots \psi^{(1)*}_{0}|0\rangle \\ =&(-1)^k\psi^{(2)}_{n}\psi^{(2)}_{n-1} \ldots \psi^{(2)}_{1}\psi^{(2)*}_{-n+1} \ldots \psi^{(2)*}_{-n+k}|0\rangle.\end{aligned}$$

Therefore, under the specialization $x^{(1)}_i$ to 0 and $x^{(2)}_i$ to x, the only term that survives is

$$(-1)^k u_1^{-(n-k)}\sigma^{(2)}_{n-k}\left(\psi^{(2)}_{n}\psi^{(2)}_{n-1} \ldots \psi^{(2)}_{1}\psi^{(2)*}_{-n+1} \ldots \psi^{(2)*}_{-n+k}|0\rangle\right) = (-1)^{nk}s_\lambda(x) \otimes q^{n-k}$$

where $\lambda = \{n-(n-k), n-1-(n-k)+1, \ldots, -n+k+1+n-1\} = k^{2n-k}$.

Case 2. $2n \ge k > n$. Observe that we should have *at least* $k-n$ positive s_j's as otherwise the number of non-positive s_j's is more than n, they then must repeat. If $s_k - (k-n) > 0$, then $\sigma^{(1)}_{k-n}(\psi^{(1)}_{s_1}\psi^{(1)}_{s_2} \ldots \psi^{(1)}_{s_k}\psi^{(1)*}_{-n+1} \ldots \psi^{(1)*}_{0}|0\rangle)$ vanishes under the specialization, since it is a positive degree polynomial. Hence $s_k = k-n$ for a non-vanishing term. It follows that we must have exactly $k-n$ positive s_j's for a non-vanishing term and $s_k = k-n, s_{k-1} = k-n-1, \ldots, s_{n+1} = 1$. Then the non-positive s_j's must exhaust all the integers $-n+1, -n+2, \ldots, -1, 0$. Hence we only need to take care of the term

$$\psi^{(1)}_{-n+1}\psi^{(2)*}_{-n+1} \cdots \psi^{(1)}_{0}\psi^{(2)*}_{0}\psi^{(1)}_{1}\psi^{(2)*}_{1} \ldots \psi^{(1)}_{k-n}\psi^{(2)*}_{k-n}\Phi^{(2)}_{n}\Phi^{(1)}_{-n}|0\rangle.$$

Again, by calculation, we find that the corresponding polynomial is $(-1)^{nk}s_\lambda(x) \otimes q^{n-k}$. This compteles the proof of the theorem.

Therefore, on each $\mathbf{C}[x_1, x_2, \dots] \otimes \mathbf{C}q^\alpha$, we are able to construct the highest weight vectors for Vir, namely, the polynomial $s_{(\beta-\alpha)^{\alpha+\beta}}(x) \otimes q^\alpha$ with $\beta \geq \alpha \geq -\beta$. Each of these vectors has energy $\beta^2 = \frac{(2\beta)^2}{4}$. Now by the Kac character formula, each β contributes an irreducible Vir-module $L^{(\alpha)}(\beta^2, 1)$ whose z-dimension is (assume $\beta > 0$)

$$\frac{z^{\beta^2}(1 - z^{2\beta+1})}{\phi(z)}.$$

Notice that

$$\sum_{\beta \geq |\alpha|} \frac{z^{\beta^2}(1 - z^{2\beta+1})}{\phi(z)} = \frac{z^{\alpha^2}}{\phi(z)}$$

is exactly the z-dimension of $\mathbf{C}[x_1, x_2, \dots] \otimes \mathbf{C}q^\alpha$. Therefore, we have the following

Theorem 2. *With the same notation as above, the irreducible decomposition of $L(\Lambda_0)$ under the Virasoro algebra Vir is described as follows:*

1) $\mathbf{C}[x_1, x_2, \dots] \otimes \mathbf{C}q^\alpha = \sum_{\beta \geq |\alpha|} L^{(\alpha)}(\beta^2, 1)$, *where each $L^{(\alpha)}(\beta^2, 1)$ is generated by the highest weight vector $s_{(\beta-\alpha)^{\alpha+\beta}}(x) \otimes q^\alpha$. Hence*

$$L(\Lambda_0) = \sum_\alpha \mathbf{C}[x_1, x_2, \dots] \otimes \mathbf{C}q^\alpha = \sum_{\alpha \in \mathbf{Z}, \beta \geq |\alpha|} \mathrm{Vir} \cdot \left(s_{(\beta-\alpha)^{\alpha+\beta}}(x) \otimes q^\alpha\right).$$

2) $L(\Lambda_0) = \sum_{n \geq 0} V(2n) \otimes L(n^2, 1)$ *(as a $sl_2\times$ Vir-module).*

Remark. We can describe the Vir-irreducible module $L^{(\alpha)}(n^2, 1)$ alternatively. For $0 \leq k \leq \sqrt{n}$, let $V_k^{(n)}$ be the $V(2k)$-isotypic component for the action of sl_2 in $\mathcal{H}(n)$, so that

$$\mathcal{H}(n) = \sum_{0 \leq k \leq [\sqrt{n}]} V_k^{(n)}.$$

Then, we have, for $|\alpha| \leq n$,

$$\begin{aligned} L^{(\alpha)}(n^2, 1) &= \sum_{k \geq n^2} V_n^{(k)} \cap (\mathbf{C}[X] \otimes q^\alpha) \\ &= \{f \otimes q^\alpha : \Omega(f \otimes q^\alpha) = n(n+1)q^\alpha\}, \end{aligned}$$

where $\Omega = EF + FE + \frac{1}{2}H^2$ is the Casimir operator of sl_2.

From the representation theory of Kac-Moody algebras, we know that there is a *unique* contravariant hermitian inner product on $L(\Lambda_0)$, up to constant multiple [G]. For our realization, the inner product is normalized so that $\|1\|^2 = 1$. It is known that this inner product saitisfies the following important relation: $(L_n x, y) = (x, L_{-n} y)$. For the application to the conformal field theory, it is very important to find the norm of the highest vectors for the Virasoro algebra Vir, since then the inner product for all pairs of vectors can be computed by only using the commutator relations among L_n's.

Theorem 3. *With the same notation as above, the Garland norm of the highest weight vector (for Vir)* $s_{(\beta-\alpha)^{(\beta+\alpha)}}(x) \otimes q^{\alpha}$ is given by

$$\|s_{(\beta-\alpha)^{(\beta+\alpha)}}(x) \otimes q^{\alpha}\|^2 = \|s_{(\beta-\alpha)^{(\beta+\alpha)}}(x)\|^2 = \binom{\beta+\alpha}{\beta-\alpha}.$$

Proof. The contravariant property gives

$$(H(n)u, v) = (u, H(-n)v).$$

Since $\|1\| = 1$, by using the realization theorem, we have for $f(x), g(x) \in \mathbf{C}[x]$,

$$(f(x), g(x)) = f(2\tilde{\partial})g(x)|_{x=0}$$

where $\tilde{\partial} := (\partial_1, \frac{1}{2}\partial_2, \frac{1}{2}\partial_3, \ldots)$ and

$$\|f(x) \otimes q^{\alpha}\|^2 = \|f(x)\|^2 \cdot \|1 \otimes q^{\alpha}\|^2.$$

In particular, as a formal power series in t and u, we have

$$(\exp[\sum x_n t^n], \exp[\sum x_n u^n]) = \exp[\sum_n \frac{2}{n}(tu)^n] = \frac{1}{(1-tu)^2} = \sum_{i=0}^{\infty}(i+1)(tu)^i,$$

which shows for $n \in \mathbf{Z}_{\geq 0}$,

$$\|s_n\|^2 = n+1. \qquad (**)$$

Lemma 3. $\|1 \otimes q^{\alpha}\|^2 = 1$.

Proof. If $\alpha > 0$, we prove by induction. Notice that $F(1 \otimes q) = s_1(x) = x_1$, $\|x_1\|^2 = 2$. Then

$$(F(1 \otimes q), F(1 \otimes q)) = (1 \otimes q, EF(1 \otimes q)) = (1 \otimes q, H(1 \otimes q)) = 2\|1 \otimes q\|^2.$$

Hence $\|1 \otimes q\|^2 = 1$.

Assume $\|1 \otimes q^k\|^2 = 1$. Then since $F(1 \otimes q) = s_{1^{2k+1}} \otimes q^k$, notice that we have an "involution" on $\mathbf{C}[x]$, $s_{\lambda}(x) \mapsto s_{\lambda'}(x)$ which is an isometry, where λ' is the conjugate partition of λ, it follows that

$$\|F(1 \otimes q^{k+1})\|^2 = \|s_{1^{2k+1}}\|^2 \cdot \|1 \otimes q^k\|^2 = 2k + 2.$$

But

$$\begin{aligned}\|F(1 \otimes q^{k+1})\|^2 =&(1 \otimes q^{k+1}, EF(1 \otimes q^{k+1}))\\ =&(1 \otimes q^{k+1}, H(1 \otimes q^{k+1})) = 2(k+1)\|1 \otimes q^{k+1}\|^2.\end{aligned}$$

Hence, $\|1 \otimes q^{k+1}\|^2 = 1$.

For $\alpha < 0$, we use the operator E instead of F to complete the proof of our lemma.

To complete the proof of our theorem, we recall that for $\alpha \geq 0$, $1 \otimes q^\alpha, F(1 \otimes q^\alpha), \ldots, F^{2\alpha}(1 \otimes q^\alpha)$ form a basis of the sl_2-module $V(2\alpha)$. We have $\|1 \otimes q^\alpha\|^2 = 1$ and

$$\|F^k(1 \otimes q^\alpha)\|^2 = (F^{k-1}(1 \otimes q^\alpha), EF^k(1 \otimes q^\alpha)).$$

Using

$$[E, F^k] = \sum_{i=0}^{k-1} F^i H F^{k-i-1}$$

and an inductive argument leads

$$\begin{aligned}\|\frac{F^k(1 \otimes q^\alpha)}{k!}\|^2 =&\frac{1}{k!^2}(F^{k-1}(1 \otimes q^\alpha), EF^k(1 \otimes q^\alpha))\\ =&\frac{1}{k!^2}(F^{k-1}(1 \otimes q^\alpha), \sum_{i=0}^{k-1} F^i H F^{k-i-1}(1 \otimes q^\alpha))\\ =&\frac{1}{k!^2}\sum_{i=0}^{k-1}(F^{k-1}(1 \otimes q^\alpha), 2(n-k+i+1)F^{k-1}(1 \otimes q^\alpha))\\ =&\frac{1}{k!^2}(2nk - k^2 + k)(F^{k-1}(1 \otimes q^\alpha), F^{k-1}(1 \otimes q^\alpha)),\end{aligned}$$

the last two formulae follow from the fact that $F^a(1 \otimes q^n) = f(x) \otimes q^{n-a}$ and $H(f(x) \otimes q^{n-a}) = 2(n-a)f(x) \otimes q^{n-a}$ for some polynomial $f(x)$. Therefore, by an induction on k, using Theorem 1, we have

$$\|s_{k^{2\alpha-k}}(x)\|^2 = \|\frac{F^k(1 \otimes q^\alpha)}{k!}\|^2 = \binom{2\alpha}{k}.$$

This completes the proof of the theorem.

Remark. The computation shows that the inner product takes integer values on the lattice spanned by Schur functions. For instance,

$$(s_{1^n}, s_n) = \begin{cases} 1, & \text{if } n = 0, 2, \\ 2, & \text{if } n = 1, \\ 0, & \text{if } n \geq 3; \end{cases}$$

$$\|s_{n,1}\|^2 = 3n; \quad \|s_{n,2}\|^2 = 6n - 6.$$

We guess that this lattice is the lattice introduced by Garland (associated to the standard Chevalley basis) ([G]).

Next we consider another level one module $L(\Lambda_1)$. With the same discussion as above, we have

Theorem 4. *The irreducible decomposition of $L(\Lambda_1)$ under the Virasoro algebra Vir is described as follows:*

1) *$L_n = L_n(H)$, so that each $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^\alpha$ is invariant under Vir.*

2) $\mathbf{C}[x_1, x_2, \ldots] \otimes \mathbf{C}q^\alpha = \sum_{\beta \geq |\alpha|} L^{(\alpha)}(\frac{(2\beta-1)^2}{4}, 1)$, *each* $L^{(\alpha)}(\frac{(2\beta-1)^2}{4}, 1)$ *is an irreducible* Vir*-module, and is generated by the highest weight vector* $\frac{F^k(1 \otimes q^\beta)}{k!}$, *where* $k = \beta - \alpha$.

3) $L(\Lambda_1) = \sum_{\alpha \geq 1} V(2\alpha - 1) \otimes L(\frac{(2\alpha-1)^2}{4}, 1)$ *where $V(2\alpha - 1)$ is the irreducible sl_2-module of dimension 2α, linear spanned by* $\{\frac{F^k(1\otimes q^\alpha)}{k!}\}$ *and*

$$\frac{F^k(1 \otimes q^\alpha)}{k!} = (-1)^{(\alpha+1)-k} s_{k(2\alpha-k-1)}(x) \otimes q^{\alpha-k}$$

for $k = 0, 1, \ldots, 2\alpha - 1$.

4) *The Garland norm of the highest weight vector (for* Vir*) $s_{(\beta-\alpha)^{(\beta+\alpha)}}(x) \otimes q^\alpha$ is given by*

$$\|s_{(\beta-\alpha)^{(\beta+\alpha)}}(x) \otimes q^\alpha\| = \|s_{(\beta-\alpha)^{(\beta+\alpha)}}(x)\| = \binom{\beta+\alpha}{\beta-\alpha}.$$

Here $\alpha + \beta$ is an odd integer.

Corollary. (Kac) *As $sl_2\times$ Vir-module, we have the decomposition*

$$L(\Lambda_0) \oplus L(\Lambda_1) = \sum_{n \geq 0} V(n) \otimes L(n^2/4, 1).$$

For level two modules (three of them), we have the (homogeneous) Lepowsky-Primc realization. Here the situation is much more complicated. In fact, unlike the level one case, where the highest weight vectors for Vir only occur in $\mathcal{H}(n)$ for n a perfect square (resp. $n = k(k+1)$) for $L(\Lambda_0)$ (resp. for $L(\Lambda_1)$), the highest weight vectors for Vir occur almost for every n. Using the realization and the Kac character formula, we are able to find the multiplicity of the irreducible Vir modules in terms of certain generating functions. Apart from that, we obtain many highest weight vectors. However, the sl_2-trick used in this paper fails to generate all the highest weight vectors in this case. In fact, for level one, Vir and sl_2 are the so-called *commuting pair*. For instance, $L(\Lambda_0)^{sl_2}$ is the universal enveloping algebra of the Lie algebra linearly spanned by $L(-2)$, $L(-3), \ldots$. However, for level two, this is not true. As a matter of fact, $L(\Lambda_0)^{sl_2}$ is a so-called W-algebra which is larger than the Virasoro algebra. The complete answer for the level two modules may naturally come from a sort of 'super-Schur' polynomials, which involves both commuting and anti-commuting variables.

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Received: February 1997

Revised: April 1997