[1]

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E

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Abstract. A selection of 128 elements form Weyl groups for exceptional groups of type E_8 with 696,729,600 elements is given. As a direct consequence, we then prove the weak Riemann Hypothesis for the so-called Weng zeta functions of E_8. Moreover, we explain what is the meaning of these $128\,(=2^7)$ elements in terms of the masses of semi-stable principal E_8-lattices, following our earlier conjecture on Stability, Parabolic Reduction, and the Masses. (This final part to be extended)

1. Root Systems

Following structures of the associated Weyl groups, to realize the set of simple roots for the exceptional group of types E_n $(n = 6, 7, 8)$, we choose an oriented basis of $\mathbb{R}^8$ as follows:

$$\begin{aligned}
\alpha_1 =&(0,0,0,-1,1,0,0,0), \quad \alpha_2 = (0,0,-1,1,0,0,0,0),\\
\alpha_3 =&(0,-1,1,0,0,0,0,0), \quad \alpha_4 = (1,1,0,0,0,0,0,0),\\
\alpha_5 =&(-1,1,0,0,0,0,0,0), \quad \alpha_6 = \frac{1}{2}(1,-1,-1,-1,-1,-1,-1,1),\\
\alpha_7 =&(0,0,0,0,-1,1,0,0), \quad \alpha_8 = (0,0,0,0,0,-1,1,0).
\end{aligned}$$

Accordingly, we will take the sets Δ_n $(n = 6, 7, 8)$ of simple roots for root systems of Φ_n for types E_n, to be

$$\begin{aligned}
\Delta_6 =&\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\},\\
\Delta_7 =&\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\},\\
\Delta_8 =&\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}.
\end{aligned}$$

Clearly, the corresponding Dynkin diagrams are given as follows:

$$E_6 \qquad \begin{array}{ccccccccc} & & & & \circ & & & & \\ & & & & |\ \alpha_4 & & & & \\ \circ & — & \circ & — & \circ & — & \circ & — & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_5 & & \alpha_6 \end{array}$$

$$E_7 \qquad \begin{array}{ccccccccccc} & & & & & & \circ & & & & \\ & & & & & & |\ \alpha_4 & & & & \\ \circ & — & \circ & — & \circ & — & \circ & — & \circ & — & \circ \\ \alpha_7 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_5 & & \alpha_6 \end{array}$$

$$E_8 \qquad \begin{array}{ccccccccccccc} & & & & & & & & \circ & & & & \\ & & & & & & & & |\ \alpha_4 & & & & \\ \circ & — & \circ & — & \circ & — & \circ & — & \circ & — & \circ & — & \circ \\ \alpha_8 & & \alpha_7 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_5 & & \alpha_6 \end{array}$$

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2. **Weyl Groups**

2.1. **Weyl Group for D_5.** Let W_n be the Weyl groups of the exceptional group of type E_n, $n = 6, 7, 8$. Then natural inclusions $\Phi_6 \subset \Phi_7 \subset \Phi_8$ of the roos systems give rise to natural inclusions $W_6 \leq W_7 \leq W_8$ of groups. If, in addition, we set $\Delta_5 := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Then Δ_5 can be understood as a set of simple roots for the D_5 of D-series group with the Dynkin diagram

$$D_5 \qquad \begin{array}{ccccccc} & & & & \circ & & \\ & & & & \vert\ \alpha_4 & & \\ \circ & - & \circ & - & \circ & - & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_5 \end{array}$$

Denote its associated roots system by Φ_5 and Weyl group by W_5 . We have two chains $\Phi_5 \subset \Phi_6 \subset \Phi_7 \subset \Phi_8$ and $W_5 \leq W_6 \leq W_7 \leq W_8$. Our method to build W_n starts with W_5, which is known to be the semi-product $\mathfrak{S}_5 \rtimes \mathrm{Sign}_{\mathrm{ev}}$ where $\mathfrak{S}_5$ denotes the 5-th symmetric group acting as follows: for any $\sigma \in \mathfrak{S}_5$,

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(4)}, x_6, x_7, x_8),$$

and $\mathrm{Sign}_{\mathrm{ev}}$ consists of even sign changes for the first 5 coordinates. Thus, totally, $\#W_5 = \#\mathfrak{S}_5 \times 2^{\binom{5}{0}+\binom{5}{2}+\binom{5}{4}} = 1920$.

2.2. **Weyl Group for E_6.** From now on, we will view all groups $W_n, n = 5, 6, 7, 8$ as subgroups of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^8)$. Thus, to obtain the group W_6, we only need to find out what are the representatives of the coset W_6/W_5. For this, we first set $\Phi_{6,5}^+ := \{\alpha \in \Phi_6^+ : \alpha \notin \Phi_5^+\}$. It consists of 16 elements. Namely,

$$\left\{ \begin{array}{cc} (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \end{array} \right\}$$

which we denote by $\{\alpha_{6,1},\ \alpha_{6,2}, \ldots, \alpha_{6,16}\}$. Accordingly, we set the reflections associated to $\alpha_{6,i}$ by $\sigma_{6,i}$, $i = 1, 2, \ldots, 16$.

Our first result, one calculated using Mathematica together with Katayama ([Ka]), is the following:

Lemma 1. *The representatives of W_6/W_5 can be chosen as*

$$\begin{aligned} \Sigma_6 = \big\{ &\mathrm{Id}, \sigma_{6,1}, \sigma_{6,2}, \sigma_{6,3}, \sigma_{6,4}, \sigma_{6,5}, \sigma_{6,6}, \sigma_{6,7}, \sigma_{6,8}, \sigma_{6,9}, \sigma_{6,10}, \\ &\sigma_{6,11}, \sigma_{6,12},\ \sigma_{6,13}, \sigma_{6,14}, \sigma_{6,15}, \sigma_{6,16}, \\ &\sigma_{6,1}\sigma_{6,8},\ \sigma_{6,1}\sigma_{6,12},\ \sigma_{6,1}\sigma_{6,14},\ \sigma_{6,1}\sigma_{6,15}, \sigma_{6,1}\sigma_{6,16}, \\ &\sigma_{6,2}\sigma_{6,15},\ \sigma_{6,2}\sigma_{6,16},\ \sigma_{6,3}\sigma_{6,16},\ \sigma_{6,5}\sigma_{6,16},\ \sigma_{6,9}\sigma_{6,16} \big\} \end{aligned}$$

As to be expected, it consists of 27 elements. Hence $\#W_6 = 27 \times 1920 = 51,840$. Based on this calculation, we could calculate the zeta functions for

(E_6, P_1) where P_1 is the maximal parabolic subgroup of E_7 corresponding to the subset $\Delta_6 \smallsetminus \{\alpha_1\}$. For later use, we denote Σ_6 by $\{\tau_{6,i}, i = 0, 1, 2, \dots 26\}$, where $\tau_{6,i}$ stands for the $i+1$-st element in the previous display for Σ_6.

The calculation, using Mathematica, is not very difficult, even the choice at the beginning for the orders of simple roots already proves to be crucial.

2.3. **Weyl Group for E_7.** To construct the Weyl group W_7 for the exceptional group of type E_7, we use W_6 as presented in the above lemma and yet-to-be-found representatives of W_7/W_6, which consists of 56 elements.

We first introduce $\Phi^+_{7,6} := \{\alpha \in \Phi^+_7 : \alpha \notin \Phi^+_6\}$, which consists of 27 elements. Namely,

$$\left\{\begin{array}{cc}
(-1,0,0,0,0,1,0,0), & (-\frac12,-\frac12,-\frac12,-\frac12,-\frac12,\frac12,-\frac12,\frac12),\\
(-\frac12,-\frac12,-\frac12,\frac12,\frac12,\frac12,-\frac12,\frac12), & (-\frac12,-\frac12,\frac12,-\frac12,\frac12,\frac12,-\frac12,\frac12),\\
(-\frac12,-\frac12,\frac12,\frac12,-\frac12,\frac12,-\frac12,\frac12), & (-\frac12,\frac12,-\frac12,-\frac12,\frac12,\frac12,-\frac12,\frac12),\\
(-\frac12,\frac12,-\frac12,\frac12,-\frac12,\frac12,-\frac12,\frac12), & (-\frac12,\frac12,\frac12,-\frac12,-\frac12,\frac12,-\frac12,\frac12),\\
(-\frac12,\frac12,\frac12,\frac12,\frac12,\frac12,-\frac12,\frac12), & (0,-1,0,0,0,1,0,0),\\
(0,0,-1,0,0,1,0,0), & (0,0,0,-1,0,1,0,0),\\
(0,0,0,0,-1,1,0,0), & (0,0,0,0,0,0,-1,1),\\
(0,0,0,0,1,1,0,0), & (0,0,0,1,0,1,0,0),\\
(0,0,1,0,0,1,0,0), & (0,1,0,0,0,1,0,0),\\
(\frac12,-\frac12,-\frac12,-\frac12,\frac12,\frac12,-\frac12,\frac12), & (\frac12,-\frac12,-\frac12,\frac12,-\frac12,\frac12,-\frac12,\frac12),\\
(\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12,-\frac12,\frac12), & (\frac12,-\frac12,\frac12,\frac12,\frac12,\frac12,-\frac12,\frac12),\\
(\frac12,\frac12,-\frac12,-\frac12,-\frac12,\frac12,-\frac12,\frac12), & (\frac12,\frac12,-\frac12,\frac12,\frac12,\frac12,-\frac12,\frac12),\\
(\frac12,\frac12,\frac12,-\frac12,\frac12,\frac12,-\frac12,\frac12), & (\frac12,\frac12,\frac12,\frac12,-\frac12,\frac12,-\frac12,\frac12),\\
(1,0,0,0,0,1,0,0) &
\end{array}\right\}.$$

We denote them, with above displayed order, by $\{\alpha_{7,1}, \alpha_{7,2}, \dots, \alpha_{7,27}\}$, and denote the reflection associated to $\alpha_{7,j}$ by $\sigma_{7,j}$, $j = 1, 2, \dots, 27$.

Proposition 2. *A group of representatives for W_7/W_6 can be chosen as*

$$\begin{aligned}\Sigma_7 = \Big\{ & \mathrm{Id}, \sigma_{7,1}, \sigma_{7,2}, \sigma_{7,3}, \sigma_{7,4}, \sigma_{7,5}, \sigma_{7,6}, \sigma_{7,7}, \sigma_{7,8}, \sigma_{7,9}, \sigma_{7,10},\\
& \sigma_{7,11}, \sigma_{7,12}, \sigma_{7,13}, \sigma_{7,14}, \sigma_{7,15}, \sigma_{7,16}, \sigma_{7,17}, \sigma_{7,18}, \sigma_{7,19}, \sigma_{7,20},\\
& \sigma_{7,21}, \sigma_{7,22}, \sigma_{7,23}, \sigma_{7,24}, \sigma_{7,25}, \sigma_{7,26}, \sigma_{7,27},\\
& \sigma_{7,1}\sigma_{7,14}, \ \sigma_{7,1}\sigma_{7,19}, \ \sigma_{7,1}\sigma_{7,20}, \ \sigma_{7,1}\sigma_{7,21}, \ \sigma_{7,1}\sigma_{7,22},\\
& \sigma_{7,1}\sigma_{7,23}, \ \sigma_{7,1}\sigma_{7,24}, \ \sigma_{7,1}\sigma_{7,25}, \ \sigma_{7,1}\sigma_{7,26}, \ \sigma_{7,1}\sigma_{7,27},\\
& \sigma_{7,2}\sigma_{7,22}, \ \sigma_{7,2}\sigma_{7,24}, \ \sigma_{7,2}\sigma_{7,25}, \ \sigma_{7,2}\sigma_{7,26}, \ \sigma_{7,2}\sigma_{7,27},\\
& \sigma_{7,3}\sigma_{7,25}, \ \sigma_{7,3}\sigma_{7,26}, \ \sigma_{7,3}\sigma_{7,27},\\
& \sigma_{7,4}\sigma_{7,26}, \ \sigma_{7,4}\sigma_{7,27}, \ \sigma_{7,5}\sigma_{7,27}, \ \sigma_{7,6}\sigma_{7,26}, \ \sigma_{7,6}\sigma_{7,27},\\
& \sigma_{7,7}\sigma_{7,27}, \ \sigma_{7,8}\sigma_{7,27}, \ \sigma_{7,9}\sigma_{7,27}, \ \sigma_{7,14}\sigma_{7,27}, \quad \sigma_{7,1}\sigma_{7,14}\sigma_{7,27} \Big\}\end{aligned}$$

With $\#W_7 = 2,903,040$, the calculation using Mathematica is still manageable: It completed in one night and we even can store W_7 as a single set in our Power MacBook. Here, as above, the special orders for simple roots plays a very crucial role. For later use, we denote Σ_7 by $\{\tau_{7,j}, j = 0, 1, 2, \dots 55\}$, where $\tau_{7,j}$ denotes the $j+1$-st elements in the previous display for Σ_7.

2.4. **Weyl Group for E_8.** Construction of W_8, consisting of $696,729,600$ elements, proves to be extremely challenging. Even in terms of representatives of W_8/W_7, there are 240 elements to be found. With our program in Mathematica, it took a couple of weeks to find the answer.

As the first step, we calculate $\Phi_{8,7}^+ := \{\alpha \in \Phi_8^+ : \alpha \notin \Phi_7^+\}$. It consists of 57 elements. Namely,

$$\left\{\begin{array}{cc}
(-1,0,0,0,0,0,0,1), & (-1,0,0,0,0,0,1,0), \\
(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(0,-1,0,0,0,0,0,1), & (0,-1,0,0,0,0,1,0), \\
(0,0,-1,0,0,0,0,1), & (0,0,-1,0,0,0,1,0), \\
(0,0,0,-1,0,0,0,1), & (0,0,0,-1,0,0,1,0), \\
(0,0,0,0,-1,0,0,1), & (0,0,0,0,-1,0,1,0), \\
(0,0,0,0,0,-1,0,1), & (0,0,0,0,0,-1,1,0), \\
(0,0,0,0,0,0,1,1), & (0,0,0,0,0,1,0,1), \\
(0,0,0,0,0,1,1,0), & (0,0,0,0,1,0,0,1), \\
(0,0,0,0,1,0,1,0), & (0,0,0,1,0,0,0,1), \\
(0,0,0,1,0,0,1,0), & (0,0,1,0,0,0,0,1), \\
(0,0,1,0,0,0,1,0), & (0,1,0,0,0,0,0,1), \\
(0,1,0,0,0,0,1,0), & (\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\
(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & (1,0,0,0,0,0,0,1), \\
(1,0,0,0,0,0,1,0) &
\end{array}\right\}$$

We denote them, with above displayed order, by $\{\alpha_{8,1}, \alpha_{8,2}, \dots, \alpha_{8,57}\}$, and denote the reflection associated to $\alpha_{8,k}$ by $\sigma_{8,k}$, $k = 1, 2, \dots, 57$.

With these, then we are ready to state our first main result son the construction of W_8.

Theorem 3. *A group of representatives for W_8/W_7 can be chosen as*

$$\begin{aligned}\Sigma_8 = \Big\{ &\mathrm{Id}, \sigma_{8,1}, \sigma_{8,2}, \sigma_{8,3}, \sigma_{8,4}, \sigma_{8,5}, \sigma_{8,6}, \sigma_{8,7}, \sigma_{8,8}, \sigma_{8,9}, \sigma_{8,10},\\
&\sigma_{8,11}, \sigma_{8,12}, \sigma_{13}, \sigma_{8,14}, \sigma_{8,15}, \sigma_{8,16}, \sigma_{8,17}, \sigma_{8,18}, \sigma_{8,19}, \sigma_{8,20},\\
&\sigma_{8,21}, \sigma_{8,22}, \sigma_{8,23},\ 24, \sigma_{8,25}, \sigma_{8,26}, \sigma_{8,27}, \sigma_{8,28}, \sigma_{8,29}, \sigma_{8,30},\\
&\sigma_{8,31}, \sigma_{8,32}, \sigma_{8,33}, \sigma_{8,34},\ 35, \sigma_{8,36}, \sigma_{8,37}, \sigma_{8,38}, \sigma_{8,39}, \sigma_{8,40},\\
&\sigma_{8,41}, \sigma_{8,42}, \sigma_{8,43}, \sigma_{8,44}, \sigma_{8,45},\ 46, \sigma_{8,47}, \sigma_{8,48}, \sigma_{8,49}, \sigma_{8,50},\\
&\sigma_{8,51}, \sigma_{8,52}, \sigma_{8,53}, \sigma_{8,54}, \sigma_{8,55}, \sigma_{8,56}, \sigma_{57},\\
&\sigma_{8,1}\sigma_{8,20}, \sigma_{8,1}\sigma_{8,22}, \sigma_{8,1}\sigma_{8,24}, \sigma_{8,1}\sigma_{8,26}, \sigma_{8,1}\sigma_{8,28}, \sigma_{8,1}\sigma_{8,29},\\
&\sigma_{8,1}\sigma_{8,31}, \sigma_{8,1}\sigma_{8,33}, \sigma_{8,1}\sigma_{8,35}, \sigma_{8,1}\sigma_{8,37}, \sigma_{8,1}\sigma_{8,39}, \sigma_{8,1}\sigma_{8,40},\\
&\sigma_{8,1}\sigma_{8,41}, \sigma_{8,1}\sigma_{8,42}, \sigma_{8,1}\sigma_{8,43}, \sigma_{8,1}\sigma_{8,44}, \sigma_{8,1}\sigma_{8,45}, \sigma_{8,1}\sigma_{8,46},\\
&\sigma_{8,1}\sigma_{8,47}, \sigma_{8,1}\sigma_{8,48}, \sigma_{8,1}\sigma_{8,49}, \sigma_{8,1}\sigma_{8,50}, \sigma_{8,1}\sigma_{8,51}, \sigma_{8,1}\sigma_{8,52},\\
&\sigma_{8,1}\sigma_{8,53}, \sigma_{8,1}\sigma_{8,54}, \sigma_{8,1}\sigma_{8,55}, \sigma_{8,1}\sigma_{8,56}, \sigma_{8,1}\sigma_{8,57},\\
&\sigma_{8,2}\sigma_{8,29}, \sigma_{8,2}\sigma_{8,40}, \sigma_{8,2}\sigma_{8,41}, \sigma_{8,2}\sigma_{8,42}, \sigma_{8,2}\sigma_{8,43}, \sigma_{8,2}\sigma_{8,44},\\
&\sigma_{8,2}\sigma_{8,45}, \sigma_{8,2}\sigma_{8,46}, \sigma_{8,2}\sigma_{8,47}, \sigma_{8,2}\sigma_{8,48}, \sigma_{8,2}\sigma_{8,49}, \sigma_{8,2}\sigma_{8,50},\\
&\sigma_{8,2}\sigma_{8,51}, \sigma_{8,2}\sigma_{8,52}, \sigma_{8,2}\sigma_{8,53}, \sigma_{8,2}\sigma_{8,54}, \sigma_{8,2}\sigma_{8,55}, \sigma_{8,2}\sigma_{8,56}, \sigma_{8,2}\sigma_{8,57},\\
&\sigma_{8,3}\sigma_{8,29}, \sigma_{8,3}\sigma_{8,43}, \sigma_{8,3}\sigma_{8,45}, \sigma_{8,3}\sigma_{8,46}, \sigma_{8,3}\sigma_{8,47}, \sigma_{8,3}\sigma_{8,49}, \sigma_{8,3}\sigma_{8,50},\\
&\sigma_{8,3}\sigma_{8,51}, \sigma_{8,3}\sigma_{8,52}, \sigma_{8,3}\sigma_{8,53}, \sigma_{8,3}\sigma_{8,54}, \sigma_{8,3}\sigma_{8,55}, \sigma_{8,3}\sigma_{8,56}, \sigma_{8,3}\sigma_{8,57},\\
&\sigma_{8,4}\sigma_{8,29}, \sigma_{8,4}\sigma_{8,46}, \sigma_{8,4}\sigma_{8,47}, \sigma_{8,4}\sigma_{8,50}, \sigma_{8,4}\sigma_{8,51}, \sigma_{8,4}\sigma_{8,52},\\
&\sigma_{8,4}\sigma_{8,53}, \sigma_{8,4}\sigma_{8,54}, \sigma_{8,4}\sigma_{8,55}, \sigma_{8,4}\sigma_{8,56}, \sigma_{8,4}\sigma_{8,57},\\
&\sigma_{8,5}\sigma_{8,29}, \sigma_{8,5}\sigma_{8,47}, \sigma_{8,5}\sigma_{8,51}, \sigma_{8,5}\sigma_{8,52}, \sigma_{8,5}\sigma_{8,53},\\
&\sigma_{8,5}\sigma_{8,54}, \sigma_{8,5}\sigma_{8,55}, \sigma_{8,5}\sigma_{8,56}, \sigma_{8,5}\sigma_{8,57},\\
&\sigma_{8,6}\sigma_{8,29}, \sigma_{8,6}\sigma_{8,52}, \sigma_{8,6}\sigma_{8,53}, \sigma_{8,6}\sigma_{8,54}, \sigma_{8,6}\sigma_{8,55}, \sigma_{8,6}\sigma_{8,56}, \sigma_{8,6}\sigma_{8,57},\\
&\sigma_{8,7}\sigma_{8,29}, \sigma_{8,7}\sigma_{8,47}, \sigma_{8,7}\sigma_{8,51}, \sigma_{8,7}\sigma_{8,53}, \sigma_{8,7}\sigma_{8,54}, \sigma_{8,7}\sigma_{8,55}, \sigma_{8,7}\sigma_{8,56}, \sigma_{8,7}\sigma_{8,57},\\
&\sigma_{8,8}\sigma_{8,29}, \sigma_{8,8}\sigma_{8,50}, \sigma_{8,8}\sigma_{8,54}, \sigma_{8,8}\sigma_{8,55}, \sigma_{8,8}\sigma_{8,56}, \sigma_{8,8}\sigma_{8,57},\\
&\sigma_{8,9}\sigma_{8,29}, \sigma_{8,9}\sigma_{8,49}, \sigma_{8,9}\sigma_{8,55}, \sigma_{8,9}\sigma_{8,56}, \sigma_{8,9}\sigma_{8,57},\\
&\sigma_{8,10}\sigma_{8,29}, \sigma_{8,10}\sigma_{8,48}, \sigma_{8,10}\sigma_{8,56}, \sigma_{8,10}\sigma_{8,57},\\
&\sigma_{8,11}\sigma_{8,29}, \sigma_{8,11}\sigma_{8,47}, \sigma_{8,11}\sigma_{8,51}, \sigma_{8,11}\sigma_{8,53}, \sigma_{8,11}\sigma_{8,54}, \sigma_{8,11}\sigma_{8,55}, \sigma_{8,11}\sigma_{8,56}, \sigma_{8,11}\sigma_{8,57},\\
&\sigma_{8,12}\sigma_{8,29}, \sigma_{8,12}\sigma_{8,46}, \sigma_{8,12}\sigma_{8,54}, \sigma_{8,12}\sigma_{8,55}, \sigma_{8,12}\sigma_{8,56}, \sigma_{8,12}\sigma_{8,57},\\
&\sigma_{8,13}\sigma_{8,29}, \sigma_{8,13}\sigma_{8,45}, \sigma_{8,13}\sigma_{8,55}, \sigma_{8,13}\sigma_{8,56}, \sigma_{8,13}\sigma_{8,57},\\
&\sigma_{8,14}\sigma_{8,29}, \sigma_{8,14}\sigma_{8,44}, \sigma_{8,14}\sigma_{8,56}, \sigma_{8,14}\sigma_{8,57},\\
&\sigma_{8,15}\sigma_{8,29}, \sigma_{8,15}\sigma_{8,43}, \sigma_{8,15}\sigma_{8,55}, \sigma_{8,15}\sigma_{8,56}, \sigma_{8,15}\sigma_{8,57},\\
&\sigma_{8,16}\sigma_{8,29}, \sigma_{8,16}\sigma_{8,42}, \sigma_{8,16}\sigma_{8,56}, \sigma_{8,16}\sigma_{8,57},\\
&\sigma_{8,17}\sigma_{8,29}, \sigma_{8,17}\sigma_{8,41}, \sigma_{8,17}\sigma_{8,56}, \sigma_{8,17}\sigma_{8,57},\\
&\sigma_{8,18}\sigma_{8,29}, \sigma_{8,18}\sigma_{8,40}, \sigma_{8,18}\sigma_{8,56}, \sigma_{8,18}\sigma_{8,57},\\
&\sigma_{8,19}\sigma_{8,29}, \sigma_{8,19}\sigma_{8,39}, \sigma_{8,19}\sigma_{8,57}, \sigma_{8,20}\sigma_{8,29}, \sigma_{8,20}\sigma_{8,38},\\
&\sigma_{8,21}\sigma_{8,29}, \sigma_{8,21}\sigma_{8,37}, \sigma_{8,21}\sigma_{8,57}, \sigma_{8,22}\sigma_{8,29}, \sigma_{8,22}\sigma_{8,36},\\
&\sigma_{8,23}\sigma_{8,29}, \sigma_{8,23}\sigma_{8,35}, \sigma_{8,23}\sigma_{8,57}, \sigma_{8,24}\sigma_{8,29}, \sigma_{8,24}\sigma_{8,34}, \sigma_{8,25}\sigma_{8,29}, \sigma_{8,25}\sigma_{8,33}, \sigma_{8,25}\sigma_{8,57},\\
&\sigma_{8,26}\sigma_{8,29}, \sigma_{8,26}\sigma_{8,32}, \sigma_{8,27}\sigma_{8,29}, \sigma_{8,27}\sigma_{8,31}, \sigma_{8,27}\sigma_{8,57},\\
&\sigma_{8,28}\sigma_{8,29}, \sigma_{8,28}\sigma_{8,30}, \sigma_{8,30}\sigma_{8,57}, \sigma_{8,32}\sigma_{8,57}, \sigma_{8,34}\sigma_{8,57}, \sigma_{8,36}\sigma_{8,57}, \sigma_{8,38}\sigma_{8,57}\Big\}\end{aligned}$$

For later use, we denote Σ_8 by $\{\tau_{8,k}, k = 0, 1, 2, \ldots 239\}$ where $\tau_{8,k}$ denotes the $k+1$-st elements in the above display.

Theorem 4. *The Weyl groups for the exceptional groups of types E_6, E_7, E_8 are given by*

$$\begin{aligned} W_6 &= \sqcup_{i=1}^{27} \tau_{6,i} W_5, \\ W_7 &= \sqcup_{j=0}^{55} \sqcup_{i=0}^{26} \tau_{7,j}\tau_{6,i} W_5, \\ W_8 &= \sqcup_{k=0}^{239} \sqcup_{j=0}^{55} \sqcup_{i=0}^{26} \tau_{8,k}\tau_{7,j}\tau_{6,i} W_5, \end{aligned}$$

As to be expected, the special orders for simple roots at the beginning proves to be extremely crucial.

This calculation is a rather long and complicated – with a series of bugs searching and program fixing, our calculation to find all representatives above last for about a month and a half, and our program in Mathematica cannot treat W_8 as a single set to store in our Power MacBook/Power Mac. There is a huge problem when we tried to store W_8 as a single unit, which forced our work stations stop calculating repeatedly. At some point, we realized this and offered a not-perfect-but-practical solution: Instead, we introduce

$$W_{8,i} := \sqcup_{k=0}^{239} \sqcup_{j=0}^{55} \tau_{8,k}\tau_{7,j}\tau_{6,i} W_5, \qquad i = 0, 1, 2, \ldots, 26$$

and hence divide our calculations into 27 parallel units. With W_8, we were able to start our second phrase of calculations, on zetas themselves and the verifications of the weak Riemann Hypothesis for the so-called Weng zetas for E_6, E_7, E_8. This lasted for another 2 months.

3. Zeta functions for (G, P)

Let G be a connected split reductive algebraic group of rank r with a fixed Borel subgroup B and associated maximal split torus T (over a base field). Denote by

$$\Big(V, \langle\cdot,\cdot\rangle, \Phi = \Phi^+ \cup \Phi^-, \Delta = \{\alpha_1, \ldots, \alpha_r\}, \varpi := \{\varpi_1, \ldots, \varpi_r\}, W\Big)$$

the associated root system. That is, V is the real vector space defined as the $\mathbb{R}$-span of rational characters of T, and as usual, is equipped with a natural inner product $\langle\cdot,\cdot\rangle$, with which we identify V with its dual V^*, $\Phi^+ \subset V$ is the set of positive roots, $\Phi^- := -\Phi^+$ the set of negative roots, $\Delta \subset V$ the set of simple roots, $\varpi \subset V$ the set of fundamental weights, and W the Weyl group. By definition, the fundamental weights are characterized by the formula $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j = 1, 2, \ldots, r$, where $\alpha^\vee := \frac{2}{\langle\alpha,\alpha\rangle}\alpha$ denotes the coroot of a root $\alpha \in \Phi$. We also define the Weyl vector ρ by $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, and introduce a *coordinate system* on V (with respect to the base $\{\varpi_1, \ldots, \varpi_r\}$ of V and the vector ρ) by writing an element $\lambda \in V$ in the form

$$\lambda \;=\; \sum_{j=1}^{r}(1-s_j)\varpi_j \;=\; \rho - \sum_{j=1}^{r} s_j\varpi_j$$

thus fixing identifications of V and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{R}^r$ and $\mathbb{C}^r$. In addition, for each Weyl element $w \in W$, we set $\Phi_w := \Phi^+ \cap w^{-1}\Phi^-$, i.e., the collection of positive roots whose w-images are negative.

As usual, by a *standard parabolic subgroup*, we mean a parabolic subgroup of G that contains the Borel subgroup B. From Lie theory (see e.g., [Hu]), there is an one-to-one correspondence between standard parabolic subgroups P of G and subsets Δ_P of Δ. In particular, if P is maximal, we may and will write $\Delta_P = \Delta \smallsetminus \{\alpha_p\}$ for a certain unique $p = p(P) \in \{1, \ldots, r\}$. For such a standard parabolic subgroup P, denote by V_P the $\mathbb{R}$-span of rational characters of the maximal split torus T_P contained in P, by V_P^* its dual space, and by $\Phi_P \subset V_P$ the set of non-trivial characters of T_P occurring in the space V. Then, by standard theory of reductive groups (see e.g.,[Ar]), V_P admits a canonical embedding in V (and V_P^* admits a canonical embedding in V^*), which is known to be orthogonal to the fundamental weight ϖ_p, and hence Φ_P can be viewed as a subset of Φ. Set $\Phi_P^+ = \Phi^+ \cap \Phi_P$, $\rho_P = \frac{1}{2}\sum_{\alpha\in\Phi_P^+}\alpha$, and $c_P = 2\langle \varpi_p - \rho_P, \alpha_p^\vee\rangle$.

In [W?], to understand the non-abelian zeta function over a number field F, we, for a connected split reductive algebraic group G, and its standard parabolic subgroup P as above (defined over F), defined the *period of G for F* by

$$\omega_F^G(\lambda) := \sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta}\langle w\lambda-\rho, \alpha^\vee\rangle} \prod_{\alpha\in\Phi_w} \frac{\widehat{\zeta}_F(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_F(\langle\lambda,\alpha^\vee\rangle+1)}$$

and the *period of (G,P) for F* by

$$\begin{aligned}\omega_F^{G,P}(s) &:= \mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P}\omega_F^G(\lambda)\\ &= \mathrm{Res}_{s_r=0}\cdots\mathrm{Res}_{s_{p+1}=0}\mathrm{Res}_{s_{p-1}=0}\cdots\mathrm{Res}_{s_1=0}\,\omega_F^G(\lambda)\,.\end{aligned}$$

Here $\widehat{\zeta}_F(s)$ denotes the completed Dedekind zeta function for F, $s = c_p \cdot s_p$ and for the last equality, we used the fact that $\langle\rho,\alpha^\vee\rangle = 1$ for all $\alpha \in \Delta$ and the relation that $\langle\varpi_i, \alpha_j^\vee\rangle = \delta_{ij}$ for all $i, j \in \{1, \ldots, r\}$. As proved in [Ko, W?], the ordering of taking residues along singular hyperplanes $\langle\lambda - \rho, \alpha^\vee\rangle = 0$ for $\alpha \in \Delta_P$ does not affect the outcome, so that the definition is independent of the numbering of the simple roots.

To get the zeta function associated to (G, P) for F, certain normalizations should be made. For this purpose, write $\omega_F^G(\lambda) = \sum_{w\in W} T_w(\lambda)$, where, for each $w \in W$,

$$T_w(\lambda) := \frac{1}{\prod_{\alpha\in\Delta} -\langle w\lambda-\rho, \alpha^\vee\rangle} \prod_{\alpha\in\Phi_w} \frac{\widehat{\zeta}_F(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_F(\langle\lambda,\alpha^\vee\rangle+1)}.$$

We must study the residue $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P} T_w(\lambda)$.

We care only about those elements $w \in W$ (we will call them *special*) which give non-trivial residues, namely, those satisfying the condition that $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P} T_w(\lambda) \not\equiv 0$. This can happen only if all singular hyperplanes are of one of the following two forms:

(1) $\langle w\lambda - \rho, \alpha^\vee\rangle = 0$ for some $\alpha \in \Delta$, giving a simple pole of the rational factor $\frac{1}{\prod_{\alpha\in\Delta}(-\langle w\lambda-\rho,\alpha^\vee\rangle)}$;
(2) $\langle\lambda, \alpha^\vee\rangle = 1$ for some $\alpha \in \Phi_w$, giving a simple pole of the zeta factor $\widehat{\zeta}_F(\langle\lambda,\alpha^\vee\rangle)$.

For special $w \in W$, and $(k,h) \in \mathbb{Z}^2$, following [Ko], we define

$$N_{P,w}(k,h) := \#\{\alpha \in w^{-1}\Phi^- : \langle \lambda_p, \alpha^\vee \rangle,\ \langle \rho, \alpha^\vee \rangle = h\}$$
$$M_P(k,h) := \max_{w:\ \text{special}} (N_{P,w}(k,h-1) - N_{P,w}(k,h)).$$

Note that $N_P(k,h) = 0$ for almost all but finitely many pairs of integers (k,h), so it makes sense to introduce the product

$$D_F^{G,P}(s) := \prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \widehat{\zeta}_F(ks+h)^{M_P(k,h)}.$$

Then the so-called *Weng zeta function of F associated to* (G,P) is defined by

$$\widehat{\zeta}_F^{G,P}(s) := q^{(g-1)\dim N_u(B)} \cdot D^{G,P}(s) \cdot \omega_F^{G,P}(s)\,. \tag{1}$$

Here $N_u(B)$ denote the nilpotent radical of the Borel subgroup B of G.

Remark. For special $w \in W$, even after taking residues, there are some zeta factors $\widehat{\zeta}_F(ks+h)$ left in the denominator of $\mathrm{Res}_{\langle \lambda-\rho,\, \alpha^\vee \rangle = 0,\ \alpha \in \Delta_P} T_w(\lambda)$. The reason for introducing the factor $D_F^{G,P}(s)$ in our normalization of the zeta functions, based on formulas in [Ko] and [W3], is to clear up all of the zeta factors appearing in the denominators associated to special Weyl elements.

4. **Conditions for Weak Riemann Hypothesis**

In this section, we explain a numerical condition stated in the work of Ki, Komori, Suzuki ([KKS]) on the weak Riemann Hypothesis for Weng zeta functions.

Let

$$\mathfrak{W}_P := \{\, w \in W : \Delta_P \subset w^{-1}(\Delta \cup \Phi^-)\,\}.$$

For each $w \in \mathfrak{W}_P$, set

$$\delta_{\alpha,w} = \begin{cases} 1 & \alpha \in w^{-1}\Phi^+ \\ 0 & \alpha \in w^{-1}\Phi^- \end{cases},$$

and define

$$k_P(w) := \sum_{\alpha \in \Phi^+ \smallsetminus \Phi_P^+} (1 - \delta_{\alpha.w}) = \left|(\Phi^+ \smallsetminus \Phi_P^+) \cap w^{-1}\Phi^-\right|,$$
$$\mathfrak{W}_p^{\ddagger} := \{w \in \mathfrak{W}_P : k_P(w) = 0\}.$$

Set $\xi(s) := s(s-1)\widehat{\zeta}(s)$ and

$$C_{P,w} := \xi(2)^{|\Delta_P \cap w^{-1}\Phi^+|} \prod_{\alpha \in \Phi_P^+ \smallsetminus \Delta_P} \xi(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w},$$
$$\frac{1}{D_{P,w}} := 2^{|\Delta_P \cap w^{-1}\Phi^+|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \smallsetminus \Delta_P)} (\langle \rho, \alpha^\vee \rangle - 1)$$
$$\times \prod_{\alpha \in \Phi_P \smallsetminus \Delta_P} (\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w})(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w} - 1)$$

Introduce then the number

$$v_{G,P} := \sum_{\substack{w\in\mathfrak{W}_P^{\ddagger} \\ |(w^{-1}\Delta)\smallsetminus\Phi_P|=1}} \frac{1}{\langle\varpi_P,\alpha_w^\vee\rangle}\cdot C_{P,w}\cdot D_{P,w},$$

where α_w is the only element of $(w^{-1}\Delta)\smallsetminus\Phi_P$.

Theorem. *([KKS]) If $v_{G,P}\neq 0$, all but finitely many zeros of $\widehat{\zeta}_{\mathbb{Q}}^{G,P}(s)$ are simple and on the critical line of its functional equation.*

Set

$$B_{P,w} := \widehat{\zeta}(2)^{|\Delta_P\cap w^{-1}\Phi^+|}\prod_{\alpha\in(w^{-1}\Delta)\cap(\Phi_P\smallsetminus\Delta_P)}\frac{1}{\langle\rho,\alpha^\vee\rangle-1}\prod_{\alpha\in\Phi_P^+\smallsetminus\Delta_P}\widehat{\zeta}(\langle\rho,\alpha^\vee\rangle+\delta_{\alpha,w}).$$

Then by [KKS, p.153, lines 2, 3], we have

$$v_{G,P} = \sum_{\substack{w\in\mathfrak{W}_P^{\ddagger} \\ |(w^{-1}\Delta)\smallsetminus\Phi_P|=1}} \frac{B_{P,w}}{\langle\varpi_P,\alpha_w^\vee\rangle}. \tag{2}$$

Here $\{\varpi_1,\varpi_2,\dots,\varpi_n\}$ is the dual bases of $\{\alpha_1^\vee,\alpha_2^\vee,\dots,\alpha_n^\vee\}$ for $n=6,7,8$, and α_w is the unique element in $(w^{-1}\Delta)\smallsetminus\Phi_P$.

5. Special and Very Special Weyl Elements

For each $w\in W$, we set

$$A_w := \{\alpha\in\Delta_P : w\alpha\in\Delta\},\qquad B_w := \{\alpha\in\Delta_P : w\alpha\in\Phi^-\}.$$

Clearly, $A_w\cup B_w\subseteq\Delta_P$. We call w *special*, if $A_w\cup B_w=\Delta_P$. Denote by

$$\begin{aligned} W_P^{\mathrm{sp}} :=& \{w\in W : w \text{ special}\},\\ W_P^{\dagger} :=& \{w\in W^{\mathrm{sp}} : |\Delta\smallsetminus w\Phi_P|=1\},\\ W_P^{\ddagger} :=& \{w\in W^{\dagger} : |w(\Phi^+\smallsetminus\Phi_P^+)\cap\Phi^-|=0\}.\end{aligned}$$

Then $W^{\mathrm{sp}}=\mathfrak{W}_P$ since $A_w\cup B_w=\Delta_P\iff\Delta_P\subset w^{-1}(\Delta\cup\Phi^-\}$, and, moreover,

$$W_P^{\dagger} = \{w\in W^{\mathrm{sp}} : |\Delta\cap w\Phi_P|=7\}.$$

To prove $v_{E_n,P}\neq 0$ for every $n=6,7,8$ and all associated maximal parabolic subgroups, we first search on the selections of Weyl elements belonging to W^{sp}, $W^{\dagger}$ and $W^{\ddagger}$ respectively. Accordingly, we divide our calculations in Mathematica into three steps:
Step 1. Calculate elements of W_P^{sp}, denoted sp. This is very long.
Step 2. Calculate elements of $W_P^{\dagger}$,denotedsps. This is rather long.
Step 3. Calculate elements of $W_P^{\ddagger}$, called very special. This is very pleasent.

Our calculation can be summarized as follows:

Theorem 5. *For an exceptional group of type E_n, $n=6,7,8$, we have*

$$W_{6,P_i}^{\ddagger}\le W_{7,P_i}^{\ddagger}\qquad \textit{and}\qquad W_{7,P_j}^{\ddagger}\le W_{8_j}^{\dagger}\le W_{8_i}^{\ddagger}\qquad (1\le i\le 6,\quad 1\le j\le 7).$$

Moreover,

$$\#W_{n,P}^{\dagger}=2^n,\qquad \#W_{n,P}^{\ddagger}=2^{n-1}.$$

To state an example with $(G,P)=(E_8,P_1)$, we denote $\tau_{5,s}\tau_{6,i}\tau_{7,j}\tau_{8,k}$ as $(s+1,i+1,j+1,k+1)$, where $w_{5,s}\in W_5$. Then, we get, from ddag.nb

$$\begin{aligned}W^\ddagger=\big\{&(1,1,1,1),(1,1,1,27),(1,1,14,27),(1,1,14,29),\\&(1375,3,1,1),(1375,3,1,27),(1375,3,14,27),(1375,3,14,29),\\&(1,9,1,1),(1,9,1,27),(1,9,14,27),(1,9,14,29),\\&(193,9,1,1),(193,9,1,27),(193,9,14,27),(193,9,14,29),\\&(301,9,1,1),(301,9,1,27),(301,9,14,27),(301,9,14,29),\\&(325,9,1,1),(325,9,1,27),(325,9,14,27),(325,9,14,29),\\&(393,9,1,1),(393,9,1,27),(393,9,14,27),(393,9,14,29),\\&(423,9,1,1),(423,9,1,27),(423,9,14,27),(423,9,14,29),\\&(441,9,1,1),(441,9,1,27),(441,9,14,27),(441,9,14,29),\\&(447,9,1,1),(447,9,1,27),(447,9,14,27),(447,9,14,29),\\&(663,13,1,1),(663,13,1,27),(663,13,14,27),(663,13,14,29),\\&(687,13,1,1),(687,13,1,27),(687,13,14,27),(687,13,14,29),\\&(781,13,1,1),(781,13,1,27),(781,13,14,27),(781,13,14,29),\\&(805,13,1,1),(805,13,1,27),(805,13,14,27),(805,13,14,29),\\&(963,15,1,1),(963,15,1,27),(963,15,14,27),(963,15,14,29),\\&(1375,16,1,1),(1375,16,1,27),(1375,16,14,27),(1375,16,14,29),\\&(1,23,1,1),(1,23,1,27),(1,23,14,27),(1,23,14,29),\\&(361,23,1,1),(361,23,1,27),(361,23,14,27),(361,23,14,29),\\&(687,23,1,1),(687,23,1,27),(687,23,14,27),(687,23,14,29),\\&(775,23,1,1),(775,23,1,27),(775,23,14,27),(775,23,14,29),\\&(961,23,1,1),(961,23,1,27),(961,23,14,27),(961,23,14,29),\\&(963,23,1,1),(963,23,1,27),(963,23,14,27),(963,23,14,29),\\&(985,23,1,1),(985,23,1,27),(985,23,14,27),(985,23,14,29),\\&(987,23,1,1),(987,23,1,27),(987,23,14,27),(987,23,14,29),\\&(1321,23,1,1),(1321,23,1,27),(1321,23,14,27),(1321,23,14,29),\\&(1323,23,1,1),(1323,23,1,27),(1323,23,14,27),(1323,23,14,29),\\&(1327,23,1,1),(1327,23,1,27),(1327,23,14,27),(1327,23,14,29),\\&(1335,23,1,1),(1335,23,1,27),(1335,23,14,27),(1335,23,14,29),\\&(1345,23,1,1),(1345,23,1,27),(1345,23,14,27),(1345,23,14,29),\\&(1347,23,1,1),(1347,23,1,27),(1347,23,14,27),(1347,23,14,29),\\&(1375,23,1,1),(1375,23,1,27),(1375,23,14,27),(1375,23,14,29),\\&(1407,23,1,1),(1407,23,1,27),(1407,23,14,27),(1407,23,14,29)\big\}\end{aligned}$$

Totally, there are $128=2^7$ of them. For other cases, please refer to the appendix where $W^\ddagger$ is hidden in Appendix C.

As we will see later, this calculation is also compatible with the Conjecture on Parabolic Reduction, Stability and the Masses [We]. Besides, even by examining the components of each element, ew can clear see natural structures for very special Weyl elements. This deserves to be further investigated.

6. Constants $\langle \varpi_P, \alpha_w^\vee \rangle$

Very special Weyl elements are these which appeared in (2) giving non-trivial contributions to $v_{G,P}$. To calculate $v_{G,P}$ for $G = E_{6,7,8}$, we here calculate the constant $\langle \varpi_P, \alpha_w^\vee \rangle$. Here, for each very special Weyl element w, α_w is the only element in $w^{-1}\Delta \smallsetminus \Phi_P$.

Proposition 6. *For all very special elements $w \in W^\ddagger$,*

$$\langle \varpi_P, \alpha_w^\vee \rangle = 1.$$

Proof. Fix a very special Weyl element $w \in W^\ddagger$. Note that $\langle \cdot, \cdot \rangle$ is w-invariant, it is sufficient to calculate $\langle w\varpi_P, \beta_w^\vee \rangle$ where β_w is the unique element in $\Delta \smallsetminus w\Phi_P$. Our calculation then shows that

Lemma 7. *For all $w \in W_{n,P}^\ddagger$, when applicable,*

$$\begin{aligned} w\varpi_1 &= (0,0,0,0,1,1,1,3), & w\varpi_2 &= (0,0,0,1,1,1,1,4), \\ w\varpi_3 &= (0,0,1,1,1,1,1,5), & w\varpi_4 &= (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{2}), \\ w\varpi_5 &= (-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{7}{2}), & w\varpi_6 &= (0,0,0,0,0,0,0,2), \\ w\varpi_7 &= (0,0,0,0,0,1,1,2), & w\varpi_8 &= (0,0,0,0,0,0,1,1) \end{aligned}$$

To go further, we calculate β_w which is listed in the program as the set titled 'com'. Finally, we verify the following

$$\langle w\varpi_P, \beta_w^\vee \rangle = 1 \qquad \forall w \in W_{n,P}^\ddagger.$$

□

7. Constants $v_{E_n,P}, n = 6,7,8$

Recall that, see e.g. Equation (2) in §4

$$v_{G,P} = \sum_{\substack{w \in \mathfrak{W}_P^\ddagger \\ |(w^{-1}\Delta) \smallsetminus \Phi_P| = 1}} \frac{B_{P,w}}{\langle \varpi_P, \alpha_w^\vee \rangle}. \tag{3}$$

where

$$B_{P,w} := \widehat{\zeta}(2)^{|\Delta_P \cap w^{-1}\Phi^+|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \smallsetminus \Delta_P)} \frac{1}{\langle \rho, \alpha^\vee \rangle - 1} \prod_{\alpha \in \Phi_P^+ \smallsetminus \Delta_P} \widehat{\zeta}(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w}).$$

By Proposition 6 in §6, $\langle \varpi_P, \alpha_w^\vee \rangle = 1$. Hence, only $B_{P,w}$ becomes essential now. Thus to calculate the constants $v_{E_n,P}, n = 6,7,8$ for all associated maximal parabolic subgroups, we need to further calculate, for each very special $w \in W^\ddagger$,

(A) The cardinal number of $\Delta_P \cap w^{-1}\Phi^+$;
(B) The product $\prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \smallsetminus \Delta_P)} \frac{1}{\langle \rho, \alpha^\vee \rangle - 1}$;
(C) The constant $\delta_{\alpha,w}$ for $\sigma \in \Phi^+ \smallsetminus \Delta_P$;
(D) The product $\prod_{\alpha \in \Phi_P^+ \smallsetminus \Delta_P} \widehat{\zeta}(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w})$.

All this is done in the Appendix at the end of this paper. With all this, now we are ready to state explicit formulas for the constants $v_{E_n,P}, n = 6,7,8$ obtained using Mathematica.

Theorem 8. *The constant $v_{G,P}$ for $G = E_n, n = 6,7,8$ is given by the following lists:*

(E_8, P_1)

$$\begin{aligned} v_{E_8,P_1} =& -\frac{1}{3628800}\widehat{\zeta}(2)^5\widehat{\zeta}(3)^4(3+4\widehat{\zeta}(2)(-2+3\widehat{\zeta}(3)))\widehat{\zeta}(4)^3\widehat{\zeta}(5)^2\widehat{\zeta}(6)\widehat{\zeta}(7)\\ &(9450+5040\widehat{\zeta}(2)^3(-9+10\widehat{\zeta}(3))+60\widehat{\zeta}(2)^2(1435+72\widehat{\zeta}(3)(-21+10\widehat{\zeta}(4)))\\ &+7\widehat{\zeta}(2)(27\widehat{\zeta}(3)(225+16\widehat{\zeta}(4)(-19+25\widehat{\zeta}(5)))\\ &\quad+200(-37+27\widehat{\zeta}(4)\widehat{\zeta}(6)(\widehat{\zeta}(4)-8\widehat{\zeta}(5)\widehat{\zeta}(8)))))\end{aligned}$$

(E_8, P_2)

$$\begin{aligned} v_{E_8,P_2} =& \frac{1}{51840}\widehat{\zeta}(2)^5\widehat{\zeta}(3)^3\widehat{\zeta}(4)(9+18\widehat{\zeta}(2)^2-4\widehat{\zeta}(2)(8+9\widehat{\zeta}(3)(-1+2\widehat{\zeta}(4))))\\ &(-45+\widehat{\zeta}(2)(200+8\widehat{\zeta}(2)(-25+36\widehat{\zeta}(3))-9\widehat{\zeta}(3)(25+16\widehat{\zeta}(4)(-2+5\widehat{\zeta}(5)))))\end{aligned}$$

(E_8, P_3)

$$\begin{aligned} v_{E_8,P_3} =& -\frac{1}{17280}\widehat{\zeta}(2)^4(-1+2\widehat{\zeta}(2))\widehat{\zeta}(3)^2(3+4\widehat{\zeta}(2)(-2+3\widehat{\zeta}(3)))\widehat{\zeta}(4)\\ &(-45+\widehat{\zeta}(2)(200+8\widehat{\zeta}(2)(-25+36\widehat{\zeta}(3))-9\widehat{\zeta}(3)(25+16\widehat{\zeta}(4)(-2+5\widehat{\zeta}(5)))))\end{aligned}$$

(E_8, P_4)

$$\begin{aligned} v_{E_8,P_4} =& -\frac{1}{12700800}\widehat{\zeta}(2)^6\widehat{\zeta}(3)^5\widehat{\zeta}(4)^4\widehat{\zeta}(5)^3\widehat{\zeta}(6)^2\widehat{\zeta}(7)\\ &(99225+2\widehat{\zeta}(2)(99225\widehat{\zeta}(2)^3+44100(-8+9\widehat{\zeta}(3))\\ &-1568\widehat{\zeta}(2)^2(425+18\widehat{\zeta}(3)(-46+24\widehat{\zeta}(3)+25\widehat{\zeta}(4)))\\ &+42\widehat{\zeta}(2)(19600+45\widehat{\zeta}(3)^2(385+12\widehat{\zeta}(4)(-64+35\widehat{\zeta}(4)+70\widehat{\zeta}(5)))\\ &+24\widehat{\zeta}(3)(-1540+\widehat{\zeta}(4)(1568+25\widehat{\zeta}(5)(-64+63\widehat{\zeta}(6)))))\\ &-144\widehat{\zeta}(3)\widehat{\zeta}(4)(3528+25\widehat{\zeta}(5)(-196+9\widehat{\zeta}(6)(32+49\widehat{\zeta}(7)(-1+4\widehat{\zeta}(8)))))))\end{aligned}$$

(E_8, P_5)

$$\begin{aligned} v_{E_8,P_5} =& \frac{1}{1814400}\widehat{\zeta}(2)^5(-1+2\widehat{\zeta}(2))\widehat{\zeta}(3)^4\widehat{\zeta}(4)^3\widehat{\zeta}(5)^2\widehat{\zeta}(6)\\ &(14175+\widehat{\zeta}(2)(11025(-8+9\widehat{\zeta}(3))+4(882\widehat{\zeta}(2)^2(-25+36\widehat{\zeta}(3))\\ &+5\widehat{\zeta}(2)(8330-9\widehat{\zeta}(3)(15\widehat{\zeta}(3)(-49+96\widehat{\zeta}(4))+2(833+16\widehat{\zeta}(4)(-49+45\widehat{\zeta}(5)))))\\ &+36\widehat{\zeta}(3)\widehat{\zeta}(4)(-882+25\widehat{\zeta}(5)(49+36\widehat{\zeta}(6)(-2+7\widehat{\zeta}(7)))))))\end{aligned}$$

(E_8, P_6)

$$
\begin{aligned}
v_{E_8,P_6} =& \frac{1}{4191264000}\widehat{\zeta}(2)^6\widehat{\zeta}(3)^6\widehat{\zeta}(4)^5\widehat{\zeta}(5)^5\widehat{\zeta}(6)^4\widehat{\zeta}(7)^3\widehat{\zeta}(8)^2\widehat{\zeta}(9)^2\widehat{\zeta}(10)\widehat{\zeta}(11) \\
&(1164240\widehat{\zeta}(2)^4(-215+264\widehat{\zeta}(3)) + 41580\widehat{\zeta}(2)^3(14980+3\widehat{\zeta}(3)(-6139+3840\widehat{\zeta}(4))) \\
&\quad - 10914750(3+16\widehat{\zeta}(3)^2\widehat{\zeta}(4)\widehat{\zeta}(5)) \\
&\quad + 140\widehat{\zeta}(2)^2(-14850\widehat{\zeta}(3)^2(126+\widehat{\zeta}(4)(-387+224\widehat{\zeta}(4)+252\widehat{\zeta}(5))) \\
&\quad - 99\widehat{\zeta}(3)(-58135+48\widehat{\zeta}(4)(1487-225\widehat{\zeta}(5)+630\widehat{\zeta}(4)\widehat{\zeta}(6))) \\
&\quad + 175(-24431+72\widehat{\zeta}(4)\widehat{\zeta}(6)(187\widehat{\zeta}(4)-216\widehat{\zeta}(5)\widehat{\zeta}(8)))) \\
&\quad + 99\widehat{\zeta}(2)(6048000\widehat{\zeta}(3)^2\widehat{\zeta}(4)\widehat{\zeta}(5) \\
&\qquad + \widehat{\zeta}(3)(-2149875+16\widehat{\zeta}(4)(202419-25\widehat{\zeta}(5)(2695+72\widehat{\zeta}(6)(-72+245\widehat{\zeta}(7))))) \\
&\quad - 9800(-245+9\widehat{\zeta}(4)\widehat{\zeta}(6)(15\widehat{\zeta}(4) \\
&\qquad - 8(-5\widehat{\zeta}(6)\widehat{\zeta}(9)\widehat{\zeta}(10)+3\widehat{\zeta}(8)(\widehat{\zeta}(5)+20\widehat{\zeta}(7)\widehat{\zeta}(10)\widehat{\zeta}(12)))))))
\end{aligned}
$$

(E_8, P_7)

$$
\begin{aligned}
v_{E_8,P_7} =& -\frac{1}{6652800}\widehat{\zeta}(2)^5(-1+2\widehat{\zeta}(2))\widehat{\zeta}(3)^5\widehat{\zeta}(4)^5\widehat{\zeta}(5)^4\widehat{\zeta}(6)^3\widehat{\zeta}(7)^3\widehat{\zeta}(8)^2\widehat{\zeta}(9)\widehat{\zeta}(10)\widehat{\zeta}(11) \\
&(-51975+44\widehat{\zeta}(2)^3(10255+48\widehat{\zeta}(3)(-448+225\widehat{\zeta}(3))) \\
&\quad - 440\widehat{\zeta}(2)^2(1540+3\widehat{\zeta}(3)(-819+210\widehat{\zeta}(3)+8\widehat{\zeta}(4)(39-70\widehat{\zeta}(5)))) \\
&+ 7\widehat{\zeta}(2)(47300+9(\widehat{\zeta}(3)(-4950+32\widehat{\zeta}(4)(132+25\widehat{\zeta}(5)(-11+6\widehat{\zeta}(6)))) \\
&\quad - 275\widehat{\zeta}(6)(3\widehat{\zeta}(4)^2-32\widehat{\zeta}(4)\widehat{\zeta}(5)\widehat{\zeta}(8)+192\widehat{\zeta}(5)\widehat{\zeta}(8)\widehat{\zeta}(9)\widehat{\zeta}(12)))))
\end{aligned}
$$

(E_8, P_8)

$$
\begin{aligned}
v_{E_8,P_8} =& \frac{1}{22054032000}\widehat{\zeta}(2)^6\widehat{\zeta}(3)^6\widehat{\zeta}(4)^6\widehat{\zeta}(5)^6\widehat{\zeta}(6)^5\widehat{\zeta}(7)^5\widehat{\zeta}(8)^4 \\
&\quad \widehat{\zeta}(9)^4\widehat{\zeta}(10)^3\widehat{\zeta}(11)^3\widehat{\zeta}(12)^2\widehat{\zeta}(13)^2\widehat{\zeta}(14)\widehat{\zeta}(15)\widehat{\zeta}(16)\widehat{\zeta}(17) \\
&(-172297125+91891800\widehat{\zeta}(2)^4(-9+10\widehat{\zeta}(3)) \\
&\quad - 24310\widehat{\zeta}(2)^3(-122605+36\widehat{\zeta}(3)(6153-3200\widehat{\zeta}(4)+150\widehat{\zeta}(3)(-20+21\widehat{\zeta}(4)))) \\
&\quad + 170\widehat{\zeta}(2)^2(-1081080\widehat{\zeta}(3)^2(10+3\widehat{\zeta}(4)(-5+4\widehat{\zeta}(5))) \\
&\qquad - 117\widehat{\zeta}(3)(-252175+16\widehat{\zeta}(4)(13046+175\widehat{\zeta}(5)(-55+36\widehat{\zeta}(6)))) \\
&\quad + 3850(-4745+36\widehat{\zeta}(4)\widehat{\zeta}(6)(13\widehat{\zeta}(4)-72\widehat{\zeta}(5)\widehat{\zeta}(8)))) \\
&- 39\widehat{\zeta}(2)(51\widehat{\zeta}(3)(606375+16\widehat{\zeta}(4)(-40887+25\widehat{\zeta}(5)(1925+18\widehat{\zeta}(6)(-119+110\widehat{\zeta}(7))))) \\
&\quad + 1925(-16660+3\widehat{\zeta}(6)(765\widehat{\zeta}(4)^2+288\widehat{\zeta}(4)\widehat{\zeta}(8)(-17\widehat{\zeta}(5)+20\widehat{\zeta}(6)\widehat{\zeta}(10)) \\
&\qquad + 5440\widehat{\zeta}(8)\widehat{\zeta}(12)(\widehat{\zeta}(5)\widehat{\zeta}(9)-18\widehat{\zeta}(10)\widehat{\zeta}(14)\widehat{\zeta}(18))))))
\end{aligned}
$$

(E_7, P_1)

$$\begin{aligned} v_{E_7,P_1} =& -\frac{1}{604800}\widehat{\zeta}(2)^4(-1+2\widehat{\zeta}(2))\widehat{\zeta}(3)^4\widehat{\zeta}(4)^3\widehat{\zeta}(5)^2\widehat{\zeta}(6)\widehat{\zeta}(7) \\ & (5040\widehat{\zeta}(2)^3(-9+10\widehat{\zeta}(3)) + 378(25-32\widehat{\zeta}(3)^2\widehat{\zeta}(4)) + 60\widehat{\zeta}(2)^2(1435+72\widehat{\zeta}(3)(-21+10\widehat{\zeta}(4))) \\ & + 35\widehat{\zeta}(2)(27\widehat{\zeta}(3)(45+16\widehat{\zeta}(4)(-3+5\widehat{\zeta}(5))) + 40(-37+27\widehat{\zeta}(4)\widehat{\zeta}(6)(\widehat{\zeta}(4)-8\widehat{\zeta}(5)\widehat{\zeta}(8))))) \end{aligned}$$

(E_7, P_2)

$$\begin{aligned} v_{E_7,P_2} =& -\frac{1}{8640}\widehat{\zeta}(2)^4\widehat{\zeta}(3)^2(3+4\widehat{\zeta}(2)(-2+3\widehat{\zeta}(3)))\widehat{\zeta}(4) \\ & (-45+\widehat{\zeta}(2)(200+8\widehat{\zeta}(2)(-25+36\widehat{\zeta}(3)) - 9\widehat{\zeta}(3)(25+16\widehat{\zeta}(4)(-2+5\widehat{\zeta}(5))))) \end{aligned}$$

(E_7, P_3)

$$\begin{aligned} v_{E_7,P_3} =& -\frac{1}{1728}\widehat{\zeta}(2)^3(-1+2\widehat{\zeta}(2))\widehat{\zeta}(3)(3+4\widehat{\zeta}(2)(-2+3\widehat{\zeta}(3))) \\ & (9+18\widehat{\zeta}(2)^2 - 4\widehat{\zeta}(2)(8+9\widehat{\zeta}(3)(-1+2\widehat{\zeta}(4)))) \end{aligned}$$

(E_7, P_4)

$$\begin{aligned} v_{E_7,P_4} =& \frac{1}{907200}\widehat{\zeta}(2)^5\widehat{\zeta}(3)^4\widehat{\zeta}(4)^3\widehat{\zeta}(5)^2\widehat{\zeta}(6) \\ & (14175+\widehat{\zeta}(2)(11025(-8+9\widehat{\zeta}(3)) + 4(882\widehat{\zeta}(2)^2(-25+36\widehat{\zeta}(3)) \\ & \quad + 5\widehat{\zeta}(2)(8330-9\widehat{\zeta}(3)(15\widehat{\zeta}(3)(-49+96\widehat{\zeta}(4)) + 2(833+16\widehat{\zeta}(4)(-49+45\widehat{\zeta}(5))))) \\ & + 36\widehat{\zeta}(3)\widehat{\zeta}(4)(-882+25\widehat{\zeta}(5)(49+36\widehat{\zeta}(6)(-2+7\widehat{\zeta}(7))))))) \end{aligned}$$

(E_7, P_5)

$$\begin{aligned} v_{E_7,P_5} =& \frac{1}{43200}\widehat{\zeta}(2)^4(-1+2\widehat{\zeta}(2))\widehat{\zeta}(3)^3\widehat{\zeta}(4)^2\widehat{\zeta}(5) \\ & (-675+2\widehat{\zeta}(2)(675\widehat{\zeta}(2)^2 - 40\widehat{\zeta}(2)(65+9\widehat{\zeta}(3)(-12+5\widehat{\zeta}(3)+10\widehat{\zeta}(4))) \\ & + 9(200+\widehat{\zeta}(3)(-225+16\widehat{\zeta}(4)(18+25\widehat{\zeta}(5)(-1+3\widehat{\zeta}(6))))))) \end{aligned}$$

(E_7, P_6)

$$\begin{aligned} v_{E_7,P_6} =& \frac{1}{9072000}\widehat{\zeta}(2)^5\widehat{\zeta}(3)^5\widehat{\zeta}(4)^4\widehat{\zeta}(5)^4\widehat{\zeta}(6)^2\widehat{\zeta}(7)^2\widehat{\zeta}(8)\widehat{\zeta}(9) \\ & (141750+453600\widehat{\zeta}(2)^4 - 1800\widehat{\zeta}(2)^3(833+15\widehat{\zeta}(3)(-35+48\widehat{\zeta}(4))) \\ & \quad - 500\widehat{\zeta}(2)^2(-3773+18\widehat{\zeta}(3)(245-328\widehat{\zeta}(4)+42\widehat{\zeta}(3)(-2+3\widehat{\zeta}(4))) + 2016\widehat{\zeta}(4)^2\widehat{\zeta}(6)) \\ & - 21\widehat{\zeta}(2)(43000+\widehat{\zeta}(3)(-37125+48\widehat{\zeta}(4)(1197+25\widehat{\zeta}(5)(-25+72\widehat{\zeta}(6)))) \\ & \quad - 5400\widehat{\zeta}(4)\widehat{\zeta}(6)(5\widehat{\zeta}(4)-8\widehat{\zeta}(8)(\widehat{\zeta}(5)-10\widehat{\zeta}(6)\widehat{\zeta}(10))))) \end{aligned}$$

(E_7, P_7)

$$v_{E_7,P_7} = \frac{1}{3326400}\widehat{\zeta}(2)^5\widehat{\zeta}(3)^5\widehat{\zeta}(4)^5\widehat{\zeta}(5)^4\widehat{\zeta}(6)^3\widehat{\zeta}(7)^3\widehat{\zeta}(8)^2\widehat{\zeta}(9)\widehat{\zeta}(10)\widehat{\zeta}(11)$$
$$(51975 + \widehat{\zeta}(2)(3850(-86 + 81\widehat{\zeta}(3)) - 44\widehat{\zeta}(2)^2(10255 + 48\widehat{\zeta}(3)(-448 + 225\widehat{\zeta}(3)))$$
$$+ 440\widehat{\zeta}(2)(1540 + 3\widehat{\zeta}(3)(-819 + 210\widehat{\zeta}(3) + 8\widehat{\zeta}(4)(39 - 70\widehat{\zeta}(5))))$$
$$+ 63(-32\widehat{\zeta}(3)\widehat{\zeta}(4)(132 + 25\widehat{\zeta}(5)(-11 + 6\widehat{\zeta}(6)))$$
$$+ 275\widehat{\zeta}(6)(3\widehat{\zeta}(4)^2 - 32\widehat{\zeta}(4)\widehat{\zeta}(5)\widehat{\zeta}(8) + 192\widehat{\zeta}(5)\widehat{\zeta}(8)\widehat{\zeta}(9)\widehat{\zeta}(12)))))$$

(E_6, P_1)

$$v_{E_6,P_1} = -\frac{1}{302400}\widehat{\zeta}(2)^4\widehat{\zeta}(3)^4\widehat{\zeta}(4)^3\widehat{\zeta}(5)^2\widehat{\zeta}(6)\widehat{\zeta}(7)$$
$$(9450 + 5040\widehat{\zeta}(2)^3(-9 + 10\widehat{\zeta}(3)) + 60\widehat{\zeta}(2)^2(1435 + 72\widehat{\zeta}(3)(-21 + 10\widehat{\zeta}(4)))$$
$$+ 7\widehat{\zeta}(2)(27\widehat{\zeta}(3)(225 + 16\widehat{\zeta}(4)(-19 + 25\widehat{\zeta}(5))) + 200(-37 + 27\widehat{\zeta}(4)\widehat{\zeta}(6)(\widehat{\zeta}(4) - 8\widehat{\zeta}(5)\widehat{\zeta}(8)))))$$

(E_6, P_2)

$$v_{E_6,P_2} = -\frac{1}{1440}\widehat{\zeta}(2)^3(-1 + 2\widehat{\zeta}(2))\widehat{\zeta}(3)^2\widehat{\zeta}(4)$$
$$(-45 + \widehat{\zeta}(2)(200 + 8\widehat{\zeta}(2)(-25 + 36\widehat{\zeta}(3)) - 9\widehat{\zeta}(3)(25 + 16\widehat{\zeta}(4)(-2 + 5\widehat{\zeta}(5)))))$$

(E_6, P_3)

$$v_{E_6,P_3} = \frac{1}{288}\widehat{\zeta}(2)^2(-1 + 2\widehat{\zeta}(2))(3 + 4\widehat{\zeta}(2)(-2 + 3\widehat{\zeta}(3)))^2$$

(E_6, P_4)

$$v_{E_6,P_4} = \frac{1}{21600}\widehat{\zeta}(2)^4\widehat{\zeta}(3)^3\widehat{\zeta}(4)^2\widehat{\zeta}(5)$$
$$(-675 + 2\widehat{\zeta}(2)(675\widehat{\zeta}(2)^2 - 40\widehat{\zeta}(2)(65 + 9\widehat{\zeta}(3)(-12 + 5\widehat{\zeta}(3) + 10\widehat{\zeta}(4)))$$
$$+ 9(200 + \widehat{\zeta}(3)(-225 + 16\widehat{\zeta}(4)(18 + 25\widehat{\zeta}(5)(-1 + 3\widehat{\zeta}(6)))))))$$

(E_6, P_5)

$$v_{E_6,P_5} = -\frac{1}{1440}\widehat{\zeta}(2)^3(-1 + 2\widehat{\zeta}(2))\widehat{\zeta}(3)^2\widehat{\zeta}(4)$$
$$(-45 + \widehat{\zeta}(2)(200 + 8\widehat{\zeta}(2)(-25 + 36\widehat{\zeta}(3)) - 9\widehat{\zeta}(3)(25 + 16\widehat{\zeta}(4)(-2 + 5\widehat{\zeta}(5)))))$$

(E_6, P_6)

$$v_{E_6,P_6} = -\frac{1}{302400}\widehat{\zeta}(2)^4\widehat{\zeta}(3)^4\widehat{\zeta}(4)^3\widehat{\zeta}(5)^2\widehat{\zeta}(6)\widehat{\zeta}(7)$$
$$(9450 + 5040\widehat{\zeta}(2)^3(-9 + 10\widehat{\zeta}(3)) + 60\widehat{\zeta}(2)^2(1435 + 72\widehat{\zeta}(3)(-21 + 10\widehat{\zeta}(4)))$$
$$+ 7\widehat{\zeta}(2)(27\widehat{\zeta}(3)(225 + 16\widehat{\zeta}(4)(-19 + 25\widehat{\zeta}(5)))$$
$$+ 200(-37 + 27\widehat{\zeta}(4)\widehat{\zeta}(6)(\widehat{\zeta}(4) - 8\widehat{\zeta}(5)\widehat{\zeta}(8)))))$$

Corollary 9. *$v_{E_n,p} \neq 0$ for all $n = 6, 7, 8$ and their maximal parabolic subgroups.*

Proof. This is simple. For example, with (E_8, P_1), we have

$$\begin{aligned} v_{E_8,P_1} = & -\frac{(\pi^3\zeta(3)^4\cdot(4\pi-3(3+\zeta(3)))\cdot\zeta(5)^2\cdot\zeta(7))}{75243740696496046080000000} \\ & \cdot(-567000\pi^3(-63+2\zeta(3))+7938000\pi(185+27\zeta(3)) \\ & -79380\pi^2(5125+136\zeta(3))+8\pi^8(-35+12\zeta(5)) \\ & -4465125(360+\zeta(3)(135+2\zeta(5)))) \end{aligned}$$

Approximately,

$$v_{E_8,P_1} = \frac{206392.}{75243740696496046080000000} \neq 0.$$

Our verifications for other cases are the same. □

Consequently, by the theorem recalled in §4, we have proved the following

Theorem 10. *All but finitely many zeros of* $\widehat{\zeta}_{\mathbb{Q}}^{E_8/P_1}(s)$ *are simple and lie on the line* $\mathrm{Re}(s) = -7$.

Remark: The structures for $W^{\ddagger}$ and $v_{E_n,P}$ as stated above is strikingly uniform. Besides that their cardinal numbers are 2^n, the number of non-trivial factors of v_{E_n,P_i} in fact is the same as the number of connected components of the Dynkin diagram by deleting the i-th simple root. Furthermore, there is a nice symmetry hidden in our formula. For example,

$$v_{E_6,P_1} = v_{E_6,P_6}, \qquad v_{E_6,P_2} = v_{E_6,P_5}.$$

All this is not just a coincidence: they are dominated a force exposed in our conjecture on Stability, Parabolic Reduction and Masses. For example, from $|W^{\ddagger}| = s^{n-1}$, we know that, from definition, $v_{E_n,P}$ consists of 2^{n-1} terms. This is exactly the number appeared in the Arthur's analytic truncation theory: Set of all standard parabolic subgroups is bijectively correspondence to a power set. For more details, please refer to our works.[1]

[1]This will be extended. With 16G data obtained from the calculation of a half year long, I am simply exhausted.

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