Non-Abelian Zeta Functions for Elliptic Curves and Their Zeros

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Outline

Stability

Setting

- $X/\mathbb{F}_q$: irreducible, reduced, regular proj curve
- g: genus of X
- F: function field, $\mathbb{A}$: adelic ring
- $\mathbb{M}_{X,n}$: moduli stack of rank n bdles of on X
  $$\simeq GL(n,F)\backslash GL(n,\mathbb{A})/\mathbb{K}$$
- $\mathbb{M}_{X,n}^{\mathrm{ss}}$: moduli stack of s.stable bdles of rank n
- $m_{X,n}(d) := \sum_{V\in\mathbb{M}_{X,n}(d)} \frac{1}{\#\mathrm{Aut}(V)}$, independent on degree d
- $m_{X,n}^{\mathrm{ss}}(d) := \sum_{V\in\mathbb{M}_{X,n}^{\mathrm{ss}}(d)} \frac{1}{\#\mathrm{Aut}(V)}$, dependent on degree d
- $\widehat{\zeta}_X(s)$: complete Artin zeta function of X
- If $n = n_1 + n_2 + \cdots + n_k$, $N_i := n_1 + \cdots + n_i$, $N_i' := n - N_i$

Mumford's Intersection Stability

Stability

V/X: vector bundle

V: semi-stable $\Leftrightarrow$

$$\frac{\deg(V_1)}{\mathrm{rank}(V_1)} \leq \frac{\deg(V)}{\mathrm{rank}(V)}, \qquad \forall V_1 \leq V$$

Various Spaces

- Moduli spaces do not work well
- Fat moduli spaces work well
- The best is adelic space

Tamagawa Number, Parabolic Reduction

Facts: Tamagawa Number, Parabolic Reduction

- (Weil)

$$m_{X,n}(d)=\widehat{\zeta}_X(1)\widehat{\zeta}_X(2)\cdots\widehat{\zeta}_X(n).$$

- Harder-Narasimhan, Desale-Ramanan and Zagier

$$\frac{m_{X,n}^{\mathrm{ss}}(0)}{q^{\frac{n(n-1)}{2}(g-1)}}=\sum_{k=1}^{n}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\frac{(-1)^{k-1}}{\prod_{j=1}^{k-1}(q^{n_j+n_{j+1}}-1)}\cdot\prod_{j=1}^{k}m_{X,n_j}(0).$$

- Zagier (motivated by Weng)

$$n\cdot m_{X,n}(d)=\sum_{k=1}^{n}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\sum_{\substack{\delta_i\in\{0,1,\dots,n_i-1\}\\ i=1,2,\dots,k}}\prod_{i=1}^{k-1}\frac{q^{v_iN_iN_i'}}{q^{N_iN_i'}-1}\cdot\prod_{j=1}^{k}\frac{m_{X,n_j}^{\mathrm{ss}}(\delta_j)}{q^{\frac{n_j(n_j-1)}{2}(g-1)}}.$$

Here $v_i\in[0,1)\cap\mathbb{Q}$ satisfying $v_i\equiv\frac{\delta_i}{n_i}-\frac{\delta_{i+1}}{n_{i+1}}\pmod 1$

Stable Lattices

Definition

- $\Lambda \subset \mathbb{R}^n$: rank n lattice.
- Λ semi-stable if

$$\Big(\mathrm{Vol}\,\Lambda_1\Big)^{\mathrm{rank}(\Lambda)} \geq \Big(\mathrm{Vol}\,\Lambda\Big)^{\mathrm{rank}(\Lambda_1)}, \qquad \forall \Lambda_1 \subset \Lambda.$$

- $\mathcal{M}_{\mathbb{Q},n}[1]$: moduli space of rank n lattices of vol 1 $\simeq SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n)$
- $\mathcal{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]$: (compact) subspace of s.stable lattices
- $v_{\mathbb{Q},n} := \mathrm{Vol}\Big(\mathcal{M}_{\mathbb{Q},n}[1]\Big)$
- $v_{\mathbb{Q},n}^{\mathrm{ss}} := \mathrm{Vol}\Big(\mathcal{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]\Big)$
- $\widehat{\zeta}_{\mathbb{Q}}(s)$: complete Riemann zeta function

Volume of Fund Domain, Parabolic Reduction

Facts

- (Siegel)

$$\frac{1}{n} \cdot v_{\mathbb{Q},n} = \widehat{\zeta}_{\mathbb{Q}}(1)\widehat{\zeta}_{\mathbb{Q}}(2)\cdots\widehat{\zeta}_{\mathbb{Q}}(n).$$

- (Weng, related to Kim-Weng)

$$v_{\mathbb{Q},n}^{\mathrm{ss}} = \sum_{k=1}^{n}(-1)^{k-1} \sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}} \frac{1}{\prod_{j=1}^{k-1}(n_j+n_{j+1})} \cdot \prod_{j=1}^{k} v_{\mathbb{Q},n_j}.$$

- (Kontsevich-Soibelman)

$$\frac{1}{n} \cdot v_{\mathbb{Q},n} = \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}} \frac{1}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_k)\cdots n_k} \cdot \prod_{j=1}^{k} v_{\mathbb{Q},n_j}^{\mathrm{ss}}.$$

Split Reductive Groups

Setting

- $F = \mathbb{Q}$: field of rational, $\mathbb{A}$: ring of adeles
- G/F: split reductive group
- B/F: Borel, P/F: standard parabolic subgroup
- $P = M \cdot N$: Levi decomposition w/ M Levi factor
- $M \sim \prod_{i=1}^{k} M_i$ simple decomposition w/ M_i's reductive
- $\mathcal{M}_{F,G}$: moduli space of G-lattices
  $$\simeq G(F)Z_{G(\mathbb{A})}\backslash G(\mathbb{A})^1/\mathbb{K}$$
- $\mathcal{M}_{F,G}^{\mathrm{ss}}$: (compact) subspace of s.stable G-lattices
- $v_{F,G} := \mathrm{Vol}\Big(\mathcal{M}_{F,G}\Big)$
- $v_{F,G}^{\mathrm{ss}} := \mathrm{Vol}\Big(\mathcal{M}_{F,G}^{\mathrm{ss}}\Big)$

Volumes of Fundamental Domains

Fact

- (Langlands)
$$v_{F,G} = c_G \cdot \prod_{i \geq 1} \widehat{\zeta}_F(i)^{-n_i(G)}.$$
Here $c_G = \mathrm{Vol}\Big(\Big\{ \sum_{\alpha\in\Delta} a_\alpha \alpha^\vee : a_\alpha \in [0,1]\Big\}\Big)$
$n_i(G) := \#\{\alpha > 0, \langle \rho, \alpha^\vee \rangle = i\} - \#\{\alpha > 0, \langle \rho, \alpha^\vee\rangle = i-1\}$
with Δ: simple roots, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$

Done by taking residues of Eisenstein series

Parabolic Reduction, Stability & the Volumes

Theorem? (Weng)

- Parabolic Reduction $\exists\, c_P \in \mathbb{Q}_{>0},\, e_P \in \mathbb{Q}_{>0}$

$$v_{F,G} = \sum_P e_P \cdot v_{F,P}^{\mathrm{ss}},$$

$$v_{F,G}^{\mathrm{ss}} = \sum_P \mathrm{sgn}(P) \cdot c_P \cdot v_{F,P}.$$

Here $\quad P = M \cdot N, \qquad M \sim \prod_j M_j$

and $\quad v_{F,P} := \prod_j v_{F,M_j}, \qquad v_{F,P}^{\mathrm{ss}} := \prod_j v_{F,M_j}^{\mathrm{ss}}$

• c_P, but not e_P: explicit expressions in terms of root system involved, related to Kim-Weng.

Non-Abelian Zeta Function

Definition (Weng)

Pure Non-Abelian Zeta Function of $X/\mathbb{F}_q$:

$$\widehat{\zeta}_{X,n}(s) := \sum_{V \in \mathbb{M}_{X,n}^{\mathrm{ss}}(n\mathbb{Z})} \frac{q^{h^0(X,V)} - 1}{\#\mathrm{Aut}(V)} \cdot (q^{-s})^{\chi(X,V)}\, d\mu, \quad \Re(s) > 1.$$

α, β-invariants

$$\beta_{X,n}(d) := \sum_{V \in \mathbb{M}_{X,n}^{\mathrm{ss}}(d)} \frac{1}{\#\mathrm{Aut}(V)} = m_{X,n}^{\mathrm{ss}}(d),$$

$$\alpha_{X,n}(d) := \sum_{V \in \mathbb{M}_{X,n}^{\mathrm{ss}}(d)} \frac{q^{h^0(X,V)} - 1}{\#\mathrm{Aut}(V)}$$

Zeta Facts

Then with $t = q^{-s}, T = t^n, Q = q^n$

$$\widehat{\zeta}_{X,n}(s) = \sum_{m=0}^{(g-1)-1} \alpha_{X,n}(mn) \cdot \Big(T^m + Q^{(g-1)-m} \cdot T^{2(g-1)-m} \Big)$$
$$+ \alpha_{X,n}(n(g-1)) \cdot T^{g-1} + \beta_{X,n}(0) \cdot \frac{(Q-1)T^g}{(1-QT)(1-T)}$$

Zeta Facts (Weng)

- (Initial State) $\widehat{\zeta}_{X,1}(s) = \widehat{\zeta}_X(s)$: complete Artin zeta
- (Rationality) $\widehat{\zeta}_{X,n}(s)$ is rational in T
- (Functional Eq) $\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s)$
- (Residue) $\mathrm{Res}_{s=1}\widehat{\zeta}_{X,n}(s) = \beta_{X,n}(0)$

Number Fields versus Function Fields

Parabolic Reduction & Periods

Parabolic Reduction: (i) Number Fields

$$v_{\mathbb{Q},n}^{\mathrm{ss}} = \sum_{k=1}^{n}(-1)^{k-1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1>0,\cdots,n_k>0}} \frac{1}{\prod_{j=1}^{k-1}(n_j+n_{j+1})} \cdot \prod_{j=1}^{k} v_{\mathbb{Q},n_j}.$$

(ii) Function Fields/$\mathbb{F}_q$

$$m_{E,n}^{\mathrm{ss}}(0) = \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1>0,\cdots,n_k>0}} \frac{(-1)^{k-1}}{\prod_{j=1}^{k-1}(q^{n_j+n_{j+1}}-1)} \cdot \prod_{j=1}^{k} m_{E,n_j}(0).$$

Periods of G: Number Fields (Weng, related to JLR)

$$\omega_{\mathbb{Q}}^{G}(\lambda) := \sum_{w\in W}\Big(\frac{1}{\prod_{\alpha\in\Delta}\langle w\lambda-\rho,\alpha^\vee\rangle} \cdot \prod_{\alpha>0,w\alpha<0} \frac{\widehat{\zeta}_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle+1)}\Big)$$

Period of G/X

Setting

- G/F: split reductive group, B/F: Borel
- P/F: maximal standard parabolic subgroup
- Δ: Simple roots, W: Weyl group , ρ: Weyl vector
- $\alpha_P \in \Delta$: simple for P
- $\{\beta_1, \ldots, \beta_{|P|}\} = \Delta \backslash \{\alpha_P\}$

Definition

Period of G/F:

$$\omega_X^G(\lambda) := \sum_{w \in W} \left(\frac{1}{\prod_{\alpha \in \Delta}(1 - q^{-\langle w\lambda - \rho, \alpha^\vee \rangle})} \cdot \prod_{\alpha > 0, w\alpha < 0} \frac{\widehat{\zeta}_X(\langle \lambda, \alpha^\vee \rangle)}{\widehat{\zeta}_X(\langle \lambda, \alpha^\vee \rangle + 1)} \right)$$

Period of $(G,P)/F$

Definition (Weng)

Period of $(G,P)/F$:

$$\omega_X^{G/P}(s)$$
$$:= \mathrm{Res}_{\langle\lambda-\rho,\beta_{1,P}^\vee\rangle=0,\,\langle\lambda-\rho,\beta_{2,P}^\vee\rangle=0,\,\dots,\,\langle\lambda-\rho,\beta_{|G|-1,P}^\vee\rangle=0}\Big(\omega_X^G(\lambda)\Big)$$
$$:= \mathrm{Res}_{\langle\lambda-\rho,\beta_{1,P}^\vee\rangle=0,\,\langle\lambda-\rho,\beta_{2,P}^\vee\rangle=0,\,\dots,\,\langle\lambda-\rho,\beta_{|G|-1,P}^\vee\rangle=0}$$
$$\sum_{w\in W}\left(\frac{1}{\prod_{\alpha\in\Delta}(1-q^{\langle w\lambda-\rho,\alpha^\vee\rangle})}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle+1)}\right)$$

- Various symmetries play key roles here.

Special Uniformity of Zetas

Definition (Weng)

Zeta function $\widehat{\zeta}_X^{G/P}$ for $(G,P)/F$:

$$\widehat{\zeta}_X^{G/P}(s) := \mathrm{Norm}\Big(\omega_X^{G/P}(s)\Big)$$

Zeta Facts (Weng)

- Functional Equation (Related to Komori)
$$\widehat{\zeta}_X^{G/P}(1-s) = \widehat{\zeta}_X^{G/P}(s)$$
- Special Uniformity of Zetas

$$\widehat{\zeta}_{X,n}(s) \doteq \widehat{\zeta}_X^{SL_n/P_{n-1,1}}(s)$$

The Riemann Hypothesis

The Riemann Hypothesis

$$\widehat{\zeta}_{X,n}(s) = 0 \qquad \Rightarrow \qquad \Re(s) = \frac{1}{2}$$

RH Established

- Rank two zetas: Yoshida
- Elliptic curve: Zagier (lower ranks: Weng)

Number Fields: Weak RH Established

- $SL_{2,3,4,5}$, Sp_4, G_2: Ki, Lagarias-Suzuki, Suzuki
- $(SL_n, P_{n-1,1})$ & $\widehat{\zeta}_{\mathbb{Q},n}(s)$: Weng using Ki, Kim-Weng(?)
- General (G, P): Ki-Komori-Suzuki

  assuming Parabolic Reduction

Sato-Tate

Setting

- p: prime, $N = \#X(\mathbb{F}_p)$, $a = p + 1 - N$
- $\cos\theta^E_{1,p} := \frac{p+1-N}{2\sqrt{p}}, \qquad 0 \leq \theta^E_{1,p} < \pi$
- $\cos\theta^E_{n,p} := \dfrac{-(p^n-1)\cdot\frac{\beta_{E,n}(0)}{\beta_{E,n-1}(0)}+(p^n+1)}{2\sqrt{p^n}}$

Sato-Tate Conjecture

For non CM elliptic curves E, and $0 \leq \alpha < \beta \leq \pi$,

$$\lim_{x\to\infty} \frac{\#\{p\ :\ \mathrm{prime},\ p \leq x,\ \alpha \leq \theta^E_{1,p} \leq \beta\}}{\#\{p\ :\ \mathrm{prime},\ p \leq x\}} = \frac{1}{\pi}\int_\alpha^\beta \sin^2\theta\, d\theta.$$

Higher Sato-Tate

Higher Sato-Tate (Weng)

For non CM elliptic curves E, and $0 \le \alpha < \beta \le \pi$,

$$\frac{\#\left\{p : \begin{array}{l}\text{prime},\\ p \le x,\end{array} \sin\left(\frac{\pi}{2}-\alpha\right) \overset{\bullet}{\ge} \boxed{\frac{p^{\frac{n-1}{2}}}{n-1}\cdot\left(\frac{\pi}{2}-\theta_{n,p}^{E}\right)+\frac{1}{2}\left(\sqrt{p}+\frac{1}{\sqrt{p}}\right)} \overset{\bullet}{\ge} \sin\left(\frac{\pi}{2}-\beta\right)\right\}}{\#\{p : \text{prime},\ p \le x\}}$$

$$\longrightarrow \frac{1}{\pi}\int_{\alpha}^{\beta}\sin^2\theta\, d\theta, \qquad \text{as } x \to \infty$$

Sato-Tate $\Rightarrow$ Higher Sato-Tate

- (Zagier) As $q \to \infty$,

$$\frac{\beta_{E,n}(0)}{\beta_{E,n-1}(0)} = 1 + \frac{\boxed{(n-1)(q-a+1)} - 2 + 3(a-2)/q + \cdots}{q^n} + O\left(\frac{n^2}{q^{2n+2}}\right)$$

Atiyah Bundles

Atiyah Bundles

$E/\mathbb{F}_q$: elliptic curve

Inductively define Atiyah Bundle I_n/E: $I_1 = \mathcal{O}_E$

$I_n =$ the only non-trivial extension of I_{n-1} by $\mathcal{O}_E$:

$$0 \to \mathcal{O}_E \to I_n \to I_{n-1} \to 0$$

since

$$\mathrm{Ext}^1(I_n, \mathcal{O}_E) \simeq H^1(E, I_n^\vee) \simeq H^0(E, I_n^\vee)^\vee = H^0(E, I_n) \simeq \mathbb{F}_q$$

Automorphisms

Automorphisms

Consider the bundle $\oplus_{j=1}^{s} I_{r_j}^{\oplus m_j}$ $(0 < r_1 < r_2 < \cdots < r_s)$

- (Atiyah) $h^0(E, \oplus_{j=1}^{s} I_{r_j}^{\oplus m_j}) = \sum_{j=1}^{s} m_j$
- (Weng)

$$\#\mathrm{Aut}\Big(\oplus_{j=1}^{s} I_{r_j}^{\oplus m_j}\Big) = q^{2\sum_{1\leq i<j\leq s} r_i m_i m_j}$$

$$\times \prod_{j=1}^{s} (q^{m_j}-1)(q^{m_j}-q)\cdots(q^{m_j}-q^{m_j-1})q^{m_j^2(r_j-1)}.$$

Counting Miracle I

Counting Miracle for I_n (Weng||Zagier)

$$\sum_{\sum_{i=1}^{s} r_i m_i = n+1} \frac{q^{h^0(E, \oplus_{j=1}^{s} I_{r_j}^{\oplus m_j})} - 1}{\#\mathrm{Aut}\Big(\oplus_{j=1}^{s} I_{r_j}^{\oplus m_j}\Big)}$$

$$= \sum_{\sum_{i=1}^{s} r_i m_i = n} \frac{1}{\#\mathrm{Aut}\Big(\oplus_{j=1}^{s} I_{r_j}^{\oplus m_j}\Big)}$$

$$= \frac{q^{\frac{n(n-1)}{2}}}{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}.$$

Counting Miracle II

Counting Miracle (Weng || Zagier, Sugawara)

$$\alpha_{E,n+1}(0) = \sum_{V \in \mathbb{M}^{ss}_{E,n+1}(0)} \frac{q^{h^0(E,V)} - 1}{\#\mathrm{Aut}\, V} = \sum_{V \in \mathbb{M}^{ss}_{E,n}(0)} \frac{1}{\#\mathrm{Aut}\, V} = \beta_{E,n}(0)$$

Counting Miracle on Atiyah Bundles

- (Weng) CM $\Leftrightarrow$ $\mathcal{A}(x) = \mathcal{B}(x) = \sum_{n=0}^{\infty} \varepsilon(n) \left(\frac{x}{q}\right)^n$
- (Zagier)

$$\mathcal{A}(x) = \mathcal{B}(x) = \sum_{n=0}^{\infty} \varepsilon(n) \left(\frac{x}{q}\right)^n$$

where $\varepsilon(m) := \dfrac{q^{m^2}}{(q^m - 1)(q^m - q) \cdots (q^m - q^{m-1})}$ and

Counting Miracle

Counting Miracle on Atiyah Bundles

$$\mathcal{A}(x) := \sum_{s=0}^{\infty} \sum_{\substack{0<r_1<r_2<\cdots<r_s \\ m_1>0,\, m_2>0,\, \dots,\, m_s>0}} \frac{\varepsilon(m_1)\cdots\varepsilon(m_s)}{q^{N(\mathbf{r},m)}} \cdot x^{r_1 m_1+\cdots+r_s m_s},$$

$$\mathcal{B}(x) := \frac{1}{x} \sum_{s=0}^{\infty} \sum_{\substack{0<r_1<r_2<\cdots<r_s \\ m_1>0,\, m_2>0,\, \dots,\, m_s>0}} (q^{m_1+\cdots+m_s} - 1) \frac{\varepsilon(m_1)\cdots\varepsilon(m_s)}{q^{N(\mathbf{r},m)}} \times x^{r_1 m_1+\cdots+r_s m_s}$$

$$\text{with} \qquad N(\mathbf{r}, m) := \sum_{i=1}^{s} r_i m_i (m_1 + 2m_{i+1} + \cdots + 2m_s).$$

Sugahara's Result

Sugahara's Relation

$X/\mathbb{F}_q$: irreducible, reduced, regular projective curve

$$\begin{aligned}
&\alpha_{X,n+1}(0)\\
=&\sum_{V\in\mathbb{M}^{ss}_{X,n+1}(0)} \frac{q^{h^0(X,V)}-1}{\#\mathrm{Aut}\,V}\\
=&q^{n(g-1)}\cdot \sum_{V\in\mathbb{M}^{ss}_{X,n}(0)} \frac{1}{\#\mathrm{Aut}\,V}\\
=&q^{n(g-1)}\cdot \beta_{X,n}(0)
\end{aligned}$$

Motivated by the study of Quives

Beta Invariants

Basic Relations on Beta Invariants (Zagier)

(i) (Parabolic Reduction)

$$\beta_{E,n}(0)=\sum_{k=1}^{n}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\frac{(-1)^{k-1}}{\prod_{j=1}^{k-1}(q^{n_j+n_{j+1}}-1)}\cdot\prod_{j=1}^{k}\prod_{i=1}^{n_j}\zeta_E(i).$$

(ii) (Multiplicative Structure)

$$\mathcal{B}_{E/\mathbb{F}_q}(x)=:\sum_{n=0}^{\infty}\beta_{B,n}(0)x^n=\prod_{n=0}^{\infty}\frac{(1-q^nx)(1-q^{n+1}x)}{(1-q^nax+q^{2n+1}x^2)}$$

$$\overset{x=q^{-s}}{=}\prod_{n\geq 1}\zeta_E(s+n)\qquad\text{with}\qquad \#E(\mathbb{F}_q)=:q+1-a.$$

Zagier's Proof of the RH

Proof of the RH (Zagier)

(i) (Recursion)

$$\beta_{E,n}(0)=\frac{q^n+q^{n+1}-a}{q^n-1}\beta_{E,n-1}(0)-\frac{q^{n-1}-q}{q^n-1}\beta_{E,n-2}(0).$$

(ii) (The Estimation)

$$\frac{\sqrt{q^n}-1}{\sqrt{q^n}+1}\leq\frac{\beta_{E,n}(0)}{\alpha_{E,n}(0)}\leq\frac{\sqrt{q^n}+1}{\sqrt{q^n}-1}$$

(iii) (The Riemann Hypothesis)

$$\widehat{\zeta}_{E,n}(s)=0 \qquad \Rightarrow \qquad \Re(s)=\frac{1}{2}$$

Local Zetas for nodal curves

Local zeta for nodal curve/$\mathbb{F}_q$

$X/\mathbb{F}_q$: Singular but nodal curve
$\Rightarrow$ Semi-stable bundles make sense $\Rightarrow$
Zetas for Nodal Curves:

$$\widehat{\zeta}_{X,n}(s) = \alpha_{X,n}(0) \cdot \frac{P_{X,n}(q^{-ns})}{(1-q^{-ns})(1-q^{n(1-s)})}$$

$P_{X,n}(t^n)$: deg $< 2g$ polynomial in $T = t^n$ with constant term 1

Global Zetas

Global Zeta: a definition

$X/\mathbb{Q}$: regular curve
$\mathfrak{X}/\mathbb{Z}$: a semi-stable model
X_p: semi-stable reduction at p $\Rightarrow$
New Global Zetas of $\mathfrak{X}/\mathbb{Z}$:

$$\widehat{\zeta}_{\mathfrak{X},n}(s) := \Big(\Gamma_{\mathbb{R}}(ns)\Gamma_{\mathbb{R}}(n(s-1))\Big) \cdot \prod_p P_{X_p,n}^{-1}(p^{-ns})$$

Analytic and Arithmetic Properties

Central Questions

- How can we meromorphic extend them?
- What kinds of arithmetic they offer?

Thank You

Thank You

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