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The asymptotic behavior of Green's functions for quasi-hyperbolic metrics on degenerating Riemann surfaces

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In this article, we consider a family of compact Riemann surfaces of genus $q \geq 2$ degenerating to a Riemann surface of genus $q-1$ with a non-separating node. We show that the Green's functions associated to a continuous family of quasi-hyperbolic metrics on such degenerating Riemann surfaces simply degenerate to that on the smooth part of the noded Riemann surface.

1. Introduction

Given a Hermitian metric on a complex manifold, an important analytic object associated to its Laplacian is the Green's function. In this paper, we study the asymptotic behavior of the Green's functions with respect to certain Hermitian metrics on a degenerating family of Riemann surfaces.

A degenerating family of Riemann surfaces $\{M_t\}$ can be obtained by shrinking non-trivial closed loops to points to form a noded Riemann surface M. This corresponds to a path in the moduli space $\mathcal{M}_q$ of compact Riemann surfaces of genus q leading to boundary points in its Deligne-Mumford compactification $\bar{\mathcal{M}}_q$. There are essentially two cases, depending on whether any of the nodes separate M. Throughout this paper, we will assume $q \geq 2$, so that M is stable, or equivalently, its smooth part $M^0 := M \backslash \{\text{nodes}\}$ admits the hyperbolic metric of constant sectional curvature -1.

The behaviors of the Green's functions and other spectral properties with respect to various canonical metrics on degenerating Riemann surfaces have been extensively studied (see e.g. [He], [Ji1], [Ji2], [Jo], [We], [Wo1], [Wo2]). The degenerative behavior of Green's functions for Arakelov metrics was studied by Jorgenson [Jo] and Wentworth [We]. Ji [Ji1] showed that in the case of non-separating nodes, the Green's functions for the hyperbolic metrics on $\{M_t\}$ simply degenerate to that of M^0 (see [Ji1] for more precise statements and the results in the case of separating nodes). Ji's method involved a detailed study of the resolvent kernel of the hyperbolic Laplacians and depended on Hejhal's results on regular b-group theory [He], and it does not appear to generalize directly to the case of Hermitian metrics of variable curvature.

In this article, we consider Hermitian metrics which are quasi-isometric to the hyperbolic metrics. Our first result is to prove the existence and the uniqueness of Green's functions (with certain growth conditions) on punctured Riemann surfaces for Hermitian metrics 'of hyperbolic growth near the punctures' (see Theorem 1 in Section 2). Our main result is to show that in the case of a non-separating node, the Green's functions with respect to a 'continuous family of quasi-hyperbolic metrics' on $\{M_t\}$ degenerate to that on M^0, which generalizes the above result of Ji (see Theorem 2 in Section 2).

The proof of Theorem 1 is elementary, and is given in Section 3. Our proof of Theorem 2 is different from the approach in [Ji1], and is more geometric in nature. We sketch it briefly as follows. First we use the Green's function on M^0 to construct a family of functions (with singularity) to approximate the Green's functions on $\{M_t\}$. Then we give a description of the asymptotic behavior of the family of quasi-hyperbolic metrics using Wolpert's corresponding results on hyperbolic metrics [Wo2], which allows us to show that the error term in our approximation goes to 0 as $t \to 0$. Here we make essential use of the fact that there is a positive lower bound for the first nonzero eigenvalues λ_1 of the Laplacians as $t \to 0$.

At present we do not know the precise behavior of the Green's functions in the equally interesting case of separating nodes. The analysis appears to be more difficult since $\lambda_1 \to 0$ as $t \to 0$ in this case. It is likely that the behavior exhibited by the hyperbolic Green's functions in [Ji1] may also prevail in the case of separating nodes, although our geometric approach does not seem to generalize directly to this case.

2. Notation and statement of results

(2.1) Throughout this paper, we consider the degeneration of compact Riemann surfaces of fixed genus $q \geq 2$ into a singular Riemann surface of genus $q-1$ with a single non-separating node p.

To facilitate ensuing discussion, we first recall the plumbing construction of a degenerating family of Riemann surfaces starting from M as follows (cf. e.g. [F], [Wo2]). Let $M^0 := M\backslash\{p\}$. Then M^0 is a punctured Riemann surface with two punctures p_1, p_2 in place of p, where p_1, p_2 correspond to two points in the normalization $\tilde{M}$ of M. Denote the unit disc in $\mathbf{C}$ by Δ. For $i = 1, 2$, fix a coordinate function $z_i : U_i \to \Delta$ such that $z_i(p_i) = 0$, where U_i is an open neighborhood of p_i. For each $t \in \Delta$, let $S_t := \{(x,y) \in \Delta^2 : xy = t\}$. Now for each $t \in \Delta$, remove the discs $|z_i| < |t|$, $i = 1, 2$, from M and glue the remaining surface with S_t via the identification

$$z_1 \sim (z_1, t/z_1) \quad \text{and} \quad z_2 \sim (t/z_2, z_2). \tag{2.1.1}$$

The resulting surfaces $\{M_t\}_{t\in\Delta}$ form an analytic family $\pi : \mathcal{M} \to \Delta$ with $M_0 = M$. Here π denotes the holomorphic projection map. Note that for $t \neq 0$, each fiber M_t is a compact Riemann surface of genus q. Also the node p does not disconnect the Riemann surface when removed from M. The restriction of $\ker(d\pi)$ to $\mathcal{M}\backslash\{p\}$ forms a holomorphic line bundle over $\mathcal{M}\backslash\{p\}$ such that $L|_{M_t} = TM_t$ and $L|_{M^0} = TM^0$, which will be called the vertical line bundle. Note that $\ker(d\pi)$ itself does not form a line bundle over $\mathcal{M}$ since $\ker(d\pi)$ is of rank 2 at p.

(2.2) We shall need the following definitions for subsequent discussion.

Definition 2.2.1. Let N be a punctured Riemann surface. A Hermitian metric ds^2 on N is said to be *of hyperbolic growth near the punctures* if for each puncture p, there exists a punctured coordinate disc $\Delta^* := \{z \in \mathbf{C} : 0 < |z| < 1\}$ centered at p such that for some constant $C_1 > 0$,

$$\text{(i) } ds^2 \le \frac{C_1|dz|^2}{|z|^2(\log|z|)^2} \quad \text{on } \Delta^*, \tag{2.2.1}$$

and there exists a local potential function ϕ on Δ^* satisfying $ds^2 = \frac{\partial^2\phi}{\partial z\partial\bar{z}}dz \otimes d\bar{z}$ on Δ^*, and for some constants C_2, $C_3 > 0$,

$$\text{(ii) } |\phi(z)| \le C_2\max\{1, \log(-\log|z|)\}, \text{ and} \tag{2.2.2}$$

$$\text{(iii) } \left|\frac{\partial\phi}{\partial z}\right|, \left|\frac{\partial\phi}{\partial\bar{z}}\right| \le \frac{C_3}{|z|\,|\log|z||} \quad \text{on } \Delta^*. \tag{2.2.3}$$

Now let $\pi : \{M_t\} \to \Delta$ and L be as in (2.1). For $t \neq 0$ (resp. $t = 0$), let ds_t^2 (resp. ds_0^2) be a Hermitian metric on M_t (resp. $M^0 := M\backslash\{\text{nodes}\}$).

Definition 2.2.2. $\{ds_t^2\}$ is said to be a continuous family of Hermitian metrics on $\{M_t\}$ if $\{ds_t^2\}$ form a continuous section of $L^* \otimes \bar{L}^*$.

For $t \neq 0$ (resp. $t = 0$), denote the hyperbolic metric on M_t (resp. M^0) by $ds^2_{\text{hyp},t}$ (resp. $ds^2_{\text{hyp},0}$).

Definition 2.2.3. A continuous family of quasi-hyperbolic metrics $\{ds_t^2\}$ on $\{M_t\}$ is a continuous family of Hermitian metrics (in the sense of Definition 2.2.2) such that

(i) there exist constants C_1, $C_2 > 0$ such that

$$C_1 ds^2_{\text{hyp},t} \le ds^2_t \le C_2 ds^2_{\text{hyp},t} \quad \text{for all } t \in \Delta, \tag{2.2.4}$$

(ii) ds_0^2 is of hyperbolic growth near the punctures on M^0 (cf. Definition 2.2.1).

Remark 2.2.4. (i) By a result of Wolpert [Wo2, Theorem 5.8] and well-known properties of hyperbolic metrics, the hyperbolic metrics $\{ds^2_{\text{hyp},t}\}$ form a continuous family of quasi-hyperbolic metrics on $\{M_t\}$.

(ii) Non-trivial continuous families of quasi-hyperbolic metrics can easily be constructed by the grafting procedure in [Wo2, §3, §4].

(iii) It follows easily from (2.2.1) that $\mathrm{Vol}(M^0, ds_0^2) < \infty$.

(iv) We remark that one can deduce (2.2.1) for ds_0^2 from (2.2.4) with $t = 0$.

(2.3) Let $\{M_t\}$ be as in (2.1). Next we consider the Green's function on each M_t. For $t \neq 0$, let ds_t^2 be a Hermitian metric on the compact Riemann surface M_t, whose associated Kähler form is denoted by ω_t. Denote also the normalized Kähler form by $\hat{\omega}_t := \frac{1}{\mathrm{Vol}(M_t, \omega_t)} \omega_t$. It is well-known that there exists a unique Green's function $g_t(\cdot,\cdot)$ on $M_t \times M_t \backslash D_t$, where D_t denotes the diagonal, such that the following conditions are satisfied:

(a) For fixed $x \in M_t$, and $y \neq x$ near x,

$$g_t(x,y) = -\log |f(y)|^2 + \alpha(y), \tag{2.3.1}$$

where f is a local holomorphic defining function for x, and α is some smooth function defined near x;

(b) $d_y d_y^c g_t(x,y) = \hat{\omega}_t - \delta_x$; (2.3.2)

(c) $\int_{M_t} g_t(x,y)\hat{\omega}_t = 0$; (2.3.3)

(d) $g_t(x,y) = g_t(y,x)$ for $x \neq y$; (2.3.4)

(e) $g_t(x,y)$ is smooth on $M_t \times M_t \backslash D_t$. (2.3.5)

See e.g. [L] for the above definition. Here $d_x^c := \frac{i}{4\pi}(\bar{\partial} - \partial)$ with respect to the first variable (so that $d_x d_x^c = \frac{i}{2\pi}\partial\bar{\partial}$), and δ_x is the Dirac delta function at x. Note also that the growth condition (a) is independent of the choice of the local holomorphic defining function for x.

For $t = 0$, suppose ds_0^2 is a Hermitian metric of finite volume on the punctured Riemann surface M^0 with $\omega_0, \hat{\omega}_0$ defined similarly as above. The Green's function $g_0(\cdot,\cdot)$ on M^0 with respect to ds_0^2 is a function on $M^0 \times M^0 \backslash \{\text{diagonal}\}$ satisfying conditions (a) to (e) above (with $t = 0$ and M_t replaced by M^0) and the following growth condition:

(f) Near each puncture p of M^0, there exists a punctured coordinate neighborhood Δ^* centered at p such that for fixed $x \notin \Delta^*$, there exists a constant $C > 0$ such that

$$|g_t(x,z)| \le C \max\{1, \log(-\log|z|)\} \qquad \text{on } \Delta^*. \tag{2.3.6}$$

As a motivation, we remark that (2.3.6) is obviously satisfied by the potential function $-\log(-\log|z|)$ for the Poincaré metric on Δ^*.

(2.4) Notation as in (2.1), (2.2) and (2.3). Now we state our results in this article. Our first result concerns existence and uniqueness of Green's function on a punctured Riemann surface, which we include here for the sake of completeness.

Theorem 1. *Let N be a punctured Riemann surface, and ds^2 be a Hermitian metric on N of hyperbolic growth near the punctures (see Definition 2.2.1). Then there exists a unique Green's function $g(\cdot,\cdot)$ on $N \times N\backslash\{\text{diagonal}\}$ with repect to ds^2 satisfying all the conditions (a) to (f) in (2.3).*

Our main result is the following

Theorem 2. *Let $\{M_t\}$ be a family of compact Riemann surface of genus $q \ge 2$ degenerating to a Riemann surface M of genus $q-1$ with a single non-separating node p as described in (2.1). Suppose $\{ds_t^2\}$ is a continuous family of quasi-hyperbolic metrics on $\{M_t\}$ (cf. Definition 2.2.3). Then for continuous sections x_t, y_t of $\{M_t\}$ such that $x_t \ne y_t$ for all t and $x_0, y_0 \in M^0 = M\backslash\{p\}$, we have*

$$\lim_{t\to 0} g_t(x_t, y_t) = g_0(x_0, y_0), \tag{2.4.1}$$

where $g_0(\cdot,\cdot)$ is the Green's function with respect to ds_0^2 given by Theorem 1.

Finally, we have the following

Corollary 3. *Let $\{M_t\}$ be as in Theorem 2, and let $\{ds_t^2\}$ be a continuous family of complete Hermitian metrics on $\{M_t\}$. Suppose there exist constants $C_1, C_2 > 0$ such that the sectional curvatures of $\{ds_t^2\}$ are pinched between $-C_1$ and $-C_2$ for all $t \in \Delta$, and ds_0^2 is of hyperbolic growth near the punctures on M^0. Then the conclusion of Theorem 2 remains valid.*

3. Green's function on punctured Riemann surface

(3.1) In this section, we give, for the sake of completeness, the proof of Theorem 1 following closely the approach in [L] in the compact case. As the proof is simple and elementary, some of the steps will only be sketched.

We shall need the following elementary lemma on the extension of harmonic functions, whose proof will be skipped:

Lemma 3.1.1. *Let u be a harmonic function on the punctured unit disc $\Delta^* := \{z \in \mathbf{C} : 0 < |z| < 1\}$. Suppose there exists a constant $C > 0$ such that*

$$|u(z)| \leq C\max\{1, \log(-\log|z|)\} \text{ for all } z \in \Delta^*. \tag{3.1.1}$$

Then u can be extended to a smooth harmonic function on the unit disc $\Delta := \{z \in \mathbf{C} : |z| < 1\}$.

Now we are ready to give the following

Proof of Theorem 1. I. Existence: Let N and ds^2 be as in Theorem 1, and let $\bar{N}$ be the smooth compactification of N such that $\bar{N}\backslash N$ consists of points $\{p_i\}$ corresponding to the punctures of N. It follows easily from (2.2.1) that $\mathrm{Vol}(N, ds^2) < \infty$. Denote the normalized volume form of ds^2 by $\hat{\omega}$ so that $\mathrm{Vol}(N, \hat{\omega}) = 1$, which readily implies that $\hat{\omega}$ extends trivially to a d-closed (1,1) current on $\bar{N}$, which we denote by $\tilde{\hat{\omega}}$. Moreover, the cohomology class represented by $\tilde{\hat{\omega}}$ is a positive generator of $H^2(\bar{N}, \mathbf{Z}) \simeq \mathbf{Z}$. Thus for $x \in N$, $\tilde{\hat{\omega}} - \delta_x$ is a d-exact (1,1) current on $\bar{N}$, where δ_x denotes the Dirac delta function at x. Since $\bar{N}$ is Kähler, one can find a locally integrable function $g'_x(\cdot)$ on $\bar{N}$ satisfying

$$d_y d^c_y g'_x = \tilde{\hat{\omega}} - \delta_x, \tag{3.1.2}$$

in the sense of (1,1) currents on $\bar{N}$. Near each puncture p_i, let ϕ_i be a local potential function for $\hat{\omega}$ satisfying (2.2.2) and (2.2.3) on some punctured coordinate unit disc Δ^* centered at p_i. Then it follows easily from (2.2.3) and (3.1.2) that $dd^c(g'_x - \phi_i) = 0$ as (1,1) currents on Δ. Standard elliptic regularity theory implies that

$$g'_x = \phi_i + h_i \text{ on } \Delta^*, \tag{3.1.3}$$

where h_i is some harmonic function on Δ. It follows easily from (2.2.1), (2.2.2) and (3.1.3) that $\int_N g'_x \hat{\omega}$ is finite. Define

$$g_x(\cdot) := g'_x(\cdot) - \int_N g'_x \hat{\omega} \text{ on } N\backslash\{x\}, \tag{3.1.4}$$

and define finally the Green's function by

$$g(x, y) := g_x(y) \text{ for } x \neq y \in N. \tag{3.1.5}$$

We need to verify condition (a) to (f) in (2.3). Condition (b) follows immediately from (3.1.2) and (3.1.3). Condition (a) is an immediate consequence of condition (b). Condition (c) follows easily from (3.1.4). To verify condition (d) on symmetry, we take $x \neq y \in N$. Let D_r denote the union of coordinate discs in $\bar{N}$ of radius r and with centers at $\{p_i\}, x, y$, and let C_r denote the union of the boundary circles of D_r. Then condition (d) can be verified as in the compact case in [L, Chapter II] by applying Stokes' theorem on $N\backslash D_r$ and showing that $\lim_{r\to 0} \int_{C_r} (g_x d^c g_y - g_y d^c g_x) = 0$ using the growth condition (f) for g_x, g_y and the growth condition for $d^c g_x, d^c g_y$ near the punctures implied by (2.2.3) and (3.1.3). To verify condition (e) on smoothness, we take a smooth normalized volume form $\hat{\mu}$ on $\bar{N}$, and find a locally integrable function β on $\bar{N}$ by solving the equation

$$dd^c\beta = \widetilde{\omega} - \hat{\mu} \tag{3.1.6}$$

as (1,1) currents on $\bar{N}$ and such that $\int_N \beta(\widetilde{\omega}+\hat{\mu}) = 0$. Then it is easy to see that, like $g_x(\cdot), \beta$ and $d\beta$ satisfy the growth conditions (2.2.2) and (2.2.3) near the punctures. Then one can deduce the smoothness of $g(\cdot,\cdot)$ on $N \times N \backslash \{\text{diagonal}\}$ from that of $\hat{\mu}$ as in [L, Chapter II, Proposition 1.3]. We only need to remark that with the growth conditions of β and $d\beta$, the arguments in [L, Chapter II, Proposition 1.3] involving Stokes' theorem remain valid by considering small circles of radius r centered at the punctures and then letting $r \to 0$. This finishes the proof of the existence part of Theorem 1.

II. Uniqueness: Suppose that $g(\cdot,\cdot)$ and $g'(\cdot,\cdot)$ are two functions satisfying conditions (a) to (f) in (2.3). Then for fixed $x \in N$, it follows from condition (b), condition (f) and Lemma 3.1.1 that $g(x,\cdot) - g'(x,\cdot)$ extends to a harmonic function on $\bar{N}$, which implies that $g(x,\cdot) = g'(x,\cdot) + k$ for some constant k. Then it follows from condition (c) that $k = 0$ and this proves the uniqueness of the Green's function. Thus we have completed the proof of Theorem 1.

4. Approximation of Green's functions on M_t

(4.1) Notation as in §2. Let $\mathcal{M} = \{M_t\}$, $\{ds_t^2\}$, p, x_t, y_t, $g_t(\cdot,\cdot)$ and $g_0(\cdot,\cdot)$ be as in Theorem 2. In this section we are going to construct an approximation of $g_t(\cdot, y_t)$ on M_t using $g_0(\cdot,\cdot)$. Recall from (2.1) the coordinate functions $z_i : U_i \to \Delta$, $i = 1, 2$, and that there exists an open neighborhood $\Delta^2 \simeq U_1 \times U_2$ centered at p such that $M_t \cap \Delta^2 = \{(z_1, z_2) \in \Delta^2 : z_1 z_2 = t\}$. Fix a small number $\delta > 0$. We define, for $t \in \Delta^*$,

$$\begin{aligned} \mathrm{I}_t :=&\{(z_1, t/z_1) \in \Delta^2 : |t|^{\frac{1}{2}+2\delta} < |z_1| < |t|^{\frac{1}{2}-2\delta}\} \\ =&\{(t/z_2, z_2) \in \Delta^2 : |t|^{\frac{1}{2}+2\delta} < |z_2| < |t|^{\frac{1}{2}-2\delta}\} \\ \subset& M_t, \end{aligned} \tag{4.1.1}$$

and we let $\mathrm{II}_t := M_t \backslash \mathrm{I}_t$. For each t, we let

$$i_{1,t} : \mathrm{I}_t \to U_1, \quad \text{and} \quad i_{2,t} : \mathrm{I}_t \to U_2 \tag{4.1.2}$$

denote the holomorphic maps induced by the coordinate projection maps on Δ^2. Also the plumbing construction in (2.1) induces a biholomorphism

$$i_t : \mathrm{II}_t \to W_t, \quad \text{where } W_t := M^0 \backslash \bigcup_{i=1,2} \{|z_i| < |t|^{\frac{1}{2}-2\delta}\}. \tag{4.1.3}$$

We denote the inverse of i_t by $j_t : W_t \to \mathrm{II}_t$. Note that $i_{1,t}$, $i_{2,t}$, i_t, j_t all depend analytically on t, and $\{W_t\}$ form an increasing sequence of compact subsets exhausting M^0 as $t \to 0$. It is easy to see that $x_t, y_t \notin \mathrm{I}_t$ for $|t|$ sufficiently small, since $x_0, y_0 \neq p$. Thus shrinking Δ if necessary, we may assume that $x_t, y_t \in \mathrm{II}_t$ for all $t \in \Delta$. We let

$$x_t' := i_t(x_t) \text{ and } y_t' := i_t(y_t), \qquad t \in \Delta, \tag{4.1.4}$$

denote the associated continuous curves on M^0. Note that $x_0' = x_0$ and $y_0' = y_0$. Write $a = \log|z_1|/\log|t|$. Then we have $\frac{1}{2} - 2\delta < a < \frac{1}{2} + 2\delta$ on each I_t. Now we fix $\eta = \eta(a)$ to be a smooth non-negative function such that $\mathrm{supp}(d\eta) \subset (\frac{1}{2} - \delta, \frac{1}{2} + \delta), \eta = 0$ for $a > \frac{1}{2} + \delta$ and $\eta = 1$ for $a < \frac{1}{2} - \delta$. Finally for each $t \in \Delta^*$, we define the following function on $M_t \backslash \{y_t\}$ given by

$$\tilde{g}_{t,y_t}(x) := \begin{cases} (1-\eta)g_0(i_{1,t}(x), y_t') + \eta g_0(i_{2,t}(x), y_t'), & x \in \mathrm{I}_t, \\ g_0(i_t(x), y_t'), & x \in \mathrm{II}_t. \end{cases} \tag{4.1.5}$$

It is easy to see that $\tilde{g}_{t,y_t}$ is smooth on $M_t \backslash \{y_t\}$, and we shall call $\tilde{g}_{t,y_t}$ the grafted Green's function on M_t.

(4.2) Notation as in (4.1). In this subsection, we obtain some estimates on I_t needed for ensuing discussion. Recall from (4.1) the coordinate neighborhood Δ^2 of p in $\mathcal{M}$ such that $M_t \cap \Delta^2 = \{(z_1, z_2) \in \Delta^2 : z_1 z_2 = t\}$. Thus for $t \neq 0$, z_1, z_2 provide two different coordinate functions on $M_t \cap \Delta^2$. First we have

Proposition 4.2.1. *There exist constants $C_1, C_2 > 0$ such that for all $t \in \Delta^*$, we have, on $M_t \cap \Delta^2$,*

$$C_1 \left(\frac{\pi}{\log|t|} \csc \frac{\pi \log|z_i|}{\log|t|} \frac{|dz_i|}{|z_i|} \right)^2 \leq ds_t^2 \leq C_2 \left(\frac{\pi}{\log|t|} \csc \frac{\pi \log|z_i|}{\log|t|} \frac{|dz_i|}{|z_i|} \right)^2 . \tag{4.2.1}$$

Proof. A result of Wolpert [Wo2, Expansion 4.2] implies that (4.2.1) holds for the hyperbolic metrics $\{ds^2_{\mathrm{hyp},t}\}$ on $\{M_t\}$. This, together with (2.2.4), implies Proposition 4.2.1.

Recall from (2.1) the coordinate mappings $z_i : U_i \to \Delta$ near p. For $|t|$ sufficiently small, $\{y_t'\} \subset\subset M^0$. Shrinking U_i if necessary, we may thus assume that $\{y_t'\} \cap U_i = \emptyset, i = 1, 2$. Then we have

Lemma 4.2.2. *There exist constants $C_1, C_2 > 0$ such that for all $t \in \Delta^*$ and for $i = 1, 2$,*

(i) $|g_0(z_i, y_t')| \leq C_1 \max\{1, \log(-\log|z_i|)\}$, and $\qquad$ (4.2.2)

(ii) $$\left| \frac{\partial g_0(z_i, y_t')}{\partial z_i} \right|, \left| \frac{\partial g_0(z_i, y_t')}{\partial \bar{z}_i} \right| \leq \frac{C_2}{|z_i| |\log|z_i||} \quad \text{on } U_i \backslash \{p\}. \tag{4.2.3}$$

Proof. For $i = 1, 2$, let ϕ_i be a local potential function for $\hat{\omega}_0$ satisfying (2.2.2) and (2.2.3) on $U_i \backslash \{p\}$. From conditions (b), (e) in (2.3) for $g_0(\cdot,\cdot)$, we easily have

$$g_0(z_i, y_t') = \phi_i(z_i) + h(t, z_i) \quad \text{on } U_i \backslash \{p\}, \tag{4.2.4}$$

where h is continuous in (t, z) and is harmonic in z_i. Then we can easily deduce Lemma 4.2.2 using the Poisson integral formula for $h(t, \cdot)$ and the corresponding growth conditions (2.2.2), (2.2.3) for ϕ_i.

We denote the Laplacian on M_t with respect to ds_t^2 by Δ_t. Here we adopt the notation that the Laplacian for a Hermitian metric h is given by $\Delta = -h^{z\bar{z}} \partial z \partial \bar{z}$.

Proposition 4.2.3. *Let* I_t *be as in (4.1.1). We have*

(i) $\displaystyle\int_{\mathrm{I}_t} \omega_t \to 0, \quad \int_{\mathrm{I}_t} \hat{\omega}_t \to 0,$ (4.2.5)

(ii) $\displaystyle\int_{\mathrm{I}_t} \tilde{g}_{t,y_t} \hat{\omega}_t \to 0,$ *and* (4.2.6)

(iii) $\displaystyle\int_{\mathrm{I}_t} (\Delta_t \tilde{g}_{t,y_t})^2 \omega_t \to 0$ *as* $t \to 0.$ (4.2.7)

Proof. First we note that by (4.1.1), there exist constants $C_1, C_2 > 0$ such that for $i = 1, 2$,

$$C_1 \le \frac{\log |z_i|}{\log |t|} \le C_2 \qquad \text{on } \mathrm{I}_t. \tag{4.2.8}$$

Write $z_1 = e^{\zeta \log |t|}$. Then $\zeta = a + ib, \frac{1}{2} - 2\delta < a < \frac{1}{2} + 2\delta, 0 \le b < \frac{2\pi}{|\log |t||}$, provides a parametrization for each I_t, $t \in \Delta^*$. It is well-known that $\mathrm{Vol}(M_t, ds^2_{\mathrm{hyp},t}) = 2\pi(2q - 2)$ for each t. Thus by (2.2.4), there exist constants $C_3, C_4 > 0$ such that $C_3 \le \mathrm{Vol}(M_t, ds_t^2) \le C_4$ for all $t \in \Delta$. Together with Proposition 4.2.1 and (4.2.8), it follows that there exist constants $C_5, C_6 > 0$ such that for all $t \in \Delta^*$,

$$C_5 \frac{i}{2} d\zeta \wedge d\bar{\zeta} \le \omega_t,\ \hat{\omega}_t \le C_6 \frac{i}{2} d\zeta \wedge d\bar{\zeta} \ \text{ on } \mathrm{I}_t. \tag{4.2.9}$$

Then

$$\begin{aligned} \int_{\mathrm{I}_t} \hat{\omega}_t \le & C_6 \left| \int_{\mathrm{I}_t} \frac{i}{2} d\zeta \wedge d\bar{\zeta} \right| \\ = & C_6 \int_0^{\frac{2\pi}{|\log |t||}} \int_{\frac{1}{2}-2\delta}^{\frac{1}{2}+2\delta} da db \\ = & \frac{8\pi\delta C_6}{|\log |t||} \to 0 \quad \text{as } t \to 0. \end{aligned} \tag{4.2.10}$$

Similarly one can show that $\int_{\mathrm{I}_t} \omega_t \to 0$ as $t \to 0$, and this finishes the verification of (i). To verify (ii), we take $z = (z_1, z_2) \in \mathrm{I}_t \subset \Delta^2$. Note that in terms of z_1, z_2, we have $i_{1,t}(z) = z_1, i_{2,t}(z) = z_2$ (cf. (4.1.2)). Now on I_t,

$$\begin{aligned}|\tilde{g}_{t,y_t}(z)| \leq& |g_0(i_{1,t}(z), y_t')| + |g_0(i_{2,t}(z), y_t')| \qquad \text{(by (4.1.5))}\\ \leq& C_7 \max\{1, \log(-\log|z_1|) + \log(-\log|z_2|)\} \quad \text{(by Lemma 4.2.2)}\\ \leq& C_8 \max\{1, \log(-\log|t|)\} \qquad \text{(by (4.2.8))},\end{aligned} \tag{4.2.11}$$

where $C_7, C_8 > 0$ are constants independent of t. Using (4.2.11), one can show as in (4.2.10) that for some constant $C_9 > 0$,

$$\left|\int_{\mathrm{I}_t} \tilde{g}_{t,y_t} \hat{\omega}_t\right| \leq C_9 \frac{\log(-\log|t|)}{|\log|t||} \to 0 \quad \text{as } t \to 0,$$

which verifies (ii). To verify (iii), we first note that by (4.1.5), we have, for $z = (z_1, z_2) \in \mathrm{I}_t \subset \Delta^2$,

$$\begin{aligned}\partial_\zeta \partial_{\bar\zeta} \tilde{g}_{t,y_t}(z) =& (\partial_\zeta \partial_{\bar\zeta} \eta)[g_0(i_{2,t}(z), y_t') - g_0(i_{1,t}(z), y_t')\\ &+ 2\mathrm{Re}[(\partial_{\bar\zeta}\eta)\partial_\zeta(g_0(i_{2,t}(z), y_t') - g_0(i_{1,t}(z), y_t'))]\\ &+ (1-\eta)\partial_\zeta\partial_{\bar\zeta} g_0(i_{1,t}(z), y_t') + \eta \partial_\zeta \partial_{\bar\zeta} g_0(i_{2,t}(z), y_t').\end{aligned} \tag{4.2.12}$$

Here $\partial_\zeta \partial_{\bar\zeta} \tilde{g}_{t,y_t}(z)$ denotes $\partial^2 \tilde{g}_{t,y_t}(z)/\partial\zeta\partial\bar\zeta$, etc. By construction, it is easy to see that there exists a constant C_{10} such that for all $t \in \Delta^*$, $|\eta|$, $|\partial_\zeta \eta|$, $|\partial_{\bar\zeta}\eta|$, $|\partial_\zeta\partial_{\bar\zeta}\eta| \leq C_{10}$ on I_t. Also, one can show as in (4.2.11) that there exists a constant C_{11} such that for all $t \in \Delta^*$, $|g_0(i_{1,t}(z), y_t')|$, $|g_0(i_{2,t}(z), y_t')| \leq C_{11}\max\{1, \log(-\log|t|)\}$ on I_t. For $t \in \Delta^*$, we also have

$$\begin{aligned}|\partial_\zeta g_0(i_{1,t}(z), y_t')| =& |\partial_{z_1} g_0(z_1, y_t') \cdot \partial z_1/\partial\zeta|\\ \leq& \frac{C_{12}}{|z_1||\log|z_1||} \cdot |z_1| \cdot |\log|t|| \text{ (by Lemma 4.2.2)}\\ \leq& C_{13} \text{ on } \mathrm{I}_t \text{ (by (4.2.8))}.\end{aligned} \tag{4.2.13}$$

Here $C_{12}, C_{13} > 0$ are constants independent of t. Similar inequality also holds for the quantity $|\partial_\zeta g_0(i_{2,t}(z), y_t')|$. From conditions (b) in (2.3) for $g_0(\cdot,\cdot)$ and (2.2.1) for ds_0^2, one can show as in (4.2.13) that there exists a constant $C_{14} > 0$ such that for $t \in \Delta^*$,

$$|\partial_\zeta\partial_{\bar\zeta} g_0(i_{1,t}(z), y_t')|,\ |\partial_\zeta\partial_{\bar\zeta} g_0(i_{2,t}(z), y_t')| \leq C_{14} \text{ on } \mathrm{I}_t.$$

It follows from (4.2.12) and the above considerations that there exists a constant $C_{15} > 0$ such that for $t \in \Delta^*$,

$$|\partial_\zeta \partial_{\bar\zeta} \tilde{g}_{t,y_t}| \leq C_{15} \max\{1, \log(-\log|t|)\} \text{ on } \mathrm{I}_t. \tag{4.2.14}$$

Since $\Delta_t \tilde{g}_{t,y_t} = -\dfrac{1}{\omega(\partial/\partial\zeta, \partial/\partial\zeta)} \partial_\zeta\partial_{\bar\zeta}\tilde{g}_{t,y_t}$, one can use (4.2.9), (4.2.14) and proceed as in (4.2.10) to show that there exists a constant $C_{16} > 0$ such that

$$\int_{\mathrm{I}_t} (\Delta_t \tilde{g}_{t,y_t})^2 \omega_t \leq C_{16} \frac{[\log(-\log|t|)]^2}{|\log|t||} \to 0 \quad \text{as } t \to 0.$$

This verifies (iii), and we have finished the proof of Proposition 4.2.3.

(4.3) Notation as in (4.1) and (4.2). Let $\{M_t\}$, x_t, y_t be as in Theorem 2. To compare the grafted Green's function $\tilde{g}_{t,y_t}$ in (4.1.5) with $g_t(\cdot, y_t)$, we make several definitions. First for $t \in \Delta^*$, we let $g_{t,y_t}(\cdot) := g_t(\cdot, y_t)$ on $M_t \backslash \{y_t\}$, and define the following function

$$u_{0,t} := g_{t,y_t} - \tilde{g}_{t,y_t} + C_t \quad \text{on } M_t \backslash \{y_t\}, \tag{4.3.1}$$

where C_t is the constant given by

$$C_t := \int_{M_t} \tilde{g}_{t,y_t} \hat{\omega}_t. \tag{4.3.2}$$

From the growth condition (a) in (2.3) for $g_0(\cdot, y_t')$ near y_t', it is easy to see that C_t is finite for each t. Recall that, from (4.1), we have two biholomorphisms $i_t : \text{II}_t \to W_t$, $j_t : W_t \to \text{II}_t$, where $W_t = M^0 \backslash \bigcup_{i=1,2} \{|z_i| < |t|^{\frac{1}{2}-2\delta}\}$. For $|t|$ sufficiently small, $y_t' \in W_t$. Also, it follows from (4.1.5) that $\tilde{g}_{t,y_t} = i_t^* g_0(\cdot, y_t')$ on $\text{II}_t \backslash \{y_t\}$. Since $i_t : \text{II}_t \to W_t$ is a biholomorphism, it follows that for any local holomorphic defining function h for y_t', $h \circ i_t$ is a local holomorphic defining function for y_t. Thus the growth condition (2.3.1) for $g_0(\cdot, y_t')$ near y_t' implies that $\tilde{g}_{t,y_t}$ also satisfies (2.3.1), i.e. for x near y_t, $x \neq y_t$,

$$\tilde{g}_{t,y_t}(x) = -\log |f(x)|^2 + \beta(x)$$

for any local holomorphic defining function f for y_t, where β is some smooth function defined near y_t. Together with the growth condition (2.3.1) for g_{t,y_t}, it follows from (4.3.1) that $u_{0,t}$ extends uniquely to a smooth function on M_t, which we denote by u_t. Define also the smooth function

$$\phi_t := \Delta_t u_t \quad \text{on } M_t. \tag{4.3.3}$$

First we have

Lemma 4.3.1. *For $0 < t_0 < 1$, there exist constants $C, C' > 0$ such that for all $0 < t \leq t_0$,*

$$C ds_0^2 \leq j_t^* ds_t^2 \leq C' ds_0^2 \qquad \text{on } W_t. \tag{4.3.4}$$

Proof. Let $z_i : U_i \to \Delta, i = 1, 2$, be as in (2.1). Write each $W_t = N \sqcup U_{1,t} \sqcup U_{2,t}$ where $U_{i,t} := \{z \in U_i : |t|^{\frac{1}{2}-2\delta} \leq |z_i| < 1\} \subset U_i, i = 1, 2$, and $N := W_t \backslash (U_{1,t} \sqcup U_{2,t})$. To prove Lemma 4.3.1, it suffices to show that (4.3.4) holds on $N, U_{1,t}, U_{2,t}$ respectively. First we note that the compactness of $N \times \{t \in \mathbf{C} : |t| \leq t_0\}$ and the continuity of $\{ds_t^2\}$ imply that (4.3.4) holds on N. From (2.2.4) with $t = 0$ and well-known behavior of the hyperbolic metric $ds^2_{\mathrm{hyp},0}$ near the punctures, it follows that there exist constants $C_1, C_2 > 0$ such that for $i = 1, 2$,

$$C_1 \frac{|dz_i|^2}{|z_i|^2(\log|z_i|)^2} \leq ds_0^2 \leq C_2 \frac{|dz_i|^2}{|z_i|^2(\log|z_i|)^2} \quad \text{on } U_i. \tag{4.3.5}$$

We observe that $0 < \dfrac{\pi \log|z_i|}{\log|t|} < \pi(\frac{1}{2} - 2\delta)$ on each $U_{i,t}$, and there exist constants $C_3, C_4 > 0$ such that the function $f(\theta) := \theta \csc\theta$ satisfies $C_3 < f(\theta) < C_4$ for $0 < \theta < \pi(\frac{1}{2} - 2\delta)$. Together with (4.3.5) and Proposition 4.2.1, one can easily verify that (4.3.4) also holds on each $U_{i,t}$, and this finishes the proof of Lemma 4.3.1.

Proposition 4.3.2. *We have*

(i) $\mathrm{Vol}(M_t, \omega_t) \to \mathrm{Vol}(M_0, \omega_0)$, (4.3.6)

(ii) $\int_{\mathrm{II}_t} \phi_t^2 \omega_t \to 0$, *and* (4.3.7)

(iii) $\int_{\mathrm{II}_t} \tilde{g}_{t,y_t} \hat{\omega}_t \to 0$ *as* $t \to 0$. (4.3.8)

Proof. Recall from (4.1) the decomposition $M_t = \mathrm{I}_t \sqcup \mathrm{II}_t$. Then one has $\mathrm{Vol}(M_t, \omega_t) = \int_{\mathrm{I}_t} \omega_t + \int_{\mathrm{II}_t} \omega_t$. Note that $\int_{\mathrm{II}_t} \omega_t = \int_{W_t} j_t^* \omega_t$, and the latter integral can be regarded as an integral on M^0 by letting the integrand to be zero on $M_t \backslash W_t$. Since W_t increases to M^0 as $t \to 0$, it follows from the continuity of $\{ds_t^2\}$ that $j_t^* \omega_t$ coverges pointwise to ω_0 on M^0 as $t \to 0$. Using Lemma 4.3.1 and the fact that $\int_{M^0} \omega_0 < \infty$, one deduces from the dominated convergence theorem that $\int_{W_t} j_t^* \omega_t \to \int_{M^0} \omega_0$ as $t \to 0$. Together with Proposition 4.2.3(i), one gets (i) immediately. To verify (ii), recall from (4.1.5) that $\tilde{g}_{t,y_t} = i_t^* g_0(\cdot, y_t')$ on $\mathrm{II}_t \backslash \{y_t\}$. From (4.3.3), condition (b) in (2.3) for $g_0(\cdot, y_t')$ and $g_t(\cdot, y_t)$, one gets

$$\begin{aligned} \phi_t &= -\frac{2\pi}{\omega_t} dd^c(g_{t,y_t} - \tilde{g}_{t,y_t} + C_t) \\ &= -\frac{2\pi}{\omega_t}(\hat{\omega}_t - i_t^* \hat{\omega}_0) \\ &= -2\pi \left(\frac{1}{\mathrm{Vol}(M_t, \omega_t)} - \frac{1}{\mathrm{Vol}(M^0, \omega_0)} \frac{i_t^* \omega_0}{\omega_t} \right) \quad \text{on } \mathrm{II}_t \backslash \{y_t\}. \end{aligned} \tag{4.3.9}$$

Here ratios of (1,1) forms make sense since M_t is 1-dimensional. Note also that since both ϕ_t and the last line of (4.3.9) are smooth functions on II_t, they are actually equal to each other on II_t. Then

$$\int_{\mathrm{II}_t} \phi_t^2 \omega_t = \int_{W_t} \left| \frac{1}{\mathrm{Vol}(M_t, \omega_t)} - \frac{1}{\mathrm{Vol}(M^0, \omega_0)} \frac{\omega_0}{j_t^* \omega_t} \right|^2 \frac{j_t^* \omega_t}{\omega_0} \omega_0. \tag{4.3.10}$$

By (i), Lemma 4.3.1 and the continuity of $\{ds_t^2\}$, it follows that the integrand in (4.3.10) is bounded from above by a constant independent of t and it converges pointwise to 0 on M^0 as $t \to 0$. Then as in (i), one can use the dominated convergence theorem to obtain (ii). To verify (iii), we note from (4.1.5) that

$$\int_{\mathrm{II}_t} \tilde{g}_{t,y_t}\hat{\omega}_t = \int_{W_t} g_0(\cdot, y_t')\frac{j_t^*\hat{\omega}_t}{\hat{\omega}_0}\hat{\omega}_0.$$

From (i) and the continuity of $\{ds_t^2\}$, we see that the integrand converges pointwise to $g_0(\cdot, y_0')$ on M^0 as $t \to 0$. Fix a local coordinate function near y_0', and let $\Delta(y_t', r_0)$ denote the coordinate disc centered at y_t' and of fixed radius $r_0 > 0$. Such $\Delta(y_t', r_0)$ exists for $|t|$ and r_0 sufficiently small. Let $\tau_t : \Delta(y_0', r_0) \to \Delta(y_t', r_0)$ denote the continuous family of translations sending y_0' to y_t'. Then it follows from condition (a) in (2.3) that in terms of polar coordinates (r, θ) centered at y_0', $\tau_t^* g_0(\cdot, y_t') = -\log r^2 + \alpha(t, r, \theta)$, where $\alpha(t, r, \theta)$ is continuous. Using the dominated convergence theorem, one deduces that

$$\int_{\Delta(y_t, r_0)} \tilde{g}_{t,y_t}\hat{\omega}_t \to \int_{\Delta(y_0', r_0)} \tilde{g}_0(\cdot, y_0')\hat{\omega}_0 \quad \text{as } t \to 0. \tag{4.3.11}$$

Now, by combining Lemma 4.2.1, Lemma 4.2.2, (4.3.5), and also the simple fact that

$$\int_{\Delta(\frac{1}{2})} \frac{\log(-\log|z|)}{|z|^2(\log|z|)^2} i dz \wedge d\bar{z} < \infty,$$

where $\Delta(\frac{1}{2}) := \{z \in \mathbf{C} : |z| < \frac{1}{2}\}$, one can use the dominated convergence theorem to deduce that

$$\int_{\mathrm{II}_t \setminus \Delta(y_t, r_0)} \tilde{g}_{t,y_t}\hat{\omega}_t \to \int_{M^0 \setminus \Delta(y_0', r_0)} g_0(\cdot, y_0')\hat{\omega}_0 \quad \text{as } t \to 0.$$

This, together with (4.3.11), implies (iii) immediately, and we have finished the proof of Proposition 4.3.2.

Throughout the rest of this paper, L^2 norms on M_t are always with respect to ds_t^2, and are simply denoted by $\|\cdot\|_2$. Now we summarize our results in §4 in the following

Proposition 4.3.3. *Let ϕ_t be as in (4.3.3) and C_t be as in (4.3.2). We have*

(i) $\|\phi_t\|_2 \to 0$, *and*
(ii) $C_t \to 0$ *as* $t \to 0$.

Proof. For $|t|$ sufficiently small, $y_t \notin \mathrm{I}_t$. Then we have

$$\begin{aligned}
\phi_t^2 =& (\Delta_t u_t)^2 \quad \text{(by (4.3.3))}\\
=& (\Delta_t g_{t,y_t} - \Delta_t \tilde{g}_{t,y_t})^2 \quad \text{(by (4.3.1))}\\
=& \left(\frac{2\pi}{\mathrm{Vol}(M_t, \omega_t)} - \Delta_t \tilde{g}_{t,y_t}\right)^2 \quad \text{(by (2.3.2))}\\
\le& \frac{8\pi^2}{(\mathrm{Vol}(M_t, \omega_t))^2} + 2(\Delta_t \tilde{g}_{t,y_t})^2 \quad \text{on } \mathrm{I}_t.
\end{aligned}$$

Then it follows from Proposition 4.2.3 (i), (iii) and Proposition 4.3.2(i) that $\int_{\mathrm{I}_t} \phi_t^2 \omega_t \to 0$ as $t \to 0$. This, together with Proposition 4.3.2(iii), implies (i) immediately. Finally (ii) is an easy consequence of Proposition 4.2.3(ii) and Proposition 4.3.2(iii), and this finishes the proof of Proposition 4.3.3.

5. Proof of Theorem 2

(5.1) Let $\{M_t\}$ be as in Theorem 2. For $t \in \Delta^*$, we denote by $\lambda_{1,t}$ (resp. $\lambda_{1,t}^{\mathrm{hyp}}$) the first non-zero eigenvalue of the Laplacian with respect to ds_t^2 (resp. $ds^2_{\mathrm{hyp},t}$) on M_t. We shall need the following

Lemma 5.1.1. *There exists a constant $\alpha > 0$ such that $\lambda_{1,t} \geq \alpha$ for all $t \in \Delta^*$.*

Proof. It is well-known and follows from results in [SWY] and [He] that in our case of degenerating Riemann surfaces with a non-separating node, there exists a constant $\beta > 0$ such that $\lambda_{1,t}^{\mathrm{hyp}} \geq \beta$ for $t \in \Delta^*$ (see e.g. [Ji1, Corollary 3.4]). Then by minimax principle and (2.2.4), Lemma 5.1.1 follows immediately.

Lemma 5.1.2. *Let u_t, ϕ_t be as in (4.3). Then $\|u_t\|_2 \leq \dfrac{1}{\alpha}\|\phi_t\|_2$ for all $t \in \Delta^*$, where α is as in Lemma 5.1.1.*

Proof. By (2.3.3), (4.3.1) and (4.3.2), it is easy to see that $\int_{M_t} u_t\omega_t = 0$. Together with (4.3.3), we have $u_t = G_t\phi_t$, where G_t is the Green's operator on M_t with respect to ds_t^2. It is well-known that this implies $\|u_t\|_2 \leq \dfrac{1}{\lambda_{1,t}}\|\phi_t\|_2$. This, together with Lemma 5.1.1, implies Lemma 5.1.2 immediately.

Proposition 5.1.3. *Let u_t be as in (4.3) and x_t be as in Theorem 2. Then we have $u_t(x_t) \to 0$ as $t \to 0$.*

Proof. Since $x_0 \neq p$, there exists $t_0 > 0$ such that $x_t \in \mathrm{II}_t$ for all $0 < |t| < t_0$. As in the proof of Proposition 4.3.2(iii), one can find a continuous family of coordinate discs $\Delta(x_t, r) \subset M_t$ centered at x_t and of a fixed radius $r > 0$ for $0 \leq |t| < t_0$, shrinking t_0 if necessary. Shrinking r if necessary, we may assume that for $0 < |t| < t_0$, each $\Delta(x_t, r) \subset \mathrm{II}_t$, and $y_t \notin \Delta(x_t, r)$. By the relative compactness of $\bigcup_{0\leq|t|\leq t_0} \Delta(x_t, r)$ in $\{M_t\}\backslash\{p\}$ and the continuity of $\{ds_t^2\}$, there exist constants $C_1, C_2 > 0$ such that for all $|t| < t_0$,

$$C_1 dz \otimes d\bar{z} \leq ds_t^2 \leq C_2 dz \otimes d\bar{z} \text{ on } \Delta(x_t, r). \tag{5.1.1}$$

Then using Nash-Moser iteration technique (cf. e.g. [GT, Theorem 8.24]), one can deduce from (4.3.3) and (5.1.1) that there exists a constant $C > 0$

such that for $0 < |t| < t_0$,

$$\begin{aligned}|u_t(x_t)| \leq & C\left(\sqrt{\int_{\Delta(x_t,r)} u_t^2\omega_t} + \sqrt{\int_{\Delta(x_t,r)} \phi_t^2\omega_t}\right)\\ \leq & C(\|u_t\|_2 + \|\phi_t\|_2)\\ \leq & C(\frac{1}{\alpha}\|\phi_t\|_2 + \|\phi_t\|_2) \quad \text{(by Lemma 5.1.2)}\\ \to & 0 \quad \text{as } t \to 0 \quad \text{(by Proposition 4.3.3(i))}.\end{aligned}$$

Here α is as in Lemma 5.1.2, and we have finished the proof of Proposition 5.1.3.

Finally we are ready to give the following

Proof of Theorem 2. Let x_t, y_t be as in Theorem 2, and $g_{t,y_t}, \tilde{g}_{t,y_t}, x'_t, y'_t$, u_t, C_t be as in (4.1) and (4.3). By construction, $x'_0 = x_0$ and $y'_0 = y_0$ in M^0. Also, for $|t|$ sufficiently small, $x_t, y_t \in \mathrm{II}_t$ and thus from (4.1.5), $\tilde{g}_{t,y_t}(x_t) = g_0(x'_t, y'_t)$. Then it follows from the continuity of $g_0(\cdot,\cdot)$ that

$$\tilde{g}_{t,y_t}(x_t) \to g_0(x'_0, y'_0) = g_0(x_0, y_0) \quad \text{as } t \to 0. \tag{5.1.2}$$

Then

$$\begin{aligned}g_t(x_t, y_t) = & g_{t,y_t}(x_t)\\ = & u_t(x_t) + \tilde{g}_{t,y_t}(x_t) - C_t \quad \text{(by (4.3.1))}\\ \to & 0 + g_0(x_0, y_0) - 0\\ & \quad \text{(by Proposition 5.1.3, (5.1.2) and Proposition 4.3.3(ii))}\\ = & g_0(x_0, y_0) \quad \text{as } t \to 0.\end{aligned}$$

Thus we have completed the proof of Theorem 2.

(5.2) To finish our discussion, we deduce the following

Proof of Corollary 3. Let $\{ds_t^2\}$ be as in Corollary 3. By Schwarz lemma [Y], the curvature hypothesis on $\{ds_t^2\}$ implies that $\{ds_t^2\}$ satisfies (2.2.4). Together with the hypothesis on ds_0^2 on M^0, it follows that $\{ds_t^2\}$ form a continuous family of quasi-hyperbolic metrics on $\{M_t\}$, and Corollary 3 follows readily from Theorem 2.

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