Higher Rank Zeta Functions and Riemann Hypothesis for Elliptic Curves

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Outline

Stability

Setting

- $X/\mathbb{F}_q$: irreducible, reduced, regular proj curve of genus g
- F: function field, $\mathbb{A}$: adelic ring
- $\mathbb{M}_{X,n}$: moduli stack of rank n bdles of on X

  $$GL(n,F)\backslash GL(n,\mathbb{A})/\mathbb{K}$$
- $\mathbb{M}^{\mathrm{ss}}_{X,n}(d)$: moduli stack of s.stable bdles of rank n, degree d
- $m_{X,n}(d) := \sum_{V\in\mathbb{M}_{X,n}(d)} \frac{1}{\#\mathrm{Aut}(V)}$, independent of d
- $m^{\mathrm{ss}}_{X,n}(d) := \sum_{V\in\mathbb{M}^{\mathrm{ss}}_{X,n}(d)} \frac{1}{\#\mathrm{Aut}(V)}$, dependent of d
- $\widehat{\zeta}_X(s)$: complete Artin zeta function of X
- If $n = n_1 + n_2 + \cdots + n_k$, $N_i := n_1 + \cdots + n_i$, $N_i' := n - N_i$

Mumford's Intersection Stability

Stability

V/X: vector bundle
V: semi-stable $\Leftrightarrow$

$$\frac{\deg(V_1)}{\mathrm{rank}(V_1)} \leq \frac{\deg(V)}{\mathrm{rank}(V)}, \qquad \forall V_1 \leq V$$

Various Spaces

- Moduli spaces do not work well
- Moduli stacks work
- Best one: Adelic space

Tamagawa Number, Parabolic Reduction

Theorem (Tamagawa Number, Parabolic Reduction)

- (Weil) $$m_{X,n}(d)=\widehat{\zeta}_X(1)\widehat{\zeta}_X(2)\cdots\widehat{\zeta}_X(n).$$
- (Harder-Narasimhan, Desale-Ramanan and Zagier)

$$\frac{m_{X,n}^{\mathrm{ss}}(0)}{q^{\frac{n(n-1)}{2}(g-1)}}=\sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\frac{1}{\prod_{j=1}^{k-1}(q^{n_j+n_{j+1}}-1)}\cdot\prod_{j=1}^{k}m_{X,n_j}(0).$$

- (Weng-Zagier: in preparation)

$$n\cdot m_{X,n}(d)=\sum_{k=1}^{n}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\sum_{\substack{\delta_i\in\{0,1,\ldots,n_i-1\}\\ i=1,2,\ldots,k}}\prod_{i=1}^{k-1}\frac{q^{v_iN_iN_i'}}{q^{N_iN_i'}-1}\cdot\prod_{j=1}^{k}\frac{m_{X,n_j}^{\mathrm{ss}}(\delta_j)}{q^{\frac{n_j(n_j-1)}{2}(g-1)}}.$$

Here $v_i\in[0,1)\cap\mathbb{Q}$ satisfying $v_i\equiv\frac{\delta_i}{n_i}-\frac{\delta_{i+1}}{n_{i+1}}\pmod 1)$

Split Reductive Groups: Number Fields

Setting

- $F = \mathbb{Q}$: field of rational, $\mathbb{A}$: ring of adeles
- G/F: split reductive group
- B/F: Borel, P/F: standard parabolic subgroup
- $P = M \cdot N$: Levi decomposition w/ M Levi factor
- $M \sim \prod_{i=1}^{k} M_i$ simple decomposition w/ M_i's reductive
- $\mathcal{M}_{F,G}$: moduli space of G-lattices

  $\simeq G(F)\backslash G(\mathbb{A})/\mathbb{K}$
- $\mathcal{M}_{F,G}^{\mathrm{ss}}$: (compact) subspace of s.stable G-lattices
- $\nu_{F,G} := \mathrm{Vol}\Big(\mathcal{M}_{F,G}\Big), \qquad \nu_{F,G}^{\mathrm{ss}} := \mathrm{Vol}\Big(\mathcal{M}_{F,G}^{\mathrm{ss}}\Big)$

Parabolic Reduction, Stability & the Volumes

Theorem? (Weng)

- Parabolic Reduction $\exists\ c_P \in \mathbb{Q}_{>0}, e_P \in \mathbb{Q}_{>0}$

$$\nu_{F,G} = \sum_P e_P \cdot \nu_{F,P}^{\rm ss},$$

$$\nu_{F,G}^{\rm ss} = \sum_P \mathrm{sgn}(P) \cdot c_P \cdot \nu_{F,P}.$$

Here $\quad P = M \cdot N, \qquad M \sim \prod_j M_j$
and $\quad \nu_{F,P} := \prod_j \nu_{F,M_j}, \qquad \nu_{F,P}^{\rm ss} := \prod_j \nu_{F,M_j}^{\rm ss}$

- c_P, but not yet e_P: explicit expressions in terms of root system
- Different Systems of basis: $\nu_{F,G}^{\rm ss} \Leftrightarrow \nu_{F,G}$
- Non-Abelian versus Abelian: Group structures involved at c_P

Volumes of Fundamental Domains

Theorem

- (Langlands)

$$\nu_{F,G} = c_G \cdot \prod_{i \geq 1} \widehat{\zeta}_F(i)^{-n_i(G)}.$$

Here $c_G = \mathrm{Vol}\Big(\Big\{ \sum_{\alpha \in \Delta} a_\alpha \alpha^\vee : a_\alpha \in [0,1]\Big\}\Big)$
$n_i(G) := \#\{\alpha > 0, \langle \rho, \alpha^\vee \rangle = i\} - \#\{\alpha > 0, \langle \rho, \alpha^\vee \rangle = i-1\}$
with Δ: simple roots, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$

Done by taking residues of Eisenstein series

Non-Abelian versus Abelian Invariants:

Group Structures involved !!!

Example: $\mathrm{SL}_n(\mathbb{Z})$, Stable Lattices

Definition

- $\Lambda \subset \mathbb{R}^n$: rank n lattice.
- Λ semi-stable if

$$\left(\mathrm{Vol}\,\Lambda_1\right)^{\mathrm{rank}(\Lambda)} \geq \left(\mathrm{Vol}\,\Lambda\right)^{\mathrm{rank}(\Lambda_1)}, \qquad \forall \Lambda_1 \subset \Lambda.$$

- $\mathcal{M}_{\mathbb{Q},n}[1]$: moduli space of rank n lattices of vol 1 $\simeq SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n)$
- $\mathcal{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]$: (compact) subspace of s.stable lattices
- $v_{\mathbb{Q},n} := \mathrm{Vol}\Big(\mathcal{M}_{\mathbb{Q},n}[1]\Big), \qquad v_{\mathbb{Q},n}^{\mathrm{ss}} := \mathrm{Vol}\Big(\mathcal{M}_{\mathbb{Q},n}^{\mathrm{ss}}[1]\Big)$
- $\widehat{\zeta}_{\mathbb{Q}}(s)$: complete Riemann zeta function

Volume of Fund Domain, Parabolic Reduction

Theorem

- (Siegel)

$$\frac{1}{n}\cdot v_{\mathbb{Q},n}=\widehat{\zeta}_{\mathbb{Q}}(1)\widehat{\zeta}_{\mathbb{Q}}(2)\cdots\widehat{\zeta}_{\mathbb{Q}}(n).$$

- (Weng)

$$v_{\mathbb{Q},n}^{\mathrm{ss}}=\sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\frac{1}{\prod_{j=1}^{k-1}(n_j+n_{j+1})}\cdot\prod_{j=1}^{k}v_{\mathbb{Q},n_j}.$$

- (Kontsevich-Soibelman)

$$\frac{1}{n}\cdot v_{\mathbb{Q},n}=\sum_{k=1}^{n}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\frac{1}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_k)\cdots n_k}\cdot\prod_{j=1}^{k}v_{\mathbb{Q},n_j}^{\mathrm{ss}}.$$

Non-Abelian Zeta Function

Definition (Weng)

Pure Non-Abelian Zeta Function of $X/\mathbb{F}_q$:

$$\widehat{\zeta}_{X,n}(s) := \sum_{V \in \mathbb{M}_{X,n}^{\mathrm{ss}}(n\mathbb{Z})} \frac{q^{h^0(X,V)} - 1}{\#\mathrm{Aut}(V)} \cdot (q^{-s})^{\chi(X,V)} \, d\mu, \ \Re(s) > 1.$$

α, β-invariants

$$\beta_{X,n}(d) := \sum_{V \in \mathbb{M}_{X,n}^{\mathrm{ss}}(d)} \frac{1}{\#\mathrm{Aut}(V)} = m_{X,n}^{\mathrm{ss}}(d),$$

$$\alpha_{X,n}(d) := \sum_{V \in \mathbb{M}_{X,n}^{\mathrm{ss}}(d)} \frac{q^{h^0(X,V)} - 1}{\#\mathrm{Aut}(V)}$$

Zeta Facts

Let $t = q^{-s}$, $T = t^n$, $Q = q^n$,

$$\widehat{\zeta}_{X,n}(s) = \sum_{m=0}^{(g-1)-1} \alpha_{X,n}(mn) \cdot \left(T^{m-(g-1)} + \left(\frac{1}{QT}\right)^{(g-1)-m} \right) + \alpha_{X,n}(n(g-1)) + \beta_{X,n}(0) \cdot \frac{(Q-1)T}{(1-QT)(1-T)}$$

- (Initial State) $\widehat{\zeta}_{X,1}(s) = \widehat{\zeta}_X(s)$: complete Artin zeta
- (Rationality) $\widehat{\zeta}_{X,n}(s)$ is rational in T
- (Functional Eq) $\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s)$
- (Residue) $\mathrm{Res}_{s=1}\widehat{\zeta}_{X,n}(s) = \beta_{X,n}(0)$

Number Fields versus Function Fields

Parabolic Reduction & Periods

Parabolic Reduction: (i) Number Fields (Weng)

$$v_{\mathbb{Q},n}^{\mathrm{ss}}=\sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\frac{1}{\prod_{j=1}^{k-1}(n_j+n_{j+1})}\cdot\prod_{j=1}^{k}v_{\mathbb{Q},n_j}.$$

(ii) Function Fields/$\mathbb{F}_q$ (Zagier)

$$m_{E,n}^{\mathrm{ss}}(0)=\sum_{k=1}^{n}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\frac{(-1)^{k-1}}{\prod_{j=1}^{k-1}(q^{n_j+n_{j+1}}-1)}\cdot\prod_{j=1}^{k}m_{E,n_j}(0).$$

Periods of G: Number Fields (Weng)

$$\omega_{\mathbb{Q}}^{G}(\lambda):=\sum_{w\in W}\Big(\frac{1}{\prod_{\alpha\in\Delta}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\widehat{\zeta}_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle+1)}\Big)$$

Period of G/X

Setting

- G/F: split reductive group, B/F: Borel
- P/F: maximal standard parabolic subgroup
- Δ: Simple roots, W: Weyl group , ρ: Weyl vector
- $\alpha_P \in \Delta$: simple for P
- $\{\beta_1, \ldots, \beta_{|P|}\} = \Delta \backslash \{\alpha_P\}$

Definition (Weng)

Period of G/F:

$$\omega_X^G(\lambda) := \sum_{w \in W} \left(\frac{1}{\prod_{\alpha \in \Delta}(1 - q^{-\langle w\lambda - \rho, \alpha^\vee \rangle})} \cdot \prod_{\alpha > 0, w\alpha < 0} \frac{\widehat{\zeta}_X(\langle \lambda, \alpha^\vee \rangle)}{\widehat{\zeta}_X(\langle \lambda, \alpha^\vee \rangle + 1)} \right)$$

Period of $(G, P)/F$

Definition (Weng)

Period of $(G, P)/F$:

$$\omega_X^{G/P}(s)$$

$$:= \mathrm{Res}_{\langle\lambda-\rho,\beta_{1,P}^\vee\rangle=0,\ \langle\lambda-\rho,\beta_{2,P}^\vee\rangle=0,\ \dots,\ \langle\lambda-\rho,\beta_{|G|-1,P}^\vee\rangle=0}\Big(\omega_X^G(\lambda)\Big)$$

$$:= \mathrm{Res}_{\langle\lambda-\rho,\beta_{1,P}^\vee\rangle=0,\ \langle\lambda-\rho,\beta_{2,P}^\vee\rangle=0,\ \dots,\ \langle\lambda-\rho,\beta_{|G|-1,P}^\vee\rangle=0}$$

$$\sum_{w\in W}\left(\frac{1}{\prod_{\alpha\in\Delta}(1-q^{\langle w\lambda-\rho,\alpha^\vee\rangle})}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle+1)}\right)$$

- Various symmetries play key roles here.

Special Uniformity of Zetas

Definition (Weng)

Zeta function $\widehat{\zeta}_X^{G/P}$ for $(G, P)/F$:

$$\widehat{\zeta}_X^{G/P}(s) := \mathrm{Norm}\Big(\omega_X^{G/P}(s)\Big)$$

Zeta Facts (Weng)

- Functional Equation
$$\widehat{\zeta}_X^{G/P}(1-s) = \widehat{\zeta}_X^{G/P}(s)$$
- Special Uniformity of Zetas
$$\widehat{\zeta}_{X,n}(s) \doteq \widehat{\zeta}_X^{SL_n/P_{n-1,1}}(s)$$

Example of Special Uniformity

$(\mathrm{SL}_3, P_{2,1})$

$$\widehat{\zeta}_{X/\mathbb{F}_q}^{SL_3/P_{2,1}}(s) = \frac{\widehat{\zeta}_X(2)\widehat{\zeta}_X(3s)}{1-q^{3-3s}} + \frac{\widehat{\zeta}_X(2)\widehat{\zeta}_X(3s-2)}{1-q^{3s}} + \frac{\widehat{\zeta}_X(1)\widehat{\zeta}_X(3s-1)}{(1-q^{3s})(1-q^{3-3s})} + \frac{\widehat{\zeta}_X(1)\widehat{\zeta}_X(3s-2)}{(1-q^2)(1-q^{3s-1})} + \frac{\widehat{\zeta}_X(1)\widehat{\zeta}_X(3s)}{(1-q^2)(1-q^{2-3s})}.$$

This is essentially rank 3 zeta function $\widehat{\zeta}_{X,3}(s)$.
Non-Abelian versus Abelian Invariants:

Group Structures Involved

Consequence of Special Uniformity

Alpha and Beta Invariants

α and β invariants: completely determined by
(i) the Lie structure; and
(ii) Special values of Artin zetas

Say, for rank 2:

$$\frac{\alpha_{X,2}(2m)}{\alpha_{X,2}(0)} = \sum_{i=0}^{m} q^{2(m-i)}\alpha_{X,1}(i) - \frac{1}{q^{g-1}}\sum_{i=0}^{m-1} q^{i}\alpha_{X,1}(i)$$

Intrinsic Relation among different Brill-Noether Loci

The Riemann Hypothesis

The Riemann Hypothesis

$$\widehat{\zeta}_{X,n}(s) = 0 \qquad \Rightarrow \qquad \Re(s) = \frac{1}{2}$$

Non-Abelian Zetas for elliptic curves:

$E/\mathbb{F}_q$: elliptic curve

$$\widehat{\zeta}_{E,n}(s) = \alpha_{E,n}(0) + \beta_{E,n}(0)\frac{(Q-1)T}{(1-T)(1-QT)}$$

Main Theorem (Hasse: $n = 1$, Weng-Zagier: $n \geq 2$)

$$\widehat{\zeta}_{E,n}(s) = 0 \qquad \Rightarrow \qquad \Re(s) = \frac{1}{2}$$

Counting Miracle

Theorem A (Counting Miracle: Weng-Zagier)

$$\alpha_{E,n+1}(0) = \sum_{V \in \mathbb{M}^{ss}_{E,n+1}(0)} \frac{q^{h^0(E,V)} - 1}{\#\mathrm{Aut}\,V} = \sum_{V \in \mathbb{M}^{ss}_{E,n}(0)} \frac{1}{\#\mathrm{Aut}\,V} = \beta_{E,n}(0)$$

Corollary

$E/\mathbb{F}_q$: elliptic curve

$$\widehat{\zeta}_{E,n}(s) = \beta_{E,n-1}(0) + \beta_{E,n}(0)\frac{(Q-1)T}{(1-T)(1-QT)}$$

Beta Invariants

Parabolic Reduction: Zagier's formula

$$\beta_{E,n}(0)=\sum_{k=1}^{n}\sum_{\substack{n_1+\cdots+n_k=n\\ n_1>0,\cdots,n_k>0}}\frac{(-1)^{k-1}}{\prod_{j=1}^{k-1}(q^{n_j+n_{j+1}}-1)}\cdot\prod_{j=1}^{k}\prod_{i=1}^{n_j}\zeta_E(i).$$

Theorem B (Multiplicative Structure of Beta: Weng-Zagier)

$$\mathcal{B}_{E/\mathbb{F}_q}(x)=:\sum_{n=0}^{\infty}\beta_{B,n}(0)x^n \overset{x=q^{-s}}{=} \prod_{n\geq 1}\zeta_E(s+n)$$

It is a wonderful world!!!

Proof of the RH

Proof of the RH

(i) (Recursion)

$$\beta_{E,n}(0) = \frac{q^n + q^{n+1} - a}{q^n - 1}\beta_{E,n-1}(0) - \frac{q^{n-1} - q}{q^n - 1}\beta_{E,n-2}(0).$$

(ii) (The Estimation)

$$\frac{\sqrt{q^n} - 1}{\sqrt{q^n} + 1} \leq \frac{\beta_{E,n}(0)}{\alpha_{E,n}(0)} \leq \frac{\sqrt{q^n} + 1}{\sqrt{q^n} - 1}$$

(iii) (The Riemann Hypothesis)

$$\widehat{\zeta}_{E,n}(s) = 0 \qquad \Rightarrow \qquad \Re(s) = \frac{1}{2}$$

Special Counting Miracle

Atiyah Bundles

$E/\mathbb{F}_q$: elliptic curve
Inductively define (indecomposible) Atiyah Bundle I_n/E:
$I_1 = \mathcal{O}_E$
$I_n =$ the only non-trivial extension of I_{n-1} by $\mathcal{O}_E$:

$$0 \to \mathcal{O}_E \to I_n \to I_{n-1} \to 0$$

since

$$\mathrm{Ext}^1(I_n, \mathcal{O}_E) \simeq H^1(E, I_n^\vee) \simeq H^0(E, I_n^\vee)^\vee = H^0(E, I_n) \simeq \mathbb{F}_q$$

Automorphisms

Automorphisms

Consider Atiyah bundles $\oplus_{j=1}^{s} I_{r_j}^{\oplus m_j}$ $(0 < r_1 < r_2 < \cdots < r_s)$

- (Atiyah) $h^0(E, \oplus_{j=1}^{s} I_{r_j}^{\oplus m_j}) = \sum_{j=1}^{s} m_j$
- (Weng-Zagier)

$$\#\mathrm{Aut}\Big(\oplus_{j=1}^{s} I_{r_j}^{\oplus m_j} \Big) = q^{2\sum_{1\leq i<j\leq s} r_i m_i m_j} \times \prod_{j=1}^{s} (q^{m_j}-1)(q^{m_j}-q)\cdots(q^{m_j}-q^{m_j-1})q^{m_j^2(r_j-1)}.$$

Special Counting Miracle

Theorem A′ (Counting Miracle: Weng-Zagier)

(i) (Special Counting Miracle: Combinatorial Aspect)

$$\alpha_{E,n+1}^{\mathrm{At}}(0) := \sum_{V:\text{Atiyah Bdl, rank}=n+1} \frac{q^{h^0(E,V)}-1}{\#\mathrm{Aut}(V)} = \frac{q^{\frac{n(n-1)}{2}}}{(q^n-1)(q^{n-1}-1)\cdots(q-1)} = \sum_{V:\text{Atiyah Bdl, rank}=n} \frac{1}{\#\mathrm{Aut}(V)} = \beta_{E,n}^{\mathrm{At}}(0)$$

(ii) (Geometric Aspect) Special CM $\Leftrightarrow$ General CM

General Counting Miracle

Geometric Aspect

Very difficulty to classify semi-stable vector bundles of degree 0 defined over $\mathbb{F}_q$:

Rationality Problem – intrinsic arithmetic information such as number of m-torsion points over $\mathbb{F}_q^k$ should be used.

But for V appeared in the α invariant, $h^0(E, V) \neq 0$ implies that there is at least one factor of Atiyah bundles;
Moreover, this factor admits no non-trivial morphisms to other factors

This gives the deduction without details calculation in general

Special Counting Miracle

Combinatorial Aspect: Generating Functions

We have $\mathcal{A}^{\mathrm{At}}(x) = \mathcal{B}^{\mathrm{At}}(x) = \sum_{n=0}^{\infty} \varepsilon(n)\left(\frac{x}{q}\right)^n$

where $\varepsilon(m) := \dfrac{q^{m^2}}{(q^m-1)(q^m-q)\cdots(q^m-q^{m-1})}$ and

$$\mathcal{A}^{\mathrm{At}}(x) := \sum_{s=0}^{\infty} \sum_{\substack{0<r_1<r_2<\cdots<r_s \\ m_1>0,\, m_2>0,\, \ldots,\, m_s>0}} \frac{\varepsilon(m_1)\cdots\varepsilon(m_s)}{q^{N(\mathbf{r},m)}} \cdot x^{r_1m_1+\cdots+r_sm_s},$$

$$\mathcal{B}^{\mathrm{At}}(x) := \frac{1}{x}\sum_{s=0}^{\infty} \sum_{\substack{0<r_1<r_2<\cdots<r_s \\ m_1>0,\, m_2>0,\, \ldots,\, m_s>0}} (q^{m_1+\cdots+m_s}-1)\frac{\varepsilon(m_1)\cdots\varepsilon(m_s)}{q^{N(\mathbf{r},m)}} \times x^{r_1m_1+\cdots+r_sm_s}$$

Sugahara's Result

Theorem (Sugahara)

$X/\mathbb{F}_q$: irreducible, reduced, regular projective curve of genus g:

$$\alpha_{X,n+1}(0) = q^{n(g-1)} \cdot \beta_{X,n}(0)$$

Totally different approach:

motivated by the work of Reineke on Quives

Sato-Tate

Setting

- p: prime, $N = \#X(\mathbb{F}_p)$, $a = p + 1 - N$
- $\cos\theta^E_{1,p} := \frac{p+1-N}{2\sqrt{p}}, \qquad 0 \leq \theta^E_{1,p} < \pi$
- $\cos\theta^E_{n,p} := \dfrac{-(p^n-1)\cdot\frac{\beta_{E,n}(0)}{\beta_{E,n-1}(0)} + (p^n+1)}{2\sqrt{p^n}}$

Sato-Tate Conjecture

For non CM elliptic curves E, and $0 \leq \alpha < \beta \leq \pi$,

$$\lim_{x\to\infty} \frac{\#\{p\ :\ \text{prime},\ p \leq x,\ \alpha \leq \theta^E_{1,p} \leq \beta\}}{\#\{p\ :\ \text{prime},\ p \leq x\}} = \frac{1}{\pi}\int_\alpha^\beta \sin^2\theta\, d\theta.$$

Higher Sato-Tate

Higher Sato-Tate (Weng-Zagier?)

For non CM elliptic curves E, and $0 \leq \alpha < \beta \leq \pi$,

$$\frac{\#\left\{p : \begin{matrix}\text{prime},\\ p \leq x,\end{matrix} \sin\left(\frac{\pi}{2}-\alpha\right) \stackrel{\bullet}{\geq} \boxed{\frac{p^{\frac{n-1}{2}}}{n-1}\cdot\left(\frac{\pi}{2}-\theta_{n,p}^{E}\right)+\frac{1}{2}\left(\sqrt{p}+\frac{1}{\sqrt{p}}\right)} \stackrel{\bullet}{\geq} \sin\left(\frac{\pi}{2}-\beta\right)\right\}}{\#\{p : \text{prime}, p \leq x\}}$$

$$\longrightarrow \frac{1}{\pi}\int_{\alpha}^{\beta} \sin^2\theta\, d\theta, \qquad \text{as } x \to \infty$$

Sato-Tate $\Rightarrow$ Higher Sato-Tate

- (Weng-Zagier) As $q \to \infty$,

$$\frac{\beta_{E,n}(0)}{\beta_{E,n-1}(0)} = 1 + \frac{\boxed{(n-1)(q-a+1)} - 2 + 3(a-2)/q + \cdots}{q^n} + O\left(\frac{n^2}{q^{2n-2}}\right)$$

Local Zetas for nodal curves

Local zeta for nodal curve/$\mathbb{F}_q$

$X/\mathbb{F}_q$: Singular but nodal curve
$\Rightarrow$ Semi-stable bundles make sense $\Rightarrow$
Zetas for Nodal Curves:

$$\widehat{\zeta}_{X,n}(s) = \alpha_{X,n}(0) \cdot \frac{P_{X,n}(q^{-ns})}{(1-q^{-ns})(1-q^{n(1-s)})}$$

$P_{X,n}(t^n)$: deg $< 2g$ polynomial in $T = t^n$ with constant term 1

Global Zetas

Global Zeta: a definition

$X/\mathbb{Q}$: regular curve
$\mathfrak{X}/\mathbb{Z}$: a semi-stable model
X_p: semi-stable reduction at p $\Rightarrow$
New Global Zetas of $\mathfrak{X}/\mathbb{Z}$:

$$\widehat{\zeta}_{\mathfrak{X},n}(s) := \Big(\Gamma_{\mathbb{R}}(ns)\Gamma_{\mathbb{R}}(n(s-1))\Big) \cdot \prod_p P_{X_p,n}^{-1}(p^{-ns})$$

Analytic and Arithmetic Properties

Central Questions

- How can we meromorphic extend them?
- What kinds of arithmetic they offer?

Trace Formula for Algebraic Stacks

Behrend's Conjecture

$\mathfrak{X}$: smooth algebraic $\overline{\mathbb{F}_q}$-stack defined over $\mathbb{F}_q$
Φ_q: arithmetic Frobenius
$H^*(\mathfrak{X}_{\rm sm}, \mathbb{Q}_l)$: smooth cohomology of $\mathfrak{X}$
$\#\mathfrak{X}(\mathbb{F}_q) := \sum_{\xi\in[\mathfrak{X}]} \frac{1}{\#\mathrm{Aut}(\xi)}$ = # of $\mathbb{F}_q$-rational points of $\mathfrak{X}$

$$\boxed{q^{\dim \mathfrak{X}} \cdot \mathrm{Tr}\,\Phi_q \,|\, H^*(\mathfrak{X}_{\rm sm}, \mathbb{Q}_l) = \#\mathfrak{X}(\mathbb{F}_q)}$$

• Analogue of Weil Conjecture ?!
• What are the reasons behind the RH for our pure non-abelian zeta functions ?!

β-Invariants for G-Bundles

Definition

- $(G, P_0, M_0; P, M, N; \mathfrak{a}_P = \mathfrak{a}_P^G \oplus \mathfrak{a}_G, \lambda = [\lambda]_P^G + [\lambda]_G, \mathfrak{a}_P^*;$ $\Phi_P^+ \supset \Delta_P; \rho_P = \frac{1}{2}\sum_{\alpha\in\Phi_P^+}\alpha; \alpha \in \Delta_P, \alpha^\vee$ coroot ass. to $\alpha;$ $\{\varpi_\alpha^G\}$ dual basis to $\{\alpha^\vee : \alpha \in \Delta_P\})$
- $\mathcal{P}$: collection of standard parabolic subgroups
- Beta invariant of the moduli stack $\mathbb{M}_{X,G}^{ss}(\lambda_G)$ of semi-stable G-bundles on X with slope λ_G

$$\beta_{X,G}(\lambda_G) := \#\mathbb{M}_{X,G}^{ss}(\lambda_G)$$

- $\beta_{X,G}^{\text{total}}(\lambda_G)$: *beta* invariant of the moduli stack of all G-bundles on X with slope λ_G: Independent on λ_G

β-Invariants for G-Bundles

Conjectural Formula for β-Invariants of G-Bundles (Weng)

$$\frac{\beta_{X,G}(\lambda_G)}{q^{\dim N\cdot(g-1)}} = \sum_{P\subset G:\ \text{standard parabolic}} (-1)^{\dim \mathfrak{a}_P^G} \times \sum_{\lambda\in\Lambda_P^G,\,[\lambda]_G=\lambda_G} \prod_{\alpha\in\Delta_P} \frac{q^{2\cdot\langle\rho_P,\alpha^\vee\rangle\cdot<\varpi_\alpha^G(\lambda)>}}{q^{2\cdot\langle\rho_P,\alpha^\vee\rangle}-1}\cdot\prod_i \beta_{X,M_i}^{\text{total}}(0)$$

where: $<x>=1+[x]-x$, $\Lambda_P^G := X_*(A_P')\Big/\sum_{\alpha\in\Delta_P^G}\mathbb{Z}\alpha^\vee$ and $M\sim\prod_i M_i$: induced from the Lie level simple decomposition for the Levi factor M.

Motivated also by Atiyah-Bott and Laumon-Rapoport

Key: normalization to make it independent of the environment !!!

Thank You

Thank You

Tokyo, 30, 01, 2013