

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E

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ABSTRACT. A selection of 128 elements form Weyl groups for exceptional groups of type E_8 with 696,729,600 elements is given. As a direct consequence, we then prove the weak Riemann Hypothesis for the so-called Weng zeta functions of E_8 . Moreover, we explain what is the meaning of these 128 ($= 2^7$) elements in terms of the masses of semi-stable principal E_8 -lattices, following our earlier conjecture on Stability, Parabolic Reduction, and the Masses. (This last part to be extended.)

1. Root Systems

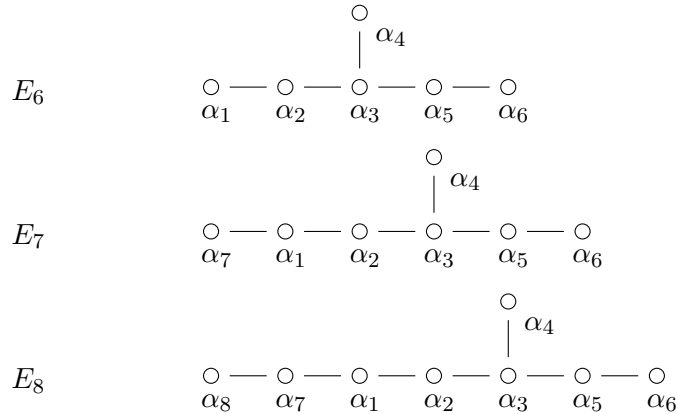
Following structures of the associated Weyl groups, to realize the set of simple roots for the exceptional group of types E_n ($n = 6, 7, 8$), we choose an oriented basis of $\mathbb{R}^8$ as follows:

$$\begin{aligned}\alpha_1 &= (0, 0, 0, -1, 1, 0, 0, 0), & \alpha_2 &= (0, 0, -1, 1, 0, 0, 0, 0), \\ \alpha_3 &= (0, -1, 1, 0, 0, 0, 0, 0), & \alpha_4 &= (1, 1, 0, 0, 0, 0, 0, 0), \\ \alpha_5 &= (-1, 1, 0, 0, 0, 0, 0, 0), & \alpha_6 &= \frac{1}{2}(1, -1, -1, -1, -1, -1, -1, 1), \\ \alpha_7 &= (0, 0, 0, 0, -1, 1, 0, 0), & \alpha_8 &= (0, 0, 0, 0, 0, -1, 1, 0).\end{aligned}$$

Accordingly, we will take the sets Δ_n ($n = 6, 7, 8$) of simple roots for root systems of Φ_n for types E_n , to be

$$\begin{aligned}\Delta_6 &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}, \\ \Delta_7 &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}, \\ \Delta_8 &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}.\end{aligned}$$

Clearly, the corresponding Dynkin diagrams are given as follows:



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2. Weyl Groups

2.1. Weyl Group for D_5 . Let W_n be the Weyl groups of the exceptional group of type E_n , $n = 6, 7, 8$. Then natural inclusions $\Phi_6 \subset \Phi_7 \subset \Phi_8$ of the roots systems give rise to natural inclusions $W_6 \leq W_7 \leq W_8$ of groups. If, in addition, we set $\Delta_5 := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Then Δ_5 can be understood as a set of simple roots for the D_5 of D -series group with the Dynkin diagram

$$D_5 \quad \begin{array}{ccccccc} & & & & \circ & & \\ & & & & | & & \\ & & & & \alpha_4 & & \\ \circ & - & \circ & - & \circ & - & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_5 \end{array}$$

Denote its associated roots system by Φ_5 and Weyl group by W_5 . We have two chains $\Phi_5 \subset \Phi_6 \subset \Phi_7 \subset \Phi_8$ and $W_5 \leq W_6 \leq W_7 \leq W_8$. Our method to build W_n starts with W_5 , which is known to be the semi-product $\mathfrak{S}_5 \rtimes \text{Sign}_{\text{ev}}$ where $\mathfrak{S}_5$ denotes the 5-th symmetric group acting as follows: for any $\sigma \in \mathfrak{S}_5$,

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_6, x_7, x_8),$$

and Sign_{ev} consists of even sign changes for the first 5 coordinates. Thus, totally, $\#W_5 = \#\mathfrak{S}_5 \times 2^{\binom{5}{0} + \binom{5}{2} + \binom{5}{4}} = 1920$.

2.2. Weyl Group for E_6 . From now on, we will view all groups W_n , $n = 5, 6, 7, 8$ as subgroups of $\text{Aut}_{\mathbb{R}}(\mathbb{R}^8)$. Thus, to obtain the group W_6 , we only need to find out what are the representatives of the coset W_6/W_5 . For this, we first set $\Phi_{6,5}^+ := \{\alpha \in \Phi_6^+ : \alpha \notin \Phi_5^+\}$. It consists of 16 elements. Namely,

$$\left\{ \begin{array}{ll} (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \end{array} \right\}$$

which we denote by $\{\alpha_{6,1}, \alpha_{6,2}, \dots, \alpha_{6,16}\}$. Accordingly, we set the reflections associated to $\alpha_{6,i}$ by $\sigma_{6,i}$, $i = 1, 2, \dots, 16$.

Our first result, one calculated using Mathematica together with Katayama ([Ka]), is the following:

Lemma 1. *The representatives of W_6/W_5 can be chosen as*

$$\Sigma_6 = \left\{ \text{Id}, \sigma_{6,1}, \sigma_{6,2}, \sigma_{6,3}, \sigma_{6,4}, \sigma_{6,5}, \sigma_{6,6}, \sigma_{6,7}, \sigma_{6,8}, \sigma_{6,9}, \sigma_{6,10}, \right. \\ \sigma_{6,11}, \sigma_{6,12}, \sigma_{6,13}, \sigma_{6,14}, \sigma_{6,15}, \sigma_{6,16}, \\ \sigma_{6,1}\sigma_{6,8}, \sigma_{6,1}\sigma_{6,12}, \sigma_{6,1}\sigma_{6,14}, \sigma_{6,1}\sigma_{6,15}, \sigma_{6,1}\sigma_{6,16}, \\ \left. \sigma_{6,2}\sigma_{6,15}, \sigma_{6,2}\sigma_{6,16}, \sigma_{6,3}\sigma_{6,16}, \sigma_{6,5}\sigma_{6,16}, \sigma_{6,9}\sigma_{6,16} \right\}$$

As to be expected, it consists of 27 elements. Hence $\#W_6 = 27 \times 1920 = 51,840$. Based on this calculation, we could calculate the zeta functions for

(E_6, P_1) where P_1 is the maximal parabolic subgroup of E_7 corresponding to the subset $\Delta_6 \setminus \{\alpha_1\}$. For later use, we denote Σ_6 by $\{\tau_{6,i}, i = 0, 1, 2, \dots, 26\}$, where $\tau_{6,i}$ stands for the $i + 1$ -st element in the previous display for Σ_6 .

The calculation, using Mathematica, is not very difficult, even the choice at the beginning for the orders of simple roots already proves to be crucial.

2.3. Weyl Group for E_7 . To construct the Weyl group W_7 for the exceptional group of type E_7 , we use W_6 as presented in the above lemma and yet-to-be-found representatives of W_7/W_6 , which consists of 56 elements.

We first introduce $\Phi_{7,6}^+ := \{\alpha \in \Phi_7^+ : \alpha \notin \Phi_6^+\}$, which consists of 27 elements. Namely,

$$\left\{ \begin{array}{ll} (-1, 0, 0, 0, 0, 1, 0, 0), & (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (0, -1, 0, 0, 0, 1, 0, 0), \\ (0, 0, -1, 0, 0, 1, 0, 0), & (0, 0, 0, -1, 0, 1, 0, 0), \\ (0, 0, 0, 0, -1, 1, 0, 0), & (0, 0, 0, 0, 0, 0, -1, 1), \\ (0, 0, 0, 0, 1, 1, 0, 0), & (0, 0, 0, 1, 0, 1, 0, 0), \\ (0, 0, 1, 0, 0, 1, 0, 0), & (0, 1, 0, 0, 0, 1, 0, 0), \\ (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ (1, 0, 0, 0, 0, 1, 0, 0) & \end{array} \right\}.$$

We denote them, with above displayed order, by $\{\alpha_{7,1}, \alpha_{7,2}, \dots, \alpha_{7,27}\}$, and denote the reflection associated to $\alpha_{7,j}$ by $\sigma_{7,j}$, $j = 1, 2, \dots, 27$.

Proposition 2. *A group of representatives for W_7/W_6 can be chosen as*

$$\begin{aligned} \Sigma_7 = \{ & \text{Id}, \sigma_{7,1}, \sigma_{7,2}, \sigma_{7,3}, \sigma_{7,4}, \sigma_{7,5}, \sigma_{7,6}, \sigma_{7,7}, \sigma_{7,8}, \sigma_{7,9}, \sigma_{7,10}, \\ & \sigma_{7,11}, \sigma_{7,12}, \sigma_{7,13}, \sigma_{7,14}, \sigma_{7,15}, \sigma_{7,16}, \sigma_{7,17}, \sigma_{7,18}, \sigma_{7,19}, \sigma_{7,20}, \\ & \sigma_{7,21}, \sigma_{7,22}, \sigma_{7,23}, \sigma_{7,24}, \sigma_{7,25}, \sigma_{7,26}, \sigma_{7,27}, \\ & \sigma_{7,1}\sigma_{7,14}, \sigma_{7,1}\sigma_{7,19}, \sigma_{7,1}\sigma_{7,20}, \sigma_{7,1}\sigma_{7,21}, \sigma_{7,1}\sigma_{7,22}, \\ & \sigma_{7,1}\sigma_{7,23}, \sigma_{7,1}\sigma_{7,24}, \sigma_{7,1}\sigma_{7,25}, \sigma_{7,1}\sigma_{7,26}, \sigma_{7,1}\sigma_{7,27}, \\ & \sigma_{7,2}\sigma_{7,22}, \sigma_{7,2}\sigma_{7,24}, \sigma_{7,2}\sigma_{7,25}, \sigma_{7,2}\sigma_{7,26}, \sigma_{7,2}\sigma_{7,27}, \\ & \sigma_{7,3}\sigma_{7,25}, \sigma_{7,3}\sigma_{7,26}, \sigma_{7,3}\sigma_{7,27}, \\ & \sigma_{7,4}\sigma_{7,26}, \sigma_{7,4}\sigma_{7,27}, \sigma_{7,5}\sigma_{7,27}, \sigma_{7,6}\sigma_{7,26}, \sigma_{7,6}\sigma_{7,27}, \\ & \sigma_{7,7}\sigma_{7,27}, \sigma_{7,8}\sigma_{7,27}, \sigma_{7,9}\sigma_{7,27}, \sigma_{7,14}\sigma_{7,27}, \sigma_{7,1}\sigma_{7,14}\sigma_{7,27} \} \end{aligned}$$

With $\#W_7 = 2,903,040$, the calculation using Mathematica is still manageable: It completed in one night and we even can store W_7 as a single set in our Power MacBook. Here, as above, the special orders for simple roots plays a very crucial role. For later use, we denote Σ_7 by $\{\tau_{7,j}, j = 0, 1, 2, \dots, 55\}$, where $\tau_{7,j}$ denotes the $j + 1$ -st elements in the previous display for Σ_7 .

2.4. Weyl Group for E_8 . Construction of W_8 , consisting of 696,729,600 elements, proves to be extremely challenging. Even in terms of representatives of W_8/W_7 , there are 240 elements to be found. With our program in Mathematica, it took a couple of weeks to find the answer.

As the first step, we calculate $\Phi_{8,7}^+ := \{\alpha \in \Phi_8^+ : \alpha \notin \Phi_7^+\}$. It consists of 57 elements. Namely,

$$\left\{ \begin{array}{ll} (-1, 0, 0, 0, 0, 0, 0, 1), & (-1, 0, 0, 0, 0, 0, 1, 0), \\ (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (0, -1, 0, 0, 0, 0, 0, 1), & (0, -1, 0, 0, 0, 0, 1, 0), \\ (0, 0, -1, 0, 0, 0, 0, 1), & (0, 0, -1, 0, 0, 0, 1, 0), \\ (0, 0, 0, -1, 0, 0, 0, 1), & (0, 0, 0, -1, 0, 0, 1, 0), \\ (0, 0, 0, 0, -1, 0, 0, 1), & (0, 0, 0, 0, -1, 0, 1, 0), \\ (0, 0, 0, 0, 0, -1, 0, 1), & (0, 0, 0, 0, 0, -1, 1, 0), \\ (0, 0, 0, 0, 0, 0, 1, 1), & (0, 0, 0, 0, 0, 1, 0, 1), \\ (0, 0, 0, 0, 0, 1, 1, 0), & (0, 0, 0, 0, 1, 0, 0, 1), \\ (0, 0, 0, 0, 1, 0, 1, 0), & (0, 0, 0, 1, 0, 0, 0, 1), \\ (0, 0, 0, 1, 0, 0, 1, 0), & (0, 0, 1, 0, 0, 0, 0, 1), \\ (0, 0, 1, 0, 0, 0, 1, 0), & (0, 1, 0, 0, 0, 0, 0, 1), \\ (0, 1, 0, 0, 0, 0, 1, 0), & (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & (1, 0, 0, 0, 0, 0, 0, 1), \\ (1, 0, 0, 0, 0, 0, 1, 0) & \end{array} \right\}$$

We denote them, with above displayed order, by $\{\alpha_{8,1}, \alpha_{8,2}, \dots, \alpha_{8,57}\}$, and denote the reflection associated to $\alpha_{8,k}$ by $\sigma_{8,k}$, $k = 1, 2, \dots, 57$.

With these, then we are ready to state our first main result on the construction of W_8 .

Theorem 3. *A group of representatives for W_8/W_7 can be chosen as*

$$\begin{aligned} \Sigma_8 = \{ & \text{Id}, \sigma_{8,1}, \sigma_{8,2}, \sigma_{8,3}, \sigma_{8,4}, \sigma_{8,5}, \sigma_{8,6}, \sigma_{8,7}, \sigma_{8,8}, \sigma_{8,9}, \sigma_{8,10}, \\ & \sigma_{8,11}, \sigma_{8,12}, \sigma_{13}, \sigma_{8,14}, \sigma_{8,15}, \sigma_{8,16}, \sigma_{8,17}, \sigma_{8,18}, \sigma_{8,19}, \sigma_{8,20}, \\ & \sigma_{8,21}, \sigma_{8,22}, \sigma_{8,23}, \quad 24, \sigma_{8,25}, \sigma_{8,26}, \sigma_{8,27}, \sigma_{8,28}, \sigma_{8,29}, \sigma_{8,30}, \\ & \sigma_{8,31}, \sigma_{8,32}, \sigma_{8,33}, \sigma_{8,34}, \quad 35, \sigma_{8,36}, \sigma_{8,37}, \sigma_{8,38}, \sigma_{8,39}, \sigma_{8,40}, \\ & \sigma_{8,41}, \sigma_{8,42}, \sigma_{8,43}, \sigma_{8,44}, \sigma_{8,45}, \quad 46, \sigma_{8,47}, \sigma_{8,48}, \sigma_{8,49}, \sigma_{8,50}, \\ & \sigma_{8,51}, \sigma_{8,52}, \sigma_{8,53}, \sigma_{8,54}, \sigma_{8,55}, \sigma_{8,56}, \sigma_{57}, \\ & \sigma_{8,1}\sigma_{8,20}, \sigma_{8,1}\sigma_{8,22}, \sigma_{8,1}\sigma_{8,24}, \sigma_{8,1}\sigma_{8,26}, \sigma_{8,1}\sigma_{8,28}, \sigma_{8,1}\sigma_{8,29}, \\ & \sigma_{8,1}\sigma_{8,31}, \sigma_{8,1}\sigma_{8,33}, \sigma_{8,1}\sigma_{8,35}, \sigma_{8,1}\sigma_{8,37}, \sigma_{8,1}\sigma_{8,39}, \sigma_{8,1}\sigma_{8,40}, \\ & \sigma_{8,1}\sigma_{8,41}, \sigma_{8,1}\sigma_{8,42}, \sigma_{8,1}\sigma_{8,43}, \sigma_{8,1}\sigma_{8,44}, \sigma_{8,1}\sigma_{8,45}, \sigma_{8,1}\sigma_{8,46}, \\ & \sigma_{8,1}\sigma_{8,47}, \sigma_{8,1}\sigma_{8,48}, \sigma_{8,1}\sigma_{8,49}, \sigma_{8,1}\sigma_{8,50}, \sigma_{8,1}\sigma_{8,51}, \sigma_{8,1}\sigma_{8,52}, \\ & \sigma_{8,1}\sigma_{8,53}, \sigma_{8,1}\sigma_{8,54}, \sigma_{8,1}\sigma_{8,55}, \sigma_{8,1}\sigma_{8,56}, \sigma_{8,1}\sigma_{8,57}, \\ & \sigma_{8,2}\sigma_{8,29}, \sigma_{8,2}\sigma_{8,40}, \sigma_{8,2}\sigma_{8,41}, \sigma_{8,2}\sigma_{8,42}, \sigma_{8,2}\sigma_{8,43}, \sigma_{8,2}\sigma_{8,44}, \\ & \sigma_{8,2}\sigma_{8,45}, \sigma_{8,2}\sigma_{8,46}, \sigma_{8,2}\sigma_{8,47}, \sigma_{8,2}\sigma_{8,48}, \sigma_{8,2}\sigma_{8,49}, \sigma_{8,2}\sigma_{8,50}, \\ & \sigma_{8,2}\sigma_{8,51}, \sigma_{8,2}\sigma_{8,52}, \sigma_{8,2}\sigma_{8,53}, \sigma_{8,2}\sigma_{8,54}, \sigma_{8,2}\sigma_{8,55}, \sigma_{8,2}\sigma_{8,56}, \sigma_{8,2}\sigma_{8,57}, \\ & \sigma_{8,3}\sigma_{8,29}, \sigma_{8,3}\sigma_{8,43}, \sigma_{8,3}\sigma_{8,45}, \sigma_{8,3}\sigma_{8,46}, \sigma_{8,3}\sigma_{8,47}, \sigma_{8,3}\sigma_{8,49}, \sigma_{8,3}\sigma_{8,50}, \\ & \sigma_{8,3}\sigma_{8,51}, \sigma_{8,3}\sigma_{8,52}, \sigma_{8,3}\sigma_{8,53}, \sigma_{8,3}\sigma_{8,54}, \sigma_{8,3}\sigma_{8,55}, \sigma_{8,3}\sigma_{8,56}, \sigma_{8,3}\sigma_{8,57}, \\ & \sigma_{8,4}\sigma_{8,29}, \sigma_{8,4}\sigma_{8,46}, \sigma_{8,4}\sigma_{8,47}, \sigma_{8,4}\sigma_{8,50}, \sigma_{8,4}\sigma_{8,51}, \sigma_{8,4}\sigma_{8,52}, \\ & \sigma_{8,4}\sigma_{8,53}, \sigma_{8,4}\sigma_{8,54}, \sigma_{8,4}\sigma_{8,55}, \sigma_{8,4}\sigma_{8,56}, \sigma_{8,4}\sigma_{8,57}, \\ & \sigma_{8,5}\sigma_{8,29}, \sigma_{8,5}\sigma_{8,47}, \sigma_{8,5}\sigma_{8,51}, \sigma_{8,5}\sigma_{8,52}, \sigma_{8,5}\sigma_{8,53}, \\ & \sigma_{8,5}\sigma_{8,54}, \sigma_{8,5}\sigma_{8,55}, \sigma_{8,5}\sigma_{8,56}, \sigma_{8,5}\sigma_{8,57}, \\ & \sigma_{8,6}\sigma_{8,29}, \sigma_{8,6}\sigma_{8,52}, \sigma_{8,6}\sigma_{8,53}, \sigma_{8,6}\sigma_{8,54}, \sigma_{8,6}\sigma_{8,55}, \sigma_{8,6}\sigma_{8,56}, \sigma_{8,6}\sigma_{8,57}, \\ & \sigma_{8,7}\sigma_{8,29}, \sigma_{8,7}\sigma_{8,47}, \sigma_{8,7}\sigma_{8,51}, \sigma_{8,7}\sigma_{8,53}, \sigma_{8,7}\sigma_{8,54}, \sigma_{8,7}\sigma_{8,55}, \sigma_{8,7}\sigma_{8,56}, \sigma_{8,7}\sigma_{8,57}, \\ & \sigma_{8,8}\sigma_{8,29}, \sigma_{8,8}\sigma_{8,50}, \sigma_{8,8}\sigma_{8,54}, \sigma_{8,8}\sigma_{8,55}, \sigma_{8,8}\sigma_{8,56}, \sigma_{8,8}\sigma_{8,57}, \\ & \sigma_{8,9}\sigma_{8,29}, \sigma_{8,9}\sigma_{8,49}, \sigma_{8,9}\sigma_{8,55}, \sigma_{8,9}\sigma_{8,56}, \sigma_{8,9}\sigma_{8,57}, \\ & \sigma_{8,10}\sigma_{8,29}, \sigma_{8,10}\sigma_{8,48}, \sigma_{8,10}\sigma_{8,56}, \sigma_{8,10}\sigma_{8,57}, \\ & \sigma_{8,11}\sigma_{8,29}, \sigma_{8,11}\sigma_{8,47}, \sigma_{8,11}\sigma_{8,51}, \sigma_{8,11}\sigma_{8,53}, \sigma_{8,11}\sigma_{8,54}, \sigma_{8,11}\sigma_{8,55}, \sigma_{8,11}\sigma_{8,56}, \sigma_{8,11}\sigma_{8,57}, \\ & \sigma_{8,12}\sigma_{8,29}, \sigma_{8,12}\sigma_{8,46}, \sigma_{8,12}\sigma_{8,54}, \sigma_{8,12}\sigma_{8,55}, \sigma_{8,12}\sigma_{8,56}, \sigma_{8,12}\sigma_{8,57}, \\ & \sigma_{8,13}\sigma_{8,29}, \sigma_{8,13}\sigma_{8,45}, \sigma_{8,13}\sigma_{8,55}, \sigma_{8,13}\sigma_{8,56}, \sigma_{8,13}\sigma_{8,57}, \\ & \sigma_{8,14}\sigma_{8,29}, \sigma_{8,14}\sigma_{8,44}, \sigma_{8,14}\sigma_{8,56}, \sigma_{8,14}\sigma_{8,57}, \\ & \sigma_{8,15}\sigma_{8,29}, \sigma_{8,15}\sigma_{8,43}, \sigma_{8,15}\sigma_{8,55}, \sigma_{8,15}\sigma_{8,56}, \sigma_{8,15}\sigma_{8,57}, \\ & \sigma_{8,16}\sigma_{8,29}, \sigma_{8,16}\sigma_{8,42}, \sigma_{8,16}\sigma_{8,56}, \sigma_{8,16}\sigma_{8,57}, \\ & \sigma_{8,17}\sigma_{8,29}, \sigma_{8,17}\sigma_{8,41}, \sigma_{8,17}\sigma_{8,56}, \sigma_{8,17}\sigma_{8,57}, \\ & \sigma_{8,18}\sigma_{8,29}, \sigma_{8,18}\sigma_{8,40}, \sigma_{8,18}\sigma_{8,56}, \sigma_{8,18}\sigma_{8,57}, \\ & \sigma_{8,19}\sigma_{8,29}, \sigma_{8,19}\sigma_{8,39}, \sigma_{8,19}\sigma_{8,57}, \sigma_{8,20}\sigma_{8,29}, \sigma_{8,20}\sigma_{8,38}, \\ & \sigma_{8,21}\sigma_{8,29}, \sigma_{8,21}\sigma_{8,37}, \sigma_{8,21}\sigma_{8,57}, \sigma_{8,22}\sigma_{8,29}, \sigma_{8,22}\sigma_{8,36}, \\ & \sigma_{8,23}\sigma_{8,29}, \sigma_{8,23}\sigma_{8,35}, \sigma_{8,23}\sigma_{8,57}, \sigma_{8,24}\sigma_{8,29}, \sigma_{8,24}\sigma_{8,34}, \sigma_{8,25}\sigma_{8,29}, \sigma_{8,25}\sigma_{8,33}, \sigma_{8,25}\sigma_{8,57}, \\ & \sigma_{8,26}\sigma_{8,29}, \sigma_{8,26}\sigma_{8,32}, \sigma_{8,27}\sigma_{8,29}, \sigma_{8,27}\sigma_{8,31}, \sigma_{8,27}\sigma_{8,57}, \\ & \sigma_{8,28}\sigma_{8,29}, \sigma_{8,28}\sigma_{8,30}, \sigma_{8,30}\sigma_{8,57}, \sigma_{8,32}\sigma_{8,57}, \sigma_{8,34}\sigma_{8,57}, \sigma_{8,36}\sigma_{8,57}, \sigma_{8,38}\sigma_{8,57} \} \end{aligned}$$

For later use, we denote Σ_8 by $\{\tau_{8,k}, k = 0, 1, 2, \dots, 239\}$ where $\tau_{8,k}$ denotes the $k + 1$ -st elements in the above display.

Theorem 4. *The Weyl groups for the exceptional groups of types E_6, E_7, E_8 are given by*

$$\begin{aligned} W_6 &= \sqcup_{i=1}^{27} \tau_{6,i} W_5, \\ W_7 &= \sqcup_{j=0}^{55} \sqcup_{i=0}^{26} \tau_{7,j} \tau_{6,i} W_5, \\ W_8 &= \sqcup_{k=0}^{239} \sqcup_{j=0}^{55} \sqcup_{i=0}^{26} \tau_{8,k} \tau_{7,j} \tau_{6,i} W_5, \end{aligned}$$

As to be expected, the special orders for simple roots at the beginning proves to be extremely crucial.

This calculation is a rather long and complicated – with a series of bugs searching and program fixing, our calculation to find all representatives above last for about a month and a half, and our program in Mathematica cannot treat W_8 as a single set to store in our Power MacBook/Power Mac. There is a huge problem when we tried to store W_8 as a single unit, which forced our work stations stop calculating repeatedly. At some point, we realized this and offered a not-perfect-but-practical solution: Instead, we introduce

$$W_{8,i} := \sqcup_{k=0}^{239} \sqcup_{j=0}^{55} \tau_{8,k} \tau_{7,j} \tau_{6,i} W_5, \quad i = 0, 1, 2, \dots, 26$$

and hence divide our calculations into 27 parallel units. With W_8 , we were able to start our second phrase of calculations, on zetas themselves and the verifications of the weak Riemann Hypothesis for the so-called Weng zetas for E_6, E_7, E_8 . This lasted for another 2 months.

3. Zeta functions for (G, P)

Let G be a connected split reductive algebraic group of rank r with a fixed Borel subgroup B and associated maximal split torus T (over a base field). Denote by

$$(V, \langle \cdot, \cdot \rangle, \Phi = \Phi^+ \cup \Phi^-, \Delta = \{\alpha_1, \dots, \alpha_r\}, \varpi := \{\varpi_1, \dots, \varpi_r\}, W)$$

the associated root system. That is, V is the real vector space defined as the $\mathbb{R}$ -span of rational characters of T , and as usual, is equipped with a natural inner product $\langle \cdot, \cdot \rangle$, with which we identify V with its dual V^* , $\Phi^+ \subset V$ is the set of positive roots, $\Phi^- := -\Phi^+$ the set of negative roots, $\Delta \subset V$ the set of simple roots, $\varpi \subset V$ the set of fundamental weights, and W the Weyl group. By definition, the fundamental weights are characterized by the formula $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j = 1, 2, \dots, r$, where $\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle} \alpha$ denotes the coroot of a root $\alpha \in \Phi$. We also define the Weyl vector ρ by $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$, and introduce a *coordinate system* on V (with respect to the base $\{\varpi_1, \dots, \varpi_r\}$ of V and the vector ρ) by writing an element $\lambda \in V$ in the form

$$\lambda = \sum_{j=1}^r (1 - s_j) \varpi_j = \rho - \sum_{j=1}^r s_j \varpi_j$$

thus fixing identifications of V and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{R}^r$ and $\mathbb{C}^r$. In addition, for each Weyl element $w \in W$, we set $\Phi_w := \Phi^+ \cap w^{-1} \Phi^-$, i.e., the collection of positive roots whose w -images are negative.

As usual, by a *standard parabolic subgroup*, we mean a parabolic subgroup of G that contains the Borel subgroup B . From Lie theory (see e.g., [Hu]), there is an one-to-one correspondence between standard parabolic subgroups P of G and subsets Δ_P of Δ . In particular, if P is maximal, we may and will write $\Delta_P = \Delta \setminus \{\alpha_p\}$ for a certain unique $p = p(P) \in \{1, \dots, r\}$. For such a standard parabolic subgroup P , denote by V_P the $\mathbb{R}$ -span of rational characters of the maximal split torus T_P contained in P , by V_P^* its dual space, and by $\Phi_P \subset V_P$ the set of non-trivial characters of T_P occurring in the space V . Then, by standard theory of reductive groups (see e.g., [Ar]), V_P admits a canonical embedding in V (and V_P^* admits a canonical embedding in V^*), which is known to be orthogonal to the fundamental weight ϖ_p , and hence Φ_P can be viewed as a subset of Φ . Set $\Phi_P^+ = \Phi^+ \cap \Phi_P$, $\rho_P = \frac{1}{2} \sum_{\alpha \in \Phi_P^+} \alpha$, and $c_P = 2\langle \varpi_p - \rho_P, \alpha_p^\vee \rangle$.

In [W?], to understand the non-abelian zeta function over a number field F , we, for a connected split reductive algebraic group G , and its standard parabolic subgroup P as above (defined over F), defined the *period of G for F* by

$$\omega_F^G(\lambda) := \sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta} \langle w\lambda - \rho, \alpha^\vee \rangle} \prod_{\alpha \in \Phi_w} \frac{\widehat{\zeta}_F(\langle \lambda, \alpha^\vee \rangle)}{\widehat{\zeta}_F(\langle \lambda, \alpha^\vee \rangle + 1)}$$

and the *period of (G, P) for F* by

$$\begin{aligned} \omega_F^{G,P}(s) &:= \text{Res}_{\langle \lambda - \rho, \alpha^\vee \rangle = 0, \alpha \in \Delta_P} \omega_F^G(\lambda) \\ &= \text{Res}_{s_r=0} \cdots \text{Res}_{s_{p+1}=0} \text{Res}_{s_{p-1}=0} \cdots \text{Res}_{s_1=0} \omega_F^G(\lambda). \end{aligned}$$

Here $\widehat{\zeta}_F(s)$ denotes the completed Dedekind zeta function for F , $s = c_p \cdot s_p$ and for the last equality, we used the fact that $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta$ and the relation that $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, r\}$. As proved in [Ko, W?], the ordering of taking residues along singular hyperplanes $\langle \lambda - \rho, \alpha^\vee \rangle = 0$ for $\alpha \in \Delta_P$ does not affect the outcome, so that the definition is independent of the numbering of the simple roots.

To get the zeta function associated to (G, P) for F , certain normalizations should be made. For this purpose, write $\omega_F^G(\lambda) = \sum_{w \in W} T_w(\lambda)$, where, for each $w \in W$,

$$T_w(\lambda) := \frac{1}{\prod_{\alpha \in \Delta} -\langle w\lambda - \rho, \alpha^\vee \rangle} \prod_{\alpha \in \Phi_w} \frac{\widehat{\zeta}_F(\langle \lambda, \alpha^\vee \rangle)}{\widehat{\zeta}_F(\langle \lambda, \alpha^\vee \rangle + 1)}.$$

We must study the residue $\text{Res}_{\langle \lambda - \rho, \alpha^\vee \rangle = 0, \alpha \in \Delta_P} T_w(\lambda)$.

We care only about those elements $w \in W$ (we will call them *special*) which give non-trivial residues, namely, those satisfying the condition that $\text{Res}_{\langle \lambda - \rho, \alpha^\vee \rangle = 0, \alpha \in \Delta_P} T_w(\lambda) \neq 0$. This can happen only if all singular hyperplanes are of one of the following two forms:

- (1) $\langle w\lambda - \rho, \alpha^\vee \rangle = 0$ for some $\alpha \in \Delta$, giving a simple pole of the rational factor $\frac{1}{\prod_{\alpha \in \Delta} (-\langle w\lambda - \rho, \alpha^\vee \rangle)}$;
- (2) $\langle \lambda, \alpha^\vee \rangle = 1$ for some $\alpha \in \Phi_w$, giving a simple pole of the zeta factor $\widehat{\zeta}_F(\langle \lambda, \alpha^\vee \rangle)$.

For special $w \in W$, and $(k, h) \in \mathbb{Z}^2$, following [Ko], we define

$$N_{P,w}(k, h) := \#\{\alpha \in w^{-1}\Phi^- : \langle \lambda_p, \alpha^\vee \rangle, \langle \rho, \alpha^\vee \rangle = h\}$$

$$M_P(k, h) := \max_{w: \text{ special}} (N_{P,w}(k, h-1) - N_{P,w}(k, h)).$$

Note that $N_P(k, h) = 0$ for almost all but finitely many pairs of integers (k, h) , so it makes sense to introduce the product

$$D_F^{G,P}(s) := \prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \widehat{\zeta}_F(ks+h)^{M_P(k,h)}.$$

Then the so-called *Weng zeta function of F associated to (G, P)* is defined by

$$\widehat{\zeta}_F^{G,P}(s) := q^{(g-1)\dim N_u(B)} \cdot D_F^{G,P}(s) \cdot \omega_F^{G,P}(s). \quad (1)$$

Here $N_u(B)$ denote the nilpotent radical of the Borel subgroup B of G .

Remark. For special $w \in W$, even after taking residues, there are some zeta factors $\widehat{\zeta}_F(ks+h)$ left in the denominator of $\text{Res}_{\langle \lambda-\rho, \alpha^\vee \rangle=0, \alpha \in \Delta_P} T_w(\lambda)$. The reason for introducing the factor $D_F^{G,P}(s)$ in our normalization of the zeta functions, based on formulas in [Ko] and [W3], is to clear up all of the zeta factors appearing in the denominators associated to special Weyl elements.

4. Conditions for Weak Riemann Hypothesis

In this section, we explain a numerical condition stated in the work of Ki, Komori, Suzuki ([KKS]) on the weak Riemann Hypothesis for Weng zeta functions.

Let

$$\mathfrak{W}_P := \{w \in W : \Delta_P \subset w^{-1}(\Delta \cup \Phi^-)\}.$$

For each $w \in \mathfrak{W}_P$, set

$$\delta_{\alpha,w} = \begin{cases} 1 & \alpha \in w^{-1}\Phi^+ \\ 0 & \alpha \in w^{-1}\Phi^- \end{cases},$$

and define

$$k_P(w) := \sum_{\alpha \in \Phi^+ \setminus \Phi_P^+} (1 - \delta_{\alpha,w}) = |(\Phi^+ \setminus \Phi_P^+) \cap w^{-1}\Phi^-|,$$

$$\mathfrak{W}_P^\dagger := \{w \in \mathfrak{W}_P : k_P(w) = 0\}.$$

Set $\xi(s) := s(s-1)\widehat{\zeta}(s)$ and

$$C_{P,w} := \xi(2)^{|\Delta_P \cap w^{-1}\Phi^+|} \prod_{\alpha \in \Phi_P^+ \setminus \Delta_P} \xi(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w}),$$

$$\frac{1}{D_{P,w}} := 2^{|\Delta_P \cap w^{-1}\Phi^+|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \setminus \Delta_P)} (\langle \rho, \alpha^\vee \rangle - 1)$$

$$\times \prod_{\alpha \in \Phi_P \setminus \Delta_P} (\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w})(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w} - 1)$$

Introduce then the number

$$v_{G,P} := \sum_{\substack{w \in \mathfrak{W}_P^\dagger \\ |(w^{-1}\Delta) \setminus \Phi_P| = 1}} \frac{1}{\langle \varpi_P, \alpha_w^\vee \rangle} \cdot C_{P,w} \cdot D_{P,w},$$

where α_w is the only element of $(w^{-1}\Delta) \setminus \Phi_P$.

Theorem. ([KKS]) *If $v_{G,P} \neq 0$, all but finitely many zeros of $\widehat{\zeta}_{\mathbb{Q}}^{G,P}(s)$ are simple and on the critical line of its functional equation.*

Set

$$B_{P,w} := \widehat{\zeta}(2)^{|\Delta_P \cap w^{-1}\Phi^+|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \setminus \Delta_P)} \frac{1}{\langle \rho, \alpha^\vee \rangle - 1} \prod_{\alpha \in \Phi_P^+ \setminus \Delta_P} \widehat{\zeta}(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w}).$$

Then by [KKS, p.153, lines 2, 3], we have

$$v_{G,P} = \sum_{\substack{w \in \mathfrak{W}_P^\dagger \\ |(w^{-1}\Delta) \setminus \Phi_P| = 1}} \frac{B_{P,w}}{\langle \varpi_P, \alpha_w^\vee \rangle}. \quad (2)$$

Here $\{\varpi_1, \varpi_2, \dots, \varpi_n\}$ is the dual bases of $\{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\}$ for $n = 6, 7, 8$, and α_w is the unique element in $(w^{-1}\Delta) \setminus \Phi_P$.

5. Special and Very Special Weyl Elements

For each $w \in W$, we set

$$A_w := \{\alpha \in \Delta_P : w\alpha \in \Delta\}, \quad B_w := \{\alpha \in \Delta_P : w\alpha \in \Phi^-\}.$$

Clearly, $A_w \cup B_w \subseteq \Delta_P$. We call w *special*, if $A_w \cup B_w = \Delta_P$. Denote by

$$W_P^{\text{sp}} := \{w \in W : w \text{ special}\},$$

$$W_P^\dagger := \{w \in W^{\text{sp}} : |\Delta \setminus w\Phi_P| = 1\},$$

$$W_P^\ddagger := \{w \in W^\dagger : |w(\Phi^+ \setminus \Phi_P^+) \cap \Phi^-| = 0\}.$$

Then $W^{\text{sp}} = \mathfrak{W}_P$ since $A_w \cup B_w = \Delta_P \iff \Delta_P \subset w^{-1}(\Delta \cup \Phi^-)$, and, moreover,

$$W_P^\dagger = \{w \in W^{\text{sp}} : |\Delta \cap w\Phi_P| = 7\}.$$

To prove $v_{E_n,P} \neq 0$ for every $n = 6, 7, 8$ and all associated maximal parabolic subgroups, we first search on the selections of Weyl elements belonging to W^{sp} , $W^\dagger$ and $W^\ddagger$ respectively. Accordingly, we divide our calculations in Mathematica into three steps:

Step 1. Calculate elements of W_P^{sp} , denoted sp. This is very long.

Step 2. Calculate elements of $W_P^\dagger$, denoted sps. This is rather long.

Step 3. Calculate elements of $W_P^\ddagger$, called very special. This is very pleasant.

Our calculation can be summarized as follows:

Theorem 5. *For an exceptional group of type E_n , $n = 6, 7, 8$, we have*

$$W_{6,P_i}^\ddagger \leq W_{7,P_i}^\ddagger \quad \text{and} \quad W_{7,P_j}^\ddagger \leq W_{8_j}^\ddagger \leq W_{8_i}^\ddagger \quad (1 \leq i \leq 6, \quad 1 \leq j \leq 7).$$

Moreover,

$$\#W_{n,P}^\dagger = 2^n, \quad \#W_{n,P}^\ddagger = 2^{n-1}.$$

To state an example with $(G, P) = (E_8, P_1)$, we denote $\tau_{5,s}\tau_{6,i}\tau_{7,j}\tau_{8,k}$ as $(s+1, i+1, j+1, k+1)$, where $w_{5,s} \in W_5$. Then, we get, from `ddag.nb`

$$\begin{aligned}
W^\dagger = \{ & (1, 1, 1, 1), (1, 1, 1, 27), (1, 1, 14, 27), (1, 1, 14, 29), \\
& (1375, 3, 1, 1), (1375, 3, 1, 27), (1375, 3, 14, 27), (1375, 3, 14, 29), \\
& (1, 9, 1, 1), (1, 9, 1, 27), (1, 9, 14, 27), (1, 9, 14, 29), \\
& (193, 9, 1, 1), (193, 9, 1, 27), (193, 9, 14, 27), (193, 9, 14, 29), \\
& (301, 9, 1, 1), (301, 9, 1, 27), (301, 9, 14, 27), (301, 9, 14, 29), \\
& (325, 9, 1, 1), (325, 9, 1, 27), (325, 9, 14, 27), (325, 9, 14, 29), \\
& (393, 9, 1, 1), (393, 9, 1, 27), (393, 9, 14, 27), (393, 9, 14, 29), \\
& (423, 9, 1, 1), (423, 9, 1, 27), (423, 9, 14, 27), (423, 9, 14, 29), \\
& (441, 9, 1, 1), (441, 9, 1, 27), (441, 9, 14, 27), (441, 9, 14, 29), \\
& (447, 9, 1, 1), (447, 9, 1, 27), (447, 9, 14, 27), (447, 9, 14, 29), \\
& (663, 13, 1, 1), (663, 13, 1, 27), (663, 13, 14, 27), (663, 13, 14, 29), \\
& (687, 13, 1, 1), (687, 13, 1, 27), (687, 13, 14, 27), (687, 13, 14, 29), \\
& (781, 13, 1, 1), (781, 13, 1, 27), (781, 13, 14, 27), (781, 13, 14, 29), \\
& (805, 13, 1, 1), (805, 13, 1, 27), (805, 13, 14, 27), (805, 13, 14, 29), \\
& (963, 15, 1, 1), (963, 15, 1, 27), (963, 15, 14, 27), (963, 15, 14, 29), \\
& (1375, 16, 1, 1), (1375, 16, 1, 27), (1375, 16, 14, 27), (1375, 16, 14, 29), \\
& (1, 23, 1, 1), (1, 23, 1, 27), (1, 23, 14, 27), (1, 23, 14, 29), \\
& (361, 23, 1, 1), (361, 23, 1, 27), (361, 23, 14, 27), (361, 23, 14, 29), \\
& (687, 23, 1, 1), (687, 23, 1, 27), (687, 23, 14, 27), (687, 23, 14, 29), \\
& (775, 23, 1, 1), (775, 23, 1, 27), (775, 23, 14, 27), (775, 23, 14, 29), \\
& (961, 23, 1, 1), (961, 23, 1, 27), (961, 23, 14, 27), (961, 23, 14, 29), \\
& (963, 23, 1, 1), (963, 23, 1, 27), (963, 23, 14, 27), (963, 23, 14, 29), \\
& (985, 23, 1, 1), (985, 23, 1, 27), (985, 23, 14, 27), (985, 23, 14, 29), \\
& (987, 23, 1, 1), (987, 23, 1, 27), (987, 23, 14, 27), (987, 23, 14, 29), \\
& (1321, 23, 1, 1), (1321, 23, 1, 27), (1321, 23, 14, 27), (1321, 23, 14, 29), \\
& (1323, 23, 1, 1), (1323, 23, 1, 27), (1323, 23, 14, 27), (1323, 23, 14, 29), \\
& (1327, 23, 1, 1), (1327, 23, 1, 27), (1327, 23, 14, 27), (1327, 23, 14, 29), \\
& (1335, 23, 1, 1), (1335, 23, 1, 27), (1335, 23, 14, 27), (1335, 23, 14, 29), \\
& (1345, 23, 1, 1), (1345, 23, 1, 27), (1345, 23, 14, 27), (1345, 23, 14, 29), \\
& (1347, 23, 1, 1), (1347, 23, 1, 27), (1347, 23, 14, 27), (1347, 23, 14, 29), \\
& (1375, 23, 1, 1), (1375, 23, 1, 27), (1375, 23, 14, 27), (1375, 23, 14, 29), \\
& (1407, 23, 1, 1), (1407, 23, 1, 27), (1407, 23, 14, 27), (1407, 23, 14, 29) \}
\end{aligned}$$

Totally, there are $128 = 2^7$ of them. For other cases, please refer to the appendix where $W^\dagger$ is hidden in Appendix C.

As we will see later, this calculation is also compatible with the Conjecture on Parabolic Reduction, Stability and the Masses [We]. Besides,

even by examining the components of each element, we can clearly see natural structures for very special Weyl elements. This deserves to be further investigated.

6. Constants $\langle \varpi_P, \alpha_w^\vee \rangle$

Very special Weyl elements are those which appeared in (2) giving non-trivial contributions to $v_{G,P}$. To calculate $v_{G,P}$ for $G = E_{6,7,8}$, we here calculate the constant $\langle \varpi_P, \alpha_w^\vee \rangle$. Here, for each very special Weyl element w , α_w is the only element in $w^{-1}\Delta \setminus \Phi_P$.

Proposition 6. *For all very special elements $w \in W^\ddagger$,*

$$\langle \varpi_P, \alpha_w^\vee \rangle = 1.$$

Proof. Fix a very special Weyl element $w \in W^\ddagger$. Note that $\langle \cdot, \cdot \rangle$ is w -invariant, it is sufficient to calculate $\langle w\varpi_P, \beta_w^\vee \rangle$ where β_w is the unique element in $\Delta \setminus w\Phi_P$. Our calculation then shows that

Lemma 7. *For all $w \in W_{n,P}^\ddagger$, when applicable,*

$$\begin{aligned} w\varpi_1 &= (0, 0, 0, 0, 1, 1, 1, 3), & w\varpi_2 &= (0, 0, 0, 1, 1, 1, 1, 4), \\ w\varpi_3 &= (0, 0, 1, 1, 1, 1, 1, 5), & w\varpi_4 &= (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}), \\ w\varpi_5 &= (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{2}), & w\varpi_6 &= (0, 0, 0, 0, 0, 0, 0, 2), \\ w\varpi_7 &= (0, 0, 0, 0, 0, 1, 1, 2), & w\varpi_8 &= (0, 0, 0, 0, 0, 0, 1, 1) \end{aligned}$$

To go further, we calculate β_w which is listed in the program as the set titled 'com'. Finally, we verify the following

$$\langle w\varpi_P, \beta_w^\vee \rangle = 1 \quad \forall w \in W_{n,P}^\ddagger.$$

□

7. Constants $v_{E_n,P}, n = 6, 7, 8$

Recall that, see e.g. Equation (2) in §4

$$v_{G,P} = \sum_{\substack{w \in W_P^\ddagger \\ |(w^{-1}\Delta) \setminus \Phi_P| = 1}} \frac{B_{P,w}}{\langle \varpi_P, \alpha_w^\vee \rangle}. \quad (3)$$

where

$$B_{P,w} := \widehat{\zeta}(2)^{|\Delta_P \cap w^{-1}\Phi^+|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \setminus \Delta_P)} \frac{1}{\langle \rho, \alpha^\vee \rangle - 1} \prod_{\alpha \in \Phi_P^+ \setminus \Delta_P} \widehat{\zeta}(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w}).$$

By Proposition 6 in §6, $\langle \varpi_P, \alpha_w^\vee \rangle = 1$. Hence, only $B_{P,w}$ becomes essential now. Thus to calculate the constants $v_{E_n,P}, n = 6, 7, 8$ for all associated maximal parabolic subgroups, we need to further calculate, for each very special $w \in W^\ddagger$,

- (A) The cardinal number of $\Delta_P \cap w^{-1}\Phi^+$;
- (B) The product $\prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \setminus \Delta_P)} \frac{1}{\langle \rho, \alpha^\vee \rangle - 1}$;
- (C) The constant $\delta_{\alpha,w}$ for $\sigma \in \Phi^+ \setminus \Delta_P$;
- (D) The product $\prod_{\alpha \in \Phi_P^+ \setminus \Delta_P} \widehat{\zeta}(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w})$.

All this is done in the Appendix at the end of this paper. With all this, now we are ready to state explicit formulas for the constants $v_{E_n, P}$, $n = 6, 7, 8$ obtained using Mathematica.

Theorem 8. *The constant $v_{G, P}$ for $G = E_n$, $n = 6, 7, 8$ is given by the following lists:*

(E_8, P_1)

$$\begin{aligned} v_{E_8, P_1} = & -\frac{1}{3628800} \widehat{\zeta}(2)^5 \widehat{\zeta}(3)^4 (3 + 4\widehat{\zeta}(2)(-2 + 3\widehat{\zeta}(3))) \widehat{\zeta}(4)^3 \widehat{\zeta}(5)^2 \widehat{\zeta}(6) \widehat{\zeta}(7) \\ & (9450 + 5040\widehat{\zeta}(2)^3 (-9 + 10\widehat{\zeta}(3)) + 60\widehat{\zeta}(2)^2 (1435 + 72\widehat{\zeta}(3)(-21 + 10\widehat{\zeta}(4))) \\ & + 7\widehat{\zeta}(2)(27\widehat{\zeta}(3)(225 + 16\widehat{\zeta}(4)(-19 + 25\widehat{\zeta}(5))) \\ & + 200(-37 + 27\widehat{\zeta}(4)\widehat{\zeta}(6)(\widehat{\zeta}(4) - 8\widehat{\zeta}(5)\widehat{\zeta}(8)))))) \end{aligned}$$

(E_8, P_2)

$$\begin{aligned} v_{E_8, P_2} = & \frac{1}{51840} \widehat{\zeta}(2)^5 \widehat{\zeta}(3)^3 \widehat{\zeta}(4)(9 + 18\widehat{\zeta}(2)^2 - 4\widehat{\zeta}(2)(8 + 9\widehat{\zeta}(3)(-1 + 2\widehat{\zeta}(4)))) \\ & (-45 + \widehat{\zeta}(2)(200 + 8\widehat{\zeta}(2)(-25 + 36\widehat{\zeta}(3)) - 9\widehat{\zeta}(3)(25 + 16\widehat{\zeta}(4)(-2 + 5\widehat{\zeta}(5)))))) \end{aligned}$$

(E_8, P_3)

$$\begin{aligned} v_{E_8, P_3} = & -\frac{1}{17280} \widehat{\zeta}(2)^4 (-1 + 2\widehat{\zeta}(2)) \widehat{\zeta}(3)^2 (3 + 4\widehat{\zeta}(2)(-2 + 3\widehat{\zeta}(3))) \widehat{\zeta}(4) \\ & (-45 + \widehat{\zeta}(2)(200 + 8\widehat{\zeta}(2)(-25 + 36\widehat{\zeta}(3)) - 9\widehat{\zeta}(3)(25 + 16\widehat{\zeta}(4)(-2 + 5\widehat{\zeta}(5)))))) \end{aligned}$$

(E_8, P_4)

$$\begin{aligned} v_{E_8, P_4} = & -\frac{1}{12700800} \widehat{\zeta}(2)^6 \widehat{\zeta}(3)^5 \widehat{\zeta}(4)^4 \widehat{\zeta}(5)^3 \widehat{\zeta}(6)^2 \widehat{\zeta}(7) \\ & (99225 + 2\widehat{\zeta}(2)(99225\widehat{\zeta}(2)^3 + 44100(-8 + 9\widehat{\zeta}(3)) \\ & - 1568\widehat{\zeta}(2)^2 (425 + 18\widehat{\zeta}(3)(-46 + 24\widehat{\zeta}(3) + 25\widehat{\zeta}(4))) \\ & + 42\widehat{\zeta}(2)(19600 + 45\widehat{\zeta}(3)^2 (385 + 12\widehat{\zeta}(4)(-64 + 35\widehat{\zeta}(4) + 70\widehat{\zeta}(5))) \\ & + 24\widehat{\zeta}(3)(-1540 + \widehat{\zeta}(4)(1568 + 25\widehat{\zeta}(5)(-64 + 63\widehat{\zeta}(6)))))) \\ & - 144\widehat{\zeta}(3)\widehat{\zeta}(4)(3528 + 25\widehat{\zeta}(5)(-196 + 9\widehat{\zeta}(6)(32 + 49\widehat{\zeta}(7)(-1 + 4\widehat{\zeta}(8)))))) \end{aligned}$$

(E_8, P_5)

$$\begin{aligned} v_{E_8, P_5} = & \frac{1}{1814400} \widehat{\zeta}(2)^5 (-1 + 2\widehat{\zeta}(2)) \widehat{\zeta}(3)^4 \widehat{\zeta}(4)^3 \widehat{\zeta}(5)^2 \widehat{\zeta}(6) \\ & (14175 + \widehat{\zeta}(2)(11025(-8 + 9\widehat{\zeta}(3)) + 4(882\widehat{\zeta}(2)^2 (-25 + 36\widehat{\zeta}(3)) \\ & + 5\widehat{\zeta}(2)(8330 - 9\widehat{\zeta}(3)(15\widehat{\zeta}(3)(-49 + 96\widehat{\zeta}(4)) + 2(833 + 16\widehat{\zeta}(4)(-49 + 45\widehat{\zeta}(5)))))) \\ & + 36\widehat{\zeta}(3)\widehat{\zeta}(4)(-882 + 25\widehat{\zeta}(5)(49 + 36\widehat{\zeta}(6)(-2 + 7\widehat{\zeta}(7)))))) \end{aligned}$$

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_3

(E_8, P_6)

$$\begin{aligned}
 v_{E_8, P_6} = & \frac{1}{4191264000} \widehat{\zeta}(2)^6 \widehat{\zeta}(3)^6 \widehat{\zeta}(4)^5 \widehat{\zeta}(5)^5 \widehat{\zeta}(6)^4 \widehat{\zeta}(7)^3 \widehat{\zeta}(8)^2 \widehat{\zeta}(9)^2 \widehat{\zeta}(10) \widehat{\zeta}(11) \\
 & (1164240 \widehat{\zeta}(2)^4 (-215 + 264 \widehat{\zeta}(3)) + 41580 \widehat{\zeta}(2)^3 (14980 + 3 \widehat{\zeta}(3) (-6139 + 3840 \widehat{\zeta}(4))) \\
 & - 10914750 (3 + 16 \widehat{\zeta}(3)^2 \widehat{\zeta}(4) \widehat{\zeta}(5)) \\
 & + 140 \widehat{\zeta}(2)^2 (-14850 \widehat{\zeta}(3)^2 (126 + \widehat{\zeta}(4) (-387 + 224 \widehat{\zeta}(4) + 252 \widehat{\zeta}(5))) \\
 & - 99 \widehat{\zeta}(3) (-58135 + 48 \widehat{\zeta}(4) (1487 - 225 \widehat{\zeta}(5) + 630 \widehat{\zeta}(4) \widehat{\zeta}(6))) \\
 & + 175 (-24431 + 72 \widehat{\zeta}(4) \widehat{\zeta}(6) (187 \widehat{\zeta}(4) - 216 \widehat{\zeta}(5) \widehat{\zeta}(8)))) \\
 & + 99 \widehat{\zeta}(2) (6048000 \widehat{\zeta}(3)^2 \widehat{\zeta}(4) \widehat{\zeta}(5) \\
 & + \widehat{\zeta}(3) (-2149875 + 16 \widehat{\zeta}(4) (202419 - 25 \widehat{\zeta}(5) (2695 + 72 \widehat{\zeta}(6) (-72 + 245 \widehat{\zeta}(7)))))) \\
 & - 9800 (-245 + 9 \widehat{\zeta}(4) \widehat{\zeta}(6) (15 \widehat{\zeta}(4) \\
 & - 8 (-5 \widehat{\zeta}(6) \widehat{\zeta}(9) \widehat{\zeta}(10) + 3 \widehat{\zeta}(8) (\widehat{\zeta}(5) + 20 \widehat{\zeta}(7) \widehat{\zeta}(10) \widehat{\zeta}(12))))))
 \end{aligned}$$

(E_8, P_7)

$$\begin{aligned}
 v_{E_8, P_7} = & -\frac{1}{6652800} \widehat{\zeta}(2)^5 (-1 + 2 \widehat{\zeta}(2)) \widehat{\zeta}(3)^5 \widehat{\zeta}(4)^5 \widehat{\zeta}(5)^4 \widehat{\zeta}(6)^3 \widehat{\zeta}(7)^3 \widehat{\zeta}(8)^2 \widehat{\zeta}(9) \widehat{\zeta}(10) \widehat{\zeta}(11) \\
 & (-51975 + 44 \widehat{\zeta}(2)^3 (10255 + 48 \widehat{\zeta}(3) (-448 + 225 \widehat{\zeta}(3))) \\
 & - 440 \widehat{\zeta}(2)^2 (1540 + 3 \widehat{\zeta}(3) (-819 + 210 \widehat{\zeta}(3) + 8 \widehat{\zeta}(4) (39 - 70 \widehat{\zeta}(5)))) \\
 & + 7 \widehat{\zeta}(2) (47300 + 9 (\widehat{\zeta}(3) (-4950 + 32 \widehat{\zeta}(4) (132 + 25 \widehat{\zeta}(5) (-11 + 6 \widehat{\zeta}(6)))) \\
 & - 275 \widehat{\zeta}(6) (3 \widehat{\zeta}(4)^2 - 32 \widehat{\zeta}(4) \widehat{\zeta}(5) \widehat{\zeta}(8) + 192 \widehat{\zeta}(5) \widehat{\zeta}(8) \widehat{\zeta}(9) \widehat{\zeta}(12))))
 \end{aligned}$$

(E_8, P_8)

$$\begin{aligned}
 v_{E_8, P_8} = & \frac{1}{22054032000} \widehat{\zeta}(2)^6 \widehat{\zeta}(3)^6 \widehat{\zeta}(4)^6 \widehat{\zeta}(5)^6 \widehat{\zeta}(6)^5 \widehat{\zeta}(7)^5 \widehat{\zeta}(8)^4 \\
 & \widehat{\zeta}(9)^4 \widehat{\zeta}(10)^3 \widehat{\zeta}(11)^3 \widehat{\zeta}(12)^2 \widehat{\zeta}(13)^2 \widehat{\zeta}(14) \widehat{\zeta}(15) \widehat{\zeta}(16) \widehat{\zeta}(17) \\
 & (-172297125 + 91891800 \widehat{\zeta}(2)^4 (-9 + 10 \widehat{\zeta}(3)) \\
 & - 24310 \widehat{\zeta}(2)^3 (-122605 + 36 \widehat{\zeta}(3) (6153 - 3200 \widehat{\zeta}(4) + 150 \widehat{\zeta}(3) (-20 + 21 \widehat{\zeta}(4)))) \\
 & + 170 \widehat{\zeta}(2)^2 (-1081080 \widehat{\zeta}(3)^2 (10 + 3 \widehat{\zeta}(4) (-5 + 4 \widehat{\zeta}(5))) \\
 & - 117 \widehat{\zeta}(3) (-252175 + 16 \widehat{\zeta}(4) (13046 + 175 \widehat{\zeta}(5) (-55 + 36 \widehat{\zeta}(6)))) \\
 & + 3850 (-4745 + 36 \widehat{\zeta}(4) \widehat{\zeta}(6) (13 \widehat{\zeta}(4) - 72 \widehat{\zeta}(5) \widehat{\zeta}(8)))) \\
 & - 39 \widehat{\zeta}(2) (51 \widehat{\zeta}(3) (606375 + 16 \widehat{\zeta}(4) (-40887 + 25 \widehat{\zeta}(5) (1925 + 18 \widehat{\zeta}(6) (-119 + 110 \widehat{\zeta}(7)))) \\
 & + 1925 (-16660 + 3 \widehat{\zeta}(6) (765 \widehat{\zeta}(4)^2 + 288 \widehat{\zeta}(4) \widehat{\zeta}(8) (-17 \widehat{\zeta}(5) + 20 \widehat{\zeta}(6) \widehat{\zeta}(10)) \\
 & + 5440 \widehat{\zeta}(8) \widehat{\zeta}(12) (\widehat{\zeta}(5) \widehat{\zeta}(9) - 18 \widehat{\zeta}(10) \widehat{\zeta}(14) \widehat{\zeta}(18))))))
 \end{aligned}$$

(E_7, P_1)

$$\begin{aligned} v_{E_7, P_1} = & -\frac{1}{604800} \widehat{\zeta}(2)^4 (-1 + 2\widehat{\zeta}(2)) \widehat{\zeta}(3)^4 \widehat{\zeta}(4)^3 \widehat{\zeta}(5)^2 \widehat{\zeta}(6) \widehat{\zeta}(7) \\ & (5040 \widehat{\zeta}(2)^3 (-9 + 10\widehat{\zeta}(3)) + 378(25 - 32\widehat{\zeta}(3)^2 \widehat{\zeta}(4)) + 60\widehat{\zeta}(2)^2 (1435 + 72\widehat{\zeta}(3)(-21 + 10\widehat{\zeta}(4))) \\ & + 35\widehat{\zeta}(2)(27\widehat{\zeta}(3)(45 + 16\widehat{\zeta}(4)(-3 + 5\widehat{\zeta}(5))) + 40(-37 + 27\widehat{\zeta}(4)\widehat{\zeta}(6)(\widehat{\zeta}(4) - 8\widehat{\zeta}(5)\widehat{\zeta}(8)))))) \end{aligned}$$

(E_7, P_2)

$$\begin{aligned} v_{E_7, P_2} = & -\frac{1}{8640} \widehat{\zeta}(2)^4 \widehat{\zeta}(3)^2 (3 + 4\widehat{\zeta}(2)(-2 + 3\widehat{\zeta}(3))) \widehat{\zeta}(4) \\ & (-45 + \widehat{\zeta}(2)(200 + 8\widehat{\zeta}(2)(-25 + 36\widehat{\zeta}(3)) - 9\widehat{\zeta}(3)(25 + 16\widehat{\zeta}(4)(-2 + 5\widehat{\zeta}(5)))))) \end{aligned}$$

(E_7, P_3)

$$\begin{aligned} v_{E_7, P_3} = & -\frac{1}{1728} \widehat{\zeta}(2)^3 (-1 + 2\widehat{\zeta}(2)) \widehat{\zeta}(3)(3 + 4\widehat{\zeta}(2)(-2 + 3\widehat{\zeta}(3))) \\ & (9 + 18\widehat{\zeta}(2)^2 - 4\widehat{\zeta}(2)(8 + 9\widehat{\zeta}(3)(-1 + 2\widehat{\zeta}(4)))) \end{aligned}$$

(E_7, P_4)

$$\begin{aligned} v_{E_7, P_4} = & \frac{1}{907200} \widehat{\zeta}(2)^5 \widehat{\zeta}(3)^4 \widehat{\zeta}(4)^3 \widehat{\zeta}(5)^2 \widehat{\zeta}(6) \\ & (14175 + \widehat{\zeta}(2)(11025(-8 + 9\widehat{\zeta}(3)) + 4(882\widehat{\zeta}(2)^2(-25 + 36\widehat{\zeta}(3)) \\ & + 5\widehat{\zeta}(2)(8330 - 9\widehat{\zeta}(3)(15\widehat{\zeta}(3)(-49 + 96\widehat{\zeta}(4)) + 2(833 + 16\widehat{\zeta}(4)(-49 + 45\widehat{\zeta}(5)))))) \\ & + 36\widehat{\zeta}(3)\widehat{\zeta}(4)(-882 + 25\widehat{\zeta}(5)(49 + 36\widehat{\zeta}(6)(-2 + 7\widehat{\zeta}(7)))))) \end{aligned}$$

(E_7, P_5)

$$\begin{aligned} v_{E_7, P_5} = & \frac{1}{43200} \widehat{\zeta}(2)^4 (-1 + 2\widehat{\zeta}(2)) \widehat{\zeta}(3)^3 \widehat{\zeta}(4)^2 \widehat{\zeta}(5) \\ & (-675 + 2\widehat{\zeta}(2)(675\widehat{\zeta}(2)^2 - 40\widehat{\zeta}(2)(65 + 9\widehat{\zeta}(3)(-12 + 5\widehat{\zeta}(3) + 10\widehat{\zeta}(4))) \\ & + 9(200 + \widehat{\zeta}(3)(-225 + 16\widehat{\zeta}(4)(18 + 25\widehat{\zeta}(5)(-1 + 3\widehat{\zeta}(6)))))) \end{aligned}$$

(E_7, P_6)

$$\begin{aligned} v_{E_7, P_6} = & \frac{1}{9072000} \widehat{\zeta}(2)^5 \widehat{\zeta}(3)^5 \widehat{\zeta}(4)^4 \widehat{\zeta}(5)^4 \widehat{\zeta}(6)^2 \widehat{\zeta}(7)^2 \widehat{\zeta}(8) \widehat{\zeta}(9) \\ & (141750 + 453600\widehat{\zeta}(2)^4 - 1800\widehat{\zeta}(2)^3 (833 + 15\widehat{\zeta}(3)(-35 + 48\widehat{\zeta}(4))) \\ & - 500\widehat{\zeta}(2)^2 (-3773 + 18\widehat{\zeta}(3)(245 - 328\widehat{\zeta}(4) + 42\widehat{\zeta}(3)(-2 + 3\widehat{\zeta}(4))) + 2016\widehat{\zeta}(4)^2 \widehat{\zeta}(6)) \\ & - 21\widehat{\zeta}(2)(43000 + \widehat{\zeta}(3)(-37125 + 48\widehat{\zeta}(4)(1197 + 25\widehat{\zeta}(5)(-25 + 72\widehat{\zeta}(6)))) \\ & - 5400\widehat{\zeta}(4)\widehat{\zeta}(6)(5\widehat{\zeta}(4) - 8\widehat{\zeta}(8)(\widehat{\zeta}(5) - 10\widehat{\zeta}(6)\widehat{\zeta}(10)))))) \end{aligned}$$

(E_7, P_7)

$$v_{E_7, P_7} = \frac{1}{3326400} \widehat{\zeta}(2)^5 \widehat{\zeta}(3)^5 \widehat{\zeta}(4)^5 \widehat{\zeta}(5)^4 \widehat{\zeta}(6)^3 \widehat{\zeta}(7)^3 \widehat{\zeta}(8)^2 \widehat{\zeta}(9) \widehat{\zeta}(10) \widehat{\zeta}(11) \\ (51975 + \widehat{\zeta}(2)(3850(-86 + 81\widehat{\zeta}(3)) - 44\widehat{\zeta}(2)^2(10255 + 48\widehat{\zeta}(3)(-448 + 225\widehat{\zeta}(3))) \\ + 440\widehat{\zeta}(2)(1540 + 3\widehat{\zeta}(3)(-819 + 210\widehat{\zeta}(3) + 8\widehat{\zeta}(4)(39 - 70\widehat{\zeta}(5)))) \\ + 63(-32\widehat{\zeta}(3)\widehat{\zeta}(4)(132 + 25\widehat{\zeta}(5)(-11 + 6\widehat{\zeta}(6))) \\ + 275\widehat{\zeta}(6)(3\widehat{\zeta}(4)^2 - 32\widehat{\zeta}(4)\widehat{\zeta}(5)\widehat{\zeta}(8) + 192\widehat{\zeta}(5)\widehat{\zeta}(8)\widehat{\zeta}(9)\widehat{\zeta}(12))))))$$

 (E_6, P_1)

$$v_{E_6, P_1} = -\frac{1}{302400} \widehat{\zeta}(2)^4 \widehat{\zeta}(3)^4 \widehat{\zeta}(4)^3 \widehat{\zeta}(5)^2 \widehat{\zeta}(6) \widehat{\zeta}(7) \\ (9450 + 5040\widehat{\zeta}(2)^3(-9 + 10\widehat{\zeta}(3)) + 60\widehat{\zeta}(2)^2(1435 + 72\widehat{\zeta}(3)(-21 + 10\widehat{\zeta}(4))) \\ + 7\widehat{\zeta}(2)(27\widehat{\zeta}(3)(225 + 16\widehat{\zeta}(4)(-19 + 25\widehat{\zeta}(5))) + 200(-37 + 27\widehat{\zeta}(4)\widehat{\zeta}(6)(\widehat{\zeta}(4) - 8\widehat{\zeta}(5)\widehat{\zeta}(8))))))$$

 (E_6, P_2)

$$v_{E_6, P_2} = -\frac{1}{1440} \widehat{\zeta}(2)^3(-1 + 2\widehat{\zeta}(2))\widehat{\zeta}(3)^2\widehat{\zeta}(4) \\ (-45 + \widehat{\zeta}(2)(200 + 8\widehat{\zeta}(2)(-25 + 36\widehat{\zeta}(3)) - 9\widehat{\zeta}(3)(25 + 16\widehat{\zeta}(4)(-2 + 5\widehat{\zeta}(5))))))$$

 (E_6, P_3)

$$v_{E_6, P_3} = \frac{1}{288} \widehat{\zeta}(2)^2(-1 + 2\widehat{\zeta}(2))(3 + 4\widehat{\zeta}(2)(-2 + 3\widehat{\zeta}(3)))^2$$

 (E_6, P_4)

$$v_{E_6, P_4} = \frac{1}{21600} \widehat{\zeta}(2)^4 \widehat{\zeta}(3)^3 \widehat{\zeta}(4)^2 \widehat{\zeta}(5) \\ (-675 + 2\widehat{\zeta}(2)(675\widehat{\zeta}(2)^2 - 40\widehat{\zeta}(2)(65 + 9\widehat{\zeta}(3)(-12 + 5\widehat{\zeta}(3) + 10\widehat{\zeta}(4))) \\ + 9(200 + \widehat{\zeta}(3)(-225 + 16\widehat{\zeta}(4)(18 + 25\widehat{\zeta}(5)(-1 + 3\widehat{\zeta}(6))))))$$

 (E_6, P_5)

$$v_{E_6, P_5} = -\frac{1}{1440} \widehat{\zeta}(2)^3(-1 + 2\widehat{\zeta}(2))\widehat{\zeta}(3)^2\widehat{\zeta}(4) \\ (-45 + \widehat{\zeta}(2)(200 + 8\widehat{\zeta}(2)(-25 + 36\widehat{\zeta}(3)) - 9\widehat{\zeta}(3)(25 + 16\widehat{\zeta}(4)(-2 + 5\widehat{\zeta}(5))))))$$

 (E_6, P_6)

$$v_{E_6, P_6} = -\frac{1}{302400} \widehat{\zeta}(2)^4 \widehat{\zeta}(3)^4 \widehat{\zeta}(4)^3 \widehat{\zeta}(5)^2 \widehat{\zeta}(6) \widehat{\zeta}(7) \\ (9450 + 5040\widehat{\zeta}(2)^3(-9 + 10\widehat{\zeta}(3)) + 60\widehat{\zeta}(2)^2(1435 + 72\widehat{\zeta}(3)(-21 + 10\widehat{\zeta}(4))) \\ + 7\widehat{\zeta}(2)(27\widehat{\zeta}(3)(225 + 16\widehat{\zeta}(4)(-19 + 25\widehat{\zeta}(5))) \\ + 200(-37 + 27\widehat{\zeta}(4)\widehat{\zeta}(6)(\widehat{\zeta}(4) - 8\widehat{\zeta}(5)\widehat{\zeta}(8))))))$$

Corollary 9. $v_{E_n, p} \neq 0$ for all $n = 6, 7, 8$ and their maximal parabolic subgroups.

Proof. This is simple. For example, with (E_8, P_1) , we have

$$\begin{aligned} v_{E_8, P_1} = & - \frac{(\pi^3 \zeta(3)^4 \cdot (4\pi - 3(3 + \zeta(3))) \cdot \zeta(5)^2 \cdot \zeta(7))}{75243740696496046080000000} \\ & \cdot (-567000\pi^3(-63 + 2\zeta(3)) + 7938000\pi(185 + 27\zeta(3)) \\ & - 79380\pi^2(5125 + 136\zeta(3)) + 8\pi^8(-35 + 12\zeta(5)) \\ & - 4465125(360 + \zeta(3)(135 + 2\zeta(5)))) \end{aligned}$$

Approximately,

$$v_{E_8, P_1} = \frac{206392.}{75243740696496046080000000} \neq 0.$$

Our verifications for other cases are the same. $\square$

Consequently, by the theorem recalled in §4, we have proved the following

Theorem 10. *All but finitely many zeros of $\widehat{\zeta}_{\mathbb{Q}}^{E_8/P_1}(s)$ are simple and lie on the line $\operatorname{Re}(s) = -7$.*

Remark: The structures for $W^\dagger$ and $v_{E_n, P}$ as stated above is strikingly uniform. Besides that their cardinal numbers are 2^n , the number of non-trivial factors of v_{E_n, P_i} in fact is the same as the number of connected components of the Dynkin diagram by deleting the i -th simple root. Furthermore, there is a nice symmetry hidden in our formula. For example,

$$v_{E_6, P_1} = v_{E_6, P_6}, \quad v_{E_6, P_2} = v_{E_6, P_5}.$$

All this is not just a coincidence: they are dominated a force exposed in our conjecture on Stability, Parabolic Reduction and Masses. For example, from $|W^\dagger| = s^{n-1}$, we know that, from definition, $v_{E_n, P}$ consists of 2^{n-1} terms. This is exactly the number appeared in the Arthur's analytic truncation theory: Set of all standard parabolic subgroups is bijectively correspondence to a power set. For more details, please refer to our works.¹

APPENDIX A. Cardinal Number of $\Delta_P \cap w^{-1}\Phi^+$

Next, we give the cardinal number of the set $\Delta_P \cap w^{-1}\Phi_n^+$ for each very special element $w \in W_{n, P}^\dagger$, appeared as the power of $\widehat{\zeta}(2)$. Since w is a bijection for the root system, it suffices to calculate the cardinal number of $w\Delta_P \cap \Phi_n^+$. Our calculations give the following

Proposition 11. *With the order of $w \in \mathfrak{W}_{n, P}^\dagger$ as in `ddag.nb`, $|\Delta_P \cap w^{-1}\Phi^+|$ for $w \in W_{n, P}^\dagger$ is given by the following lists:*

$(W_{8, 1}^\dagger)$

7, 5, 6, 6, 6, 4, 5, 5, 6, 4, 5, 5, 5, 3, 4, 4, 5, 3, 4, 4, 4, 2, 3, 3, 5, 3, 4, 4, 4, 2, 3, 3,
4, 2, 3, 3, 3, 1, 2, 2, 5, 3, 4, 4, 6, 4, 5, 5, 4, 2, 3, 3, 5, 3, 4, 4, 5, 3, 4, 4, 6, 4, 5, 5,
6, 4, 5, 5, 5, 3, 4, 4, 5, 3, 4, 4, 4, 2, 3, 3, 4, 2, 3, 3, 5, 3, 4, 4, 3, 1, 2, 2, 4, 2, 3, 3,
2, 0, 1, 1, 3, 1, 2, 2, 3, 1, 2, 2, 4, 2, 3, 3, 3, 1, 2, 2, 4, 2, 3, 3, 4, 2, 3, 3, 5, 3, 4, 4

¹This will be extended. With 16G data obtained from the calculation of a half year long, I am simply exhausted.

$(W_{8,2}^{\dagger})$

5, 3, 4, 2, 4, 3, 3, 4, 6, 4, 5, 3, 5, 4, 4, 5, 4, 2, 3, 1, 3, 2, 2, 3, 5, 3, 4, 2, 4, 3, 3, 4,
 7, 5, 6, 4, 6, 5, 5, 6, 5, 3, 4, 2, 4, 3, 3, 4, 6, 4, 5, 3, 5, 4, 4, 5, 6, 4, 5, 3, 5, 4, 4, 5,
 5, 3, 4, 2, 4, 3, 3, 4, 4, 2, 3, 1, 3, 2, 2, 3, 4, 2, 3, 1, 3, 2, 2, 3, 3, 1, 2, 0, 2, 1, 1, 2,
 4, 2, 3, 1, 3, 2, 2, 3, 5, 3, 4, 2, 4, 3, 3, 4, 5, 3, 4, 2, 4, 3, 3, 4, 6, 4, 5, 3, 5, 4, 4, 5

 $(W_{8,3}^{\dagger})$

7, 5, 5, 3, 4, 4, 4, 4, 6, 5, 6, 6, 5, 6, 5, 5, 6, 4, 4, 2, 3, 3, 3, 3, 5, 4, 5, 5, 4, 5, 4, 4,
 5, 3, 3, 1, 2, 2, 2, 2, 4, 3, 4, 4, 3, 4, 3, 3, 6, 4, 4, 2, 3, 3, 3, 3, 5, 4, 5, 5, 4, 5, 4, 4,
 5, 3, 3, 1, 2, 2, 2, 2, 4, 3, 4, 4, 3, 4, 3, 3, 4, 2, 2, 0, 1, 1, 1, 1, 3, 2, 3, 3, 2, 3, 2, 2,
 6, 4, 4, 2, 3, 3, 3, 3, 5, 4, 5, 5, 4, 5, 4, 4, 5, 3, 3, 1, 2, 2, 2, 2, 4, 3, 4, 4, 3, 4, 3, 3

 $(W_{8,4}^{\dagger})$

7, 5, 3, 5, 2, 4, 2, 4, 1, 3, 2, 4, 1, 3, 1, 3, 0, 2, 4, 3, 2, 2, 1, 3, 3, 4, 2, 3, 3, 2, 3, 4,
 3, 4, 2, 2, 1, 3, 2, 3, 4, 5, 4, 4, 3, 3, 2, 5, 4, 3, 6, 4, 5, 5, 5, 5, 6, 4, 4, 3, 4, 4, 5, 4,
 3, 6, 4, 5, 5, 5, 6, 4, 4, 3, 4, 3, 6, 4, 5, 5, 5, 5, 4, 4, 3, 3, 2, 4, 3, 3, 2, 5, 5, 6, 5, 4,
 4, 4, 3, 4, 4, 3, 3, 3, 3, 2, 2, 3, 3, 2, 2, 2, 2, 1, 1, 5, 5, 6, 4, 4, 3, 4, 4, 3, 4, 3, 3, 2

 $(W_{8,5}^{\dagger})$

7, 5, 3, 5, 2, 4, 2, 4, 1, 3, 4, 3, 2, 3, 3, 4, 3, 4, 2, 3, 4, 4, 4, 3, 3, 3, 3, 2, 2, 4, 3, 4,
 6, 5, 5, 4, 4, 3, 6, 5, 4, 3, 4, 5, 5, 4, 5, 6, 4, 4, 5, 5, 6, 5, 5, 6, 4, 4, 3, 5, 5, 4, 6, 5,
 6, 4, 2, 4, 1, 3, 1, 3, 0, 2, 3, 2, 1, 2, 2, 3, 2, 3, 1, 2, 3, 3, 3, 2, 2, 2, 2, 1, 1, 3, 2, 3,
 5, 4, 4, 3, 3, 2, 5, 4, 3, 2, 3, 4, 4, 3, 4, 5, 3, 3, 4, 4, 5, 4, 4, 5, 3, 3, 2, 4, 4, 3, 5, 4

 $(W_{8,6}^{\dagger})$

7, 6, 6, 6, 5, 6, 5, 5, 5, 4, 4, 5, 5, 6, 5, 4, 5, 3, 4, 4, 5, 4, 5, 3, 4, 4, 5, 4, 3, 4, 4, 5,
 2, 3, 3, 4, 4, 3, 4, 3, 5, 5, 5, 4, 4, 1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5, 5, 4, 2,
 3, 3, 4, 4, 3, 3, 4, 3, 2, 4, 3, 3, 2, 4, 3, 4, 3, 5, 6, 6, 4, 4, 2, 1, 3, 2, 3, 2, 4, 3, 1, 0,
 2, 1, 2, 1, 3, 2, 2, 1, 3, 2, 3, 2, 4, 5, 5, 3, 2, 1, 3, 2, 3, 2, 4, 3, 3, 2, 4, 3, 4, 3, 5, 4

 $(W_{8,7}^{\dagger})$

7, 6, 6, 5, 6, 5, 5, 4, 4, 3, 3, 2, 4, 3, 3, 2, 2, 1, 3, 2, 4, 3, 3, 2, 2, 1, 3, 2, 2, 1, 1, 0,
 2, 1, 3, 2, 4, 3, 5, 4, 3, 2, 2, 1, 3, 2, 4, 3, 4, 3, 3, 2, 4, 3, 5, 4, 3, 2, 4, 3, 4, 3, 5, 4,
 4, 3, 5, 4, 5, 4, 6, 5, 6, 5, 4, 3, 3, 2, 4, 3, 3, 2, 2, 1, 3, 2, 4, 3, 5, 4, 4, 3, 3, 2, 4, 3,
 5, 4, 4, 3, 5, 4, 5, 4, 6, 5, 4, 3, 3, 2, 4, 3, 5, 4, 5, 4, 4, 3, 5, 4, 6, 5, 4, 3, 5, 4, 5, 4

 $(W_{8,8}^{\dagger})$

5, 7, 6, 6, 5, 4, 3, 4, 4, 3, 2, 3, 2, 3, 3, 4, 1, 2, 2, 3, 0, 1, 1, 2, 1, 2, 2, 5, 5, 3, 1, 2,
 2, 3, 2, 3, 3, 4, 5, 4, 4, 3, 4, 5, 4, 3, 4, 5, 2, 3, 3, 4, 3, 4, 4, 5, 4, 5, 5, 6, 4, 5, 5, 4,
 4, 3, 4, 3, 3, 2, 5, 5, 4, 3, 4, 4, 1, 2, 2, 3, 2, 3, 3, 6, 4, 2, 3, 3, 4, 3, 4, 4, 5, 5, 6, 4,
 5, 4, 5, 3, 4, 6, 5, 4, 6, 4, 3, 3, 4, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4, 5, 5, 4, 4, 3, 2, 3, 3, 4

$(W_{7,1}^\dagger)$

6, 5, 5, 4, 5, 4, 4, 3, 4, 3, 3, 2, 4, 3, 4, 3, 3, 2, 3, 2, 2, 1, 4, 3, 5, 4, 4, 3, 3, 2, 3, 2,
4, 3, 3, 2, 4, 3, 4, 3, 2, 1, 3, 2, 1, 0, 2, 1, 2, 1, 3, 2, 2, 1, 3, 2, 5, 4, 5, 4, 3, 2, 4, 3

 $(W_{7,2}^\dagger)$

6, 4, 4, 2, 5, 5, 3, 3, 3, 1, 2, 2, 3, 1, 2, 2, 2, 0, 1, 1, 3, 1, 2, 2, 5, 3, 4, 2, 4, 4, 3, 3,
4, 2, 4, 2, 3, 3, 3, 3, 5, 3, 4, 4, 4, 2, 3, 3, 5, 3, 4, 4, 3, 1, 2, 2, 4, 2, 5, 3, 3, 3, 4, 4

 $(W_{7,3}^\dagger)$

6, 4, 4, 2, 3, 1, 5, 4, 3, 2, 4, 5, 2, 3, 5, 3, 5, 3, 5, 3, 2, 2, 4, 3, 4, 3, 3, 4, 3, 4, 4, 4,
4, 2, 4, 2, 1, 1, 3, 2, 3, 2, 2, 3, 2, 3, 3, 5, 3, 3, 1, 2, 0, 4, 3, 2, 1, 3, 4, 1, 2, 4, 2

 $(W_{7,4}^\dagger)$

6, 4, 4, 3, 3, 2, 3, 4, 4, 5, 3, 3, 4, 5, 2, 4, 4, 3, 3, 5, 4, 2, 5, 3, 2, 4, 4, 1, 3, 3, 2, 3,
4, 5, 4, 3, 2, 3, 4, 3, 2, 3, 5, 1, 2, 3, 2, 2, 3, 4, 4, 1, 2, 3, 2, 1, 2, 3, 3, 0, 1, 2, 2, 1

 $(W_{7,5}^\dagger)$

6, 4, 5, 3, 3, 2, 3, 2, 2, 1, 3, 2, 2, 1, 2, 1, 1, 0, 5, 4, 4, 3, 3, 2, 3, 2, 2, 1, 4, 4, 3, 3,
3, 2, 5, 4, 4, 4, 3, 3, 3, 2, 5, 4, 4, 3, 4, 5, 3, 4, 3, 2, 3, 2, 2, 1, 5, 4, 4, 3, 4, 3, 3, 2

 $(W_{7,6}^\dagger)$

6, 5, 5, 4, 5, 4, 4, 3, 4, 3, 4, 2, 3, 4, 3, 3, 5, 4, 4, 1, 2, 2, 3, 2, 3, 3, 4, 4, 2, 3, 3, 4,
3, 3, 4, 2, 3, 2, 3, 3, 4, 4, 1, 2, 2, 3, 0, 1, 1, 2, 1, 2, 2, 5, 5, 3, 1, 2, 2, 3, 2, 3, 3, 4

 $(W_{7,7}^\dagger)$

6, 5, 3, 2, 3, 4, 2, 1, 2, 3, 3, 4, 2, 1, 2, 1, 0, 1, 2, 3, 3, 2, 3, 4, 4, 3, 2, 3, 2, 1, 2, 3,
4, 2, 1, 2, 3, 3, 2, 3, 4, 3, 2, 3, 4, 4, 3, 4, 5, 2, 3, 3, 4, 3, 4, 5, 5, 4, 3, 4, 5, 4, 4, 5

 $(W_{6,1}^\dagger)$

5, 4, 4, 3, 3, 2, 3, 3, 2, 2, 1, 3, 4, 3, 2, 2, 3, 2, 3, 3, 1, 2, 0, 1, 1, 2, 1, 2, 4, 4, 2, 3

 $(W_{6,2}^\dagger)$

5, 3, 4, 2, 2, 1, 2, 1, 1, 0, 2, 1, 4, 3, 3, 2, 3, 3, 2, 2, 4, 3, 3, 2, 4, 3, 2, 1, 3, 4, 2, 3

 $(W_{6,3}^\dagger)$

5, 3, 4, 2, 4, 2, 3, 1, 4, 4, 3, 3, 3, 3, 2, 2, 3, 3, 2, 2, 2, 2, 1, 1, 4, 2, 3, 1, 3, 1, 2, 0

 $(W_{6,4}^\dagger)$

5, 3, 2, 3, 4, 2, 1, 2, 3, 3, 4, 2, 1, 2, 1, 0, 1, 2, 3, 3, 2, 3, 4, 4, 3, 2, 3, 2, 1, 2, 3, 4

 $(W_{6,5}^\dagger)$

5, 4, 4, 3, 4, 3, 3, 2, 4, 3, 3, 2, 3, 2, 2, 1, 4, 3, 3, 2, 3, 2, 2, 1, 3, 2, 2, 1, 2, 1, 1, 0

 $(W_{6,6}^\dagger)$

5, 4, 3, 4, 4, 3, 2, 3, 2, 3, 3, 4, 1, 2, 2, 3, 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4

APPENDIX B. $\prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \setminus \Delta_P)} \frac{1}{\langle \rho, \alpha^\vee \rangle - 1}$

In this section, for each very special Weyl element $w \in \mathfrak{W}_{n,P}^\dagger$, we give the rational factors contribution $\prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \setminus \Delta_P)} \frac{1}{\langle \rho, \alpha^\vee \rangle - 1}$. Our calculation gives the following

Proposition 12. *For very special Weyl elements $w \in W_{n,P}^\dagger$, the corresponding constants $\prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \setminus \Delta_P)} (\langle \rho, \alpha^\vee \rangle - 1)$ are given by the following lists.*

$(W_{8,1}^\dagger)$

1, 4, -3, -3, -8, -32, 24, 24, -7, -28, 21, 21, 20, 80, -60, -60, 12, 48, -36, -36,
- 18, -72, 54, 54, 20, 80, -60, -60, -18, -72, 54, 54, -36, -144, 108, 108, 24, 96,
- 72, -72, 10, 40, -30, -30, -6, -24, 18, 18, -16, -64, 48, 48, 10, 40, -30, -30, 25,
100, -75, -75, -8, -32, 24, 24, -8, -32, 24, 24, 30, 120, -90, -90, 12, 48, -36, -36,
- 32, -128, 96, 96, -24, -96, 72, 72, 15, 60, -45, -45, 24, 96, -72, -72, -16, -64,
48, 48, -32, -128, 96, 96, 24, 96, -72, -72, 54, 216, -162, -162, -32, -128, 96, 96,
36, 144, -108, -108, -24, -96, 72, 72, -64, -256, 192, 192, 30, 120, -90, -90

$(W_{8,2}^\dagger)$

9, 81, -36, -72, -36, 54, 54, -36, -5, -45, 20, 40, 20, -30, -30, 20, -12, -108,
48, 96, 48, -72, -72, 48, 12, 108, -48, -96, -48, 72, 72, -48, 1, 9, -4, -8, -4, 6,
6, -4, 8, 72, -32, -64, -32, 48, 48, -32, -5, -45, 20, 40, 20, -30, -30, 20, -5, -45,
20, 40, 20, -30, -30, 20, 16, 144, -64, -128, -64, 96, 96, -64, -18, -162, 72, 144,
72, -108, -108, 72, -18, -162, 72, 144, 72, -108, -108, 72, 16, 144, -64, -128,
- 64, 96, 96, -64, -12, -108, 48, 96, 48, -72, -72, 48, 12, 108, -48, -96, -48, 72,
72, -48, 8, 72, -32, -64, -32, 48, 48, -32, -5, -45, 20, 40, 20, -30, -30, 20

$(W_{8,3}^\dagger)$

1, 16, 9, 16, -18, -18, -12, -12, -5, 12, -5, -5, 12, -5, 8, 8, -2, -32, -18, -32,
36, 36, 24, 24, 10, -24, 10, 10, -24, 10, -16, -16, 4, 64, 36, 64, -72, -72, -48,
- 48, -20, 48, -20, -20, 48, -20, 32, 32, -3, -48, -27, -48, 54, 54, 36, 36, 15,
- 36, 15, 15, -36, 15, -24, -24, 6, 96, 54, 96, -108, -108, -72, -72, -30, 72, -30,
- 30, 72, -30, 48, 48, -8, -128, -72, -128, 144, 144, 96, 96, 40, -96, 40, 40,
- 96, 40, -64, -64, -3, -48, -27, -48, 54, 54, 36, 36, 15, -36, 15, 15, -36, 15,
- 24, -24, 6, 96, 54, 96, -108, -108, -72, -72, -30, 72, -30, -30, 72, -30, 48, 48

$(W_{8,4}^\dagger)$

1, 49, 100, 36, -128, -60, -144, -64, 144, 72, -128, -60, 144, 81, 144, 72,
 - 128, -72, -72, 120, -162, -144, 144, 100, 120, -90, -128, 120, 100, -128,
 144, -96, 120, -72, -144, -162, 144, 144, -162, 144, -90, 25, -32, -48, 60, 72,
 - 64, 20, -60, 72, -8, -64, 35, 14, 30, 42, -8, -45, -48, 48, -24, -75, 20, -36, 40,
 - 8, -45, 28, 21, 24, -8, -30, -36, 40, -80, 96, -8, -24, 21, 28, 18, 18, -45, -48,
 72, 60, -64, -75, 108, 108, -108, 14, 35, -8, 24, -30, -64, -60, 72, -60, -60,
 80, 96, 96, 96, -96, -108, 96, 80, -108, -108, -108, -96, 96, 96, 30, 42, -8, -48,
 - 45, 48, -75, -80, 96, -75, 108, 108, -108

 $(W_{8,5}^\dagger)$

1, 36, 64, 25, -72, -36, -72, -36, 64, 36, -50, 72, -72, 64, 64, -60, 72, -50,
 - 72, 81, -60, -40, -40, 48, 54, 54, 48, -48, -48, -48, 54, -48, -7, 30, 20, -24,
 - 48, 54, -7, 15, -30, 32, -36, 12, 24, -48, 15, -7, -20, -20, 18, 18, -7, 20, 30,
 - 7, -24, -30, 32, 24, 12, -36, -7, 16, -2, -72, -128, -50, 144, 72, 144, 72, -128,
 - 72, 100, -144, 144, -128, -128, 120, -144, 100, 144, -162, 120, 80, 80, -96,
 - 108, -108, -96, 96, 96, 96, -108, 96, 14, -60, -40, 48, 96, -108, 14, -30, 60,
 - 64, 72, -24, -48, 96, -30, 14, 40, 40, -36, -36, 14, -40, -60, 14, 48, 60, -64,
 - 48, -24, 72, 14, -32

 $(W_{8,6}^\dagger)$

1, -11, -12, -9, 27, -10, 20, 28, 20, -30, -54, 24, 24, -8, 32, -32, 14, 40, -54,
 - 30, 28, -108, 42, 144, -64, -24, 14, -96, 72, -32, -60, 25, -108, 48, 96, -40,
 - 64, 96, -36, 48, 28, 49, 35, -42, -60, 144, -64, -144, 60, -243, 108, 240, -96,
 - 108, 48, 96, -40, 144, -64, -108, 42, 56, -96, -128, 180, 96, -96, -90, 120,
 100, -60, 96, -128, -80, 120, 48, -64, -48, 72, -45, 60, 30, -12, -12, -216, -48,
 - 72, 96, 72, -108, 72, -96, -50, 80, 96, -128, -96, 144, -108, 144, 80, -128,
 - 162, 216, 144, -216, 192, -256, -150, 70, 70, 250, -72, 96, 72, -108, 72, -96,
 - 50, 80, 96, -128, -80, 120, -90, 120, 56, -96

 $(W_{8,7}^\dagger)$

1, -2, -11, 22, -9, 18, 16, -32, -150, 300, 128, -256, -80, 160, 72, -144, -72,
 144, 54, -108, -30, 60, 128, -256, -108, 216, 72, -144, -72, 144, 64, -128, -48,
 96, 32, -64, -50, 100, 48, -96, 48, -96, -48, 96, 36, -72, -20, 40, -45, 90, 48,
 - 96, -36, 72, 18, -36, 64, -128, -40, 80, -60, 120, 16, -32, -60, 120, 35, -70,
 64, -128, -12, 24, -12, 24, -80, 160, 72, -144, -48, 96, 54, -108, -48, 96, 36,
 - 72, -24, 48, 30, -60, -36, 72, 36, -72, -27, 54, 15, -30, -40, 80, 25, -50, 35, -70,
 - 9, 18, -30, 60, 32, -64, -24, 48, 12, -24, 18, -36, -20, 40, 15, -30, -7, 14, -50,
 100, 30, -60, 48, -96

$(W_{8,8}^\dagger)$

56, 1, -17, -13, 120, -128, 120, -105, -288, 300, -256, 240, -96, 108, 90, -72,
 144, -162, -144, 120, -128, 144, 144, -128, 216, -216, -256, 20, 70, 250, 96,
 - 108, -96, 80, -128, 120, 120, -96, 30, -36, -60, 48, -96, 18, -30, 100, -90, 42,
 - 64, 72, 60, -48, 40, -45, -36, 28, -32, 20, 14, -8, -32, 27, 42, -100, -48, 72,
 - 60, 54, 108, -72, 77, 21, -24, 40, -36, -72, 96, -108, -108, 96, -144, 144, 160,
 - 11, -150, -72, 81, 72, -60, 80, -75, -70, 54, 24, -10, -75, 35, -40, 24, 48, -30,
 - 14, 72, -90, -18, -72, 96, 72, -60, -72, 64, 96, -96, -162, 160, -108, 96, 192,
 - 180, 24, 40, -48, -48, 48, -64, 90, 72, -96

 $(W_{7,1}^\dagger)$

1, -2, -7, 14, -8, 16, 20, -40, 12, -24, -18, 36, 30, -60, 20, -40, -18,
 36, -36, 72, 24, -48, 10, -20, -6, 12, 12, -24, -32, 64, -16, 32, 10,
 - 20, -24, 48, 25, -50, 15, -30, 24, -48, -16, 32, -32, 64, 24, -48,
 54, -108, -32, 64, 36, -72, -24, 48, -8, 16, -8, 16, -64, 128, 30, -60

 $(W_{7,2}^\dagger)$

1, 4, 16, 64, -3, -3, -48, -48, -18, -72, 54, 54, -18, -72, 54, 54, 16,
 64, -48, -48, -12, -48, 36, 36, -5, -20, 12, 48, 15, 15, -36, -36, 8,
 32, 8, 32, -24, -24, -24, -24, -5, -20, 15, 15, 9, 36, -27, -27, -5,
 - 20, 15, 15, -12, -48, 36, 36, 12, 48, -5, -20, -36, -36, 15, 15

 $(W_{7,3}^\dagger)$

1, 9, 4, 36, -8, -32, -4, 6, -16, 24, 6, -4, 24, -16, -4, -16, -3, -27,
 - 3, -27, 24, 24, 12, -18, 12, -18, -18, 12, -18, 12, 12, 12, 6, 54, 6,
 54, -48, -48, -24, 36, -24, 36, 36, -24, 36, -24, -24, -24, -2, -18,
 - 8, -72, 16, 64, 8, -12, 32, -48, -12, 8, -48, 32, 8, 32

 $(W_{7,4}^\dagger)$

1, 36, 25, -36, -36, 36, -50, 20, 30, -7, -24, -60, 15, -7, 64, 12, 24,
 - 48, -60, -7, 15, 72, -7, -48, 81, 18, 18, -72, -20, -36, 54, -50,
 30, -7, 20, -24, 64, -40, 16, -40, 72, -48, -7, -72, 48, -30, 54, 64,
 - 48, 24, 12, -72, 54, -30, 48, -72, 54, -36, -20, 64, -48, 32, 32, -48

 $(W_{7,5}^\dagger)$

1, 25, -2, -50, -32, 64, -36, 72, 36, -72, -32, 64, 36, -72, 36, -72,
 - 32, 64, -6, 20, 12, -40, -24, 48, -27, 54, 24, -48, 10, 15, -20, -30,
 - 16, 32, -6, 12, 15, 10, -30, -20, -16, 32, -6, 12, 12, -24, 20, -6,
 - 40, 12, -27, 54, -24, 48, 24, -48, -6, 12, 12, -24, 16, -32, -18, 36

$(W_{7,6}^\dagger)$

1, -9, -10, 18, -8, 16, 36, -48, 24, -24, 20, 36, -32, 16, -24, -24, -7,
 21, 12, -48, 48, 81, -80, 36, -32, -48, 36, 48, 32, -45, -24, 24, -60,
 -20, 12, 64, -60, 32, -36, -30, 36, 24, -48, 54, 48, -40, 64, -72, -72,
 64, -108, 108, 128, -10, -10, -125, -48, 54, 48, -40, 64, -60, -60, 48

$(W_{7,7}^\dagger)$

1, -11, -150, 128, -80, 48, 72, -72, 54, -30, -50, 30, 128, -108, 72,
 -72, 64, -48, 32, -50, -30, 32, -24, 12, 48, -80, 72, -48, 54, -48, 36,
 -24, 30, 48, -48, 36, -20, -36, 36, -27, 15, -45, 48, -36, 18, 18, -20,
 15, -7, 64, -40, -40, 25, -60, 35, -12, -9, 16, -60, 35, -9, 16, 64, -12

$(W_{6,1}^\dagger)$

1, -7, -8, 20, 12, -18, 30, 20, -18, -36, 24, 10, -6, 12, -32, -16,
 10, -24, 25, 15, 24, -16, -32, 24, 54, -32, 36, -24, -8, -8, -64, 30

$(W_{6,2}^\dagger)$

1, 16, -2, -32, -18, 36, -18, 36, 16, -32, -12, 24, -5, 12, 10, -24,
 8, 8, -16, -16, -5, 10, 9, -18, -5, 10, -12, 24, 12, -5, -24, 10

$(W_{6,3}^\dagger)$

1, 4, -3, -12, -3, -12, 4, 16, -3, -3, 9, 9, 9, 9, -12, -12, 6, 6, -18,
 -18, -18, -18, 24, 24, -2, -8, 6, 24, 6, 24, -8, -32

$(W_{6,4}^\dagger)$

1, 25, -32, 20, -6, -36, 36, -27, 15, 10, -6, -32, 36, -24, 36, -32,
 24, -16, 10, 15, -16, 12, -6, -6, 20, -24, 16, -27, 24, -18, 12, -6

$(W_{6,5}^\dagger)$

1, -2, -5, 10, -5, 10, 8, -16, -5, 10, 9, -18, 12, -24, -12, 24, -5, 10,
 8, -16, 12, -24, -12, 24, 16, -32, -18, 36, -18, 36, 16, -32

$(W_{6,6}^\dagger)$

1, -8, 12, -7, -8, 30, -32, 20, -16, 12, 15, -6, 24, -18, -24, 10, -32,
 24, 36, -16, 54, -36, -64, 25, 24, -18, -24, 10, -32, 20, 30, -8

APPENDIX C. Probability $\delta_{\alpha,w}$

Before we are able to calculate the product $\prod_{\alpha \in \Phi_P^+ \setminus \Delta_P} \widehat{\zeta}(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha,w})$, we in this section calculate $\delta_{\alpha,w}$ for $\alpha \in \Phi^+ \setminus \Delta_P$ and $w \in \mathfrak{W}^\dagger$, by detecting whether $w\alpha \in \Phi^+$. Our calculations give the follows, corresponding to cg[2] in the program. (We remark that $W^\dagger$ in all cases are also given heres.)

Proposition 13. $(w, \delta_{\alpha,w})$ for $\alpha \in \Phi^+ \setminus \Delta_P$ and $w \in \mathfrak{W}^\dagger$ is given by:

$(W_{8,1}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 1, 1, 27), (1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 1, 14, 27), (1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 1, 14, 29), (1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1375, 3, 1, 1), (0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1)),$
 $((1375, 3, 1, 27), (0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1)),$
 $((1375, 3, 14, 27), (0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1)),$
 $((1375, 3, 14, 29), (0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1)),$
 $((1, 9, 1, 1), (0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((1, 9, 1, 27), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((1, 9, 14, 27), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((1, 9, 14, 29), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((193, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((193, 9, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((193, 9, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((193, 9, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((301, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((301, 9, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((301, 9, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((301, 9, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((325, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((325, 9, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((325, 9, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((325, 9, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((393, 9, 1, 1), (0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((393, 9, 1, 27), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((393, 9, 14, 27), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((393, 9, 14, 29), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((423, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((423, 9, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((423, 9, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((423, 9, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((441, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((441, 9, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((441, 9, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((441, 9, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((447, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((447, 9, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((447, 9, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((447, 9, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$

$((663, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((663, 13, 1, 27), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((663, 13, 14, 27), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((663, 13, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((687, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((687, 13, 1, 27), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((687, 13, 14, 27), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((687, 13, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((781, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((781, 13, 1, 27), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((781, 13, 14, 27), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((781, 13, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((805, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((805, 13, 1, 27), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((805, 13, 14, 27), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((805, 13, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 15, 1, 1), (0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 15, 1, 27), (0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 15, 14, 27), (0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 15, 14, 29), (0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 16, 1, 1), (1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 16, 1, 27), (1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 16, 14, 27), (1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 16, 14, 29), (1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1, 23, 1, 1), (1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1))$
 $((1, 23, 1, 27), (1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1))$
 $((1, 23, 14, 27), (1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1))$
 $((1, 23, 14, 29), (1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1))$
 $((361, 23, 1, 1), (1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((361, 23, 1, 27), (1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((361, 23, 14, 27), (1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((361, 23, 14, 29), (1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((687, 23, 1, 1), (0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1))$
 $((687, 23, 1, 27), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1))$
 $((687, 23, 14, 27), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1))$
 $((687, 23, 14, 29), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1))$
 $((775, 23, 1, 1), (0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((775, 23, 1, 27), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((775, 23, 14, 27), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((775, 23, 14, 29), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$

$((961, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((961, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((961, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((961, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((985, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((985, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((985, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((985, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((987, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((987, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((987, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((987, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1321, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1321, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1321, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1321, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1323, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1323, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1323, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1323, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1327, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1327, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1327, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1327, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1335, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((1335, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((1335, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((1335, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((1345, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1345, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1345, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1345, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1347, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1347, 23, 1, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1347, 23, 14, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1347, 23, 14, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$

$((1375, 23, 1, 1), (0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 23, 1, 27), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 23, 14, 27), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 23, 14, 29), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1407, 23, 1, 1), (0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1))$
 $((1407, 23, 1, 27), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1))$
 $((1407, 23, 14, 27), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1))$
 $((1407, 23, 14, 29), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1)).$

$(W_{8,2}^\dagger)$

$((607, 4, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((607, 4, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((607, 4, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((607, 4, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((608, 4, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((608, 4, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((608, 4, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((608, 4, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((625, 4, 1, 1), (0, 0, 1, 1, 1, 1, 0, 0, 0)),$
 $((625, 4, 1, 25), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((625, 4, 13, 27), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((625, 4, 14, 25), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((626, 4, 13, 25), (0, 0, 1, 0, 0, 1, 0, 0, 0)),$
 $((626, 4, 13, 27), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((626, 4, 14, 25), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((626, 4, 14, 29), (0, 0, 1, 0, 1, 0, 0, 0, 0)),$
 $((649, 4, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((649, 4, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((649, 4, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((649, 4, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((650, 4, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((650, 4, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((650, 4, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((650, 4, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((655, 4, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((655, 4, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((655, 4, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((655, 4, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((656, 4, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((656, 4, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((656, 4, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((656, 4, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 1, 1, 25), (1, 1, 1, 0, 0, 0, 1, 1, 1)),$
 $((1, 1, 13, 27), (1, 1, 1, 0, 0, 0, 1, 1, 1)),$
 $((1, 1, 14, 25), (1, 1, 1, 0, 0, 0, 1, 1, 1)),$
 $((2, 1, 13, 25), (1, 1, 1, 0, 0, 1, 1, 1, 1)),$
 $((2, 1, 13, 27), (1, 1, 1, 0, 0, 0, 1, 1, 1)),$
 $((2, 1, 14, 25), (1, 1, 1, 0, 0, 0, 1, 1, 1)),$
 $((2, 1, 14, 29), (1, 1, 1, 0, 1, 0, 1, 1, 1)),$

$((241, 6, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 1)),$
 $((241, 6, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((241, 6, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((241, 6, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((242, 6, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 1)),$
 $((242, 6, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((242, 6, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((242, 6, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 1)),$
 $((265, 6, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 1)),$
 $((265, 6, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((265, 6, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((265, 6, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((266, 6, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 1)),$
 $((266, 6, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((266, 6, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((266, 6, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 1)),$
 $((175, 10, 1, 1), (1, 0, 0, 1, 1, 1, 1, 0, 1)),$
 $((175, 10, 1, 25), (1, 0, 0, 0, 0, 0, 1, 0, 1)),$
 $((175, 10, 13, 27), (1, 0, 0, 0, 0, 0, 1, 0, 1)),$
 $((175, 10, 14, 25), (1, 0, 0, 0, 0, 0, 1, 0, 1)),$
 $((176, 10, 13, 25), (1, 0, 0, 0, 0, 1, 1, 0, 1)),$
 $((176, 10, 13, 27), (1, 0, 0, 0, 0, 0, 1, 0, 1)),$
 $((176, 10, 14, 25), (1, 0, 0, 0, 0, 0, 1, 0, 1)),$
 $((176, 10, 14, 29), (1, 0, 0, 0, 1, 0, 1, 0, 1)),$
 $((1, 16, 1, 1), (1, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((1, 16, 1, 25), (1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1, 16, 13, 27), (1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1, 16, 14, 25), (1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((2, 16, 13, 25), (1, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((2, 16, 13, 27), (1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((2, 16, 14, 25), (1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((2, 16, 14, 29), (1, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((31, 16, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((31, 16, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((31, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((31, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((32, 16, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((32, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((32, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((32, 16, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0)),$

$((49, 16, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((49, 16, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((49, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((49, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((50, 16, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((50, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((50, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((50, 16, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((55, 16, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((55, 16, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((55, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((55, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((56, 16, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((56, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((56, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((56, 16, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((151, 16, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((151, 16, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((151, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((151, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((152, 16, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((152, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((152, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((152, 16, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((175, 16, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0)),$
 $((175, 16, 1, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((175, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((175, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((176, 16, 13, 25), (0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((176, 16, 13, 27), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((176, 16, 14, 25), (0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((176, 16, 14, 29), (0, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((241, 16, 1, 1), (0, 0, 1, 1, 1, 1, 0, 0, 0)),$
 $((241, 16, 1, 25), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((241, 16, 13, 27), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((241, 16, 14, 25), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((242, 16, 13, 25), (0, 0, 1, 0, 0, 1, 0, 0, 0)),$
 $((242, 16, 13, 27), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((242, 16, 14, 25), (0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((242, 16, 14, 29), (0, 0, 1, 0, 1, 0, 0, 0, 0)),$

$((655, 16, 1, 1), (1, 1, 1, 1, 1, 1, 0, 0, 0)),$
 $((655, 16, 1, 25), (1, 1, 1, 0, 0, 0, 0, 0, 0)),$
 $((655, 16, 13, 27), (1, 1, 1, 0, 0, 0, 0, 0, 0)),$
 $((655, 16, 14, 25), (1, 1, 1, 0, 0, 0, 0, 0, 0)),$
 $((656, 16, 13, 25), (1, 1, 1, 0, 0, 1, 0, 0, 0)),$
 $((656, 16, 13, 27), (1, 1, 1, 0, 0, 0, 0, 0, 0)),$
 $((656, 16, 14, 25), (1, 1, 1, 0, 0, 0, 0, 0, 0)),$
 $((656, 16, 14, 29), (1, 1, 1, 0, 1, 0, 0, 0, 0))$

$(W_{8,3}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 1, 1, 23), (1, 0, 0, 0, 0, 1, 0)),$
 $((1, 1, 12, 25), (1, 0, 0, 0, 0, 0, 0)),$
 $((1, 1, 13, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((2, 1, 13, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((2, 1, 14, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((3, 1, 12, 25), (1, 0, 0, 0, 0, 0, 0)),$
 $((3, 1, 13, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((4, 1, 12, 23), (1, 0, 0, 0, 1, 1, 1)),$
 $((4, 1, 12, 25), (1, 0, 0, 0, 0, 0, 0)),$
 $((4, 1, 13, 27), (1, 0, 0, 1, 0, 0, 0)),$
 $((5, 1, 12, 25), (1, 0, 0, 0, 0, 0, 1)),$
 $((5, 1, 14, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((5, 1, 14, 29), (1, 0, 1, 1, 0, 1, 0)),$
 $((6, 1, 13, 23), (1, 0, 0, 0, 0, 0, 1)),$
 $((6, 1, 13, 27), (1, 0, 0, 1, 0, 0, 0)),$
 $((145, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1)),$
 $((145, 1, 1, 23), (1, 0, 0, 0, 0, 1, 0)),$
 $((145, 1, 12, 25), (1, 0, 0, 0, 0, 0, 0)),$
 $((145, 1, 13, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((146, 1, 13, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((146, 1, 14, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((147, 1, 12, 25), (1, 0, 0, 0, 0, 0, 0)),$
 $((147, 1, 13, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((148, 1, 12, 23), (1, 0, 0, 0, 1, 1, 1)),$
 $((148, 1, 12, 25), (1, 0, 0, 0, 0, 0, 0)),$
 $((148, 1, 13, 27), (1, 0, 0, 1, 0, 0, 0)),$
 $((149, 1, 12, 25), (1, 0, 0, 0, 0, 0, 1)),$
 $((149, 1, 14, 23), (1, 0, 0, 0, 0, 0, 0)),$
 $((149, 1, 14, 29), (1, 0, 1, 1, 0, 1, 0)),$
 $((150, 1, 13, 23), (1, 0, 0, 0, 0, 0, 1)),$
 $((150, 1, 13, 27), (1, 0, 0, 1, 0, 0, 0)),$
 $((1, 6, 1, 1), (0, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 6, 1, 23), (0, 0, 0, 0, 0, 1, 0)),$
 $((1, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((1, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((2, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((2, 6, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((3, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((3, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$

$((4, 6, 12, 23), (0, 0, 0, 0, 1, 1, 1)),$
 $((4, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((4, 6, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((5, 6, 12, 25), (0, 0, 0, 0, 0, 0, 1)),$
 $((5, 6, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((5, 6, 14, 29), (0, 0, 1, 1, 0, 1, 0)),$
 $((6, 6, 13, 23), (0, 0, 0, 0, 0, 0, 1)),$
 $((6, 6, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((25, 6, 1, 1), (0, 1, 1, 1, 1, 1, 1)),$
 $((25, 6, 1, 23), (0, 0, 0, 0, 0, 1, 0)),$
 $((25, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((25, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((26, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((26, 6, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((27, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((27, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((28, 6, 12, 23), (0, 0, 0, 0, 1, 1, 1)),$
 $((28, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((28, 6, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((29, 6, 12, 25), (0, 0, 0, 0, 0, 0, 1)),$
 $((29, 6, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((29, 6, 14, 29), (0, 0, 1, 1, 0, 1, 0)),$
 $((30, 6, 13, 23), (0, 0, 0, 0, 0, 0, 1)),$
 $((30, 6, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((121, 6, 1, 1), (0, 1, 1, 1, 1, 1, 1)),$
 $((121, 6, 1, 23), (0, 0, 0, 0, 0, 1, 0)),$
 $((121, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((121, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((122, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((122, 6, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((123, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((123, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((124, 6, 12, 23), (0, 0, 0, 0, 1, 1, 1)),$
 $((124, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((124, 6, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((125, 6, 12, 25), (0, 0, 0, 0, 0, 0, 1)),$
 $((125, 6, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((125, 6, 14, 29), (0, 0, 1, 1, 0, 1, 0)),$
 $((126, 6, 13, 23), (0, 0, 0, 0, 0, 0, 1)),$
 $((126, 6, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_{33}

$((145, 6, 1, 1), (0, 1, 1, 1, 1, 1, 1)),$
 $((145, 6, 1, 23), (0, 0, 0, 0, 0, 1, 0)),$
 $((145, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((145, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((146, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((146, 6, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((147, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((147, 6, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((148, 6, 12, 23), (0, 0, 0, 0, 1, 1, 1)),$
 $((148, 6, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((148, 6, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((149, 6, 12, 25), (0, 0, 0, 0, 0, 0, 1)),$
 $((149, 6, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((149, 6, 14, 29), (0, 0, 1, 1, 0, 1, 0)),$
 $((150, 6, 13, 23), (0, 0, 0, 0, 0, 0, 1)),$
 $((150, 6, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((25, 10, 1, 1), (0, 1, 1, 1, 1, 1, 1)),$
 $((25, 10, 1, 23), (0, 0, 0, 0, 0, 1, 0)),$
 $((25, 10, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((25, 10, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((26, 10, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((26, 10, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((27, 10, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((27, 10, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((28, 10, 12, 23), (0, 0, 0, 0, 1, 1, 1)),$
 $((28, 10, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((28, 10, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((29, 10, 12, 25), (0, 0, 0, 0, 0, 0, 1)),$
 $((29, 10, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((29, 10, 14, 29), (0, 0, 1, 1, 0, 1, 0)),$
 $((30, 10, 13, 23), (0, 0, 0, 0, 0, 0, 1)),$
 $((30, 10, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((121, 10, 1, 1), (0, 1, 1, 1, 1, 1, 1)),$
 $((121, 10, 1, 23), (0, 0, 0, 0, 0, 1, 0)),$
 $((121, 10, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((121, 10, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((122, 10, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((122, 10, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((123, 10, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((123, 10, 13, 23), (0, 0, 0, 0, 0, 0, 0)),$

$((124, 10, 12, 23), (0, 0, 0, 0, 1, 1, 1)),$
 $((124, 10, 12, 25), (0, 0, 0, 0, 0, 0, 0)),$
 $((124, 10, 13, 27), (0, 0, 0, 1, 0, 0, 0)),$
 $((125, 10, 12, 25), (0, 0, 0, 0, 0, 0, 1)),$
 $((125, 10, 14, 23), (0, 0, 0, 0, 0, 0, 0)),$
 $((125, 10, 14, 29), (0, 0, 1, 1, 0, 1, 0)),$
 $((126, 10, 13, 23), (0, 0, 0, 0, 0, 0, 1)),$
 $((126, 10, 13, 27), (0, 0, 0, 1, 0, 0, 0))$

$$(W_{8,4}^\dagger)$$
[illegible]

$((113, 1, 12, 4), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((115, 1, 13, 4), (0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((1, 2, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((3, 2, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((19, 2, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1)),$
 $((22, 2, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((23, 2, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((24, 2, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((34, 2, 11, 23), (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((46, 2, 11, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((48, 2, 11, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((65, 2, 12, 25), (0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1)),$
 $((71, 2, 12, 4), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((91, 2, 13, 4), (0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1)),$
 $((91, 2, 13, 27), (0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0)),$
 $((97, 2, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1)),$
 $((97, 2, 14, 4), (0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0)),$
 $((97, 2, 14, 29), (0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0)),$
 $((100, 2, 3, 3), (0, 1)),$
 $((101, 2, 3, 3), (0, 1)),$
 $((102, 2, 3, 3), (0, 1)),$
 $((71, 3, 12, 25), (0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((91, 3, 13, 4), (0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((97, 3, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((101, 3, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((102, 3, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((106, 3, 11, 23), (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((108, 3, 11, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((113, 3, 12, 4), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((113, 3, 12, 25), (0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((115, 3, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((115, 3, 13, 27), (0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0)),$
 $((116, 3, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((42, 4, 11, 23), (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((48, 4, 11, 23), (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((65, 4, 12, 4), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((71, 4, 12, 4), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((91, 4, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1)),$
 $((92, 4, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1)),$
 $((94, 4, 11, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1)),$
 $((94, 4, 11, 23), (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)),$

$((95, 4, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1)),$
 $((95, 4, 12, 25), (0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((108, 4, 11, 23), (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((10, 6, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((18, 6, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((22, 6, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((24, 6, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((34, 6, 11, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((42, 6, 11, 4), (0, 0)),$
 $((46, 6, 11, 4), (0, 0)),$
 $((48, 6, 11, 4), (0, 0)),$
 $((65, 6, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((65, 6, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0)),$
 $((66, 6, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((66, 6, 11, 23), (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((72, 6, 11, 23), (0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((92, 6, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)),$
 $((100, 6, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((102, 6, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1, 10, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0)),$
 $((1, 10, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0)),$
 $((10, 10, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((10, 10, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((17, 10, 2, 4), (0, 0)),$
 $((17, 10, 3, 3), (0, 0)),$
 $((18, 10, 2, 4), (0, 0)),$
 $((18, 10, 3, 3), (0, 0)),$
 $((19, 10, 2, 4), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)),$
 $((19, 10, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)),$
 $((22, 10, 2, 4), (0, 0)),$
 $((22, 10, 3, 3), (0, 0)),$
 $((23, 10, 2, 4), (0, 0)),$
 $((23, 10, 3, 3), (0, 0)),$
 $((24, 10, 2, 4), (0, 0)),$
 $((24, 10, 3, 3), (0, 0)),$
 $((34, 10, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0)),$
 $((34, 10, 3, 3), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0)),$
 $((34, 10, 3, 4), (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((42, 10, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((46, 10, 2, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$

[illegible]

$(W_{8,5}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 1, 1, 58), (0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((1, 1, 11, 58), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1, 1, 28, 21), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((4, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((4, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((5, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((5, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((6, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((6, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((10, 1, 11, 58), (0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((10, 1, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((12, 1, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((17, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((17, 1, 13, 58), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((18, 1, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((19, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((19, 1, 14, 58), (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((20, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((22, 1, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((23, 1, 13, 58), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((145, 1, 11, 58), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((145, 1, 28, 21), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((148, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((148, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((149, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((149, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((150, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((150, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((172, 1, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((174, 1, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((197, 1, 13, 58), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((217, 1, 14, 29), (0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1)),$
 $((217, 1, 14, 58), (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((217, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((218, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((283, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((284, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$

$((331, 1, 13, 27), (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((331, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((343, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)),$
 $((344, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)),$
 $((350, 1, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)),$
 $((355, 1, 13, 27), (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((355, 1, 13, 58), (0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)),$
 $((401, 1, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((425, 1, 12, 25), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((425, 1, 28, 21), (0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((449, 1, 12, 25), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((453, 1, 11, 58), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((455, 1, 12, 25), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((455, 1, 12, 58), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((473, 1, 12, 25), (0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1)),$
 $((514, 1, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((514, 1, 28, 21), (0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((514, 1, 28, 58), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((516, 1, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((538, 1, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((540, 1, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((546, 1, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1)),$
 $((546, 1, 11, 58), (0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((550, 1, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((574, 1, 11, 23), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1)),$
 $((586, 1, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1)),$
 $((1, 10, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 10, 1, 58), (0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((1, 10, 11, 58), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1, 10, 28, 21), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((4, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((4, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((5, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((5, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((6, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((6, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((10, 10, 11, 58), (0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((10, 10, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((12, 10, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_8

$((17, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((17, 10, 13, 58), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((18, 10, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((19, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((19, 10, 14, 58), (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((20, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((22, 10, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((23, 10, 13, 58), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((145, 10, 11, 58), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((145, 10, 28, 21), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((148, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((148, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((149, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((149, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((150, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((150, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((172, 10, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((174, 10, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((197, 10, 13, 58), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((217, 10, 14, 29), (0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1)),$
 $((217, 10, 14, 58), (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((217, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)),$
 $((218, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)),$
 $((283, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((284, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((331, 10, 13, 27), (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((331, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((343, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((344, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((350, 10, 12, 58), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((355, 10, 13, 27), (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((355, 10, 13, 58), (0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((401, 10, 28, 21), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)),$
 $((425, 10, 12, 25), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((425, 10, 28, 21), (0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((449, 10, 12, 25), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((453, 10, 11, 58), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((455, 10, 12, 25), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((455, 10, 12, 58), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((473, 10, 12, 25), (0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1)),$

$((514, 10, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((514, 10, 28, 21), (0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0)),$
 $((514, 10, 28, 58), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((516, 10, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((538, 10, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((540, 10, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((546, 10, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1)),$
 $((546, 10, 11, 58), (0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)),$
 $((550, 10, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((574, 10, 11, 23), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1)),$
 $((586, 10, 11, 23), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1))$

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_3
$$(W_{8,6}^\dagger)$$
[illegible]

[illegible]

((1827, 1, 1, 32), (0, 0)),
((1827, 1, 38, 106), (0, 0)),
((1830, 1, 1, 32), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((1830, 1, 38, 106), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((1855, 1, 1, 32), (0, 0, 0, 0, 1, 0)),
((1855, 1, 38, 106), (0, 0, 0, 0, 1, 0)),
((1856, 1, 1, 32), (0, 0, 0, 0, 1, 0)),
((1856, 1, 38, 106), (0, 0, 0, 0, 1, 0)),
((1887, 1, 1, 32), (0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((1887, 1, 38, 106), (0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((1920, 1, 1, 32), (0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((1920, 1, 38, 106), (0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))

$$(W_{8,7}^\dagger)$$
[illegible]

[illegible]

[illegible]

((238, 25, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((238, 25, 1, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((239, 25, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((239, 25, 1, 29), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((65, 26, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((65, 26, 1, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((66, 26, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((66, 26, 1, 29), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((34, 27, 1, 1), (1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),
((34, 27, 1, 29), (1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_8
$$(W_{8,8}^\dagger)$$
[illegible][illegible]
$$((1, 1, 15, 1), (1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, \\ 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1))),$$
$$((1, 1, 44, 1), (0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1))),$$
$$((1, 1, 56, 1), (0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, \\ 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1))),$$
$$((481, 1, 56, 1), (0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$$
$$((1201, 1, 56, 1), (0, 0, 0, 1, 0, \\ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$$
$$((1202, 1, 56, 1), (0, 0, 0, 1, 0, \\ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$$
$$((1440, 1, 56, 1), (0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$$
$$((1527, 1, 56, 1), (0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$$
[illegible][illegible][illegible][illegible][illegible]
$$((1686, 1, 56, 1), (0, 1, 0, 0, 0, 0, 0, \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$$
$$((1705, 1, 56, 1), (0, \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$$
[illegible][illegible]

[illegible]

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_3 [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$(W_{7,1}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1))$
 $((1, 1, 14, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1))$
 $((1, 9, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1))$
 $((1, 9, 14, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1))$
 $((1, 23, 1, 1), (1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1))$
 $((1, 23, 14, 1), (1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1))$
 $((193, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((193, 9, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((301, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((301, 9, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((325, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((325, 9, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((361, 23, 1, 1), (1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((361, 23, 14, 1), (1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((393, 9, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((393, 9, 14, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((423, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((423, 9, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((441, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((441, 9, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((447, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((447, 9, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((663, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((663, 13, 14, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((687, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((687, 13, 14, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((687, 23, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1))$
 $((687, 23, 14, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1))$
 $((775, 23, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((775, 23, 14, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((781, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((781, 13, 14, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((805, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((805, 13, 14, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((961, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((961, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 15, 1, 1), (0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 15, 14, 1), (0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((963, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_9

$((985, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((985, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((987, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((987, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1321, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1321, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1323, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1323, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1327, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1327, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1335, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((1335, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((1345, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1345, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1347, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1347, 23, 14, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 3, 1, 1), (0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1))$
 $((1375, 3, 14, 1), (0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1))$
 $((1375, 16, 1, 1), (1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 16, 14, 1), (1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 23, 1, 1), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1375, 23, 14, 1), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((1407, 23, 1, 1), (0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1))$
 $((1407, 23, 14, 1), (0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1))$

$(W_{7,2}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1)), ((1, 1, 13, 1), (1, 1, 1, 0, 1, 1, 1)),)$
 $((1, 16, 1, 1), (1, 0, 0, 1, 0, 0, 0)), ((1, 16, 13, 1), (1, 0, 0, 0, 0, 0, 0)),)$
 $((2, 1, 13, 1), (1, 1, 1, 0, 1, 1, 1)), ((2, 1, 14, 1), (1, 1, 1, 0, 1, 1, 1)),)$
 $((2, 16, 13, 1), (1, 0, 0, 0, 0, 0, 0)), ((2, 16, 14, 1), (1, 0, 0, 0, 0, 0, 0)),)$
 $((31, 16, 1, 1), (0, 0, 0, 1, 0, 0, 0)), ((31, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((32, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)), ((32, 16, 14, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((49, 16, 1, 1), (0, 0, 0, 1, 0, 0, 0)), ((49, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((50, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)), ((50, 16, 14, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((55, 16, 1, 1), (0, 0, 0, 1, 0, 0, 0)), ((55, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((56, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)), ((56, 16, 14, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((151, 16, 1, 1), (0, 0, 0, 1, 0, 0, 0)), ((151, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((152, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)), ((152, 16, 14, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((175, 10, 1, 1), (1, 0, 0, 1, 1, 0, 1)), ((175, 10, 13, 1), (1, 0, 0, 0, 1, 0, 1)),)$
 $((175, 16, 1, 1), (0, 0, 0, 1, 0, 0, 0)), ((175, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((176, 10, 13, 1), (1, 0, 0, 0, 1, 0, 1)), ((176, 10, 14, 1), (1, 0, 0, 0, 1, 0, 1)),)$
 $((176, 16, 13, 1), (0, 0, 0, 0, 0, 0, 0)), ((176, 16, 14, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((241, 6, 1, 1), (0, 0, 0, 1, 0, 0, 1)), ((241, 6, 13, 1), (0, 0, 0, 0, 0, 0, 1)),)$
 $((241, 16, 1, 1), (0, 0, 1, 1, 0, 0, 0)), ((241, 16, 13, 1), (0, 0, 1, 0, 0, 0, 0)),)$
 $((242, 6, 13, 1), (0, 0, 0, 0, 0, 0, 1)), ((242, 6, 14, 1), (0, 0, 0, 0, 0, 0, 1)),)$
 $((242, 16, 13, 1), (0, 0, 1, 0, 0, 0, 0)), ((242, 16, 14, 1), (0, 0, 1, 0, 0, 0, 0)),)$
 $((265, 6, 1, 1), (0, 0, 0, 1, 0, 0, 1)), ((265, 6, 13, 1), (0, 0, 0, 0, 0, 0, 1)),)$
 $((266, 6, 13, 1), (0, 0, 0, 0, 0, 0, 1)), ((266, 6, 14, 1), (0, 0, 0, 0, 0, 0, 1)),)$
 $((607, 4, 1, 1), (0, 0, 0, 1, 0, 0, 0)), ((607, 4, 13, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((608, 4, 13, 1), (0, 0, 0, 0, 0, 0, 0)), ((608, 4, 14, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((625, 4, 1, 1), (0, 0, 1, 1, 0, 0, 0)), ((625, 4, 13, 1), (0, 0, 1, 0, 0, 0, 0)),)$
 $((626, 4, 13, 1), (0, 0, 1, 0, 0, 0, 0)), ((626, 4, 14, 1), (0, 0, 1, 0, 0, 0, 0)),)$
 $((649, 4, 1, 1), (0, 0, 0, 1, 0, 0, 0)), ((649, 4, 13, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((650, 4, 13, 1), (0, 0, 0, 0, 0, 0, 0)), ((650, 4, 14, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((655, 4, 1, 1), (0, 0, 0, 1, 0, 0, 0)), ((655, 4, 13, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((655, 16, 1, 1), (1, 1, 1, 1, 0, 0, 0)), ((655, 16, 13, 1), (1, 1, 1, 0, 0, 0, 0)),)$
 $((656, 4, 13, 1), (0, 0, 0, 0, 0, 0, 0)), ((656, 4, 14, 1), (0, 0, 0, 0, 0, 0, 0)),)$
 $((656, 16, 13, 1), (1, 1, 1, 0, 0, 0, 0)), ((656, 16, 14, 1), (1, 1, 1, 0, 0, 0, 0))$

$(W_{7,3}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1)), ((1, 1, 12, 1), (1, 0, 0, 0)), ((1, 6, 1, 1), (0, 1, 1, 1))$
 $((1, 6, 12, 1), (0, 0, 0, 0)), ((2, 1, 12, 1), (1, 0, 0, 0)), ((2, 6, 12, 1), (0, 0, 0, 0))$
 $((4, 1, 12, 1), (1, 0, 0, 1)), ((4, 1, 13, 1), (1, 0, 0, 0)), ((4, 6, 12, 1), (0, 0, 0, 1))$
 $((4, 6, 13, 1), (0, 0, 0, 0)), ((5, 1, 12, 1), (1, 0, 0, 0)), ((5, 1, 14, 1), (1, 0, 1, 0))$
 $((5, 6, 12, 1), (0, 0, 0, 0)), ((5, 6, 14, 1), (0, 0, 1, 0)), ((6, 1, 13, 1), (1, 0, 0, 0))$
 $((6, 6, 13, 1), (0, 0, 0, 0)), ((25, 6, 1, 1), (0, 1, 1, 1)), ((25, 6, 12, 1), (0, 0, 0, 0))$
 $((25, 10, 1, 1), (0, 1, 1, 1)), ((25, 10, 12, 1), (0, 0, 0, 0)), ((26, 6, 12, 1), (0, 0, 0, 0))$
 $((26, 10, 12, 1), (0, 0, 0, 0)), ((28, 6, 12, 1), (0, 0, 0, 1)), ((28, 6, 13, 1), (0, 0, 0, 0))$
 $((28, 10, 12, 1), (0, 0, 0, 1)), ((28, 10, 13, 1), (0, 0, 0, 0)), ((29, 6, 12, 1), (0, 0, 0, 0))$
 $((29, 6, 14, 1), (0, 0, 1, 0)), ((29, 10, 12, 1), (0, 0, 0, 0)), ((29, 10, 14, 1), (0, 0, 1, 0))$
 $((30, 6, 13, 1), (0, 0, 0, 0)), ((30, 10, 13, 1), (0, 0, 0, 0)), ((121, 6, 1, 1), (0, 1, 1, 1))$
 $((121, 6, 12, 1), (0, 0, 0, 0)), ((121, 10, 1, 1), (0, 1, 1, 1)), ((121, 10, 12, 1), (0, 0, 0, 0))$
 $((122, 6, 12, 1), (0, 0, 0, 0)), ((122, 10, 12, 1), (0, 0, 0, 0)), ((124, 6, 12, 1), (0, 0, 0, 1))$
 $((124, 6, 13, 1), (0, 0, 0, 0)), ((124, 10, 12, 1), (0, 0, 0, 1)), ((124, 10, 13, 1), (0, 0, 0, 0))$
 $((125, 6, 12, 1), (0, 0, 0, 0)), ((125, 6, 14, 1), (0, 0, 1, 0)), ((125, 10, 12, 1), (0, 0, 0, 0))$
 $((125, 10, 14, 1), (0, 0, 1, 0)), ((126, 6, 13, 1), (0, 0, 0, 0)), ((126, 10, 13, 1), (0, 0, 0, 0))$
 $((145, 1, 1, 1), (1, 1, 1, 1)), ((145, 1, 12, 1), (1, 0, 0, 0)), ((145, 6, 1, 1), (0, 1, 1, 1))$
 $((145, 6, 12, 1), (0, 0, 0, 0)), ((146, 1, 12, 1), (1, 0, 0, 0)), ((146, 6, 12, 1), (0, 0, 0, 0))$
 $((148, 1, 12, 1), (1, 0, 0, 1)), ((148, 1, 13, 1), (1, 0, 0, 0)), ((148, 6, 12, 1), (0, 0, 0, 1))$
 $((148, 6, 13, 1), (0, 0, 0, 0)), ((149, 1, 12, 1), (1, 0, 0, 0)), ((149, 1, 14, 1), (1, 0, 1, 0))$
 $((149, 6, 12, 1), (0, 0, 0, 0)), ((149, 6, 14, 1), (0, 0, 1, 0)), ((150, 1, 13, 1), (1, 0, 0, 0),)$
 $((150, 6, 13, 1), (0, 0, 0, 0)));$

$(W_{7,4}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1))$
 $((1, 1, 3, 1), (0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0))$
 $((1, 2, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((9, 2, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((13, 2, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((15, 2, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((34, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0))$
 $((34, 2, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0))$
 $((34, 10, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0))$
 $((34, 10, 3, 1), (1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1))$
 $((40, 2, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0))$
 $((65, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0))$
 $((65, 2, 12, 1), (0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0))$
 $((65, 6, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1))$
 $((66, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0))$
 $((66, 6, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1))$
 $((66, 6, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0))$
 $((66, 10, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0))$
 $((91, 1, 3, 1), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((91, 2, 13, 1), (0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 0))$
 $((91, 4, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1))$
 $((94, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((94, 4, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1))$
 $((94, 10, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((95, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((95, 4, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1))$
 $((95, 4, 12, 1), (0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0))$
 $((96, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((96, 4, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1))$
 $((96, 6, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0))$
 $((96, 10, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((97, 1, 3, 1), (0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((97, 2, 3, 1), (0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((97, 2, 14, 1), (0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0))$
 $((97, 3, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((99, 3, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_6

$((106, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((106, 2, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((106, 3, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1))$
 $((106, 10, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((113, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((113, 2, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((113, 3, 12, 1), (0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1))$
 $((114, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((114, 2, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((114, 6, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0))$
 $((114, 10, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((115, 1, 3, 1), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((115, 2, 3, 1), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((115, 3, 3, 1), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((115, 3, 13, 1), (0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0))$
 $((118, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((118, 2, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((118, 3, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((118, 10, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((119, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((119, 2, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((119, 3, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((119, 4, 12, 1), (0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0))$
 $((120, 1, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((120, 2, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((120, 3, 3, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((120, 6, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0))$
 $((120, 10, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$

$(W_{7,5}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1))$
 $((1, 1, 28, 1), (0, 1, 1, 0, 1, 0, 0, 0, 0, 0))$
 $((1, 10, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1))$
 $((1, 10, 28, 1), (0, 1, 1, 0, 1, 0, 0, 0, 0, 0))$
 $((10, 1, 28, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0))$
 $((10, 10, 28, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0))$
 $((17, 1, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((17, 10, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((18, 1, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((18, 10, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((19, 1, 28, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0))$
 $((19, 10, 28, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0))$
 $((22, 1, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((22, 10, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((23, 1, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((23, 10, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((24, 1, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((24, 10, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((217, 1, 14, 1), (0, 1, 1, 0, 1, 0, 0, 1, 1, 1))$
 $((217, 1, 28, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0))$
 $((217, 10, 14, 1), (0, 1, 1, 0, 1, 0, 0, 1, 1, 1))$
 $((217, 10, 28, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0))$
 $((220, 1, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((220, 10, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((221, 1, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((221, 10, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((222, 1, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((222, 10, 28, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((331, 1, 13, 1), (0, 0, 1, 0, 0, 0, 0, 0, 1, 1))$
 $((331, 1, 28, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0, 0))$
 $((331, 10, 13, 1), (0, 0, 1, 0, 0, 0, 0, 0, 1, 1))$
 $((331, 10, 28, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0, 0))$
 $((332, 1, 28, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0, 0))$
 $((332, 10, 28, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0, 0))$
 $((355, 1, 13, 1), (0, 0, 1, 0, 0, 0, 0, 0, 1, 1))$
 $((355, 10, 13, 1), (0, 0, 1, 0, 0, 0, 0, 0, 1, 1))$

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_5

$((425, 1, 12, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((425, 1, 28, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0))$
 $((425, 10, 12, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((425, 10, 28, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0))$
 $((431, 1, 12, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((431, 10, 12, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((455, 1, 12, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((455, 10, 12, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((473, 1, 12, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((473, 10, 12, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1))$
 $((514, 1, 11, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0))$
 $((514, 1, 28, 1), (1, 1, 1, 1, 1, 1, 0, 0, 0))$
 $((514, 10, 11, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0))$
 $((514, 10, 28, 1), (1, 1, 1, 1, 1, 1, 0, 0, 0))$
 $((522, 1, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((522, 10, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((526, 1, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((526, 10, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((528, 1, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((528, 10, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((546, 1, 11, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0))$
 $((546, 10, 11, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0))$
 $((574, 1, 11, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((574, 10, 11, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0))$
 $((586, 1, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((586, 10, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((588, 1, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$
 $((588, 10, 11, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0))$

[illegible]

$(W_{6,1}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((1, 9, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1)),$
 $((1, 23, 1, 1), (1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1)),$
 $((193, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((301, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((325, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((361, 23, 1, 1), (1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)),$
 $((393, 9, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)),$
 $((423, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((441, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((447, 9, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((663, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((687, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((687, 23, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1)),$
 $((775, 23, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)),$
 $((781, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((805, 13, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((961, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((963, 15, 1, 1), (0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((963, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((985, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((987, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1321, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1323, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1327, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1335, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((1345, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1347, 23, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1375, 3, 1, 1), (0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1)),$
 $((1375, 16, 1, 1), (1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1375, 23, 1, 1), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1407, 23, 1, 1), (0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1))$

ZEROS OF ZETA FUNCTIONS FOR EXCEPTIONAL GROUPS OF TYPE E_7

$(W_{6,2}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1)), ((1, 16, 1, 1), (1, 0, 0, 0, 0, 0)),$
 $((2, 1, 1, 1), (1, 1, 1, 1, 1, 1)), ((2, 16, 1, 1), (1, 0, 0, 0, 0, 0)),$
 $((31, 16, 1, 1), (0, 0, 0, 0, 0, 0)), ((32, 16, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((49, 16, 1, 1), (0, 0, 0, 0, 0, 0)), ((50, 16, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((55, 16, 1, 1), (0, 0, 0, 0, 0, 0)), ((56, 16, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((151, 16, 1, 1), (0, 0, 0, 0, 0, 0)), ((152, 16, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((175, 10, 1, 1), (1, 0, 0, 1, 0, 1)), ((175, 16, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((176, 10, 1, 1), (1, 0, 0, 1, 0, 1)), ((176, 16, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((241, 6, 1, 1), (0, 0, 0, 0, 0, 1)), ((241, 16, 1, 1), (0, 0, 1, 0, 0, 0)),$
 $((242, 6, 1, 1), (0, 0, 0, 0, 0, 1)), ((242, 16, 1, 1), (0, 0, 1, 0, 0, 0)),$
 $((265, 6, 1, 1), (0, 0, 0, 0, 0, 1)), ((266, 6, 1, 1), (0, 0, 0, 0, 0, 1)),$
 $((607, 4, 1, 1), (0, 0, 0, 0, 0, 0)), ((608, 4, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((625, 4, 1, 1), (0, 0, 1, 0, 0, 0)), ((626, 4, 1, 1), (0, 0, 1, 0, 0, 0)),$
 $((649, 4, 1, 1), (0, 0, 0, 0, 0, 0)), ((650, 4, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((655, 4, 1, 1), (0, 0, 0, 0, 0, 0)), ((655, 16, 1, 1), (1, 1, 1, 0, 0, 0)),$
 $((656, 4, 1, 1), (0, 0, 0, 0, 0, 0)), ((656, 16, 1, 1), (1, 1, 1, 0, 0, 0)).$

$(W_{6,3}^\dagger)$

$((1, 1, 1, 1), (1, 1)), ((1, 6, 1, 1), (0, 1)), ((4, 1, 1, 1), (1, 0)), ((4, 6, 1, 1), (0, 0)),$
 $((5, 1, 1, 1), (1, 0)), ((5, 6, 1, 1), (0, 0)), ((6, 1, 1, 1), (1, 0)), ((6, 6, 1, 1), (0, 0)),$
 $((25, 6, 1, 1), (0, 1)), ((25, 10, 1, 1), (0, 1)), ((28, 6, 1, 1), (0, 0)), ((28, 10, 1, 1), (0, 0)),$
 $((29, 6, 1, 1), (0, 0)), ((29, 10, 1, 1), (0, 0)), ((30, 6, 1, 1), (0, 0)), ((30, 10, 1, 1), (0, 0)),$
 $((121, 6, 1, 1), (0, 1)), ((121, 10, 1, 1), (0, 1)), ((124, 6, 1, 1), (0, 0)), ((124, 10, 1, 1), (0, 0)),$
 $((125, 6, 1, 1), (0, 0)), ((125, 10, 1, 1), (0, 0)), ((126, 6, 1, 1), (0, 0)), ((126, 10, 1, 1), (0, 0)),$
 $((145, 1, 1, 1), (1, 1)), ((145, 6, 1, 1), (0, 1)), ((148, 1, 1, 1), (1, 0)), ((148, 6, 1, 1), (0, 0)),$
 $((149, 1, 1, 1), (1, 0)), ((149, 6, 1, 1), (0, 0)), ((150, 1, 1, 1), (1, 0)), ((150, 6, 1, 1), (0, 0)).$

$(W_{6,4}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)), ((1, 2, 1, 1), (0, 1, 1, 0, 0, 0, 0, 0, 1, 0)),$
 $((33, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0)), ((34, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0)),$
 $((34, 10, 1, 1), (1, 1, 1, 0, 0, 0, 0, 1, 1, 1)), ((61, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((63, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)), ((64, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((65, 2, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0)), ((65, 6, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((66, 6, 1, 1), (0, 0, 0, 0, 0, 0, 0, 1, 1, 1)), ((73, 2, 1, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0)),$
 $((81, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)), ((82, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((85, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)), ((87, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((88, 2, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)), ((89, 2, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0)),$
 $((91, 2, 1, 1), (0, 0, 1, 0, 0, 1, 1, 0, 0, 0)), ((91, 4, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((93, 4, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)), ((94, 4, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((95, 4, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 1)), ((97, 2, 1, 1), (0, 1, 1, 0, 1, 1, 1, 0, 1, 0)),$
 $((97, 3, 1, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0)), ((105, 3, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((106, 3, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)), ((109, 3, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((111, 3, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)), ((112, 3, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((113, 3, 1, 1), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0)), ((115, 3, 1, 1), (0, 0, 1, 0, 0, 1, 1, 0, 0, 0))$

 $(W_{6,5}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)), ((1, 10, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((217, 1, 1, 1), (0, 1, 0, 0, 1, 1)), ((217, 10, 1, 1), (0, 1, 0, 0, 1, 1)),$
 $((331, 1, 1, 1), (0, 0, 0, 0, 0, 1)), ((331, 10, 1, 1), (0, 0, 0, 0, 0, 1)),$
 $((355, 1, 1, 1), (0, 0, 0, 0, 0, 1)), ((355, 10, 1, 1), (0, 0, 0, 0, 0, 1)),$
 $((425, 1, 1, 1), (0, 0, 1, 0, 0, 0)), ((425, 10, 1, 1), (0, 0, 1, 0, 0, 0)),$
 $((455, 1, 1, 1), (0, 0, 0, 0, 0, 0)), ((455, 10, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((473, 1, 1, 1), (0, 0, 0, 0, 0, 0)), ((473, 10, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((479, 1, 1, 1), (0, 0, 0, 0, 0, 0)), ((479, 10, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((514, 1, 1, 1), (1, 1, 1, 0, 0, 0)), ((514, 10, 1, 1), (1, 1, 1, 0, 0, 0)),$
 $((546, 1, 1, 1), (0, 0, 1, 0, 0, 0)), ((546, 10, 1, 1), (0, 0, 1, 0, 0, 0)),$
 $((574, 1, 1, 1), (0, 0, 0, 0, 0, 0)), ((574, 10, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((576, 1, 1, 1), (0, 0, 0, 0, 0, 0)), ((576, 10, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((586, 1, 1, 1), (0, 1, 0, 0, 0, 0)), ((586, 10, 1, 1), (0, 1, 0, 0, 0, 0)),$
 $((594, 1, 1, 1), (0, 0, 0, 0, 0, 0)), ((594, 10, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((598, 1, 1, 1), (0, 0, 0, 0, 0, 0)), ((598, 10, 1, 1), (0, 0, 0, 0, 0, 0)),$
 $((600, 1, 1, 1), (0, 0, 0, 0, 0, 0)), ((600, 10, 1, 1), (0, 0, 0, 0, 0, 0))$

$(W_{6,6}^\dagger)$

$((1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)),$
 $((481, 1, 1, 1), (0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1)),$
 $((1201, 1, 1, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1)),$
 $((1202, 1, 1, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1)),$
 $((1440, 1, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1)),$
 $((1527, 1, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)),$
 $((1615, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((1616, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)),$
 $((1681, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1682, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1683, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1686, 1, 1, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1705, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1706, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1707, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1710, 1, 1, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1801, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1802, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1803, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1806, 1, 1, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1807, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1808, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1815, 1, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1824, 1, 1, 1), (0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1825, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1826, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1827, 1, 1, 1), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1830, 1, 1, 1), (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1855, 1, 1, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1856, 1, 1, 1), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1887, 1, 1, 1), (0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$
 $((1920, 1, 1, 1), (1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0))$

APPENDIX D. $\prod_{\alpha \in \Phi_P^+ \setminus \Delta_P} \widehat{\zeta}(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha, w})$

Now we calculate $\prod_{\alpha \in \Phi_P^+ \setminus \Delta_P} \widehat{\zeta}(\langle \rho, \alpha^\vee \rangle + \delta_{\alpha, w})$ for $w \in \mathfrak{W}^\dagger$. This is give in the table $kg[1]$ and the result can be rephrased as the following (with $G[s] = \widehat{\zeta}(s)$)

$$(W_{8,1}^\dagger)$$

[illegible]

$(W_{8,2}^\dagger)$

$$\begin{aligned}
& G[2]^3 G[3]^4 G[4]^2, G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], \\
& G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], G[2]^2 G[3]^5 G[4]^2, G[2]^4 G[3]^4 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^5 G[4], G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], \\
& G[2]^3 G[3]^5 G[4], G[2]^3 G[3]^4 G[4]^2, G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^4 G[4]^2, \\
& G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], G[2]^5 G[3]^3 G[4], \\
& G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], G[3]^5 G[4]^3 G[5], G[2]^2 G[3]^4 G[4]^2 G[5], \\
& G[2]^2 G[3]^4 G[4]^2 G[5], G[2]^2 G[3]^4 G[4]^2 G[5], G[2] G[3]^5 G[4]^2 G[5], G[2]^2 G[3]^4 G[4]^2 G[5], \\
& G[2]^2 G[3]^4 G[4]^2 G[5], G[2] G[3]^5 G[4]^2 G[5], G[2]^2 G[3]^5 G[4]^2, G[2]^4 G[3]^4 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^5 G[4], G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], \\
& G[2]^3 G[3]^5 G[4], G[2]^2 G[3]^5 G[4]^2, G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], \\
& G[2]^3 G[3]^5 G[4], G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^5 G[4], G[2] G[3]^5 G[4]^3, \\
& G[2]^3 G[3]^4 G[4]^2, G[2]^3 G[3]^4 G[4]^2, G[2]^3 G[3]^4 G[4]^2, G[2]^2 G[3]^5 G[4]^2, \\
& G[2]^3 G[3]^4 G[4]^2, G[2]^3 G[3]^4 G[4]^2, G[2]^2 G[3]^5 G[4]^2, G[2]^2 G[3]^5 G[4]^2, G[2]^4 G[3]^4 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^5 G[4], G[2]^4 G[3]^4 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^5 G[4], G[2]^3 G[3]^4 G[4]^2, G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], \\
& G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^4 G[4]^2, G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^4 G[4]^2, G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^4 G[4]^2, G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], \\
& G[2]^3 G[3]^4 G[4]^2, G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], \\
& G[2]^5 G[3]^3 G[4], G[2]^5 G[3]^3 G[4], G[2]^4 G[3]^4 G[4], G[2]^2 G[3]^5 G[4]^2, G[2]^4 G[3]^4 G[4], \\
& G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], G[2]^3 G[3]^5 G[4], G[2]^4 G[3]^4 G[4], G[2]^4 G[3]^4 G[4], \\
& G[2]^3 G[3]^5 G[4], G[2] G[3]^5 G[4]^3, G[2]^3 G[3]^4 G[4]^2, G[2]^3 G[3]^4 G[4]^2, \\
& G[2]^3 G[3]^4 G[4]^2, G[2]^2 G[3]^5 G[4]^2, G[2]^3 G[3]^4 G[4]^2, G[2]^3 G[3]^4 G[4]^2, G[2]^2 G[3]^5 G[4]^2
\end{aligned}$$

$(W_{8,3}^\dagger)$

$$\begin{aligned}
& G[3]^4 G[4]^2 G[5], G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2] G[3]^4 G[4]^2, G[2]^3 G[3]^3 G[4], \\
& G[2]^2 G[3]^4 G[4], G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], G[2] G[3]^4 G[4]^2, G[2]^2 G[3]^4 G[4], \\
& G[2]^2 G[3]^4 G[4], G[3]^4 G[4]^2 G[5], G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2] G[3]^4 G[4]^2, \\
& G[2]^3 G[3]^3 G[4], G[2]^2 G[3]^4 G[4], G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], G[2] G[3]^4 G[4]^2, \\
& G[2]^2 G[3]^4 G[4], G[2]^2 G[3]^4 G[4], G[2] G[3]^3 G[4]^2 G[5], G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^2 G[3]^3 G[4]^2, G[2]^4 G[3]^2 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^2 G[3]^3 G[4]^2, G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2] G[3]^3 G[4]^2 G[5], G[2]^3 G[3]^3 G[4], \\
& G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^4 G[3]^2 G[4], G[2]^2 G[3]^3 G[4]^2, G[2]^4 G[3]^2 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^4 G[3]^2 G[4], G[2]^2 G[3]^3 G[4]^2, G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2] G[3]^3 G[4]^2 G[5], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^2 G[3]^3 G[4]^2, G[2]^4 G[3]^2 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^2 G[3]^3 G[4]^2, G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2] G[3]^3 G[4]^2 G[5], G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^2 G[3]^3 G[4]^2, \\
& G[2]^4 G[3]^2 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^2 G[3]^3 G[4]^2, \\
& G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2] G[3]^3 G[4]^2 G[5], G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^2 G[3]^3 G[4]^2, G[2]^4 G[3]^2 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^4 G[3]^2 G[4], G[2]^2 G[3]^3 G[4]^2, G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4]
\end{aligned}$$

[illegible]

$$\begin{aligned}
& G[2]^3 G[3]^6 G[4]^5 G[5]^4 G[6]^2 G[7], G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], \\
& G[2]^2 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7], G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^2 G[3]^7 G[4]^5 G[5]^4 G[6]^2 G[7], G[2]^3 G[3]^6 G[4]^5 G[5]^4 G[6]^2 G[7], \\
& G[2]^3 G[3]^6 G[4]^5 G[5]^4 G[6]^2 G[7], G[2]^3 G[3]^7 G[4]^5 G[5]^3 G[6]^2 G[7], \\
& G[2]^3 G[3]^6 G[4]^5 G[5]^4 G[6]^2 G[7], G[2]^3 G[3]^6 G[4]^5 G[5]^4 G[6]^2 G[7], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], G[2]^4 G[3]^7 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^2 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7], \\
& G[2]^3 G[3]^7 G[4]^5 G[5]^3 G[6]^2 G[7], G[2]^2 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], \\
& G[2]^4 G[3]^7 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^4 G[3]^7 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^3 G[3]^6 G[4]^5 G[5]^4 G[6]^2 G[7], G[2] G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^2, \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^4 G[3]^6 G[4]^5 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^5 G[3]^6 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], \\
& G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^6 G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7]
\end{aligned}$$

$(W_{8,5}^\dagger)$

$$\begin{aligned}
& G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2] G[3]^5 G[4]^4 G[5]^3 G[6]^2, G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6], G[2]^3 G[3]^6 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], \\
& G[2]^2 G[3]^6 G[4]^4 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], \\
& G[2] G[3]^5 G[4]^4 G[5]^3 G[6]^2, G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^3 G[3]^6 G[4]^3 G[5]^2 G[6], G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6],
\end{aligned}$$

$$\begin{aligned}
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^2 G[3]^6 G[4]^4 G[5]^2 G[6], G[2]^3 G[3]^6 G[4]^3 G[5]^2 G[6], \\
& G[3]^5 G[4]^4 G[5]^3 G[6]^2 G[7], G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2] G[3]^5 G[4]^4 G[5]^3 G[6]^2, \\
& G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^5 G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6], \\
& G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6], G[2]^3 G[3]^6 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6], \\
& G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], \\
& G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2]^2 G[3]^6 G[4]^4 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^3 G[3]^5 G[4]^4 G[5]^2 G[6], G[2] G[3]^5 G[4]^4 G[5]^3 G[6]^2, G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], \\
& G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^3 G[3]^6 G[4]^3 G[5]^2 G[6], \\
& G[2]^2 G[3]^5 G[4]^4 G[5]^3 G[6], G[2]^4 G[3]^5 G[4]^3 G[5]^2 G[6], G[2]^2 G[3]^6 G[4]^4 G[5]^2 G[6], \\
& G[2]^3 G[3]^6 G[4]^3 G[5]^2 G[6]
\end{aligned}$$

$(W_{8,6}^\dagger)$

$$\begin{aligned}
& G[3]^6 G[4]^6 G[5]^5 G[6]^5 G[7]^4 G[8]^3 G[9]^2 G[10]^2 G[11] G[12], \\
& G[2]^2 G[3]^6 G[4]^6 G[5]^6 G[6]^5 G[7]^3 G[8]^3 G[9]^2 G[10] G[11], \\
& G[2] G[3]^6 G[4]^6 G[5]^5 G[6]^6 G[7]^3 G[8]^2 G[9]^3 G[10]^2 G[11], \\
& G[2]^2 G[3]^8 G[4]^7 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^7 G[5]^5 G[6]^5 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^2 G[3]^7 G[4]^7 G[5]^5 G[6]^5 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^2 G[3]^6 G[4]^6 G[5]^6 G[6]^5 G[7]^3 G[8]^3 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^8 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^8 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^7 G[5]^5 G[6]^5 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^2 G[3]^8 G[4]^6 G[5]^6 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^8 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^7 G[5]^5 G[6]^5 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^2 G[3]^8 G[4]^6 G[5]^6 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^8 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^8 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^8 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^2 G[3]^8 G[4]^6 G[5]^6 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^2 G[3]^8 G[4]^6 G[5]^6 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^8 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11],
\end{aligned}$$

[illegible]

$$\begin{aligned}
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^2 G[3]^7 G[4]^6 G[5]^6 G[6]^5 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^2 G[3]^7 G[4]^6 G[5]^6 G[6]^5 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^6 G[3]^6 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^5 G[3]^7 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^4 G[3]^7 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^7 G[4]^6 G[5]^6 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11], \\
& G[2]^3 G[3]^7 G[4]^6 G[5]^6 G[6]^4 G[7]^3 G[8]^2 G[9]^2 G[10] G[11]
\end{aligned}$$

[illegible]

$$\begin{aligned}
& G[2]^2 G[3]^6 G[4]^6 G[5]^5 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^7 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^7 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^7 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^7 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^2 G[3]^6 G[4]^6 G[5]^5 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^2 G[3]^6 G[4]^6 G[5]^5 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11]
\end{aligned}$$

[illegible]

$(W_{7,2}^\dagger)$

$$\begin{aligned}
& G[3]^4 G[4]^2 G[5], G[2] G[3]^3 G[4]^2 G[5], G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2] G[3]^3 G[4]^2 G[5], G[2] G[3]^3 G[4]^2 G[5], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2] G[3]^4 G[4]^2, G[2]^2 G[3]^3 G[4]^2, G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^2 G[3]^3 G[4]^2, G[2]^2 G[3]^3 G[4]^2, G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^2 G[3]^4 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], G[2]^3 G[3]^3 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], \\
& G[2]^3 G[3]^3 G[4], G[2]^4 G[3]^2 G[4], G[2] G[3]^4 G[4]^2, G[2]^2 G[3]^3 G[4]^2, \\
& G[2]^4 G[3]^2 G[4], G[2]^4 G[3]^2 G[4], G[2]^2 G[3]^3 G[4]^2, G[2]^2 G[3]^3 G[4]^2
\end{aligned}$$

 $(W_{7,3}^\dagger)$

$$\begin{aligned}
& G[3]^3 G[4], G[2]^2 G[3]^2, G[2] G[3]^2 G[4], G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], G[2] G[3]^3, \\
& G[2]^2 G[3]^2, G[2]^2 G[3]^2, G[2]^3 G[3], G[2]^2 G[3]^2, G[2] G[3]^3, G[2]^3 G[3], G[2]^2 G[3]^2, \\
& G[2]^2 G[3]^2, G[2]^3 G[3], G[2] G[3]^2 G[4], G[2]^3 G[3], G[2] G[3]^2 G[4], G[2]^3 G[3], G[2]^3 G[3], \\
& G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], G[2]^3 G[3], G[2]^2 G[3]^2, \\
& G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], G[2]^3 G[3], G[2] G[3]^2 G[4], G[2]^3 G[3], G[2] G[3]^2 G[4], \\
& G[2]^3 G[3], G[2]^3 G[3], G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], \\
& G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], G[2]^3 G[3], G[3]^3 G[4], G[2]^2 G[3]^2, \\
& G[2] G[3]^2 G[4], G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^3 G[3], G[2] G[3]^3, G[2]^2 G[3]^2, G[2]^2 G[3]^2, \\
& G[2]^3 G[3], G[2]^2 G[3]^2, G[2] G[3]^3, G[2]^3 G[3], G[2]^2 G[3]^2, G[2]^2 G[3]^2, G[2]^3 G[3]
\end{aligned}$$

[illegible]

$(W_{7,5}^\dagger)$

$G[3]^4G[4]^3G[5]^2G[6], G[2]^2G[3]^4G[4]^3G[5], G[3]^4G[4]^3G[5]^2G[6], G[2]^2G[3]^4G[4]^3G[5],$
 $G[2]^3G[3]^4G[4]^2G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5],$
 $G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]^3G[3]^4G[4]^2G[5],$
 $G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5],$
 $G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]G[3]^4G[4]^3G[5]^2, G[2]^3G[3]^4G[4]^2G[5],$
 $G[2]G[3]^4G[4]^3G[5]^2, G[2]^3G[3]^4G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5],$
 $G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5],$
 $G[2]^2G[3]^4G[4]^3G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]^2G[3]^4G[4]^3G[5], G[2]^3G[3]^4G[4]^2G[5],$
 $G[2]^3G[3]^4G[4]^2G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]^2G[3]^4G[4]^3G[5], G[2]^2G[3]^4G[4]^3G[5],$
 $G[2]^3G[3]^4G[4]^2G[5], G[2]^2G[3]^4G[4]^3G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]^2G[3]^4G[4]^3G[5],$
 $G[2]^3G[3]^4G[4]^2G[5], G[2]^2G[3]^4G[4]^3G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]^2G[3]^5G[4]^2G[5],$
 $G[2]^2G[3]^5G[4]^2G[5], G[2]^2G[3]^5G[4]^2G[5], G[2]^2G[3]^5G[4]^2G[5], G[2]^2G[3]^5G[4]^2G[5],$
 $G[2]^3G[3]^4G[4]^2G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]G[3]^4G[4]^3G[5]^2,$
 $G[2]^3G[3]^4G[4]^2G[5], G[2]G[3]^4G[4]^3G[5]^2, G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5],$
 $G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5],$
 $G[2]^2G[3]^4G[4]^3G[5], G[2]^2G[3]^4G[4]^3G[5], G[2]^3G[3]^4G[4]^2G[5], G[2]^3G[3]^4G[4]^2G[5],$
 $G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5], G[2]^4G[3]^3G[4]^2G[5]$

$(W_{7,7}^\dagger)$

$$\begin{aligned}
& G[3]^5 G[4]^5 G[5]^5 G[6]^4 G[7]^3 G[8]^3 G[9]^2 G[10] G[11] G[12], \\
& G[2] G[3]^6 G[4]^6 G[5]^5 G[6]^4 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^2 G[3]^6 G[4]^6 G[5]^5 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^7 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^2 G[3]^6 G[4]^6 G[5]^5 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^5 G[3]^5 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^4 G[3]^6 G[4]^5 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11], \\
& G[2]^3 G[3]^6 G[4]^6 G[5]^4 G[6]^3 G[7]^3 G[8]^2 G[9] G[10] G[11],
\end{aligned}$$

[illegible]

$(W_{6,1}^\dagger)$

$$\begin{aligned}
& G[3]^4 G[4]^4 G[5]^3 G[6]^2 G[7] G[8], G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], G[2] G[3]^4 G[4]^5 G[5]^2 G[6]^2 G[7], \\
& G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2] G[3]^5 G[4]^4 G[5]^3 G[6] G[7], G[2] G[3]^5 G[4]^4 G[5]^3 G[6] G[7], \\
& G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7]
\end{aligned}$$

 $(W_{6,2}^\dagger)$

$$\begin{aligned}
& G[3]^3 G[4]^2 G[5], G[2]^2 G[3]^3 G[4], G[3]^3 G[4]^2 G[5], G[2]^2 G[3]^3 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], \\
& G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], \\
& G[2] G[3]^3 G[4]^2, G[2]^3 G[3]^2 G[4], G[2] G[3]^3 G[4]^2, G[2]^3 G[3]^2 G[4], G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], \\
& G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], \\
& G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2] G[3]^3 G[4]^2, \\
& G[2]^3 G[3]^2 G[4], G[2] G[3]^3 G[4]^2
\end{aligned}$$

 $(W_{6,3}^\dagger)$

$$\begin{aligned}
& G[3]^2, G[2] G[3], G[2] G[3], G[2]^2, G[2] G[3], G[2]^2, G[2] G[3], G[2]^2, G[2] G[3], G[2] G[3], \\
& G[2]^2, G[2]^2, G[2]^2, G[2]^2, G[2]^2, G[2]^2, G[2] G[3], G[2] G[3], G[2]^2, G[2]^2, G[2]^2, G[2]^2, \\
& G[2]^2, G[2]^2, G[3]^2, G[2] G[3], G[2] G[3], G[2]^2, G[2] G[3], G[2]^2, G[2] G[3], G[2]^2
\end{aligned}$$

 $(W_{6,4}^\dagger)$

$$\begin{aligned}
& G[3]^4 G[4]^3 G[5]^2 G[6], G[2]^2 G[3]^4 G[4]^3 G[5], G[2]^3 G[3]^4 G[4]^2 G[5], G[2]^3 G[3]^4 G[4]^2 G[5], \\
& G[2] G[3]^4 G[4]^3 G[5]^2, G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], \\
& G[2]^3 G[3]^4 G[4]^2 G[5], G[2]^2 G[3]^4 G[4]^3 G[5], G[2]^2 G[3]^4 G[4]^3 G[5], G[2]^3 G[3]^4 G[4]^2 G[5], \\
& G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], \\
& G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^3 G[3]^4 G[4]^2 G[5], G[2]^2 G[3]^4 G[4]^3 G[5], G[2]^3 G[3]^4 G[4]^2 G[5], \\
& G[2]^3 G[3]^4 G[4]^2 G[5], G[2]^3 G[3]^4 G[4]^2 G[5], G[2]^2 G[3]^5 G[4]^2 G[5], G[2] G[3]^4 G[4]^3 G[5]^2, \\
& G[2]^3 G[3]^4 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], \\
& G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^4 G[3]^3 G[4]^2 G[5], G[2]^3 G[3]^4 G[4]^2 G[5], G[2]^2 G[3]^4 G[4]^3 G[5]
\end{aligned}$$

$(W_{6,5}^\dagger)$

$$\begin{aligned}
& G[3]^3 G[4]^2 G[5], G[3]^3 G[4]^2 G[5], G[2] G[3]^3 G[4]^2, G[2] G[3]^3 G[4]^2, G[2]^2 G[3]^3 G[4], \\
& G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], \\
& G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], \\
& G[2]^3 G[3]^2 G[4], G[2] G[3]^3 G[4]^2, G[2] G[3]^3 G[4]^2, G[2]^2 G[3]^3 G[4], G[2]^2 G[3]^3 G[4], \\
& G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^2 G[3]^3 G[4], \\
& G[2]^2 G[3]^3 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4], \\
& G[2]^3 G[3]^2 G[4], G[2]^3 G[3]^2 G[4]
\end{aligned}$$

 $(W_{6,6}^\dagger)$

$$\begin{aligned}
& G[3]^4 G[4]^4 G[5]^3 G[6]^2 G[7] G[8], G[2] G[3]^4 G[4]^5 G[5]^2 G[6]^2 G[7], \\
& G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], \\
& G[2] G[3]^5 G[4]^4 G[5]^3 G[6] G[7], G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], \\
& G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^4 G[3]^4 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], G[2]^3 G[3]^5 G[4]^3 G[5]^2 G[6] G[7], \\
& G[2]^2 G[3]^5 G[4]^4 G[5]^2 G[6] G[7], G[2] G[3]^5 G[4]^4 G[5]^3 G[6] G[7]
\end{aligned}$$

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