

Lin Weng

**Zeta Functions of
Reductive Groups
and Their Zeros**

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Five Essays on Arithmetic Cohomology
(Joint with K. Sugahara)

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