Cohen-Lenstra Heuristics for Relative Shafarevich-Tate Groups

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China-Japan Number Theory Seminar 2013

Description

Joint Work

- The contents of today's talk are works in progress

jointly with Kotaro SUGAHARA

Relative Shafarevich-Tate Group

Notations

- K: number field, $\mathcal{O}_K$: ring of integers
- $G_K := \mathrm{Gal}(\overline{K}/K)$: absolute Galois group of K
- $S(K) = S_{\mathrm{fin}}$: collection of inequivalent finite valuations
- K_v: v-completion of K, $\mathcal{O}_v$: maximal ideal of K_v, $v \in S_{\mathrm{fin}}$
- E/K: elliptic curve defined over K
- $\mathcal{E} \to \mathrm{Spec}\, \mathcal{O}_K$: integral model of E/K
- E_v/K_v: special fiber over $v \in S_{\mathrm{fin}}$
- $E_{(v)}/k_{(v)}$: reduction of E_v with $k_{(v)}$ residue field of v
- $\zeta_{E_{(v)}}(s)$: Artin zeta function for $E_{(v)}/k_{(v)}$
- $\zeta_{\mathcal{E}/K} = \prod_v \zeta_{E_{(v)}}(s)$: Hasse-Weil zeta function for E/K

Relative Shafarevich-Tate Group

Definition

- L/K: finite Galois extension
- $E := E(\overline{K})$
- $w \in S(L)$: $w | v \in S(K)$
- $G_w := G_{L_w}$: decomposition group at w
- $Ш(E/L) := \mathrm{Ker}\Big(H^1(G_L, E) \to \prod_w H^1(G_w, E_v)\Big)$

Inflation-Restriction Exact Sequence

- $0 \to H^1(G_{L/K}, E(L)) \to H^1(G_K, E) \to H^1(G_L, E)$
- $0 \to H^1(G_{L_w/K_v}, E(L_w)) \to H^1(G_v, E_v) \to H^1(G_w, E_v)$
- Not easy to see the relative Shafarevich-Tate group $Ш(E/L)$

Pure torsion sheaves

Pure Torsion Sheaves

- X/K: arithmetic variety
- $\mathfrak{X}/\mathrm{Spec}\mathcal{O}_K$: integral model
- $T_v := (T_{v,Q})_{Q\in X_{(v)}}$: pure torsion sheaf on $X_{(v)}$ if
  (i) $T_{v,Q}$ coherent torsion $\mathcal{O}_{X_{(v)},Q}$ sheaf
  (ii) $\exists$ injection of bundle map $\phi : V_1 \to V_2$ of same rank on $X_{(v)}$ such that $\mathrm{Coker}(\phi) = T_v$
  (iii) for all but finitely many closed points Q of $X_{(v)}$, $T_{v,Q} = 0$
- $\mathrm{Aut}(T_v) = \mathrm{Aut}_{\mathcal{O}_{X_{(v)}}}(T_v)$: auto group as $\mathcal{O}_{X_{(v)}}$-module
- $N(T_v) := \prod_Q \# T_{v,Q}$: Norm of T_v
- $T = (T_v)$: torsion sheaves on $\mathfrak{X}$ (finitely many v: nontrivial)
- $\mathrm{Aut}(T) := \prod_v \mathrm{Aut}(T_v)$, $\qquad N(T) := \prod_v N(T_v)$

Zeta functions for torsion sheaves

Torsion Zeta Function

Torsion zeta function for X/K:

$$\zeta_{\mathfrak{X}}^{T}(s) := \sum_{T/\mathfrak{X}:\ \text{torsion sheaves}} \frac{1}{\#\mathrm{Aut}(T)} \cdot N(T)^{-s}$$

Theorem (Zetas and Strange Duality)

$$(i) \qquad \zeta_{\mathfrak{X}}^{T}(s) = \prod_{k \geq 1} \zeta_X(s+k),$$

$$(ii) \qquad \sum_{T_v/E_{(v)}} \frac{1}{\#\mathrm{Aut}(T)} = \sum_{V/E_{(v)}} \frac{1}{\#\mathrm{Aut}(V)}$$

w/ T_v: torsion sheaves on $E_{(v)}$ of degree n, i.e., norm q_v^n
V: rank n semi-stable bundles of degree 0 on $E_{(v)}$

α and β invariants

β invariants (Weil-Siegel, Harder-Narasimhan)

β invariants for curves $X/\mathbb{F}_q$:

$$\beta_{X,n}(d) = \sum_V \frac{1}{\#\mathrm{Aut}(V)}$$

w/ V: rank n semi-stable bundles on $X/\mathbb{F}_q$ of degree $d \in \mathbb{Z}$.

$$\beta_{X,n}(d) = \beta_{X,n}(0).$$

α invariants (Weng)

$$\alpha_{X,n}(d) = \sum_V \frac{\#q_v^{h^0(E_{(v)},V)} - 1}{\#\mathrm{Aut}(V)}$$

w/ V: rank n semi-stable bundles on $X/\mathbb{F}_q$ of degree $d \in \mathbb{Z}$.

Non-abelian zeta for elliptic curves and the RH

Definition (Weng)

Non-abelian zeta function curves $X/\mathbb{F}_q$:

$$\zeta_{X,n}(s) = \sum_V \frac{\# q_v^{h^0(X,V)} - 1}{\#\mathrm{Aut}(V)} \cdot q_v^{-s\cdot \deg V}$$

w/ V: rank n semi-stable bundles on X of degree $\in n\mathbb{Z}$

Example

$X = E$: an elliptic curve

$$\zeta_{E,n}(s) = \alpha_{E,n}(0) + \beta_{E,n}(0) \cdot \frac{(Q-1)\cdot T}{(1-T)(1-QT)}$$

with $Q = q^n$, $T = t^n$, $t = q^{-s}$

Non-abelian zeta for elliptic curves and the RH

Theorem (Weng-Zagier)

- (WZ: $E_{(v)}$; Sugahara: $X/\mathbb{F}_q$)

$$\alpha_{X,n+1}(0) = q^{n(g-1)} \cdot \beta_{X,n}(0).$$

- $$\sum_{n\geq 0} \beta_{E_{(v)},n}(0) \cdot q_v^{-ns} = \prod_{k\geq 1} \zeta_{E_{(v)}}(s+k)$$

- $$1 < \frac{\beta_{E_{(v)},n}(0)}{\beta_{E_{(v)},n-1}(0)} < \frac{q^{n/2}+1}{q^{n/2}-1}$$

- The Riemann Hypothesis holds for $\zeta_{E_{(v)},n}(s)$.

Elliptic Average

Elliptic Average

- $E/\mathbb{Q}$: elliptic curve
- f: $\mathbb{C}$-valued function on iso classes of finite groups
- $$\mathcal{M}_{E,r}(f) := \lim_{x\to\infty} \frac{\sum_{|D_L|\leq x} f(Ш(E/L))}{\sum_{|D_L|\leq x} 1}$$

  w/ L: quadratic field satisfying $\mathrm{rk}\big(E(L)\big) = r$

Torsion Sheaf Average: Special Case

- F : $\mathbb{C}$-valued function on iso classes of torsion sheaves
- $$M_0(F) := \lim_{x\to\infty} \frac{\sum_{N(T)\leq x} \frac{1}{\#\mathrm{Aut}(T)} \cdot F(T)}{\sum_{N(T)\leq x} \frac{1}{\#\mathrm{Aut}(T)}}$$

Torsion Sheaf Average

Torsion Sheaf Average: General Case

- $E/\mathbb{Q}$: elliptic curve
- $r \in \mathbb{Z}_{\geq 0}$
- F : $\mathbb{C}$-valued function on iso classes of torsion sheaves
- $$M_r(F) := \lim_{x\to\infty} \frac{\sum_{N(T)\leq x} \frac{1}{\#\mathrm{Aut}(T)} \cdot \sum_{\varphi\colon \mathcal{O}^r \to T} F(T/\mathrm{Im}\varphi)}{\sum_{N(T)\leq x} \frac{1}{\#\mathrm{Aut}(T)}}$$

Probabilities

F: characteristic function of certain property P,

$\Longrightarrow$ $M_r(F) =:$ the probability of P

Heuristics for $Ш(E/L)$

Heuristic Assumption for $Ш(E/L)$

$\exists$ natural $f \mapsto F$ for certain restricted type f
s.t. if $E/\mathbb{Q}$ is an elliptic curve

$$\mathcal{M}_{E,r}(f) = M_r(F)$$

Example

Example

- f_0: characteristic function of the odd part of G being acyclic
- F_0: characteristic function of the odd part of T being simple
- $T = (T_{v,Q})$ is called simple if $T_{v,Q} = \mathcal{O}_{v,Q}/\mathfrak{m}_{v,Q}^{n_{v,Q}}$

Theorem

Under the Heuristic Assumption, for a fixed $E/\mathbb{Q}$,
the probability of $Ш^{\mathrm{odd}}(E/L)$ being cyclic
among quadratic extensions $L/\mathbb{Q}$ satisfying $\mathrm{rank}(E(L)) = 0$ is

$$\left(\prod_{p\text{: bad}} \frac{\zeta_{E_{(p)}}(6)}{\zeta_{E_{(p)}}(2)\zeta_{E_{(p)}}(3)} \right) \cdot \frac{\zeta_{\mathcal{E}}(2)\zeta_{\mathcal{E}}(3)}{\zeta_{\mathcal{E}}(6)} \cdot C_\infty(E)$$

w/ $C_\infty(E) := \mathrm{Res}_{s=0} \prod_{k\geq 1} \zeta_{\mathcal{E}}(s+k)$

Why Torsions (I)

Torsion points and Bundles

Rank two s. stable bundles V of determinant $\mathcal{O}_E$:

- Over $\overline{\mathbb{F}_q}$: $Gr(V) = \lambda \oplus \lambda^{-1},\ \lambda/\overline{\mathbb{F}_q}$
- Over $\mathbb{F}_q$
  - (i.a) $V = \lambda_2 \oplus \lambda_2$ (i.b) $V = I_2 \otimes \lambda_2,$ $\lambda_2 \in E(\mathbb{Q})[2]$
  - (ii) $V = \lambda \oplus \lambda^{-1},$ $\lambda \neq \lambda^{-1},\ \lambda \in \mathrm{Pic}^0(E)(\mathbb{Q})$
  - (iii) $V = \lambda \oplus \lambda^{\sigma},$ $\lambda^{\sigma} = \lambda^{-1},\ \lambda \in \mathrm{Pic}^0(E)(L)\backslash\mathrm{Pic}^0(E)(\mathbb{Q})$

Bundles and Torsion Sheaves

Strange Duality: Over elliptic curve $E_{(v)}/\mathbb{F}_{q_v}$

$$\sum_{T_v,\deg(T_v)=n} \frac{1}{\#\mathrm{Aut}(T_v)} = \sum_{V:\text{s.stable rk}=n,\ \deg=0} \frac{1}{\#\mathrm{Aut}(V)}.$$

Why Torsions (II)

Cohen-Lenstra: Number Fields

$$\sum_G \frac{1}{|\operatorname{Aut} G|} \cdot |G|^{-s} = \prod_{k \geq 1} \zeta(s+k).$$

Torsion Zeta: Varieties

$$\sum_{T/\mathfrak{X}} \frac{1}{|\operatorname{Aut}(T)|} \cdot N(T)^{-s} = \prod_{k \geq 1} \zeta_{\mathfrak{X}}(s+k).$$

Why Relative Shafarevich-Tate

BSD Conjecture

$\mathrm{rk} E(\mathbb{Q}) = \mathrm{ord}_{s=1} \zeta_{\mathcal{E}}(s) =: r$ and

$$\lim_{s\to 1} \frac{1}{(s-1)^r} \cdot \zeta_{\mathcal{E}}(s) = \#\text{Ш}(E/\mathbb{Q}) \cdot \frac{R_E \cdot \mathrm{Tam}(E) \cdot \Omega_E^+}{\# E(\mathbb{Q})_{\mathrm{tor}}}.$$

Dedekind Theorem

$$\lim_{s\to 1} (s-1)\zeta_K(s) = \#\mathrm{CL}(K) \cdot \frac{2^{r_1}(2\pi)^{r_2} R}{w\sqrt{|D_K|}}.$$

Cohen-Lenstra: Number Fields

Cohen-Lenstra: Number Fields

Rank of the unit group of the field $L = \mathbb{Q}(\sqrt{D})$ plays a key role

Number Fields versus Elliptic Curves: Zeta Analogues

Number Field: w versus $E(\mathbb{Q})_{\mathrm{tor}}$: Elliptic Curves
$\Longrightarrow$ Unit group versus Mordell-Weil Group

Our Heuristics: Elliptic Curves

Rank of Mordell-Weil group $E(L = \mathbb{Q}(\sqrt{D})$ should play key role

Thank You

THANK YOU!

FUKUOKA, Oct 28, 2013