Bott-Chern Secondary Characteristic Objects and Arithmetic Riemann-Roch Theorem
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and

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Part I

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INTRODUCTION

The main purpose of this book is to give basic concepts, techniques, and results of arithmetic geometry in the sense of Arakelov. In particular, we will give the arithmetic Riemann-Roch theorem for local complete intersection morphisms, but with one technical condition that the morphism at infinite place is smooth.

In this introduction, we will illustrate the theory with one simplest example: the situation in the category of complex projective manifolds.

We start with the following Riemann-Roch theorem in the sense of Gorthendieck:

Let $f : X \to Y$ be a smooth morphism of complex manifold. Then for any vector sheaf $\mathcal{E}$ on X, we may define the push-out morphism of $\mathcal{E}$ in the sense of K-theory: $f_K(\mathcal{E}) := \sum_j (-1)^j R^i f_*(\mathcal{E})$. Then, at the cohomology class level, we have the following **Grothendieck-Riemann-Roch theorem with respect to smooth morphisms:**

$$f_{\mathrm{CH}}(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(\mathcal{T}_f)) = \mathrm{ch}(f_K(\mathcal{E})),$$

where f_{CH} is the natural push-out morphism of algebraic cycles, td is the Todd characteristic class, and $\mathcal{T}_f$ is the relative tangent sheaf of f.

On the other hand, let $i : X \hookrightarrow Z$ be a closed immersion of complex manifolds. There exists a natural exact sequence:

$$0 \to \mathcal{T}_X \to i^*\mathcal{T}_Z \to \mathcal{N}_i \to 0.$$

For any vector sheaf $\mathcal{E}$ on X, the direct image $i_*\mathcal{E}$ is a coherent sheaf on Z. By classical sheaf theory, there exists a vector sheaf resolution of $i_*\mathcal{E}$ on Z:

$$\mathcal{E}. : 0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to i_*\mathcal{E} \to 0.$$

Then the **Grothendieck-Riemann-Roch theorem with respect to closed immersions** says that we have the following equality at the level of cohomology classes, i.e. in $\mathrm{CH}(Z)_{\mathbf{Q}}$,

$$\mathrm{ch}(i_*\mathcal{E}) = i_*(\mathrm{td}(\mathcal{N})^{-1}\,\mathrm{ch}(\mathcal{E})).$$

Thus by the fact that $i_*\eta = \sum_j (-1)^j \mathcal{E}_j$, we have

$$\mathrm{ch}(\mathcal{E}.) = i_*(\mathrm{td}(\mathcal{N})^{-1}\,\mathrm{ch}(\mathcal{E})).$$

Put the above two situations together, we may have the **Grothendieck-Riemann-Roch theorem with respect to l.c.i. morphisms:**

Let $f : X \to Y$ be a l.c.i. morphism of complex manifold. Then the following diagram is commutative:

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}()\mathrm{td}(T_f)} & \mathrm{CH}(X)_{\mathbf{Q}}. \\ f_K \downarrow & & \downarrow f_{\mathrm{CH}} \\ K_0(Y) & \xrightarrow{\mathrm{ch}} & \mathrm{CH}(Y)_{\mathbf{Q}}. \end{array}$$

With above, put the situation in a simplest way, we may say that that arithmetic Riemann-Roch theorem should be a natural generalization of the above commutative diagram so that the above diagram is still commutative after we put the arithmetic notation at the suitable place correspondingly, that is, the arithmetic Riemann-Roch theorem should become the following commutative diagram

$$\begin{array}{ccc} K_0^{\mathrm{Ar}}(X) & \xrightarrow{\mathrm{ch}_{\mathrm{Ar}}()\mathrm{td}_{\mathrm{Ar}}(T_f^{\mathrm{Ar}})} & \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}}. \\ f_K^{\mathrm{Ar}} \downarrow & & \downarrow f_{\mathrm{CH}}^{\mathrm{Ar}} \\ K_0^{\mathrm{Ar}}(Y) & \xrightarrow{\mathrm{ch}} & \mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}}. \end{array}$$

With this in mind, the first thing we need to do is to give the fundamental concepts and results in the above picture, such as arithmetic intersection theory, arithemtic characteristic classes, etc.. In three fundamental papers [GS 90], [GS 91b] and [GS 91c], Gillet and Soulé give the arithmetic intersection theory and arithmetic characteristic classes. Next let us expose them in our situation here.

By a standard result, we know that under the natural Chern character, the algebraic K-group is isomorphic to the Chow group for any regular variety. In particular, the algebraic intersection can be introduced use the topological property of algebraic K-theory. Among others, let us just mention that the divisors corresponds to the line sheaves, and all the theory may be deduced from this very special situation, as we have the splitting principle for vector sheaves. More precisely, let $\mathcal{L}$ be a line sheaf on X, then its associated algebraic cycle may be defined by $\mathrm{div}(s)$ for a non-trivial rational section s of $\mathcal{L}$: This is a natural correspondence at the level of (de Rham) cohomology classes. In order to go further, let us consider the situation at the level of differential forms. One then knows that we may put hermitian metrics on $\mathcal{L}$. Choose one hermitian metric, say ρ. Then we have the first Chern characteristic form $c_1(\mathcal{L}, \rho)$. It is well-known that $c_1(\mathcal{L}, \rho)$ is a closed differential form and its de Rham class is just the corresponding algebraic cycle class, which is then of course independent on the choice of the metric. On the other hand, *the form itself does depend on the metric*. Furthermore, we have the following Poincaré-Lelong equation:

$$dd^c[-\log|s|_\rho^2] = [c_1(\mathcal{L}, \rho] - \delta_{\mathrm{div}(s)},$$

where Δ is the Dirac symbol. With this, a natural idea is to choose $(\mathrm{div}(s), -\log|s|_\rho^2)$ to define $c_{1\,\mathrm{Ar}}(\mathcal{L}, \rho)$. Thus, by the splitting principle, we may lead to define the arithmetic cycle in general. In practice, following [GS 90], we define an arithmetic cycle as a pair (Z, g_Z) such that Z is an algebraic cycle, and $dd^c g_Z + \Delta_Z$ is a smooth form. Usually, we call g_Z as a Green's current for Z. Also we have the arithmetic Chow group $CH_{\mathrm{Ar}}(X)$ by letting it be a quotient group generated by the above pairs modulo the relations given by $(\mathrm{div}(f), -[\log|f|^2])$, where f denotes the rational function of certain irreducible subvarieties.

With this definition about arithmetic cycles, we may introduce the arithmetic intersection. The story about the algebraic cycle is rather obvious: We may choose the algebraic intersection among them. But the situation for the currents is rather complicated: In general, Green's currents may have very bad singularities. To control this, from the Poincaré-Lelong equation, we may introduce Green's currents with logarithmic singularities around Z. It may be shown that for each classes modulo the exact currents for ∂ and $\bar{\partial}$, there is a representative of Green's current with logarithmic singularities. With this control of singularities, we may introduce the arithmetic intersection by a moving lemma at he level of K_1-groups. For more details, see II.2.

Basically, we may also use the splitting principle to introduce the the arithmetic characteristic classes. Since we also wnat that the arithmetic Chern character should offer a natural isomorphism between the arithmetic Chow group and the arithmetic K-group. In algebraic geometry, i.e. at the level of cohomology classes, we define the algebraic K-group $K(X)$ as the quotient of the group generated by vector sheaves on X modulo the relations $\mathcal{E}_2 - \mathcal{E}_1 - \mathcal{E}_3 = 0$ if

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is exact. Thus a natural choice for $K_0^{\mathrm{Ar}}(X)$ should be the quotient group generated by triples $(\mathcal{E}, \rho; \omega)$, where $(\mathcal{E}, \rho)$ is a hermitian vector sheaf on X, and ω is a sommth form. What should be the relations among them?

To find the relations, let us go back to the definition about the arithmetic cycles. We know that $c_1(\mathcal{L}, \rho)$ as a differential form is dependent on the choice of the hermitian metric ρ. So the relations for the arithmetic K-group should reflects this change, since we now consider the problems at the level of differential forms. Note that if in the above algebraic exact sequence, if we let $\mathcal{E}_1 = 0$, and by the fact the ch is an isomorphism between $K(X)$ and $\mathrm{CH}(X)_{\mathbf{Q}}$, we see that the above problem about the relations for the arithmetic K-group becomes the following: How to measure the difference $\mathrm{ch}(\mathcal{E}, \rho) - \mathrm{ch}(\mathcal{E}, \tau)$?

To answer the latest problem, we come to a famous theorem given by Bott and Chern. Around 1968, in the paper [BG 68], Bott and Chern could solve the follwoing partical differential equation

$$dd^c \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho, \tau) = \mathrm{ch}(\mathcal{E}, \rho) - \mathrm{ch}(\mathcal{E}, \tau).$$

Just from this, we introduce the definition for the arithmetic K-group: The relations are

$$(\mathcal{E}_1, \rho_1; \omega_1) + (\mathcal{E}_3, \rho_3; \omega_3) = (\mathcal{E}_2, \rho_2; \omega_1 + \omega_3 - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.),$$

where as above, we have the exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

and ρ_i are hermitian metric on $\mathcal{E}_i$ and the form $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.)$ is the solution of the following equation

$$dd^c \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.) = \mathrm{ch}(\mathcal{E}_2, \rho_2) - \mathrm{ch}(\mathcal{E}_1, \rho_1) - \mathrm{ch}(\mathcal{E}_3, \rho_3).$$

thus we can finish tha process for introducing arithmetic intersection theory and arithmetic characteristic classes and hence give the notation in the two rows of the arithmetic Riemann-Roch theorem diagram above.

Next, we introduce the push-out morphism f_K^{Ar} and $f_{\mathrm{CH}}^{\mathrm{Ar}}$. For this, we need to give a factorization for our morphism f. Say, $f = g \circ i$ with $i : X \to Z$ a regular closed immersion and $g : Z \to Y$ a smooth morphism. We first consider the push-out morphism for arithmetic cycles: The algebraic cycles are settled by a standard algebraic process. For Green's current, we then need to be careful. For smooth morphisms, we just take the integration along the fibres. For closed immersion, we may use the arithmetic intersection to do so, after choosing certain arithmetic cycles for X, viewing as a subvariety in Z. Thus finally definition comes from the moving lemma at K_1-level.

In the following, we only discuss the situation for smooth morphisms to give a definition for f_K^{Ar}. For this, we recall the following theory about the classical Bott-Chern secondary characteristic forms above.

Let $\mathcal{E}$ be a vector sheaf of rank r on a complex manifold X. Put a hermitian metric ρ on $\mathcal{E}$. Then there exists a unique cononical connection Δ associated with $(\mathcal{E}, \rho)$. Hence we have its curvature Δ^2. In this way, we may define the Chern characteristic form $\mathrm{ch}(\mathcal{E}, \rho)$ by first inentifing $\mathrm{End}(\mathcal{E})$ with the metrix algabra $M_r(\mathbf{C})$, then defining it locally as $\exp(-\frac{1}{2\pi i}\Delta^2)$. By a local disçussion, we know that this offers us a global differential form on X. And from the Bianchi identity, it is closed. Also it is compatible with the pull-back by any morphism. Furthermore, from the de Rham cohomology theory, the cohomology class of $\mathrm{ch}(\mathcal{E}, \rho)$ does not depend on the choice of ρ. So this class offer us a satisfactory answer in algebraic geometry. On the other hand, the form $\mathrm{ch}(\mathcal{E}, \rho)$ itself does depend on ρ. Such a kind of dependence is given by the classical Bott-Chern secondary characteristic forms:

We first introduce axioms for the **classical Bott-Chern secondary characteristic form**, $\phi_{\mathrm{BC}}(\mathcal{E}., \rho.)$, with respect to any power series ϕ, a short exact sequence of vector sheaves

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

and hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1, 2, 3$: (it is worthy to mention that here it is not necessary to assume that ρ_1 and ρ_3 are induced from ρ_2.)

Axiom 1. (**Downstairs Rule**) Let

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

be a short exact sequence of vector sheaves over a complex manifold X with hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1, 2, 3$. Then

$$d_X d_X^c \phi_{\mathrm{BC}}(\mathcal{E}., \rho.) = \phi(\mathcal{E}_2, \rho_2) - \phi(\mathcal{E}_1 \oplus \mathcal{E}_3, \rho_1 \oplus \rho_3)$$

holds in $\tilde{A}(X) := \oplus A^{p,p}(X)/\mathrm{Im}\partial + \bar{\partial}$.

Axiom 2. (**Functorial Rule**) For any morphism $f : X' \to X$ of complex manifolds, we have

$$f^* \phi_{\mathrm{BC}}(\mathcal{E}., \rho.) = \phi_{\mathrm{BC}}(f^*\mathcal{E}., f^*\rho.).$$

Axiom 3. **(Uniqueness Rule)** If $(\mathcal{E}.,\rho.)$ is split, i.e. $(\mathcal{E}_2,\rho_2) = (\mathcal{E}_1 \oplus \mathcal{E}_3, \rho_1 \oplus \rho_3)$, then

$$\phi_{\rm BC}(\mathcal{E}.,\rho.) = 0.$$

Then we may state the Bott-Chern theorem as the following

Existence Theorem for Classical Bott-Chern Secondary Characteristic Forms. Let

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

be a short exact sequence of vector sheaves on a complex manifold X with hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1, 2, 3$. Then for any symmetric power series ϕ, there exists a unique differential form $\phi_{\rm BC}(\mathcal{E}.,\rho.) \in \tilde{A}(X)$ such that $\phi_{\rm BC}(\mathcal{E}.,\rho.)$ satisfies the axioms 1, 2, and 3 above.

There are several methods to prove this theorem. The basic idea is that we first form a family of differential forms phi$(D\mathcal{E}_t, D\rho_t)$ so that, at $t=0$, it gives phi$(\mathcal{E}_2,\rho_2)$, while at $t=\infty$, it becomes $\phi(\mathcal{E}_1\oplus\mathcal{E}_3,\rho_1\oplus\rho_3)$. then the integration of this form will offer a solution. For more details, see Chapter I.1.

What should be the relation of the classical Bott-Chern secondary characteristic forms and $f_K^{\rm Ar}$? First we discuss the situation when f is smooth. In algebraic geometry, we know that $f_K(\mathcal{E}) = \sum(-1)^k R^k f_*\mathcal{E}$. But, in general, $R^k f_*\mathcal{E}$ are coherent sheaves. So to define $f_K^{\rm Ar}$, we need to note the fact that $K_(X)$ is generated by f-acyclic vector sheaves. So in the following, we will only consider the situation for such vector sheaf on X. As a consequence, we know that $f_K(\mathcal{E}) = f_*\mathcal{E}$, which is a vector sheaf on Y. Thus we have a natural element $(f_*\mathcal{E}, f_*\rho)$ in $K_0^{\rm Ar}(Y)$ for f-acyclic hermitian vector sheaf $(\mathcal{E},\rho)$ on X. On the other hand, fixed a hermitian metric on the relative tangent vector sheaf $\mathcal{T}_f$ of f, we have, by Riemann-Roch theorem, another differential form $f_*(\mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T}_f,\rho_f)$. By Grothendieck-Riemann-Roch theorem, as de Rham cohomology classes, $\mathrm{ch}(f_*\mathcal{E}, f_*\rho)$ is just $f_*(\mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T}_f,\rho_f)$. But as differential forms, they are not the same in general. Therefore, as in the situation for the classical Bott-Chern secondary characteristic forms, we may think the difference $\mathrm{ch}(f_*\mathcal{E}, f_*\rho) - f_*(\mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T}_f,\rho_f)$ as the change of $\mathcal{E},\rho)$ with the action of (f,ρ_f). This lead us to introduce the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms by the following axioms:

Axiom 1. **(Downstairs Rule)** Let $f: X \to Y$ be a smooth morphism of Kähler manifolds with a hermitian metric ρ_f on the relative tangent sheaf $\mathcal{T}_f$. Suppose $(\mathcal{E},\rho)$ is an f-acyclic hermitian vector sheaf on X, then on $\tilde{A}(Y)$, we have

$$\begin{aligned} d_Y d_Y^c \mathrm{ch}_{\rm BC}(\mathcal{E},\rho,f,\rho_f) = \\ f_*(\mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T}_f,\rho_f)) - \mathrm{ch}\,(f_*\mathcal{E}, f_*\rho). \end{aligned}$$

Axiom 2. **(Base Change Rule)** For any flat base change $g: Y' \to Y$, we have

$$g^*\mathrm{ch}_{\rm BC}(\mathcal{E},\rho;f,\rho_f) = \mathrm{ch}_{\rm BC}(g_f^*\mathcal{E}, g_f^*\rho; f_g, \rho_{f_g}).$$

Here g_f denotes the induced morphism of g with respect to f, and similarly for f_g. That is, we have the following commutative diagram:

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{g_f} & X \\ f_g \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Here ρ_{f_g} is the natural metric induced by the flat base change g from ρ_f.

Axiom 3. (Uniqueness With Respect To Vector Sheaves) For any short exact sequence of f-acyclic vector sheaves

$$\mathcal{E}.: \quad 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

with hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1, 2, 3$, let

$$f_*\mathcal{E}.: \quad 0 \to f_*\mathcal{E}_1 \to f_*\mathcal{E}_2 \to f_*\mathcal{E}_3 \to 0$$

be the direct image of $\mathcal{E}.$ with associated hermitian metrics $f_*\rho_j$ on $f_*\mathcal{E}_j$ for $j = 1, 2, 3$. Then

$$\begin{aligned} &\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_2, \rho_2; f, \rho_f) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_1, \rho_1; f, \rho_f) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_3, \rho_3; f, \rho_f) \\ &\quad = f_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.)\, \mathrm{td}(\mathcal{T}_f, \rho_f)) - \mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E}., f_*\rho.). \end{aligned}$$

Axiom 4. (Uniqueness With Respect To Morphisms) Let $f : X \to Y$ and $g : Y \to W$ be two smooth morphisms of Kähler manifolds. Let $(\mathcal{E}, \rho)$ be an f-acyclic hermitian vector sheaf on X such that $f_*\mathcal{E}$ is g-acyclic. Then

$$\begin{aligned} &\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; g \circ f, \rho_{g \circ f}) - \mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E}, f_*\rho; g, \rho_g) - g_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; f, \rho_f)\, \mathrm{td}(\mathcal{T}_g, \rho_g)) \\ &\quad = (g \circ f)_*(\mathrm{ch}(\mathcal{E}, \rho)\, \mathrm{td}_{\mathrm{BC}}(f, g)). \end{aligned}$$

Here $\mathrm{td}_{\mathrm{BC}}(f, g)$ denotes the classical Bott-Chern secondary characteristic form associated with the following short exact sequence of the relative hermitian tangent sheaves:

$$0 \to \mathcal{T}_f \to \mathcal{T}_{g \circ f} \to f^*\mathcal{T}_g \to 0.$$

With this, similarly as the situation for the classical Bott-Chern secondary characteristic forms, we may also have the following

Existence Theorem Of Relative Bott-Chern Secondary Characteristic Forms With Respect To Smooth Morphisms.
Let $f : X \to Y$ be a smooth morphism of Kähler manifolds with a hermitian metric ρ_f on the relative tangent sheaf $\mathcal{T}_f$. Then for any f-acyclic hermitian vector sheaf $(\mathcal{E}, \rho)$, there exists a unique element $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho, f, \rho_f)$ in $\tilde{A}(N)$ which satisfies the axioms above.

For the proof of this theorem, we may imitate the one for the classical one. But now we are working in a infinite dimensional situation, i.e. on $C^\infty(Y, f_*\mathcal{E})$. Then we meet certain problems. The most important one is that about the convergence. We know that the natural L^2-connection will not offer us a (good) trace class. Fortunately, by the work of Bismut about the local index theorem, we may choose the Bismut superconnection as an alternative object.

Basically, the Bismut superconnection is the limit of the Dirac operator along the fibre of f with a blowing up metric proces for the base, i.e. the change of the metric is given by $\rho_f + s^{-1}\rho_Y$. Thus, we may have the associated heat kernel $\exp(-\mathbf{B}_t^2)$. Here the parameter may be introduced by rescaling the total metric with a factor t. Then Bismut's local index theorem may be roughly stated as that when $t \to 0^+$, the supertrace of the restriction of the above heat kernel to the diagonal gives us the differential form $f_*(\mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T}_f,\rho_f))$. On the other hand, a result of Berline and Vergne asserts that when $t \to \infty$, the same date offer us the differential form $\mathrm{ch}\,(f_*\mathcal{E}, f_*\rho)$. Therefore, it is possible to give the above existence theorem for the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms. Surely, during this process, we need a kind of local double transgression formula, for which we need to introduce the number operator form an intrinsic point of view. Also in order to overcome the difficult about the convergence for the trace class in question, we need to use a cone construction following Faltings [Fa 92].

The above process coming from the index theorem is rather complicated, which is originally given by Bismut with certain technique from stochastic integration. Here we use the heat kernel approach following Berline, Getzler and Vergne [BGS 92]. In this book, we devote it with several chapters: From I.2 to I.6.

With the above work about Bott-Chen secondary characteristic objects, for any power series $R(x) \in R[[x]]$, we may define the associated push-out morphism for the arithmetic K-group by $f_K^{R,\mathrm{Ar}}(\mathcal{E},\rho)$ for f-acyclic hermitian vector $(\mathcal{E},\rho)$ with

$$(f_*\mathcal{E}, f_*\rho) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho; f,\rho_f) + f_*(0, \mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T}_f,\rho_f)\,\omega(R(\mathcal{T}_f))),$$

where $R(\mathcal{E})$ is a additive characteristic class defined by the power series $R(x)$, and $\omega(Z, g_Z) := dd^c g_Z + \delta_Z$. Then the arithmetic Riemann-Roch theorem may be stated as follows

Arithmetic Riemann-Roch theorem For Smooth Morphism:([Fa 92]) There exists unique power series $R(x)$ such that for any smooth morphism of regular arithmetic varieties $f: X \to Y$, the following diagram

$$\begin{array}{ccc} K_0^{\mathrm{Ar}}(X) & \xrightarrow{\mathrm{ch}_{\mathrm{Ar}}()\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_f^{\mathrm{Ar}})} & \mathrm{CH}_{\mathrm{Ar}}(X)_{\mathbf{Q}}. \\ f_K^{\mathrm{Ar}} \downarrow & & \downarrow f_{\mathrm{CH}}^{\mathrm{Ar}} \\ K_0^{\mathrm{Ar}}(Y) & \xrightarrow{\mathrm{ch}} & \mathrm{CH}_{\mathrm{Ar}}(Y)_{\mathbf{Q}} \end{array}$$

commutes, where $f_K^{\mathrm{Ar}} := f_K^{R,\mathrm{Ar}}$.

The proof of this theorem may be divided into two steps. First, we consider the situation for smooth morphisms. We follow [Fa 92]. With the same notation as above, for any power series P, for any smooth morphism $f: X \to Y$ of regular arithmetic varieties over an arithmetic ring (A, Σ, F_∞), any f-acyclic hermitian vector sheaf $(\mathcal{E},\rho)$ on X, let

$$\mathrm{Err}(\mathcal{E},\rho; f,\rho_f; P) := \mathrm{ch}_{\mathrm{Ar}}(f_K(\mathcal{E},\rho)) - f_{\mathrm{CH}}(\mathrm{ch}_{\mathrm{Ar}}(\mathcal{E},\rho)\mathrm{Td}_{\mathrm{Ar}}^{\mathrm{P}}(f,\rho_f)).$$

To prove the theorem, it is sufficient to show that there exists a unique power series $R(x)$ so that

$$\mathrm{Err}(\mathcal{E},\rho; f,\rho_f; R) = 0,$$

for any f-acyclic hermitian vector sheaf $(\mathcal{E}, \rho)$. For this we need some intermediary results.

Proposition 1. Let $f : X \to Y$ be a smooth morphism of regular arithmetic varieties with an F_∞-invariant hermitian metric ρ_f on the relative tangent vector sheaf of f. Then for any short exact sequence of f-acyclic hermitian vector sheaves

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

with F_∞-invariant hermitian metrics ρ_i on $\mathcal{E}_i$ for $i = 1, 2, 3$, we have

$$\operatorname{Err}(\mathcal{E}_1, \rho_1; f, \rho_f; P) + \operatorname{Err}(\mathcal{E}_3, \rho_3; f, \rho_f; P) = \operatorname{Err}(\mathcal{E}_2, \rho_2; f, \rho_f; P).$$

In particular, $\operatorname{Err}(\mathcal{E}, \rho; f, \rho_f; P)$ does not depend on the metric ρ. Moreover, $\operatorname{Err}(\mathcal{E}, \rho; f, \rho_f; P)$ lies in the a-image of harmonic forms.

Proposition 2. Let $f : X \to Y$ and $g : Y \to Z$ be two smooth morphisms of regular arithmetic varieties which have F_∞-invariant hermitian metrics ρ_f, ρ_g and $\rho_{g\circ f}$ on the relative tangent vector sheaves of f, g and $g \circ f$ respectively. Let $(\mathcal{E}, \rho)$ be an f-acyclic hermitian vector sheaf on X such that $f_*\mathcal{E}$ is g-acyclic. Then

$$\operatorname{Err}(\mathcal{E}, \rho; g \circ f, \rho_{g\circ f}; P) = \operatorname{Err}(f_*\mathcal{E}, f_*\rho; g, \rho_g; P) + g_*(\operatorname{Err}(\mathcal{E}, \rho; f, \rho_f; P)\operatorname{Td}^{\mathrm{P}}_{\mathrm{Ar}}(g, \rho_g)).$$

In particular, $\operatorname{Err}(\mathcal{E}, \rho; f, \rho_f; P)$ does not depend on the metric ρ_f.

Remark. Because of these two propositions, we denote $\operatorname{Err}(\mathcal{E}, \rho; f, \rho_f; P)$ simply as $\operatorname{Err}(\mathcal{E}; f; P)$.

Proposition 3. There is a natural morphism

$$\operatorname{Err} : K(X_F) \to H(X_{\mathbf{R}})/\rho(\mathrm{CH}^{(1,0)}(Y))_{\mathbf{Q}},$$

such that $\operatorname{Err}(\mathcal{E}; P) = \operatorname{Err}(E; f; P)$.

Proposition 4. Let $f : X \to Y$ be a smooth morphism of regular arithmetic varieties with an F_∞-invariant hermitian metric ρ_f on the relative tangent vector sheaf of f. Then for any flat base change $g : Z \to Y$, we have

$$g^* \operatorname{Err}(\mathcal{E}; f; P) = \operatorname{Err}(g_f^*\mathcal{E}; f_g; P).$$

Here we use the following diagram

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{g_f} & X \\ f_g \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & Y. \end{array}$$

Proposition 5. There is a unique power series $R(x)$ such that for any $\mathbf{P}^1$-bundle

$$p : X = \mathbf{P}_Y(\mathcal{F}) \to Y,$$

$\operatorname{Err}(\mathcal{E}; p; R) = 0$.

Finally, we consider Err for closed immersions. In this case, we have to introduce a new Err term. That is, let $i : X \hookrightarrow Z$ be a closed immersion with the smooth structure morphisms $f : X \to Y$ and $g : Z \to Y$ of regular arithmetic varieties, then we define

$$\mathrm{Err}(\mathcal{E}; i; P) := \mathrm{Err}(\mathcal{E}; f; P) - \mathrm{Err}(i_*\mathcal{E}; g; P).$$

By Proposition 3, this deffnition makes sense, even through $i_*\mathcal{E}$ is usually only a coherent sheaf.

Proposition 6. Let $i : X \hookrightarrow Z$ be a *codimension-one* regular closed immersion of regular arithmetic varieties over an arithmetic variety Y with smooth structure morphisms $f : X \to Y$ and $g : Z \to Y$. Let $(\mathcal{E}, \rho)$ be an f-acyclic hermitian vector sheaf on X such that $i_*\mathcal{E}$ is g-acyclic, then

$$\mathrm{Err}(\mathcal{E}; i; P) = 0.$$

With this, note that by the deformation to the normal cone theory, at the level of algabraic K-theory, any closed immersion may be deduced from codimension one closed immersions and the zero section of projctive bundle, thus by Proposition 3 and the fact that arithmetic Riemann-Roch theorem holds for identity morphisms, we see that it sufficient for us to prove the theorem for projective bundles. Hence we may use the induction on the relative dimension to deduce the result.

In particular, in proving Proposition 6, we need to use a result about the deformation theory for the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms. This finally leads us to introduce an axiom for the so-called Bott-Chern ternary characteristic objects. Roughly speaking, the ternary objects measure the change of secondary characteristic objects. We will not give more details for them, which will be found in Chapter I.9.

In order to discuss the arithmetic Riemann-Roch theorem for l.c.i. morphisms of arithmetic varieties, we need to have a similar discussion as above for closed immersions.

First, we consider the difference given by the Grothendieck-Riemann-Roch theorem at the level of differential forms. Put metrics on the exact sequence of normal sheaves. Also, even through $i_*\mathcal{E}$ is only a coherent sheaf, we may still put the metrics on $\mathcal{E}_j$. Just as for smooth morphisms, a natural question is how we can measure the change of (η, g_η), after the action of the closed immersion i, at the level of differential forms. Similarly, it is for this reason that we introduce the relative Bott-Chern secondary characteristic currents with respect to closed immersions, $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; i, g_i)$, which is originally given by Bismut, Gillet and Soule [BGS 91]. (Here we have to use the language of current, as at least formally, the i_*-image of a form may be written as the product of this form with the Dirac symbol δ_X of X in Z.) So we have the downstairs rule as follows:

$$dd^c \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; i, g_i) = \mathrm{td}(\mathcal{N}, g_{\mathcal{N}})^{-1}\, \mathrm{ch}(\eta, g_\eta)\, \delta_X - \mathrm{ch}(\mathcal{E}_., \rho_.).$$

But the situation is not so simple. We know that the metrics on $\mathcal{E}_.$ are not unique and, in general, we cannot control them very well. In order to introduce the relative Bott-Chern secondary characteristic currents with respect to closed immersions, we need a technical assumption on the metrics, which is nothing but the so-called Bismut condition (A), which gives certain compatibility condition for the associated metrics. We may also give the axioms for the relative Bott-Chern secondary characteristic currents with respect to closed immersions. Similarly, we have the existence theorem for them. For more details, see Chapter I.7 and Chapter I.8.

On the other hand, for any closed immersion i, we may deform it to the zero section of projective bundles. Therefore, one may also hope that there is a ternary theory for closed immmersions. At the same time, the zero section is rather simple, it suggests us to use the Koszul complex to make the calculation in a quite precise form. All of this will given in Chapter I.9, and was first given in [BGS 91].

Once we have the relative Bott-Chern secondary characteristic objects and a special Bott-Chern ternary characteristic objects, we may finally give the arithmetic Riemann-Roch theorem for l.c.i. morphisms. But since at finite place, we cannot only use the deformation to the normal cone, it is quite natural for us to use the MacPherson's Grassmannian construction to achieve the final result. For more details, see Chapter II.5, Chapter II.6 and Chapter II.7.

At the end of this book, we propose a definition for higher arithmetic K-groups by Quillen's construction. This will finally offer us a global triangle relation between arithmetc K-theory, algebraic K-theory and certain analytic homology theory, and hence give the regulator morphisms with their more general meaning: The global morphisms which relate the properties of the finite part and the properties of the infinite part for an arithmetic object.

Part I

Bott-Chern Secondary Characteristic Objects

Chapter I.1
Classical Bott-Chern Secondary Characteristic Forms

In this chapter, we recall basic concepts and results associated with the classical Bott-Chern secondary characteristic forms, which were first given by Bott and Chern around 1968 [BC 68].

The classical Bott-Chern secondary characteristic form measures the change of various characteristic forms with respect to different metrics. In this sense, we may think of the classical Bott-Chern secondary characteristic form as a refined version of characteristic forms in the theory of Chern-Weil, i.e. we discuss the Chern-Weil theory at the level of differential forms.

§I.1.1. Characteristic Forms

I.1.1.a. Connections, Curvatures, and the Canonical Connection

We start with the situation over a real manifold.

Let M be an n-dimensional real C^∞ manifold and E a C^∞ complex vector bundle of rank r over M. As usual, we let

$A^p(M)$:= the complex vector space of C^∞ complex p-forms over M;

$A^p(E)$:= the complex vector space of C^∞ complex p-forms over M with values in E.

By definition, a **connection** ∇ on E is a homomorphism

$$\nabla : A^0(E) \longrightarrow A^1(E)$$

over $\mathbf{C}$ such that, for $f \in A^0(M)$, $\alpha \in A^0(E)$,

$$\nabla(f\alpha) = \alpha\, df + f\nabla\alpha. \tag{1}$$

The connection ∇ above may be realized locally as follows:

Let $s = (s_1, \ldots, s_r)$ be a local frame field of E over an open subset $U \subset M$, so that
(i) $s_j \in A^0(E|_U)$ for $j = 1, \ldots, r$;
(ii) $(s_1(x), \ldots, s_r(x))$ is a basis of the fiber E_x of E at x for each $x \in U$. Then for the connection ∇,

$$\nabla s_j = \sum_{k=1}^{r} s_k \omega_j^k, \tag{2}$$

with $\omega_j^k \in A^1(U)$. We call the matrix of 1-forms $\omega := (\omega_j^k)$ the **connection form** of ∇ with respect to the local frame field s. Obviously, if s' is another local frame field over U, and if ω' is the connection form of ∇ with respect to s', then there is a C^∞ matrix-valued function $a : U \to \mathrm{GL}(r; \mathbf{C})$ such that

$$s = s'a, \tag{3}$$

and

$$\omega = a^{-1}\omega' a + a^{-1}\, da. \tag{4}$$

We may extend the connection ∇ to a $\mathbf{C}$-linear morphism

$$\nabla : A^p(E) \longrightarrow A^{p+1}(E)$$

for $p \geq 0$ by setting

$$\nabla(\alpha\phi) := \nabla(\alpha) \wedge \phi + \alpha\, d\phi$$

for $\alpha \in A^0(E), \phi \in A^p(M)$.

We define the **curvature** of ∇ to be

$$\nabla^2 := \nabla \circ \nabla : A^0(E) \longrightarrow A^2(E); \tag{5}$$

and let $R = \frac{1}{2\pi i}\nabla^2$. Then R is $A^0(M)$-linear. Hence, R is a 2-form on M with the value in End(E). Using the matrix notation, the **curvature form** Ω of ∇ with respect to the frame field s is defined by

$$s\,\Omega := \nabla^2 s.$$

Thus $\Omega = d\omega + \omega \wedge \omega$, and there follows easily the **Bianchi identity:**

$$d\Omega = \Omega \wedge \omega - \omega \wedge \omega.$$

Furthermore, if we let Ω' be the curvature form of ∇ with respect to the local frame s', then

$$\Omega = a^{-1}\Omega' a. \tag{6}$$

Globally, let $\{U, V, \ldots\}$ be an open covering of M with a local frame field s_U on each U. If $U \cap V \neq \emptyset$, then on $U \cap V$, $s_U = s_v g_{VU}$, with $g_{VU} : U \cap V \to \mathrm{GL}(r; \mathbf{C})$ a C^∞ map, called a **transition function**. Let ω_U be the connection form on U with respect to s_U, then on $U \cap V$, we have

$$\omega_U = g_{VU}^{-1}\,\omega\, g_{VU} + g_{VU}^{-1}\, dg_{VU}. \tag{7}$$

Conversely, given a system of $gl(r;\mathbf{C})$-valued 1-forms ω_U on U satisfying (7), we may obtain a connection ∇ on E having ω_U as its connection form. Also, if Ω_U is the curvature form of ∇ with respect to s_U, then we have

$$\Omega_U = g_{VU}^{-1}\,\Omega_V\, g_{VU} \tag{8}$$

on $U \cap V$.

From now on, we assume that M is a complex manifold and E is a C^∞ complex vector bundle of rank r over M. We let

$A^{p,q}(M)$:= the vector space of C^∞ complex (p,q)-forms over M;

$A^{p,q}(E)$:= the vector space of C^∞ complex (p,q)-forms over M with values in E.

Thus

$$A^k(M) = \sum_{p+q=k} A^{p,q}(M), \quad A^k(E) = \sum_{p+q=k} A^{p,q}(E),$$

and there are natural operators

$$\partial : A^{p,q}(M) \to A^{p+1,q}(M), \quad \bar\partial : A^{p,q}(M) \to A^{p,q+1}(M).$$

So

$$d = \partial + \bar\partial.$$

Usually, we also introduce d^c as follows:

$$d^c := \frac{1}{4\pi i}(\partial - \bar\partial).$$

Hence,

$$dd^c = \frac{i}{2\pi}\partial\bar\partial = \frac{1}{2\pi i}\bar\partial\partial$$

is a real operator. Let

$$A(M) := \oplus_k A^k(M), \quad \tilde{A}(M) := A(M)/(\mathrm{Im}\partial + \mathrm{Im}\bar\partial).$$

Let ∇ be a connection of E as above. We may write $\nabla = \nabla^{1,0} + \nabla^{0,1}$ with

$$\nabla^{1,0} : A^{p,q}(E) \to A^{p+1,q}(E), \quad \nabla^{0,1} : A^{p,q}(E) \to A^{p,q+1}(E).$$

Hence

$$\nabla^2 = \nabla^{1,0}\circ\nabla^{1,0} + (\nabla^{1,0}\circ\nabla^{0,1} + \nabla^{0,1}\circ\nabla^{1,0}) + \nabla^{0,1}\circ\nabla^{0,1},$$

where

$(\nabla^{1,0})^2 \in A^{2,0}(\mathrm{End}(E))$, $\nabla^{1,0} \circ \nabla^{0,1} + \nabla^{0,1} \circ \nabla^{1,0} \in A^{1,1}(\mathrm{End}(E))$, $(\nabla^{0,1})^2 \in A^{0,2}(\mathrm{End}(E))$.

In the language of differential forms, we have the corresponding decompositions

$$\omega = \omega^{1,0} + \omega^{0,1}, \quad \Omega = \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}.$$

The complex vector bundle which admits a holomorphic structure is characterised as follows:

Proposition 1 (Newlander-Nirenberg) Let E be a C^∞ complex vector bundle over a complex manifold M. Then E admits a holomorphic structure, i.e. the transition functions are holomorphic, if and only if there exists a connection $\nabla = \nabla^{0,1} + \nabla^{1,0}$ such that

$$(\nabla^{0,1})^2 = \nabla^{0,1} \circ \nabla^{0,1} = 0.$$

Furthermore, we have $\nabla^{0,1} = \bar{\partial}$.

The proof of this standard result may be found in any textbook on differential geometry.

Let E be a C^∞ complex vector bundle over a (real or complex) manifold M. A **hermitian metric** ρ on E is a C^∞ hermitian inner product on the fibers of E; usually, we write this as a pair (E, ρ). Given a local frame field $s_U = (s_1, \ldots, s_r)$ of E on U, we let

$$h_{i\bar{j}} := (s_i, s_j)_\rho, \text{ and } H_U := (h_{i\bar{j}}).$$

Then H_U is a positive definite hermitian matrix at each point of U. We say that s_U is a **unitary field** or an **orthonormal frame field** if H_U is the identity matrix. Under a change of local frame fields $s_U = s_V\, g$, where $g = g_{UV}$ is the transition function, we have $H_U = g^t\, H_V\, \bar{g}$.

A connection ∇ of (E, ρ) is called a **hermitian connection** if ∇ preserves ρ (or makes ρ parallel) in the following sense:

$$d(\xi, \eta)_\rho = (\nabla\xi, \eta)_\rho + (\xi, \nabla\eta)_\rho \tag{9}$$

for any $\xi, \eta \in A(E)$.

Proposition 2. Let (E, ρ) be a holomorphic hermitian vector bundle on a complex manifold M. Then there exists a unique hermitian connection which preserves the holomorphic structure on E with respect to ρ. We call this connection the **canonical connection** of (E, ρ). The curvature of this canonical connection is of type $(1,1)$.

From now on, we will assume that every vector bundle is a holomorphic vector bundle and that the connection for a hermitian vector bundle is the canonical connection. We will also use the terminology of vector sheaves for locally free sheaves, and will make no difference when we use vector bundles and vector sheaves in the sequel.

I.1.1.b. **Characteristic Forms**

We start with a fact from algebra: Let $B \subset \mathbf{R}$ be a subring, and let $\phi \in B[[T_1, \ldots, T_r]]$ be any symmetric power series. For every $k \geq 0$; let $\phi_{[k]}$ be the degree k homogeneous component of ϕ. Then there exists a unique polynomial map

$$\Phi_{[k]} : M_r(\mathbf{C}) \to \mathbf{C}$$

such that

(1) $\Phi_{[k]}$ is invariant under the conjugation of $\mathrm{GL}_n(\mathbf{C})$.
(2) $\Phi_{[k]}(\mathrm{diag}(a_1, \ldots, a_r)) = \phi_{[k]}(a_1, \ldots, a_r)$.

More generally, for any B-algebra A, we define

$$\Phi = \oplus_{k\geq 0}\Phi_{[k]} : M_r(A) \to A.$$

Furthermore, if I is a nilpotent subalgebra of A, then we may also define

$$\Phi = \oplus_{k\geq 0}\Phi_{[k]} : M_r(I) \to A.$$

Thus, if $(\mathcal{E}, \rho)$ is a hermitian vector sheaf of rank r on a complex manifold M, and ϕ is as above, we define

$$\phi(\mathcal{E}, \rho) := \Phi(-R_{\mathcal{E},\rho}) \in A(M)$$

as follows:

First, identify $\mathrm{End}(\mathcal{E})$ with $M_r(\mathbf{C})$ locally and then apply the construction above to

$$I := \oplus_{p\geq 1} A^{p,p}(M).$$

Note that by the results in the previous subsection, especially the equality a.8, we know that the above procedure for $\phi(\mathcal{E}, \rho)$ is well-defined, since Φ is invariant under the conjugation. Moreover, we have the following

Proposition 1. With the same notation as above,
(1) $\phi(\mathcal{E}, \rho)$ is a closed form on M, i.e. $d\,\phi(\mathcal{E}, \rho) = 0$.
(2) For any morphism $f : N \to M$, $f^*(\phi(\mathcal{E}, \rho)) = \phi(f^*\mathcal{E}, f^*\rho)$.
(3) The de Rham cohomology class of $\phi(\mathcal{E}, \rho)$ *does not* depend on the choice of ρ, but the form $\phi(\mathcal{E}, \rho)$ itself *does* depend on ρ.

The proof of this proposition can be found in any standard textbook which contains the theory of characteristic forms. (For example, assertion 1 comes from the Bianchi identity.)

Here, instead of giving a proof, we would like to mention the following fact: Part **3** of this proposition is the starting point to introduce the whole story in this book: We understand everything at the level of differential forms rather than at the level of cohomology classes. Roughly speaking, the refined version of the characteristic form $\phi(\mathcal{E},\rho)$ for finite dimensional vector sheaves and infinite dimensional vector sheaves, i.e. the classical Bott-Chern secondary characteristic forms and the relative Bott-Chern secondary characteristic objects for both smooth morphisms and closed immersions are the central parts of our theory.

§I.1.2. Classical Bott-Chern Secondary Characteristic Forms

I.1.2.a. Axioms for Classical Bott-Chern Secondary Characteristic Forms

From above, we know that for any hermitian vector sheaf $(\mathcal{E},\rho)$ on a complex manifold M, we can define the associated characteristic form for any symmetric power series ϕ. We also know that the de Rham cohomology class of this form does not depend on the choice of the metric, but the form itself does depend on the metric. With this in mind, the first thing we have to understand is how the characteristic forms change with respect to hermitian metrics. Around 1968, Bott and Chern first solved this problem by considering the second order partial differential equation:

$$d_M d_M^c \eta = \phi(\mathcal{E},\rho) - \phi(\mathcal{E},\rho').$$

They found that in fact one can solve this differential equation in $\tilde{A}(M)$. Hence, they gave the classical Bott-Chern secondary characteristic forms [BC 68].

Now we introduce axioms for the **classical Bott-Chern secondary characteristic form**, $\phi_{\mathrm{BC}}(\mathcal{E}.,\rho.)$, with respect to any power series ϕ as in section 1.a, a short exact sequence of vector sheaves

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

and hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1,2,3$: (it is worthy to mention that here it is not necessary to assume that ρ_1 and ρ_3 are induced from ρ_2.)

Axiom 1. (Downstairs Rule) Let

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

be a short exact sequence of vector sheaves over a complex manifold M with hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1,2,3$. Then, there exists an element $\phi_{\mathrm{BC}}(\mathcal{E}.,\rho.) \in \tilde{A}(M)$, such that

$$d_M d_M^c \phi_{\mathrm{BC}}(\mathcal{E}.,\rho.) = \phi(\mathcal{E}_2,\rho_2) - \phi(\mathcal{E}_1 \oplus \mathcal{E}_3, \rho_1 \oplus \rho_3)$$

holds.

Axiom 2. (Functorial Rule) For any morphism $f : N \to M$ of complex manifolds, we have

$$f^* \phi_{\mathrm{BC}}(\mathcal{E}.,\rho.) = \phi_{\mathrm{BC}}(f^*\mathcal{E}., f^*\rho.).$$

Axiom 3. **(Uniqueness Rule)** If $(\mathcal{E}.,\rho.)$ is split, i.e. $(\mathcal{E}_2,\rho_2) = (\mathcal{E}_1 \oplus \mathcal{E}_3, \rho_1 \oplus \rho_3)$, then

$$\phi_{\mathrm{BC}}(\mathcal{E}.,\rho.) = 0.$$

Among these axioms, axiom 1 is essential. Furthermore, ϕ_{BC} measures the change of the characteristic forms with respect to metrics; Indeed, with the degenerate short exact sequence obtained by letting $\mathcal{E}_3 = 0$, axiom 1 gives the Bott-Chern equation stated at the beginning of this subsection. In that case, we denote $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho.)$ by $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_1,\rho_1,\rho_2)$.

I.1.2.b. **Existence of Classical Bott-Chern Secondary Characteristic Forms**

Next we prove the following

Existence Theorem for Classical Bott-Chern Secondary Characteristic Forms. Let

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

be a short exact sequence of vector sheaves on a complex manifold M with hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1, 2, 3$. Then for any symmetric power series ϕ, there exists a unique differential form $\phi_{\mathrm{BC}}(\mathcal{E}.,\rho.) \in \tilde{A}(M)$ such that $\phi_{\mathrm{BC}}(\mathcal{E}.,\rho.)$ satisfies the axioms 1, 2, and 3 above.

Proof: There are several different ways to prove this theorem. Here we use the $\mathbf{P}^1$-deformation technique. ($\mathbf{P}^1$ means a projective line.) Other methods will be explained in the following chapters.

The basic idea of the $\mathbf{P}^1$-deformation technique is that in order to construct certain differential forms, we introduce a new parameter in $\mathbf{P}^1$ and then try to find differential forms with parameter in $\mathbf{P}^1$. Finally, we show that our forms are nothing but the integration of the forms with parameter in $\mathbf{P}^1$ over $\mathbf{P}^1$ with respect to the current $[\log|z|^2]$.

For our purpose here, we first construct an exact sequence $D\mathcal{E}.$ on $M \times \mathbf{P}^1$, called a **$\mathbf{P}^1$-deformation** of $\mathcal{E}.$ as follows:

Let s be a section of the sheaf $\mathcal{O}_{\mathbf{P}^1}(1)$, such that s vanishes at ∞ and has the value 1 at 0. Let

$$D\mathcal{E}_1 := \mathcal{E}_1(1) := \mathcal{E}_1 \otimes \mathcal{O}_{\mathbf{P}^1}(1), \quad D\mathcal{E}_2 := (\mathcal{E}_2 \oplus \mathcal{E}_1(1))/\mathcal{E}_1, \quad D\mathcal{E}_3 := \mathcal{E}_3$$

with the natural morphism $\mathrm{Id}_{\mathcal{E}_1} \otimes s : \mathcal{E}_1 \to \mathcal{E}_1(1)$. Then we have the following exact sequence on $M \times \mathbf{P}^1$

$$D\mathcal{E}. : \quad 0 \to D\mathcal{E}_1 \to D\mathcal{E}_2 \to D\mathcal{E}_3 \to 0.$$

For any point $z \in \mathbf{P}^1$, let $i_z : M \to M \times \mathbf{P}^1$ be a morphism, defined by $i_z(x) = (x,z)$. Then we have

(1) $i_z^* D\mathcal{E}_2 \simeq \mathcal{E}_2$, if $z \neq \infty$.
(2) $i_\infty^* D\mathcal{E}_2 \simeq \mathcal{E}_1 \oplus \mathcal{E}_3$.
(3) $i_\infty^* D\mathcal{E}_1 \simeq \mathcal{E}_1$.

Using a partition of unity, we may choose a hermitian metric $D\rho_2$ on $D\mathcal{E}_2$ in such a way that the isomorphisms (1) and (2) above become isometries. Hence, we introduce a new parameter in $\mathbf{P}^1$.

Now let
$$\phi_{\mathrm{BC}}(\mathcal{E}_\cdot, \rho_\cdot) := \int_{\mathbf{P}^1} [\log|z|^2]\, \phi(D\mathcal{E}_2, D\rho_2).$$
Since
$$d_{\mathbf{P}^1} d^c_{\mathbf{P}^1} [\log|z|^2] = \delta_0 - \delta_\infty,$$
we know that
$$d_M d_M^c \phi_{\mathrm{BC}}(\mathcal{E}_\cdot, \rho_\cdot) = i_0^* \phi(D\mathcal{E}_2, D\rho_2) - i_\infty^* \phi(D\mathcal{E}_2, D\rho_2).$$
Hence by the functorial properties of characteristic forms in Prop. 1.b.(2), we have axiom 1.

In order to check axiom 2, from the construction above, by Prop. 1.b.(2) again, it is enough to prove that, in $\tilde{A}(M)$, the above construction does not depend on the choice of the metric $D\rho_2$ on $D\mathcal{E}_2$. Suppose there exists another choice $D\rho'_2$. Consider the product $M \times \mathbf{P}^1 \times \mathbf{P}^1$ with points (y, z, u). We have the following natural maps:
$$M \times \mathbf{P}^1 \overset{i_u^{12}}{\hookrightarrow} M \times \mathbf{P}^1 \times \mathbf{P}^1 \overset{p_{12}}{\rightarrow} M \times \mathbf{P}^1,$$
and
$$M \times \mathbf{P}^1 \overset{i_z^{13}}{\hookrightarrow} M \times \mathbf{P}^1 \times \mathbf{P}^1 \overset{p_{13}}{\rightarrow} M \times \mathbf{P}^1,$$
with
$$i_u^{12}(y,z) := (y,z,u), \quad p_{12}(y,z,u) := (y,z),$$
$$i_z^{13}(y,u) := (y,z,u), \quad p_{13}(y,z,u) := (y,u).$$
Also let $p_1 : M \times \mathbf{P}^1 \to M$ be the projection to the first factor. Then on the bundle $p_{12}^* D\mathcal{E}_2$, we may find a metric τ such that

(1) $(i_0^{12})^*(p_{12}^* D\mathcal{E}_2, \tau) \simeq (D\mathcal{E}_2, D\rho_2)$;
(2) $(i_\infty^{12})^*(p_{12}^* D\mathcal{E}_2, \tau) \simeq (D\mathcal{E}_2, D\rho'_2)$;
(3) $(i_0^{13})^*(p_{12}^* D\mathcal{E}_2, \tau) \simeq p_1^*(\mathcal{E}_2, \rho_2)$;
(4) $(i_\infty^{13})^*(p_{12}^* D\mathcal{E}_2, \tau) \simeq p_1^*(\mathcal{E}_1 \oplus \mathcal{E}_3, \rho_1 \oplus \rho_3)$.

Hence,
$$\begin{aligned}
&\int_{\mathbf{P}^1} [\log|z|^2]\, \phi(D\mathcal{E}_2, D\rho_2) - \int_{\mathbf{P}^1} [\log|z|^2]\, \phi(D\mathcal{E}_2, D\rho'_2) \\
&= \int_{\mathbf{P}^1} [\log|z|^2]\, (\phi(D\mathcal{E}_2, D\rho_2) - \phi(D\mathcal{E}_2, D\rho'_2)) \\
&= \int_{\mathbf{P}^1} [\log|z|^2]\, (\phi((i_0^{12})^*(p_{12}^* D\mathcal{E}_2, \tau)) - \phi((i_\infty^{12})^*(p_{12}^* D\mathcal{E}_2, \tau))).
\end{aligned}$$

Thus by Stokes' formula, we have

$$\begin{aligned}&\int_{\mathbf{P}^1}[\log|z|^2]\,\phi(D\mathcal{E}_2,D\rho_2)-\int_{\mathbf{P}^1}[\log|z|^2]\,\phi(D\mathcal{E}_2,D\rho_2')\\&=\int_{\mathbf{P}^1_z\times\mathbf{P}^1}[\log|z|^2]\,[\log|u|^2](d_u d_u^c(\phi(p_{12}^*D\mathcal{E}_2,\tau))).\end{aligned}$$

But if we let $\partial=\partial_M+\partial_z+\partial_u$ and $\bar\partial=\bar\partial_M+\bar\partial_z+\bar\partial_u$ be the differentials on $M\times\mathbf{P}^1\times\mathbf{P}^1$, then by the fact that characteristic forms are d closed, we have

$$\begin{aligned}&\int_{\mathbf{P}^1\times\mathbf{P}^1}[\log|z|^2]\,[\log|u|^2](d_u d_u^c(\phi(p_{12}^*D\mathcal{E}_2,\tau)))\\&\qquad=\int_{\mathbf{P}^1\times\mathbf{P}^1}[\log|z|^2]\,[\log|u|^2](d_z d_z^c(\phi(p_{12}^*D\mathcal{E}_2,\tau))).\end{aligned}$$

Thus, using Stokes' formula again, we have

$$\begin{aligned}&\int_{\mathbf{P}^1}[\log|z|^2]\,\phi(D\mathcal{E}_2,D\rho_2)-\int_{\mathbf{P}^1}[\log|z|^2]\,\phi(D\mathcal{E}_2,D\rho_2')\\&=\int_{\mathbf{P}^1}[\log|u|^2]\,(\phi((i_0^{13})^*(p_{12}^*D\mathcal{E}_2,\tau))-\phi((i_\infty^{13})^*(p_{12}^*D\mathcal{E}_2,\tau)))\\&=\int_{\mathbf{P}^1}[\log|u|^2]\,(p_1^*(\phi(\mathcal{E}_2,\rho_2)-\phi(\mathcal{E}_1\oplus\mathcal{E}_3,\rho_1\oplus\rho_3)))\\&=\int_{\mathbf{P}^1}[\log|u|^2]\,(p_1^*(\phi(\mathcal{E}_2,\rho_2)-\phi(\mathcal{E}_1\oplus\mathcal{E}_3,\rho_1\oplus\rho_3)))\\&=0.\end{aligned}$$

Here, in the last step, we use the fact that

$$p_1^*(\phi(\mathcal{E}_2,\rho_2)-\phi(\mathcal{E}_1\oplus\mathcal{E}_3,\rho_1\oplus\rho_3))$$

is a constant form with resprct to $\mathbf{P}^1$. So we have axiom 2.

The proof of axiom 3 is rather simple, since in the case that $(\mathcal{E}.,\rho.)$ is split, we may choose a metric $D\rho_2$ which does not depend on z.

Finally, we have to prove the uniqueness. For this, let us start from the exact sequence $D\mathcal{E}.$. By axiom 1, we know that

$$d_{M\times\mathbf{P}^1}d^c_{M\times\mathbf{P}^1}\phi_{\mathrm{BC}}(D\mathcal{E}.,D\rho.)=\phi(D\mathcal{E}_2,D\rho_2)-\phi(D\mathcal{E}_1\oplus D\mathcal{E}_3,D\rho_1\oplus D\rho_3).$$

Hence we have

$$\begin{aligned}&\int_{\mathbf{P}^1}[\log|z|^2]\,d_{M\times\mathbf{P}^1}d^c_{M\times\mathbf{P}^1}\phi_{\mathrm{BC}}(D\mathcal{E}.,D\rho.)\\&=\int_{\mathbf{P}^1}[\log|z|^2]\,\phi(D\mathcal{E}_2,D\rho_2)-\int_{\mathbf{P}^1}[\log|z|^2]\,\phi(D\mathcal{E}_1\oplus D\mathcal{E}_3,D\rho_1\oplus D\rho_3)\end{aligned}$$

But the last term does not change if we change z to z^{-1}, hence it is zero. Therefore, we have

$$\begin{aligned}
&\int_{\mathbf{P}^1} [\log|z|^2]\, d_{M\times\mathbf{P}^1} d^c_{M\times\mathbf{P}^1} \phi_{\mathrm{BC}}(D\mathcal{E}., D\rho.)\\
&= \int_{\mathbf{P}^1} [\log|z|^2]\, d_z d^c_z \phi_{\mathrm{BC}}(D\mathcal{E}., D\rho.)\\
&= i_0^*\phi_{\mathrm{BC}}(D\mathcal{E}., D\rho.) - i_\infty^*\phi_{\mathrm{BC}}(D\mathcal{E}., D\rho.)\\
&= \int_{\mathbf{P}^1} \log|z|^2\, \phi(D\mathcal{E}_2, D\rho_2).
\end{aligned}$$

Here, in the last step, we use that fact that $(D\mathcal{E}., D\rho.)$ is split at infinity. So finally, by axioms 2, 3, we know that the classical Bott-Chern secondary characteristic form associated with $(\mathcal{E}., \rho.)$ is the one constructed above.

I.1.2.c. **Properties of Classical Bott-Chern Secondary Characteristic Forms**

In this subsection, we discuss the classical Bott-Chern secondary characteristic forms in more detail.

Theorem. (1) Let ϕ_1, ϕ_2 be two symmetric power series in $\mathbf{C}[[T_1, \ldots, T_n]]$ and let

$$\mathcal{E}. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

be a short exact sequence of vector sheaves on a complex manifold with hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1, 2, 3$. Then

$$\begin{aligned}
(\phi_1 + \phi_2)_{\mathrm{BC}}(\mathcal{E}., \rho.) &= \phi_{1\,\mathrm{BC}}(\mathcal{E}., \rho.) + \phi_{2\,\mathrm{BC}}(\mathcal{E}., \rho.);\\
(\phi_1\phi_2)_{\mathrm{BC}}(\mathcal{E}., \rho.) &= \phi_{1\,\mathrm{BC}}(\mathcal{E}., \rho.)\phi_2(\mathcal{E}_2, \rho_2) + \phi_1(\mathcal{E}_1 \oplus \mathcal{E}_3, \rho_1 \oplus \rho_3)\phi_{2\,\mathrm{BC}}(\mathcal{E}., \rho.)\\
&= \phi_{1\,\mathrm{BC}}(\mathcal{E}., \rho.)\phi_2(\mathcal{E}_1 \oplus \mathcal{E}_3, \rho_1 \oplus \rho_3) + \phi_1(\mathcal{E}_2, \rho_2)\phi_{2\,\mathrm{BC}}(\mathcal{E}., \rho.).
\end{aligned}$$

(2) Let ϕ be a symmetric power series in n variables, and let $\phi_\alpha(T_1, \ldots, T_{n_1})$ and $\varphi_\alpha(T_{n_1+1}, \ldots, T_n)$ be the symmetric power series defined by

$$\phi(T_1, \ldots, T_n) = \sum_\alpha \phi_\alpha(T_1, \ldots, T_{n_1})\varphi_\alpha(T_{n_1+1}, \ldots, T_n).$$

Let

$$\mathcal{E}_{.,i} : 0 \to \mathcal{E}_{1,i} \to \mathcal{E}_{2,i} \to \mathcal{E}_{3,i} \to 0$$

be a short exact sequence of vector sheaves on a complex manifold with hermitian metrics $\rho_{j,i}$ on $\mathcal{E}_{j,i}$ and $\mathrm{rk}(\mathcal{E}_{2i}) = n_i$, for $j = 1, 2, 3$, $i = 1, 2$. Then

$$\begin{aligned}
&\phi_{\mathrm{BC}}(\mathcal{E}_{.1} \oplus \mathcal{E}_{.,2}, \rho_{.,1} \oplus \rho_{.,2})\\
&= \sum_\alpha [\phi_{\alpha\,\mathrm{BC}}(\mathcal{E}_{.,1}, \rho_{.,1})\varphi_\alpha(\mathcal{E}_{2,2}, \rho_{2,2}) + \phi_\alpha(\mathcal{E}_{1,1} \oplus \mathcal{E}_{3,1}, \rho_{1,1} \oplus \rho_{3,1})\varphi_{\alpha\,\mathrm{BC}}(\mathcal{E}_{.,2}, \rho_{.,2})].
\end{aligned}$$

(3) Let

$$\mathcal{E}_. : 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

be a short exact sequence of vector sheaves on a complex manifold with hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1,2,3$, and $\mathrm{rk}(\mathcal{E}_2) = n_1$. Let $(\mathcal{F}, \tau)$ be a hermitian vector sheaf with $\mathrm{rk}(\mathcal{F}) = n_2$, and ϕ a symmetric power series in $n_1 n_2$ variables. Define the symmetric pdwer series ϕ_β, φ_β in n_1, n_2 variables, respectively, by

$$\begin{aligned}\phi(T_1 + U_1, \dots, T_{n_1} + U_1, \dots, T_1 + U_{n_2}, \dots, T_{n_1} + U_{n_2}) \\ =: \sum_\beta \phi_\beta(T_1, \dots, T_{n_1})\varphi_\beta(U_1, \dots, U_{n_2}).\end{aligned}$$

Then

$$\phi_{\mathrm{BC}}(\mathcal{E}_. \otimes \mathcal{F}, \rho_. \otimes \tau) = \sum_\beta \phi_{\beta\,\mathrm{BC}}(\mathcal{E}_., \rho_.)\varphi_\beta(\mathcal{F}, \tau).$$

(4) (**Nine Diagram**) Let

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E}_{11} & \to & \mathcal{E}_{12} & \to & \mathcal{E}_{13} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E}_{21} & \to & \mathcal{E}_{22} & \to & \mathcal{E}_{23} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E}_{31} & \to & \mathcal{E}_{32} & \to & \mathcal{E}_{33} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

be a commutative diagram of vector sheaves with lines $\mathcal{E}_{j.}$ and columns $\mathcal{E}_{.i}$ exact. Let ρ_{ji} be hermitian metrics on $\mathcal{E}_{ji}$ for $i, j = 1,2,3$; and let ϕ be a symmetric power series in $n = \mathrm{rk}(\mathcal{E}_{22})$ variables. Then

$$\phi_{\mathrm{BC}}(\mathcal{E}_{2.}, \rho_{2.}) - \phi_{\mathrm{BC}}(\mathcal{E}_{1.} \oplus \mathcal{E}_{3.}, \rho_{1.} \oplus \rho_{3.}) = \phi_{\mathrm{BC}}(\mathcal{E}_{.2}, \rho_{.2}) - \phi_{\mathrm{BC}}(\mathcal{E}_{.1} \oplus \mathcal{E}_{.3}, \rho_{.1} \oplus \rho_{.3}).$$

Proof. (1) By the facts that, for any hermitian vector sheaf $(\mathcal{E}, \rho)$,

$$\begin{aligned}(\phi_1 + \phi_2)(\mathcal{E}, \rho) &= \phi_1(\mathcal{E}, \rho) + \phi_2(\mathcal{E}, \rho), \\ (\phi_1\phi_2)(\mathcal{E}, \rho) &= \phi_1(\mathcal{E}, \rho)\phi_2(\mathcal{E}, \rho),\end{aligned}$$

we know that both sides of the equalities satisfy the axioms for classical Bott-Chern secondary characteristic forms. By uniqueness, we have the assertion.

(2) Define

$$\phi_{\mathrm{BC}}(1,2) := \sum_\alpha [\phi_{\alpha\,\mathrm{BC}}(\mathcal{E}_{.,1}, \rho_{.,1})\varphi_\alpha(\mathcal{E}_{2,2}, \rho_{2,2}) + \phi_\alpha(\mathcal{E}_{1,1} \oplus \mathcal{E}_{3,1}, \rho_{1,1} \oplus \rho_{3,1})\varphi_{\alpha\,\mathrm{BC}}(\mathcal{E}_{.,2}, \rho_{.,2})].$$

With the same notation as in the construction of the previous subsection, we know that

$$\int_{\mathbf{P}^1}[\log|z|^2]\,\phi(D\mathcal{E}_{1,1}\oplus D\mathcal{E}_{3,1}\oplus D\mathcal{E}_{1,2}\oplus D\mathcal{E}_{3,2}, D\rho_{1,1}\oplus D\rho_{3,1}\oplus D\rho_{1,2}\oplus D\rho_{3,2})=0$$

in $\tilde{A}(M)$. In fact, the above expression is invariant on replacing z by $1/z$. Furthermore, by the fact that

$$\phi(\mathcal{E}_{2,1}\oplus\mathcal{E}_{2,2},\rho_{2,1}\oplus\rho_{2,2})=\sum_{\alpha}\phi_\alpha(\mathcal{E}_{2,1},\rho_{2,1})\varphi_\alpha(\mathcal{E}_{2,2},\rho_{2,2}),$$

we know that

$$\begin{aligned}
&dd^c\phi_{\mathrm{BC}}(1,2)\\
&=\sum_{\alpha}[\phi_\alpha(\mathcal{E}_{2,1},\rho_{2,1})\varphi_\alpha(\mathcal{E}_{2,2},\rho_{2,2})+\phi_\alpha(\mathcal{E}_{1,1}\oplus\mathcal{E}_{3,1},\rho_{1,1}\oplus\rho_{3,1})\varphi_\alpha(\mathcal{E}_{1,2}\oplus\mathcal{E}_{3,2},\rho_{1,2}\oplus\rho_{3,2})]\\
&=\phi(\mathcal{E}_{2,1}\oplus\mathcal{E}_{2,2},\rho_{2,1}\oplus\rho_{2,2})-\phi(\mathcal{E}_{1,1}\oplus\mathcal{E}_{1,3}\oplus\mathcal{E}_{1,2}\oplus\mathcal{E}_{3,2},\rho_{1,1}\oplus\rho_{1,3}\oplus\rho_{1,2}\oplus\rho_{3,2}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\phi_{\mathrm{BC}}(\mathcal{E}_{\cdot,1}\oplus\mathcal{E}_{\cdot,2},\rho_{\cdot,1}\oplus\rho_{\cdot,2})\\
&=\int_{\mathbf{P}^1}[\log|z|^2]\,\phi(D\mathcal{E}_{2,1}\oplus D\mathcal{E}_{2,2},D\rho_{2,1}\oplus D\rho_{2,2})\\
&=\int_{\mathbf{P}^1}[\log|z|^2]\,dd^c\phi_{\mathrm{BC}}(1,2)\\
&=\int_{\mathbf{P}^1}dd^c[\log|z|^2]\,\phi_{\mathrm{BC}}(D1,D2)\\
&=i_0^*\phi_{BC}(D1,D2)-i_\infty^*\phi_{\mathrm{BC}}(D1,D2)\\
&=\phi_{\mathrm{BC}}(1,2)-0=\phi_{\mathrm{BC}}(1,2).
\end{aligned}$$

The results in 3 and 4 are direct consequences of the construction stated in the last subsection. We leave the verification to the reader. In particular, we have the following

Corollary. With the same notation as above, we have

1. If $(\mathcal{L},\tau)$ is a hermitian line sheaf,

$$\phi_{\mathrm{BC}}(\mathcal{E}.\otimes\mathcal{L},\rho.\otimes\tau)=\sum_{i\geq 0}\phi_{i\,\mathrm{BC}}(\mathcal{E}.,\rho.)c_1(\mathcal{L},\tau)^i.$$

Here ϕ_i is defined by the relation:

$$\phi(T_1+T,\ldots,T_n+T)=:\sum_i\phi_i(T_1,\ldots,T_n)T^i.$$

2. $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho_1,\rho_3)=\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho_1,\rho_2)+\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho_2,\rho_3)$.

§I.1.3. Superconnections

In the above discussion, the key point is that for a given hermitian (holomorphic) vector bundle $(\mathcal{E},\rho)$, there exists a unique canonical connection $\nabla_{\mathcal{E},\rho}$. From this, we obtain the associated curvature form. By the local invariant property under conjugation, we construct the characteristic forms, which are global differential forms on M. In order to go further, it is very important for us to find what is the main ingredient in the definition of connections. It has only recently been discovered by Quillen that a key point in the definition of connections is that the connection is a special odd endomorphism of $A(M,\mathcal{E})$ which satisfies the Leibniz rule. Thus we may generalize definitions from connections to these for superconnections. This process is not difficult to understand formally, but it is very powerful and mathematicians have spent several decades to discover it.

Let $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$ be a superbundle on a complex manifold M, i.e. it is a $\mathbf{Z}_2$-graded vector sheaf. Let $A(M,\mathcal{E})$ be the space of $\mathcal{E}$-valued differential forms on M. This space has a natural $\mathbf{Z}$-grading given by the degree of differential forms. Usually, we will denote the degree i component of a differential form α by $\alpha_{[i]}$. Also if let

$$A^{\pm}(M,\mathcal{E}):=\sum_i A^{2i}(M,\mathcal{E}^{\pm})\oplus\sum_i A^{2i+1}(M,\mathcal{E}^{\mp}),$$

we have the total $\mathbf{Z}_2$-grading on $A(M,\mathcal{E})$ as follows:

$$A(M,\mathcal{E})=A^+(M,\mathcal{E})\oplus A^-(M,\mathcal{E}).$$

By definition, a **superconnection** $\mathbf{A}$ on a supervector sheaf $\mathcal{E}$ is an odd first-order differential operator

$$\mathbf{A}:A^{\pm}(M,\mathcal{E})\to A^{\mp}(M,\mathcal{E})$$

which satisfies the Leibniz rule in the $\mathbf{Z}_2$-graded sense: If $\alpha\in A(M)$ and $\theta\in A(M,\mathcal{E})$, then

$$\mathbf{A}(\alpha\wedge\theta)=d\alpha\wedge\theta+(-1)^{|\alpha|}\alpha\wedge\mathbf{A}\theta.$$

Here $|\alpha|$ denotes the degree of α.

As usual, we also define the **curvature** of a superconnection $\mathbf{A}$ to be the operator $\mathbf{A}^2$ on $A(M,\mathcal{E})$.

It is not difficult to prove the following

Proposition 1. Let $\mathcal{E}$ be a supervector sheaf on a complex manifold M. Let $\mathbf{A}$ be a superconnection on $\mathcal{E}$. Then

(a) The operator $\mathbf{A}_{[1]}$ is a covariant derivative on $\mathcal{E}$ which preserves the sub-sheaves $\mathcal{E}^+$ and $\mathcal{E}^-$. The operators $\mathbf{A}_{[i]}$ for $i\neq 1$ are given by the action of differential forms $\omega_{[i]}\in A(M,\mathrm{End}(\mathcal{E}))$ on $A(M,\mathcal{E})$, where $\omega_{[i]}\in A(M,\mathrm{End}^-(\mathcal{E}))$ if i is even, and $\omega_{[i]} \in A(M, \mathrm{End}^+(\mathcal{E}))$ if i is odd. We call $\mathbf{A}_{[1]}$ the **covariant derivative component** of $\mathbf{A}$.

(b) The space of superconnections on $\mathcal{E}$ is an affine space modelled on the vector space $A^-(M, \mathrm{End}(\mathcal{E}))$.

(c) If $\mathbf{A}_s$ is a smooth one-parameter family of superconnections on $\mathcal{E}$, $d\mathbf{A}_s/ds$ lies in $A^-(M, \mathrm{End}(\mathcal{E}))$.

In order to use superconnections to define characteristic forms, we need the supertrace mapping on the space $A(M, \mathrm{End}(\mathcal{E}))$. By definition, the **supertrace** map

$$\mathrm{Tr}_s : A(M, \mathrm{End}(\mathcal{E})) \to A(M)$$

is defined locally as follows:

On each fiber of $\mathrm{End}(\mathcal{E})$ on x, $\mathrm{Tr}_{s,x} : \mathrm{End}(\mathcal{E})_x \to \mathbf{C}$ is defined by

$$\mathrm{Tr}_{s,x} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \mathrm{Tr}(a) - \mathrm{Tr}(d).$$

Since the algebra $A(M)$ is supercommutative, this map vanishes on the supercommutators and preserves the $\mathbf{Z}_2$-gradings.

Now let ϕ be a power series as in section 1.a. For any superconnection $\mathbf{A}$ on a supervector sheaf $\mathcal{E}$, we know that $\phi(-\mathbf{A}^2)$ is in $A^+(M, \mathrm{End}(\mathcal{E}))$, since $\mathbf{A}^2$ is in $A^+(M, \mathrm{End}(\mathcal{E}))$. Thus $\mathrm{Tr}_s[\phi(-\mathbf{A}^2)]$ is an element in $A^+(M)$. We denote this element by $\mathrm{Tr}_s[\phi(\mathcal{E}, \mathbf{A})]$, and call this differential form the **complex characteristic form** of $\mathbf{A}$ with respect to the power series ϕ. Here, we need to make the following remark. Classically, as we consider the problem in the integral cohomology theory, for a covariant derivative ∇, we define the characteristic form by $\mathrm{Tr}_s\phi(-\frac{1}{2\pi i}\nabla^2)$. But when we consider the superconnection formalism in the sense of index theorem, it is not quite natural, since now we may meet certain scalar factors for appropriated degrees. To deal with this difference, we may usually introduce the following operator on forms (or on currents):

Let M be a complex manifold. Define $[2\pi \mathrm{i}]$ as an operator on $\oplus A^{p,p}(M)$ such that

$$[2\pi \mathrm{i}] \sum_p \omega_{[p]} := \sum_p (\frac{1}{2\pi i})^p \omega_{[p]},$$

for any element $\sum_p \omega_{[p]} \in \oplus_p A^{p,p}(M)$ with $\omega_{[p]} \in A^{p,p}(M)$.

We list the most important properties of such a differential form in the following

Proposition 2. Let $\mathbf{A}$ be a superconnection of a supervector sheaf $\mathcal{E}$ on a complex manifold M. Then

1. The characteristic form $\mathrm{Tr}_s[\phi(\mathcal{E}, \mathbf{A})]$ is a closed differential form of even degree.
2. (Transgression Formula) If $\mathbf{A}_t$ is a differentiable one-parameter family of superconnections on $\mathcal{E}$,

$$\frac{d}{dt}\mathrm{Tr}_s[\phi(\mathcal{E}, \mathbf{A}_t)] = d\,\mathrm{Tr}_s[\frac{d\mathbf{A}_t}{dt}\phi'(\mathcal{E}, \mathbf{A}_t)].$$

3. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are two superconnections on $\mathcal{E}$, then the differential forms

$$\mathrm{Tr}_s[\phi(\mathcal{E},\mathbf{A})],\quad \mathrm{Tr}_s[\phi(\mathcal{E},\mathbf{A})]$$

lie in the same de Rham cohomology class.

For the proof of all these properties, we only need to know that Tr_s vanishes on the supercommutators and for any $\alpha \in \mathcal{A}(M, \mathrm{End}(\mathcal{E})), d(\mathrm{Tr}_s\alpha) = \mathrm{Tr}_s([\mathbf{A},\alpha])$. Later, we will see that superconnections are very powerful.

Chapter I.2
Relative Bott-Chern Secondary Characteristic Forms For Smooth Morphisms I: Axioms

From the previous chapter, we see that the classical Bott-Chern secondary characteristic form measures the change of characteristic forms with respect to the change of metrics. What should be the corresponding objects in the relative case? To explain this in more detail, we recall the classical Grothendieck-Riemann-Roch theorem in algebraic geometry.

Let $f : X \to Y$ be a smooth morphism of regular algebraic varieties. Then for any vector sheaf $\mathcal{E}$ on X, we may define the push-out morphism of $\mathcal{E}$ in the sense of K-theory: $f_K(\mathcal{E}) := \sum_j (-1)^j R^i f_*(\mathcal{E})$. Then, at the cohomology class level, we have the following Grothendieck-Riemann-Roch theorem:

$$f_{\mathrm{CH}}(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(\mathcal{T}_f)) = \mathrm{ch}(f_K(\mathcal{E})),$$

where f_{CH} is the natural push-out morphism of algebraic cycles, td is the Todd characteristic class, and $\mathcal{T}_f$ is the relative tangent sheaf of f.

The first observation towards the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms comes from the following fact: At the level of differential forms, the similar equality

$$f_{\mathrm{CH}}(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(\mathcal{T}_f)) = \mathrm{ch}(f_K(\mathcal{E}))$$

no longer holds in general. More precisely, now we may assume that $\mathcal{E}$ is f-acyclic, that is, the higher direct images of $\mathcal{E}$ with respect to f vanish, i.e. $R^i f_* \mathcal{E} = 0$ for $i > 0$. Then $f_*\mathcal{E}$ is a vector sheaf on Y. Furthermore, with respect to a hermitian metric ρ_f on the relative tangent sheaf of f, if ρ is a hermitian metric on $\mathcal{E}$, we may naturally define the push-out matric $f_*\rho$ on $f_*\mathcal{E}$. In this way, we may get two differential forms on Y:

$$\mathrm{ch}(f_*\mathcal{E}, f_*\rho)$$

and

$$f_*(\mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T}_f,\rho_f)).$$

As an easily corollary of the Grothendieck-Riemann-Roch theorem, at the level of de Rham cohomology classes, i.e. modulo the d-exact forms, we know that they are just the same.

However, if we consider the problem at the level of differential forms, we see that the difference of these two differential forms is usually not zero. So, as in the classical (absolute) situation, we may ask how one can measure the difference of these two differential forms on Y. As one may imagine, the supposed concept about relative Bott-Chern secondary characteristic forms are distributed to measure this difference. In this sense, a key axiom of the relative Bott-Chern secondary characteristic form for $(\mathcal{E},\rho;f,\rho_f)$, denoted as $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,\rho_f)$, should be the following equation:

$$d_Y d_Y^c \mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,\rho_f) = \\ f_*(\mathrm{ch}(\mathcal{E},\rho)\mathrm{td}(\mathcal{T}_f,\rho_f)) - \mathrm{ch}(f_*\mathcal{E},f_*\rho).$$

Another way to think of the above problem is that in the hermitian K-theory, the direct image $(f_*\mathcal{E},f_*\rho)$ is not a good definition for $f_K(\mathcal{E},\rho)$, when $\mathcal{E}$ is f-acyclic, since by checking certain concrete examples such as Riemann surfaces or projective spaces, the metric $f_*\rho$ is not the right one. So we need to introduce a new metric $\rho_{f,\mathrm{RR}}$ on $f_*\mathcal{E}$, the Riemann-Roch metric associated to $(\mathcal{E},\rho;f,\rho_f)$, so that it is compatible with the refined Riemann-Roch theorem at the level of differential forms, and it should also induce the Quillen metric on the corresponding determinant line sheaf. Suppose such a metric does exist, we now may think of the reletive Bott-Chern secondary characteristic forms with respect to smooth forms as a measure for the change of Chern forms from the Riemann-Roch metric to the push-out matric on $f_*\mathcal{E}$.

This chapter consists of two sections. In the first one, we introduce the axioms for the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms. In the second, we give the existence theorem for them.

§I.2.1. Axioms Of Bott-Chern Secondary Characteristic Forms With Respect To Smooth Morphisms

I.2.1.a. Downstairs Rule

Let $f: M \to N$ be a smooth morphism of Kähler manifolds with a hermitian metric ρ_f on the relative tangent sheaf $\mathcal{T}_f$. Let $(\mathcal{E},\rho)$ be an f-acyclic vector sheaf on M. Then by the Grothendieck-Riemann-Roch theorem, we have the formula:

$$f_*(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(\mathcal{T}_f)) = \mathrm{ch}\,(f_*\mathcal{E}).$$

This formula only holds at the level of cohomology classes. Usually, if we consider the situation at the level of differential forms, the difference of the two differential forms

$$f_*(\mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T},\rho_f)) - \mathrm{ch}\,f_*(\mathcal{E},\rho)$$

is not 0. It is natural to ask how we can measure such a difference. In general, motivated by the Quillen metric on the determinant line sheaf, we also want to introduce a good metric on $f_*\mathcal{E}$, when $\mathcal{E}$ is f-acyclic. In this sense, the relative Bott-Chern secondary characteristic form measures the change of the Chern characteristic forms with respect to the new metric and the natural L^2- metric on $f_*\mathcal{E}$.

From the remark above, we are led to coinsider the following axiom for the relative Bott-Chern secondary characteristic form $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho,f,\rho_f)$ on N:

Axiom 1. (**Downstairs Rule**) Let $f : M \to N$ be a smooth morphism of Kähler manifolds with a hermitian metric ρ_f on the relative tangent sheaf $\mathcal{T}_f$. Suppose $(\mathcal{E},\rho)$ is an f-acyclic hermitian vector sheaf on M, then there exists an element $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho,f,\rho_f) \in \tilde{A}(N)$, such that

$$\begin{aligned} d_N d_N^c \mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho,f,\rho_f) = \\ f_*(\mathrm{ch}(\mathcal{E},\rho)\,\mathrm{td}(\mathcal{T}_f,\rho_f)) - \mathrm{ch}\,(f_*\mathcal{E},f_*\rho). \end{aligned}$$

I.2.1.b. **Functorial Property**

For the classical Bott-Chern secondary characteristic form, we have an axiom in the sense of functors. Now we should have a similar axiom. Since the relative Bott-Chern secondary characteristic form is defined over N, it is enough for us to consider its behavior under the base change. But for a most general base change, we know that even the cohomology groups do not behave very well. Thus we assume that our base change is a special one, say, a flat morphism. We know that in this case, everything works well. (See Proposition III.9.3 of [H 77]).

Axiom 2. (**Base Change Rule**) For any flat base change $g : M' \to N$, we have

$$g^*\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,\rho_f) = \mathrm{ch}_{\mathrm{BC}}(g_f^*\mathcal{E},g_f^*\rho;f_g,\rho_{f_g}).$$

Here g_f denotes the induced morphism of g with respect to f, and similarly for f_g. That is, we have the following commutative diagram:

$$\begin{array}{ccc} M\times_N M' & \xrightarrow{g_f} & M \\ f_g \downarrow & & \downarrow f \\ M' & \xrightarrow{g} & N. \end{array}$$

Also here ρ_{f_g} is the natural metric induced by the flat base change g from ρ_f.

I.2.1.c. **Uniqueness**

The next axiom for the classical Bott-Chern secondary characteristic form is the uniqueness rule. This rule represents the initial condition in the present context. In fact, the classical Bott-Chern secondary characteristic form involves the triangle relation in the category of hermitian sheaves. What is the situation when the original triangle degenerates? Usually this kind of initiative condition will determine the solution uniquely among general solutions.

Classically, there is only one triangle relation, i.e., the one for hermitian vector sheaves. But now, there are two triangle relations: one is for hermitian vector sheaves, while the other is for smooth morphisms. Now we give the triangle relations in a gerenal situation; later we will show that they are equivalent to some other degenerate triangle relations.

Axiom 3. (Uniqueness With Respect To Vector Sheaves) For any short exact sequence of f-acyclic vector sheaves

$$\mathcal{E}.: \quad 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

with hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 1, 2, 3$, let

$$f_*\mathcal{E}.: \quad 0 \to f_*\mathcal{E}_1 \to f_*\mathcal{E}_2 \to f_*\mathcal{E}_3 \to 0$$

be the direct image of $\mathcal{E}.$ with associated hermitian metrics $f_*\rho_j$ on $f_*\mathcal{E}_j$ for $j = 1, 2, 3$. Then

$$\begin{aligned} &\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_2, \rho_2; f, \rho_f) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_1, \rho_1; f, \rho_f) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_3, \rho_3; f, \rho_f) \\ &\quad = f_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.)\,\mathrm{td}(\mathcal{T}_f, \rho_f)) - \mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E}., f_*\rho.). \end{aligned}$$

Axiom 4. (Uniqueness With Respect To Morphisms) Let $f : M \to N$ and $g : N \to Q$ be two smooth morphisms of Kähler manifolds. Let $(\mathcal{E}, \rho)$ be an f-acyclic hermitian vector sheaf on M such that $f_*\mathcal{E}$ is g-acyclic. Then

$$\begin{aligned} &\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; g \circ f, \rho_{g\circ f}) - \mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E}, f_*\rho; g, \rho_g) - g_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; f, \rho_f)\,\mathrm{td}(\mathcal{T}_g, \rho_g)) \\ &\quad = (g \circ f)_*(\mathrm{ch}(\mathcal{E}, \rho)\,\mathrm{td}_{\mathrm{BC}}(f, g)). \end{aligned}$$

Here $\mathrm{td}_{\mathrm{BC}}(f, g)$ denotes the classical Bott-Chern secondary characteristic form associated with the following short exact sequence of the relative hermitian tangent sheaves:

$$0 \to \mathcal{T}_f \to \mathcal{T}_{g\circ f} \to f^*\mathcal{T}_g \to 0.$$

From above, we know that the axioms here for the relative Bott-Chern secondary characteristic form have a similar pattern as those for the classical Bott-Chern secondary characteristic form. They are composed by the following aspects: Downstairs Rule, Base Change Rule, and Uniqueness Rule. Now taking Proposition 1.3 into the consideration, we know that one can also define ternary objects and so on. In this way, we get a special kind of hierarchy for characteristic forms.

§I.2.2 Existence Theorem For Relative Bott-Chern Secondary Characteristic Forms With Respect To Smooth Morphisms

Note that we have set up the axioms for the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms, it is a natural question to ask whether they exist or not. If they exist, are they unique? For answering these questions, we have the following

Existence Theorem Of Relative Bott-Chern Secondary Characteristic Forms With Respect To Smooth Morphisms.
Let $f : M \to N$ be a smooth morphism of Kähler manifolds with a hermitian metric ρ_f on the relative tangent sheaf $\mathcal{T}_f$. Then for any f-acyclic hermitian vector sheaf $(\mathcal{E}, \rho)$, there exists a unique element $\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho, f, \rho_f)$ in $\tilde{A}(N)$, which satisfies the axioms in the last section.

The proof of this existence theorem has the same style as the one for the classical Bott-Chern secondary characteristic form. That is, we first introduce a new one-dimensional parameter for the connections. But in the case now, we do not have the canonical connection and its natural generalization of the Chern characteristic form. Instead, we have to introduce the so-called Bismut superconnection and the heat kernels associated with the generalized Laplacian. As a consequence, technically, instead of integrating over $\mathbf{P}^1$, we will use the Mellin transform to integrate our family over $\mathbf{R}_{\geq 0}$. All of this will be done in the following chapters.

Chapter I.3
Existence Of Heat Kernels

The heat kernels were used in the proof of the Riemann-Roch theorem is slightly strange but very powerful. Soon after Hirzebruch found his remarkable Riemann-Roch formula, Grothendieck and Atiyah-Singer generalized the formula to more general contexts. For Grothendieck, in algebraic geometry, the Riemann-Roch theorem means the following commutative diagram:

$$\begin{array}{ccc} K_0(M) & \xrightarrow{\mathrm{ch}()\mathrm{td}(\mathcal{T}_f)} & \mathrm{CH}(M)_{\mathbf{Q}}. \\ f_K \downarrow & & \downarrow f_{\mathrm{CH}} \\ K_0(N) & \xrightarrow{\mathrm{ch}} & \mathrm{CH}(N)_{\mathbf{Q}} \end{array}$$

For Atiyah-Singer, they noted that, by Hodge theory, the cohomology groups for a vector sheaf are nothing but the kernels of certain elliptic operators. Hence they can study the index of an elliptic operator over more general manifolds, say spin-manifolds. Technically, the first proof of the Hirzebruch-Riemann-Roch formula or the Atiyah-Singer index theorem is in the style of the cobordism theory. Later, Grothendieck and Atiyah-Singer gave the proofs of their theorems using algebraic K-theory. It was only after Patodi that mathematicians realized that the use of heat kernels, following Seeley and others, has greater flexibility. It is in this way that the local family index theorem can be proved at the level of differential forms. (The ordinary family index theorem may be thought of as an integration form of this local version.)

In this chapter, following some classical methods, we will prove that for a generalized Laplacian, there exist heat kernels.

Partially because the heat kernel technique is hard to understand for most of algebraic geometers and is the core of our method here, we will devote it this chapter with details. The references here are [BGV 92], [Gi 84].

§I.3.1. Sobolev Spaces

In this section, we introduce a basic tool for the study of heat kernels. We first do everything locally, i.e. we study the situation over Euclidean spaces and then note that since our manifold is compact, we may choose a finite open covering and extend the local discussion globally by using a partition of unity.

I.3.1.a. The Situation For Euclidean Spaces

For any Euclidean space $\mathbf{R}^m$, there is a natural scalar product and hence a metric: For any two vectors $x := (x_1, \ldots, x_m)$ and $y := (y_1, \ldots, y_m)$ in $\mathbf{R}^m$,

$$< x, y > := \sum_{j=1}^{m} x_j y_j, \quad \text{and} \quad |x| := < x, x >^{1/2}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{Z}^m_{\geq 0}$ be a multi-index, we let

$$|\alpha| := \sum_{j=1}^{m} \alpha_j, \quad \alpha! := \prod_{j=1}^{m} \alpha_j!, \quad x^\alpha := \prod_{j=1}^{m} x_j^{\alpha_j}.$$

Define

$$d_x^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_m}}{\partial x_n}, \quad D_x^\alpha := (-1)^{|\alpha|} d_x^\alpha.$$

We also fix a volume form dx on $\mathbf{R}^m$, which comes from the usual Lebesgue measure on $\mathbf{R}^m$, but with an additional normalizing factor $(2\pi)^{-m/2}$. Then we have

$L^2(\mathbf{R}^m)$:= the space of all complex valued functions f on $\mathbf{R}^m$ such that, under the natural L^2 inner product $(h, g) = \int_{\mathbf{R}^m} h(x)\bar{g}(x)dx, (f, f)$ is finite;

$C^\infty(\mathbf{R}^m)$:= the space of all smooth complex valued functions on $\mathbf{R}^m$;

$C_0^\infty(\mathbf{R}^m)$:= the space of all functions in $C^\infty(\mathbf{R}^m)$ with compact supports. This space is a dense subset of $L^2(\mathbf{R}^m)$.

$C^k(\mathbf{R}^m)$:= the space of continuous functions on $\mathbf{R}^m$ with continuous partial derivatives up to the order k.

We consider a special class of smooth complex valued functions on $\mathbf{R}^m$, the **Schwartz class** $\mathcal{S}$. By definition,

$\mathcal{S}$:= the space of the functions in $C^\infty(\mathbf{R}^m)$ such that for each pair of multi-index α, β, there is a constant $C_{\alpha,\beta}$ such that

$$|x^\alpha D_x^\beta f| \leq C_{\alpha,\beta}.$$

It is a basic fact that $C_0^\infty(\mathbf{R}^m) \subset \mathcal{S} \subset L^2(\mathbf{R}^m)$. Thus, $\mathcal{S}$ is dense in $L^2(\mathbf{R}^m)$.

The next key concept is that of the Fourier transform on $\mathcal{S}$. By definition, for any f in $\mathcal{S}$, the **Fourier transform** $\hat{f}$ is defined by

$$\hat{f}(\xi) := \int_{\mathbf{R}^m} e^{-i<x,\xi>} f(x)dx.$$

The fundamental dual properties for the Fourier transform are stated as follows:

$$D_\xi^\alpha \hat{f}(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha f}, \quad \xi^\alpha \hat{f}(\xi) = \widehat{D_x^\alpha f}.$$

In particular, the Fourier transform defines a map from $\mathcal{S}$ to itself. Also there is a fixed point of the Fourier transform, namely, the **Gaussian distribution** $\exp(-\frac{1}{2}|x|^2)$. Moreover, the Fourier transform is a bijection of $\mathcal{S}$, since we have

$$f(x) = \hat{\hat{f}}(-x).$$

There are two natural ring structures on $\mathcal{S}$: one is defined by the ordinary pointwise product, while the other is defined by the following convolution: for any $f, g \in \mathcal{S}$,

$$(f_* g)(x) := \int_{\mathbf{R}^m} f(x-y)\, g(y)\, dy.$$

The Fourier transform gives a homomorphism of these two rings. That is, we have

$$\hat{f} \cdot \hat{g} = \widehat{f_* g}, \quad \hat{f}_* \hat{g} = \widehat{f \cdot g}.$$

Finally, since $\mathcal{S}$ is dense in $L^2(\mathbf{R}^m)$ and $(\hat{f}, \hat{g}) = (f, g)$ for any $f, g \in \mathcal{S}$, we know that the Fourier transform may be extended to a unitary map

$$L^2(\mathbf{R}^m) \to L^2(\mathbf{R}^m).$$

This last result is usually called the **Plancherel theorem.**

With the above construction, there is the Sobolev space $H_s(\mathbf{R}^m)$ which is a measure of the L^2 derivatives. For any $s \in \mathbf{R}$ and $f \in \mathcal{S}$, we let

$$|f|_s^2 := \int_{\mathbf{R}^m} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi.$$

Then the **Sobolev space** $H_s(\mathbf{R}^m)$ is the completion of $\mathcal{S}$ with respect to the norm $|\cdot|_s$. In a certain sense, the subscript s counts the number of L^2 derivatives: If $s = n$ is a positive integer, we define the norm $|.|_n'$ by

$$|f|_n'^2 = \sum_{|\alpha| \le n} \int_{\mathbf{R}^m} |\xi^\alpha \hat{f}|^2 d\xi = \sum_{|\alpha| \le n} \int_{\mathbf{R}^m} |D_x^\alpha \hat{f}|^2 dx.$$

Obviously, this is an equivalent norm for $H_n(\mathbf{R}^m)$. With this interpretation in our mind, it is not surprising that $|\alpha|^{\text{th}}$ L^2-derivatives are lost, when we extend D_x^α to H_s. That is, D_x^α defines a continuous map

$$D_x^\alpha : H_s(\mathbf{R}^m) \to H_{s-|\alpha|}(\mathbf{R}^m).$$

(In fact, this comes from the following very simple estimation:

$$|\xi^\alpha|^2(1+|\xi|^2)^{s-|\alpha|} \le C(1+|\xi|^2)^s$$

for certain constant C which depends on the parameter (s,α) but not on f.) It also follows from the Plancherel theorem that $H_s(\mathbf{R}^m)$ is isomorphic to the L^2 space with the measure $(1+|\xi|^2)^{s/2}d\xi$.

In the following, we display a few relations among the various kinds of norms. First we introduce a new norm on $\mathcal{S}$: For any $k \in \mathbf{Z}_{\ge 0}$, and $f \in \mathcal{S}$,

$$|f|_{\infty,k} := \sup_{x\in\mathbf{R}^m} \sum_{|\alpha|\le k} |D_x^\alpha f|.$$

Obviously, the completion of $\mathcal{S}$ with respect to this norm is a subset of $C^k(\mathbf{R}^m)$.

Sobolev Lemma. Let $k \ge 0$ be a positive integer and s a real number with $s > k+\frac{m}{2}$. If $f \in H_s$, then $f \in C^k(\mathbf{R}^m)$ and

$$|f|_{\infty,k} \le C|f|_s$$

for some constant C.

Proof. First for $k=0$, since $\int_{\mathbf{R}^m}(1+|\xi|^2)^{-s}d\xi$ is bounded for $s > \frac{m}{2}$ and $f \in \mathcal{S}$, we have

$$\begin{aligned}|f(x)|^2 &= |\hat{\hat{f}}(x)|^2 \\ &= |\int_{\mathbf{R}^m}\{e^{ix\cdot\xi}\hat{f}(\xi)(1+|\xi|^2)^{s/2}\}\{(1+|\xi|^2)^{s/2}\}d\xi|^2 \\ &\le C|f|_s^2.\end{aligned}$$

Therefore,

$$|f|_{\infty,0} \le C|f|_s.$$

Since elements of H_s are the limits of elements in $\mathcal{S}$ with respect to the $|\cdot|_s$-norm, and since the uniform limit of continuous function is continuous, so the elements of H_s are continuous and the same norm estimate extends to H_s. In general, if $k>0$, we may use the following statement to obtain similar assertions: If $|\alpha| \le k$ and $s-k > \frac{m}{2}$, we have

$$|D_x^\alpha f|_{\infty,k} \le C|D_x^\alpha f|_{s-|\alpha|} \le C|f|_s.$$

This completes the proof.

Next, we consider the relation between $|\cdot|_s$ and $|\cdot|_t$ for different s and t. Since for $s > t$,

$$(1+|\xi|^2)^s \ge (1+|\xi|^2)^t,$$

we know that the identity map on $\mathcal{S}$ is actually an injection of $H_s \hookrightarrow H_t$ which is norm non-increasing. Furthermore, if we restrict the supports of our elements, this injection is compact. That is, we have the following

Rellich Lemma. Let $\{f_n\} \subset \mathcal{S}$ be a sequence of functions with supports in a compact subset K. If there is a constant C such that $|f_n|_s \leq C$ for all n, then for any $s > t$, there exists a subsequence $\{f_{n_k}\}$ which converges in H_t.

Proof. Let $g \in C_0^\infty(\mathbf{R}^m)$ be a function such that $g = 1$ in a neighborhood of K. Then $gf_n = f_n$ and

$$\begin{aligned}|\hat{f}_n(x)| &= |\int_{\mathbf{R}^m} \hat{f}_n(\xi)\hat{g}(x-\xi)d\xi| \\ &\leq \int_{\mathbf{R}^m} |\hat{f}_n(\xi)||\hat{g}(x-\xi)|d\xi \\ &\leq |f_n|_s[\int_{\mathbf{R}^m} |\hat{g}(x-\xi)|^2(1+|\xi|^2)^{-s}d\xi]^{\frac{1}{2}} \\ &\leq Ch(x),\end{aligned}$$

where h is a continuous function of x. Similar estimates hold for all derivatives of $\hat{f}_n(x)$. So there is a subsequence f_{n_k} of f_n, such that $\hat{f}_{n_k}$ converges uniformly on each compact subset of $\mathbf{R}^n$. Thus for any $r > 0$,

$$\begin{aligned}&|f_{n_j} - f_{n_k}|_t^2 = \\ &= \int_{|\xi|\leq r} |\hat{f}_{n_j} - \hat{f}_{n_k}|^2(1+|\xi|^2)^t d\xi + \int_{|\xi|\geq r} |\hat{f}_{n_j} - \hat{f}_{n_k}|^2(1+|\xi|^2)^t d\xi \\ &\leq C_r \max_{|\xi|\leq r}|\hat{f}_{n_j} - \hat{f}_{n_k}|^2 + 2C(1+r^2)^{t-s}.\end{aligned}$$

Others are trivial.

Another very useful estimation is the following

Lemma 1. If $s > t > u$, and $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that

$$|f|_t \leq \varepsilon|f|_s + C(\varepsilon)|f|_u.$$

Proof. This inequality is a direct consequence of the following

$$(1+|\xi|^2)^{2t} \leq \varepsilon(1+|\xi|^2)^{2s} + C(\varepsilon)(1+|\xi|^2)^{2u}.$$

We end the discussion of Sobolev spaces with the following

Lemma 2. The L^2 pairing on $\mathcal{S}$ extends to a perfect pairing of $H_s \times H_{-s} \to \mathbf{C}$. Thus we may identify H_{-s} with H_s^*.

Proof. By the Cauchy-Schwartz inequality, we know that for any $f, g \in \mathcal{S}$,

$$|(f,g)| \leq |f|_s|g|_{-s}.$$

Thus, for any $f \in \mathcal{S}$, let $g_f \in \mathcal{S}$ be the function defined by

$$\hat{g}_f := \hat{f}(1 + |\xi|^2)^s.$$

Then $(f, g_f) = |f|_s |g_f|_{-s}$. Moreover, for any $f \in \mathcal{S}$,

$$|f|_s = \sup_{g \in \mathcal{S}, g \neq 0} \frac{|(f,g)|}{|g|_{-s}}.$$

So we have the lemma.

I.3.1.b. Pseudo-Differential Operators On $\mathbf{R}^m$

By definition, a **linear partial differential operator of order** d is a polynomial expression $P = p(x, D) = \sum_{|\alpha| \leq d} a_\alpha(x) D_x^\alpha$, where the components of $a_\alpha(x)$ are smooth. The **symbol** of P, $\sigma P = p$, is defined by

$$\sigma P = p(x, \xi) = \sum_{|\alpha| \leq d} a_\alpha(x) \xi^\alpha,$$

which is a polynomial of degree d in the dual variable ξ. It may be helpful to regard (x, ξ) as defining a point of the cotangent space $T^*(\mathbf{R}^m)$. The **leading symbol** $\sigma_L P$ of P is the part of $p(x, \xi)$ of the highest degree.

Now we go slight further. Note that, for any $f \in \mathcal{S}$,

$$Pf(x) = \int_{\mathbf{R}^m} e^{i<x,\xi>} p(x, \xi) \hat{f}(\xi) d\xi,$$

so, we may use this formalism to define the action of pseudo-differential operators for a wider class of symbols $p(x, \xi)$ than polynomials. That is, we say that $p(x, \xi)$ is a **symbol of order** d and write $p \in S^d$ if $p(x, \xi)$ is smooth in $(x, \xi) \in \mathbf{R}^m \times \mathbf{R}^m$ with a compact x support and for all (α, β), there are constants $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|^2)^{d-|\beta|}.$$

For such a symbol, we define the associated operator $P(x, D)$ as the linear operator map $\mathcal{S} \to \mathcal{S}$ given by

$$Pf(x) := \int_{\mathbf{R}^m} e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi) d\xi.$$

Note that since for any $d \in \mathbf{R}$ and $f \in C_0^\infty(\mathbf{R}^m)$, $f(x)(1 + |\xi|^2)^{d/2} \in S^d$, we see that the order of a symbol needs not be an integer. Usually we refer to such an operator as a **pseudo-differential operator**, or shortly a ΨDO. If for all d, $f \in S^d$, we denote this by $p \in S^{-\infty}$ and say that p is **infinitely smoothing**.

We have the following

Lemma 1. For a $p \in S^d$, there is a constant C such that for $f \in \mathcal{S}$ $|Pf|_{s-d} \leq C|f|_s$. It follows that P may be extended to a continuous map $P : H_s \to H_{s-d}$ for all s.

Proof. For our purpose, let

$$q(\zeta,\xi) := \int_{\mathbf{R}^m} e^{-i<x,\zeta>} p(x,\xi)dx$$

and define

$$K(\zeta,\xi) := q(\zeta-\xi,\xi)(1+|\xi|)^{-s}(1+|\zeta|)^{s-d}.$$

Then, by the Cauchy-Schwartz inequality, we have

$$\begin{aligned}|(Pf,g)| =& |\int_{\mathbf{R}^m} K(\zeta,\xi)\hat{f}(\xi)(1+|\xi|)^s\bar{\hat{g}}(\zeta)(1+|\zeta|)^{d-s}d\zeta d\xi| \\ \leq& \{\int_{\mathbf{R}^m} |K(\zeta,\xi)||\hat{f}(\xi)|^2(1+|\xi|)^{2s}d\zeta d\xi\}^{1/2} \\ &\times \{\int_{\mathbf{R}^m} |K(\zeta,\xi)||\hat{g}(\zeta)|^2(1+|\zeta|)^{2d-2s}d\zeta d\xi\}^{1/2}.\end{aligned}$$

Now the lemma is a consequence of the following two easy estimations

$$\int_{\mathbf{R}^m} K(\zeta,\xi)d\xi \leq C, \quad \int_{\mathbf{R}^m} K(\zeta,\xi)d\zeta \leq C.$$

Next we discuss the smoothing approximation for a ΨDO. We first introduce an **equivalence relation** on the symbols by defining $p \sim q$ if $p-q \in S^{-\infty}$. Also for any given symbols p and $p_j \in S^{d_j}$, we say that **p is approximated by $\sum_{j=1}^{\infty} p_j$**, and denote it as

$$p \sim \sum_{j=1}^{\infty} p_j,$$

if $d_j \to -\infty$ and for any d, there is an integer $k(d)$ such that $k \geq k(d)$ implies that $p - \sum_{j=1}^{k} p_j \in S^d$.

For the application to Riemannian manifolds, in the following discussion, we will need to restrict the domain and the range of our operators. Let U be an open subset of $\mathbf{R}^m$ with a compact closure. Let $\Psi_d(U)$ be the space of those operators P for which the associated symbol $p(x,\xi) \in S^d$ has support in U; thus P may be thought of as $P : C_0^\infty(U) \to C_0^\infty(U)$. We also let

$$\Psi(U) := \bigcup_d \Psi_d(U), \Psi_\infty(U) := \bigcap_d \Psi_d(U).$$

More generally, we may consider the matrix valued symbol, which lead to little additional difficulty.

Lemma 2. Let $r(x,\xi,y)$ be a matrix valued symbol which is smooth in (x,ξ,y) and has compact x support inside U. Suppose that for each multi-index (α,β,γ) there exists a constant $C_{\alpha,\beta,\gamma}$ such that

$$|D_x^\alpha D_\xi^\beta D_y^\gamma r| \le C_{\alpha,\beta,\gamma}(1+|\xi|)^{d-|\beta|}.$$

We have
(a) If f is vector valued with compact support in U, R is defined by

$$Rf(x) := \int_{\mathbf{R}^m\times\mathbf{R}^m} e^{i<x-y,\xi>} r(x,\xi,y)f(y)dy\,d\xi.$$

Then the operator R is in $\Psi_d(U)$ and

$$\sigma R(x,\xi) \sim \{\sum_\alpha d_\xi^\alpha D_y^\alpha r/\alpha!\}|_{x=y}.$$

(b) If $d < -m - k$ and

$$K(x,y) := \int_{\mathbf{R}^m} e^{i<x-y,\xi>} r(x,\xi,y)d\xi,$$

then K is C^k in x,y and $Rf(x) = \int_{\mathbf{R}^m} K(x,y)f(y)dy$.
(c) If the x-support of r is disjoint from the y-support of r. Then R is infinitely smoothing and is represented by a smooth kernel function $K(x,y)$.

Proof. (a) First we may assume that the support of r in y is also compact. This may be done by multiplying r by a cut-off function in y with a compact support and with value 1 on U. Now let $q(x,\xi,\zeta)$ be the Fourier transform of r in y, then by an easy estimation, we know that

$$Rf(x) = \int_{\mathbf{R}^m\times\mathbf{R}^m} e^{i<x,\xi>} q(x,\xi,\xi-\zeta)\hat{f}(\zeta)d\xi\,d\zeta.$$

Thus, if we let

$$p(x,\zeta) := \int_{\mathbf{R}^m} e^{i<x,\xi-\zeta>} q(x,\xi,\xi-\zeta)d\xi,$$

then

$$Rf(x) = \int_{\mathbf{R}^m} e^{i<x,\zeta>} p(x,\zeta)\hat{f}(\zeta)d\zeta$$

is a ΨDO.

But by the Taylor expression of q on the middle variable, we have

$$q(x,\xi+\zeta,\xi) = \sum_{|\alpha|\le k} \frac{d_\zeta^\alpha q(x,\zeta,\xi)\xi^\alpha}{\alpha!} + q_k(x,\zeta,\xi).$$

Note that since q_k decays to arbitrarily high order in (ξ, ζ) and gives a symbol in S^{d-k} after integration, we have

$$\begin{aligned} p(x,\zeta) &= \int_{\mathbf{R}^m} e^{i<x,\xi>} q(x, \xi + \zeta, \xi) d\xi \\ &\doteq \int_{\mathbf{R}^m} e^{i<x,\xi>} \sum_{|\alpha| \le k} \frac{d_\zeta^\alpha q(x,\zeta,\xi)\xi^\alpha}{\alpha!} d\xi + \text{remainder} \\ &= \sum_{|\alpha|\le k} \frac{d_\zeta^\alpha D_y^\alpha r(x,\zeta,y)}{\alpha!}|_{x=y} + \text{remainder}. \end{aligned}$$

Hence we have (a).

(b) By our condition, we know that K is well-defined. Now the result is a consequence of Fubini's theorem.

(c) In this case, we can not define K as in (b), since the integration here does not converge. But there is another definition for K. In fact, from our assumption, we know that on the support of r, $|x - y| \ge \varepsilon > 0$. Define the Laplacian $\Delta_\xi := \sum_\nu D_{\xi_\nu}^2$. Thus formally

$$Rf(x) = \int_{\mathbf{R}^m \times \mathbf{R}^m} e^{i<x-y,\xi>} |x-y|^{-2k} \Delta_\xi^k r(x,\xi,y)\, f(y)\, dy\, d\xi.$$

So we may define

$$K(x,y) := \int_{\mathbf{R}^m} e^{i<x-y,\xi>} |x-y|^{-2k} \Delta_\xi^k r(x,\xi,y)\, d\xi$$

for sufficiently large k. Now the assertion is a consequence of the fact that $\Delta_\xi^k r$ decays to arbitrarily high order in ξ. This completes the proof.

From the proof above, and because of the presence of the terms $|x-y|^{-2k}$, we know that, in general, $K(x,y)$ becomes singular when $x = y$. But if $K(x,y)$ is a smooth matrix-valued function with a compact x-support, we may define an operator $P(K) \in \Psi_{-\infty}$ as follows: For any f with a compact support in U,

$$P(K)(f)(x) := \int_{\mathbf{R}^m} K(x,y) f(y) dy.$$

Thus $P(K)$ defines a continuous operator $P(K) : H_s \to H_t$ for any s, t. Let $|P|_{s,t}$ denote the operator norm, then, for any $f \in \mathcal{S}$,

$$|Pf|_t \le |P|_{s,t} |f|_s$$

and if $k \in \mathbf{Z}_{\ge 0}$,

$$|K|_{\infty,k} \le C(k) |P|_{-k,k}.$$

Finally, we give a result on the smoothing approximation for a ΨDO.

Lemma 3. Let $p_j \in S^{d_j}(U)$ with $d_j > d_{j+1}, d_j \to -\infty$. Then there exists a symbol $p \in S^{d_0}$ such that $p \sim \sum_j p_j$. Moreover, p is unique modulo $S^{-\infty}$.

Proof. We only need to prove the existence. For this, we introduce a cut-off function ϕ: $0 \le \phi \le 1$, and $\phi(\xi) = 0$ (resp. 1) for $|\xi| \le 1$ (resp. ≥ 2.) Thus we may cut away the support near $\xi = 0$. Let

$$p(x,\xi) := \sum_j \phi(t_j\xi)p_j(x,\xi).$$

Now note that since $p_j - \phi(t_j\xi)p_j \in S^{-\infty}$, by using a diagonalization argument, we have the assertion.

I.3.1.c. Situation For Manifolds

In this subsection, we extend the above discussion globally by using a partition of unity.

Let M be a smooth compact m-dimensional Riemannian manifold without boundary. Let $d\mu$ denote the Riemannian measure on M. Let $C^\infty(M)$ be the space of smooth functions on M.

A linear operator $P : C^\infty(M) \to C^\infty(M)$ is called a **ΨDO of order** d if for every open chart U on M and every $\phi, \varphi \in C_0^\infty(U)$, $\phi P \varphi \in \Psi_d(U)$, denoted by $P \in \Psi_d(M)$. Let

$$\Psi(M) = \bigcup_d \Psi_d(M), \quad \Psi_{-\infty}(M) := \bigcap_d \Psi_d(M).$$

In any coordinate system, we define $\sigma(P)$ as one for $\phi P \phi$ with $\phi = 1$ near the point in question; this is unique modulo $S^{-\infty}$. In the same way, we may define the leading term. It is obvious that the leading symbol is invariantly defined on the cotangent bundle $T^*(M)$.

We define $L^2(M)$ using the natural L^2 inner product, that is, $L^2(M)$ is the completion of $C^\infty(M)$ with respect to the L^2 norm. For $P : C^\infty(M) \to C^\infty(M)$, we let P^* be defined by $(Pf, g) = (f, P^*g)$, if such a P^* exists.

Now we use a partition of unity to define the **Sobolev space** $H_s(M)$: Take a cover of M by a finite number of coordinate charts U_i; for each i there is a diffeomorphism $h_i : O_i \to U_i$, where O_i is an open subset of $\mathbf{R}^m$ with compact closure. For $f \in C_0^\infty(U_i)$, define

$$|f|_s^{(i)} := |h_i^* f|_s.$$

Let $\{\phi_i\}$ be a partition of unity associated to this covering. Then define

$$|f|_s := \sum_i |\phi_i f|_s^{(i)}.$$

Obviously, $|\ |_s$ is a well-defined norm. In this way, we have the Sobolev space for M. Since M is a compact manifold, it is not difficult to generalize all the results in the previous subsection to the similar results for M.

§I.3.2. Elliptic Operators And Fredholm Operators

I.3.2.a. Elliptic Operators

Let M be a compact m-dimensional Riemannian manifold. Take a finite covering of M by coordinate charts $\{U_i\}$. For simplicity, we may think of $U = U_i$ as an open subset of $\mathbf{R}^m$.

Let $p \in S^d(U)$ be a square matrix and U_1 an open subset with $\bar{U}_1 \subset U$. We say that p is **elliptic** on U_1 if there exists an open subset U_2 with $\bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset U$ and if there exists $q \in S^{-d}$ such that $pq - I \in S^{-\infty}$ and $qp - I \in S^{-\infty}$ over U_2. (Recall that if $r \in S^{-\infty}$ then for every $\phi \in C_0^\infty$, $r\phi \in S^{-\infty}$.) For example, for any $\phi(x) \in C_0^\infty$, the symbol $\phi(x)(1+|\xi|^2)^{d/2} I$ is an elliptic symbol of order d whenever $\phi(x) \neq 0$. Globally on M, we say that P is **elliptic** if for any $\phi, \varphi \in C_0^\infty(U)$, $\phi P \varphi$ is elliptic whenever $\phi\varphi(x) \neq 0$.

The basic properties of elliptic operators are contained in the following

Lemma. Let $P \in \Psi_d$ be elliptic. Then
(1) There exists $Q \in \Psi_{-d}$ such that

$$PQ - I \in \Psi_{-\infty}, \quad QP - I \in \Psi_{-\infty}.$$

(2) P is hyperelliptic. That is, $f \in H_s$ and Pf is smooth implies f is smooth.
(3) (**Gãrding's Inequality**) There exists a constant C such that for any $f \in C_0^\infty(M)$,

$$|f|_d \leq C(|f|_0 + |Pf|_0).$$

Proof. By using a partition of unity associated with a finite covering of M by coordinate charts, we know that it is enough to prove the statements locally. Let U be an open subset of $\mathbf{R}^m$. First note that if $P \in \Psi_d$, $Q \in \Psi_e$, then $PQ \in \Psi_{d+e}$ and $\sigma(PQ) \sim \sum_\alpha d_\xi^\alpha p D_x^\alpha q/\alpha!$. Thus by the recursion, we may let

$$q_k := -q \sum_{|\alpha|+j=k, j<k} d_\xi^\alpha p D_\xi^\alpha q_j/\alpha!$$

and $q_0 = q$. From this, we know that if Q has the symbol $q_0 + q_1 + \ldots$, then we have the conclusion in (1).

(2) is trivial.

Finally for (3) we choose a cut-off function $\phi \in C_0^\infty$ with value 1 on U_1. Then if $f \in C_0^\infty$,

$$|f|_d = |\phi f|_d \leq |\phi(I - QP)f|_d + |\phi QPf|_d.$$

Since $\phi(I - QP)$ is an infinite smoothing operator, $|\phi(I - QP)f| \leq C|f|_0$. Now (3) comes from the fact that ϕQ is a bounded map from L^2 to H_d.

Remark. From this lemma, we know that for $d > 0$, one may define H_d by using the norm $|f|_0 + |Pf|_0$ and define H_{-d} by the dual of H_d.

I.3.2.b. **Fredholm Operators**

From above, we know that an elliptic ΨDO is invertible modulo compact operators. Takeing this fact out, we introduce the concept for Fredholm operators.

Let $\mathcal{H}$ be a Hilbert space and End($\mathcal{H}$) the space of all bounded linear maps $\mathcal{H} \to \mathcal{H}$. With the natural norm

$$|T| =: \sup_{x \in \mathcal{H}-\{0\}} \frac{|Tx|}{|x|},$$

End($\mathcal{H}$) is a Banach space. Let GL($\mathcal{H}$) be the subset of End($\mathcal{H}$) consisting of maps T which are bijections. By the inverse boundedness theorem, we know that GL($\mathcal{H}$) is an open subset of End($\mathcal{H}$) and is a topological group.

By definition, $T \in \text{End}(\mathcal{H})$ is **compact** if $\{x_n\}$ is a bounded sequence, i.e. $|x_n| \leq C$ for any n, there is a subsequence $\{x_{n_k}\}$ so that $Tx_{n_k} \to y$ for some $y \in \mathcal{H}$. Let Cpt($\mathcal{H}$) denote the set of all compact maps. An easy statement is that Cpt($\mathcal{H}$) is a closed 2-sided ideal of End($\mathcal{H}$).

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, then Hom($\mathcal{H}_1, \mathcal{H}_2$) is the Banach space of all bounded linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$ with the operator norm. For any $T \in \text{Hom}(\mathcal{H}_1, \mathcal{H}_2)$, let

$$N(T) := \{e \in \mathcal{H}_1 : T(e) = 0\} \quad \text{and} \quad R(E) := \{f \in \mathcal{H}_2 : f = T(e) \text{ for some } e \in \mathcal{H}_1\}.$$

If $\perp$ denotes the operation of taking orthogonal complement, then $R(T)^{\perp} = N(T^*)$. For any T, $N(T)$ is closed.

Let Fred($\mathcal{H}_1, \mathcal{H}_2$) be defined by the element $T \in \text{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ so that there is $S \in \text{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ such that $ST - I \in \text{Cpt}(\mathcal{H}_1), TS - I \in \text{Cpt}(\mathcal{H}_2)$. An element in this space is called a **Fredholm operator**.

Lemma 1. The element $T \in \text{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ is a Fredholm operator if and only if T is such that $R(T)$, $R(T^*)$ are closed, and

$$\dim N(T) < \infty, \quad \dim N(T^*) < \infty.$$

Thus, if T is Fredholm, so is T^*; if T_1, T_2 are Fredholm, so is $T_2 \circ T_1$.

Proof. Let $T \in \text{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ and let $x_n \in N(T)$ be such that $|x_n| = 1$. Then

$$x_n = (I - S_1 T)x_n = Cx_n$$

with C being compact. So we have a convergent subsequence. Hence the unit sphere in $N(T)$ is compact. As a consequence, $N(T)$ is finite dimensional. Next, we prove that $R(T)$ is closed. For this, let $y_n = Tx_n$ and $y_n \to y$. Without loss of generality, we may also assume that $x_n \in N(T)^{\perp}$. There are two possibilities.

(1) $\{x_n\}$ is a bounded sequence. Then by the fact that

$$x_n = S_1 y_n + (I - S_1 T)x_n, \quad S_1 y_n \to S_1 y,$$

we may further assume that $x_n \to x$. Hence,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tx,$$

which is in the range of T.

(2) $|x_n| \to \infty$. For this case, let $x'_n := x_n/|x_n|$. Then

$$Tx'_n \to y_n/|x_n| \to 0.$$

Using the same argument as above, we may assume that $x'_n \to x$ with $Tx = 0, |x| = 1$ and $x \in N(T)^{\perp}$. This is impossible.

Thus, $R(T)$ is closed. By duality, we easily have the same assertion for T^*.

Conversely, suppose $N(T)$ and $N(T^*)$ are finite dimensional and $R(T)$ is closed. We have the decomposition

$$\mathcal{H}_1 = N(T) \oplus N(T)^{\perp}, \quad \mathcal{H}_2 = N(T^*) \oplus R(T).$$

Now the assertion is a consequence of the fact that

$$T : N(T)^{\perp} \to R(T)$$

is a bijection.

For any Fredholm operator T, we define the **index** of T by

$$\mathrm{Ind}\,(T) := \dim N(T) - \dim N(T^*).$$

Obviously, $\mathrm{Ind}\,(T) = -\mathrm{Ind}\,(T^*)$ and $\mathrm{Ind}\,(T_2 \circ T_1) = \mathrm{Ind}\,(T_2) + \mathrm{Ind}\,(T_1)$. In particular, for elliptic ΨDOes, we have the following

Lemma 2. Let $P : C^\infty(M) \to C^\infty(M)$ be an elliptic ΨDO of order d over a compact manifold without boundary. Then
(a) The dimension of $N(P)$ is finite.
(b) $P : H_s(M) \to H_{s-d}(M)$ is a Fredholm operator and $\mathrm{Ind}\,(P)$ does not depend on s.
(c) $\mathrm{Ind}\,(P)$ only depends on the homotopy type of $\sigma_L P$.

Proof. Only (b) needs to be proved. Note that since there exists an elliptic ΨDO Q of order $-d$ such that $QP - I$ and $PQ - I$ are infinitely smoothing operators, $P : H_s \to H_{d-s}$ and $Q : H_{d-s} \to H_s$ are continuous. Thus $QP - I$ and $PQ - I$ are continuous, hence they are compact.

I.3.2.c. **Examples**

Let $\mathcal{E}$ be a graded vector bundle $\{\mathcal{E}_j\}_{j\in\mathbf{Z}}$ such that $\mathcal{E}_j \neq 0$ for only finite number of j's. Let $P = \{P_j\}$ be a collection of operators such that P_j is d^{th} order ΨDO with $P_j : C^\infty(\mathcal{E}_j) \to C^\infty(\mathcal{E}_{j+1})$. We say that $(\mathcal{E}, P)$ is a **complex** if

$$P_{j+1} \circ P_j = 0 \quad \text{and} \quad \sigma_L P_{j+1} \circ \sigma_L P_j = 0.$$

A complex $(\mathcal{E}, P)$ is called an **elliptic complex** if

$$N(\sigma_L P_j)(x,\xi) = R(\sigma_L P_{j-1})(x,\xi)$$

for $\xi \neq 0$. As usual, we define the cohomology by

$$H^j(\mathcal{E}, P) := N(P_j)/R(P_{j-1}).$$

If these cohomology groups are all finite dimensional, we define

$$\operatorname{Ind}(P) := \sum_j (-1)^j \dim H^j(V, P)$$

as the **Euler characteristic** of the complex $(\mathcal{E}, P)$.

Choose a fixed hermitian inner product on the fibers of $\mathcal{E}$. We may define $L^2(\mathcal{E})$. Let P^* be the adjoint of P with respect to this structure. If $(\mathcal{E}, P)$ is elliptic, we have a self-adjoint Laplacian

$$\Delta_j := (P^* P)_j := P_j^* P_j + P_{j-1} P_{j-1}^*.$$

For $p_j := \sigma_L P_j$, we have $\sigma_L(\Delta_j) = p_j^* p_j + p_{j-1} p_{j-1}^*$. By definition, we know that $(\mathcal{E}, P)$ is an elliptic d^{th} order partial differential complex if and only if Δ_j is an elliptic operator of order $2d$ for all j. The most important result in this direction is the following

Hodge Decomposition Theorem. Let $(\mathcal{E}, P)$ be an elliptic d^{th} order ΨDO complex. Then
(1) $L^2(\mathcal{E}_j) = N(\Delta_j) \oplus R(P_{j-1}) \oplus R(P_j^*)$ as an orthogonal direct sum.
(2) $N(\Delta_j)$ is a finite dimensional vector space and there is a natural isomorphism of $H^j(\mathcal{E}, P) \simeq N(\Delta_j)$. The elements in $N(\Delta_j)$ are smooth sections of $\mathcal{E}_j$.

Proof. Think of Δ_j as an operator $\Delta_j : H_{2d}(\mathcal{E}_j) \to L^2(\mathcal{E}_j)$. Since this is elliptic, we know that
(1) $N(\Delta_j)$ consists of smooth sections of $\mathcal{E}_j$;
(2) $N(\Delta_j)$ is finite dimensional;
(3) $R(\Delta_j)$ is closed;
(4) $L^2(\mathcal{E}_j) = N(\Delta_j) \oplus R(\Delta_j)$.

Since $P_j P_{j-1} = 0$, $R(P_{j-1})$ and $R(P_j^*)$ are orthogonal. But for $f \in N(\Delta_j)$,

$$0 = (\Delta_j f, f) = (P_j f, P_j f) + (P_{j-1}^* f, P_{j-1}^* f),$$

so $N(\Delta_j) = N(P_j) \cap N(P_{j-1}^*)$. Thus $R(\Delta_j)$ contains the span of $R(P_{j-1})$ and $R(P_j^*)$. Hence we have (1).

The natural inclusion of $N(\Delta_j)$ into $N(P_j)$ defines a map $N(\Delta_j) \to H^j(\mathcal{E}, P)$. If $f \in C^\infty(\mathcal{E}_j)$ and $P_j f = 0$, there is a $f_0 \in N(\Delta_j)$ such that $f = f_0 + \Delta f_1$ with $f_1 \in C^\infty(\mathcal{E}_j)$. Note that since $P_j \Delta_j f_1 = 0$, we have

$$0 = (P_j f_1, P_j P_j^* P_j f_1 + P_j P_{j-1} P_{j-1}^* f_1) = (P_j^* P_j f_1, P_j^* P_j f_1).$$

Thus $P_j^* P_j f_1 = 0$, so $\Delta_j f_1 = P_{j-1} P_{j-1}^* f_1 \in R(P_{j-1})$. Therefore the mapping $N(\Delta_j) \to H^j(\mathcal{E}, P)$ is surjective. This completes the proof.

Remark. As a more concrete example, consider the de Rham complex: Here $\mathcal{E}_j := C^\infty(\wedge^j(T^*M))$, $P_j := d$ the total derivative. Obviously, if for any $\xi \in T^*M$, let $\mathrm{ext}(\xi) : \wedge^j(T^*M) \to \wedge^{j+1}(T^*M)$ be the exteriour multiplication, so that $\mathrm{ext}(\xi)\omega := \xi \wedge \omega$, then $\sigma_L(d)(x, \xi) = i\,\mathrm{ext}(\xi)$. Now we know that the de Rham complex is an elliptic complex. Thus if we denote the adjoint of d as δ, $\Delta = \delta d + d\delta = (d+\delta)^2$ with $\sigma_L \Delta = |\xi|^2$. Hence, by Hodge decomposition theorem, we know that

$$H^p(M, \mathbf{C}) \simeq N(d_p)/R(d_{p-1}) \simeq N(\Delta_p)$$

is the space of **harmonic p-forms**. As a consequence, we know that $\mathrm{Ind}\,(d) = \chi(M)$, the Eular-Poincaré characteristic of M. Actually, if we let $* : \wedge^p(T^*M) \to \wedge^{m-p}(T^*M)$ be the Hodge star operator, defined by $\omega \wedge *\omega := (\omega, \omega)\, d\mu$, then by Stokes' theorem,

$$** = (-1)^{p(m-p)}, \quad \delta = (-1)^{mp+m+1} * d * .$$

Therefore, we further have the Poincaré duality: $* : N(\Delta_p) \simeq N(\Delta_{m-p})$.

§I.3.3. Existence Of Heat Kernels: I

In this section, we will prove the existence of heat kernels associated with a self-adjoint elliptic ΨDO.

We begin with the spectral theory. Let $T \in \mathrm{Cpt}(\mathcal{H})$ be a self-adjoint compact operator on the Hilbert space $\mathcal{H}$. Let

$$\mathrm{Spec}(T) := \{\lambda \in \mathbf{C} : T - \lambda I \notin \mathrm{GL}(\mathcal{H})\}.$$

It is an easy exercise to show that $\mathrm{Spec}(T)$ is a closed subset of $\mathbf{C}$ which is contained in the closed interval $[-|T|, |T|]$. For any $\lambda \in [-|T|, |T|]$, let $E(\lambda) := \{x \in \mathcal{H} : Tx = \lambda x\}$ be the eigen-space. If $\lambda \neq 0$, $Tx = \lambda x$ implies that the unit disk in $E(\lambda)$ is compact. Hence $E(\lambda)$ is finite dimensional. On the other hand, $\dim\{E(-|T|) \oplus E(|T|)\}$ is not 0. Indeed, suppose $T \neq 0$, we choose x_n so that $|x_n| = 1$ and $|Tx_n| \to |T|$. We may assume $\{x_n\}$ is the subsequence for which $Tx_n \to y$. Then

$$|T^2 x_n - |T|^2 x_n|^2 = |T^2 x_n|^2 + ||T|^2 x_n|^2 - 2|T|^2(T^2 x_n, x_n) \leq 2|T|^4 - 2|T|^2 |Tx_n|^2 \to 0.$$

Therefore, $x_n \to x := Ty/|T|^2 \neq 0$. Hence $|T^2x - |T|^2x| = 0$, i.e.

$$(T - |T|)(T + |T|)x = 0.$$

From this relation, we easily have the assertion.

In this way, we have a non-trivial decomposition $\mathcal{H} = E(-|T|) \oplus E(|T|) \oplus \mathcal{H}_1$. Let $T_1 := T|_{\mathcal{H}_1}$, so that $|T_1| \leq |T|$. Then we may decompose $\mathcal{H}_1$ in the same manner. Continuing this process inductively, we may find T_n so that $|T_n| \leq |T_{n-1}|$.

Claim. $|T_n| \to 0$.

In fact, for any n, choose x_n so that $Tx_n = \pm|T_n|x_n$ and $|x_n| = 1$. Thus $|x_i - x_j| = \sqrt{2}$. On the other hand, since T is compact, we may choose a convergent subsequence $|T_n|x_n \to y$. From here, we know $|T_n| \to 0$. That is, we have the following

Lemma 1. Let $T \in \mathrm{Cpt}(\mathcal{H})$ be self-adjoint. Then

$$\mathcal{H} = \bigoplus_k E(\lambda_k) \oplus E(0).$$

Hence we can find a complete orthonormal system for $\mathcal{H}$ consisting of eigenvectors of T.

As an application, we have the following

Lemma 2. Let $P : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ be an elliptic self-adjoint ΨDO of order $d > 0$. Then

(a) There exists a complete orthonormal basis $\{\phi_n\}$ for $L^2(\mathcal{E})$ so that $P\phi_n = \lambda_n\phi_n$.
(b) ϕ_n is smooth and $\lim_{n\to\infty}|\lambda_n| = \infty$.
(c) If we let the λ_n be such that $|\lambda_1| \leq |\lambda_2| \leq \dots$, then there exists constants $C, \delta > 0$ such that $|\lambda_n| \geq Cn^\delta$ for sufficiently large n.

Proof. Note that since $P : H_d(\mathcal{E}) \hookrightarrow L^2(\mathcal{E})$ is a Fredholm operator, by Gärding's inequality, we know that $P : N(P)^\perp \bigcap H_d(\mathcal{E}) \to N(P)^\perp \bigcap L^2(\mathcal{E})$ is a bijection. Define the **Green operator** $G = G(P)$ as the inverse of this map and extend G to be the zero on $N(P)$. Since $H_d(\mathcal{E}) \hookrightarrow L^2(\mathcal{E})$ is compact, G is a compact self-adjoint operator. Let $\{\phi_n\}$ be a complete orthonormal basis of eigenvectors of G with $G\phi_n = \mu_n\phi_n$. Note that, since $N(G) = N(P)$, if $\mu_n \neq 0$, then $P\phi_n = \mu_n^{-1}\phi$. Thus $|\lambda_n| \to \infty$. Also if k is an integer so that $dk > 1$, then $P^k - \lambda_n^k I$ is elliptic. Thus by hyperelliptic, $(P^k - \lambda_n^k I)\phi_n = 0$ implies $\phi_n \in C^\infty(\mathcal{E})$.

For (c), without loss of generality, we may assume that $\mathcal{E} = M \times \mathbf{C}$ and $d > \frac{m}{2}$. Let $F(a)$ be the space spanned by the ϕ_j with $|\lambda_j| \leq a$. Denote $n(a)$ as $\dim F(a)$. Then on $F(a)$, we have

$$\sup_{x\in M}|f(x)| \leq C|f|_d \leq C(|Pf|_0 + |f|_0) = C(1+a)|f|_0.$$

Therefore

$$\sum_{j=1}^{n(a)} \phi_j(x)\bar{\phi}_n(x) \le C^2(1+a)^2.$$

Integrating this estimate over M, we have

$$n(a) \le C^2(1+a)^2 \mathrm{vol}(M).$$

That is

$$|\lambda_{n(a)}| = a \ge C_1(n(a) - C_2)^{1/2},$$

which completes the proof.

Now let $P : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ be a self-adjoint elliptic ΨDO of order $d > 0$. We say that P has **positive definite leading symbol** if there exists $p(x,\xi) : T^*(M) \to \mathrm{End}(\mathcal{E})$ such that $p(x,\xi)$ is a positive definite hermitian matrix for $\xi \ne 0$ and $\sigma P - p \in S^{d-1}$ in any coordinate system. We claim that there is a constant C such that for every such P, $\mathrm{Spec}(P) \subset [-C, \infty)$. In fact, we may construct a Q_0 with the leading symbol $\sqrt{p}$ and if we let $Q := Q_0^* Q_0$, then $P - Q \in S^{d-1}$. Therefore

$$\begin{aligned}
(Pf, f) &= (Qf, f) + ((P-Q)f, f) \ge (Q_0 f, Q_0 f) - |((P-Q)f, f)| \\
&\ge (Q_0 f, Q_0 f) - C|f|_{d/2}|(P-Q)f|_{d/2} \\
&\ge (Q_0 f, Q_0 f) - C|f|_{d/2}|f|_{d/2-1} \\
&\ge (Q_0 f, Q_0 f) - C|f|_{d/2}(\varepsilon|Pf|_{d/2} - C(\varepsilon)|f|_0) \\
&\ge (Q_0 f, Q_0 f) - 2C\varepsilon|Q_0 f|_0^2 - C(\varepsilon)|f|_0^2 \\
&\ge -C(\varepsilon)|f|_0^2,
\end{aligned}$$

with $2C\varepsilon \le 1$.

With above, we may introduce the heat kernels. Fix a P as above, so that P is a self-adjoint elliptic ΨDO of order $d > 0$. Then the associated **heat equation** is defined by the system:

$$\begin{cases} (\frac{d}{dt} + P)f(x,t) = 0, & \text{for } t \ge 0 \\ f(x,0) = f(x) & . \end{cases}$$

Formally, it has a solution $f(x,t) = e^{-tP}f(x)$: Let $f(x) = \sum c_n \phi_n$ be the generalized Fourier series and

$$K(x,t,y) := \sum_n e^{-t\lambda_n}\phi_n(x) \otimes \bar{\phi}_n(y) : \mathcal{E}_x \to \mathcal{E}_y,$$

which is usual called the **heat kernel** of P. Then

$$f(x,t) = \sum_n e^{t\lambda_n} c_n \phi_n$$

and

$$e^{-tP}f(x) = \int_M K(x,t,y)f(y)d\mu(y)$$
$$= \sum_n e^{-t\lambda_n}\phi_n(x)\int_M f(y)\phi_n(y)d\mu(y).$$

Hence we have

$$\sum e^{-t\lambda_n} = \mathrm{Tr}_{L^2}e^{-tP} = \int_M \mathrm{Tr}_{\mathcal{E}_x}K(x,t,x)d\mu(x).$$

Now we justify the above formal process. In fact, if $jd > k + \frac{m}{2}$,

$$|\phi_n|_{\infty,k} \le C(|\phi_n|_0 + |P^j\phi_n|_0) = C(1+|\lambda_n|^j).$$

Thus, by the fact that Spec $P \subset [-c,\infty)$, without loss of generality, we may assume that all $\lambda > 0$. Note that since $e^{t\lambda}\lambda^j \le t^{-j}C(j)e^{-t\lambda/2}$, we have

$$|K(x,t,y)|_{\infty,k} \le t^{-j(k)}C(k)\sum_n e^{-t\lambda_n/2}.$$

On the other hand, by Lemma 2 above, we know that for n big enough, $\lambda \ge Cn^\delta$ with $\delta > 0$, so $K(x,t,y)$ is an infinitely smooth function of (t,x,y) for $t > 0$.

Thus if $(\mathcal{E},P)$ is an elliptic complex, $e^{-t\Delta_j}$ is in Ψ_∞ with a smooth kernel function. Let

$$E_i(\lambda) := \{\phi \in L^2(\mathcal{E}_i) : \Delta_i\phi = \lambda\phi\}.$$

Then $P_i : E_i(\lambda) \to E_{i+1}(\lambda)$ defines an acyclic complex if $\lambda \ne 0$, so that $\sum(-1)^i \dim E_i(\lambda) = 0$ for $\lambda \ne 0$. Therefore, we have the following

Lemma 3. With the notation as above,

$$\mathrm{Ind}\,(P) = \sum_i (-1)^i \mathrm{Tr}(e^{-t\Delta_i}).$$

§I.3.4. The Existence Of Heat Kernels: II

In the previous section, we used the following facts to prove the existence of the heat kernels for an elliptic self-adjoint ΨDO P of degree $d > 0$ with a positive leading symbol: The spectrum $\{\lambda_n\}$ is a bounded below subset of $\mathbf{R}$; there exist positive numbers δ and C such that for sufficient large n, $\lambda_n \ge Cn^\delta$. Since P is also a Fredholm operator, we know that the index of P is well-defined. As a consequence of this theory, if $(\mathcal{E},Q)$ is an elliptic complex, then

$$\mathrm{Ind}(Q) = \sum_i (-1)^i \mathrm{Tr}(e^{-t\Delta_i}),$$

where Δ is the Laplacian associated with Q.

In this section, we consider another aspect of the heat kernels, that is, the local asymptotic expansion of heat kernels, by using the geometry of the vector bundle E and the manifold M itself. Hence we find a deep relation between the index of certain elliptic operators and the geometry of objects in question.

From now on, we only consider the objects which come from geometry. Let M be a manifold, and $\mathcal{E}$ be a vector bundle over M. The **algebra of differential operators on $\mathcal{E}$**, denoted by $\mathcal{D}(M,\mathcal{E})$, is the subalgebra of $\mathrm{End}(C^\infty(M,\mathcal{E}))$, generated by elements of $C^\infty(M,\mathrm{End}(\mathcal{E}))$ and the covariant derivatives ∇_X, where ∇ is any connection on $\mathcal{E}$ and X ranges over all vector fields on M. If ∇' is another connection on $\mathcal{E}$, $\nabla'_X - \nabla_X$ is in $C^\infty(M,\mathrm{End}(\mathcal{E}))$, $\mathcal{D}(M,\mathcal{E})$ is independent of the choice of ∇.

There is a natural filtration on $\mathcal{D}(M,\mathcal{E})$ defined by

$$\mathcal{D}_i(M,\mathcal{E}) := C^\infty(M,\mathrm{End}\,\mathcal{E})\,\mathrm{Span}\{\nabla_{X_1}\dots\nabla_{X_j} : j \le i\}.$$

We call an element of $\mathcal{D}_i$ an **i-th order differential operator**. Thus the symbol morphism is the natural morphism

$$\mathcal{D} \to \mathrm{gr}\,\mathcal{D} := \sum_{k=0}^{\infty} \mathcal{D}_k/\mathcal{D}_{k-1}.$$

By Leibniz's rule, we know that, as an associated graded algebra, $\mathrm{gr}\,\mathcal{D}$ is isomorphic to the space of sections of the bundle $S(TM)\otimes\mathrm{End}(\mathcal{E})$, where S denotes the symmetric product. Moreover, the isomorphism

$$\sigma_k : \mathrm{gr}_k\mathcal{D} \to C^\infty(M, S^k(TM)\otimes\mathrm{End}(\mathcal{E}))$$

may be given by the following formula: if $D \in \mathcal{D}_k$, then for $x \in M$ and $\xi \in T_xM$,

$$\sigma_k(D)(x,\xi) = \lim_{t\to\infty} t^{-k}(e^{-itf}De^{itf})(x) \in \mathrm{End}(\mathcal{E}_x),$$

where $f \in C^\infty(M)$ such that $df(x) = \xi$. (We may check this with $D = -i\,d_x^1$ to get a good illustration.) Also, locally, any differential operator D may be written uniquely as

$$\sum_{j=0}^{n}\sum_{|\alpha|=j} a_\alpha(x)D_x^\alpha$$

over any trivialized open subset with coordinates $(x_1,\dots,x_m)$. With this, we know that a differential operator D of order k is elliptic if the section $\sigma_k(D) \in C^\infty(T^*M,\pi^*\mathrm{End}(\mathcal{E}))$ over the cotangent bundle space is invertible over the open subset $\{(c,\xi) : \xi \ne 0\}$. Here π denotes the natural projection.

Now we fix a Riemannian metric g on M. By definition, a differential operator H on $\mathcal{E}$ is a **generalized Laplacian** if locally H has the form $-\sum_{i,j} g^{i,j}\partial_i\partial_j$, up to a differential operator of order ≤ 1. Here ∂_i denotes $\frac{\partial}{\partial x_i}$ and $g^{i,j}$ stands for $g(dx_i, dx_j)\mathrm{Id}_{\mathcal{E}}$. Thus that H is a generalized Laplacian means that H is a second-order differential operator such that $\sigma_2(H)(x,\xi) = |\xi|^2$. Or equivalently, for any $f \in C^\infty(M)$,

$$[[H,f],f] = -2|df|^2.$$

Example: Let $\mathcal{E}$ be a vector bundle on a Riemannian manifold, with a connection $\nabla^{\mathcal{E}}$. Let ∇ be the Levi-Civita connection on M. Then the usual Laplacian $\Delta^{\mathcal{E}}$ on $C^\infty(M,\mathcal{E})$ is defined by

$$\Delta^{\mathcal{E}} s := -\mathrm{Tr}(\nabla^{T^*M\otimes\mathcal{E}}\nabla^{\mathcal{E}} s).$$

Here we denote by $\mathrm{Tr}\, S \in C^\infty(M,\mathcal{E})$ the contraction of an element $S \in C^\infty(M, T^*M \otimes T^*M \otimes \mathcal{E})$ with the metric $g \in C^\infty(M, TM \otimes TM)$. Thus, for any two vector fields X, Y, we have

$$(\nabla^{T^*M\otimes\mathcal{E}}\nabla^{\mathcal{E}} s)(X,Y) = (\nabla^{\mathcal{E}}_X\nabla^{\mathcal{E}}_Y - \nabla^{\mathcal{E}}_{\nabla_X Y})s.$$

Therefore, if e_i is a local orthonormal frame of TM,

$$\Delta^{\mathcal{E}} = -\sum_i(\nabla^{\mathcal{E}}_{e_i}\nabla^{\mathcal{E}}_{e_i} - \nabla^{\mathcal{E}}_{\nabla_{e_i}e_i}).$$

On the other hand, with respect to the frame $\partial/\partial x_i$, defined by a coordinate system around a point in M, we have

$$\Delta = -\sum_{ij} g^{ij}(\xi)(\nabla^{\mathcal{E}}_{\partial/\partial x_i}\nabla^{\mathcal{E}}_{\partial/\partial x_j} - \sum_k \Gamma^k_{ij}\nabla^{\mathcal{E}}_{\partial/\partial x_k}).$$

Here the Christoffel symbols Γ^k_{ij} are defined by $\nabla_{\partial/\partial x_i}\partial/\partial x_j := \sum_k \Gamma^k_{ij}\partial/\partial x_k$. So the Laplacian $\Delta^{\mathcal{E}}$ is a generalized Laplacian. (Usually, this formula is called the **Weitzenböck formula**. Later we will give its generalization, the Lichnerowicz formula.) It is not difficult to show that any generalized Laplacian is of the form $\Delta^{\mathcal{E}} + F$, with F a section of the bundle $\mathrm{End}\,(\mathcal{E})$. Similarly, for superconnections, we may also introduce the associated Laplacians.

The next aim is to prove that for any generalized Laplacian over a compact manifold M, there exists a unique heat kernel. For this purpose, we make the following

Definition. A **heat kernel** for a generalized Laplacian H is a family of sections $p(x,t,y)$ of $\mathcal{E}_x \otimes \mathcal{E}^*_y$ depending on $t \in \mathbf{R}_{>0}$, such that the following conditions hold:
(1) The action is C^∞ at $(x,t,y) \in M \times \mathbf{R}_{>0} \times M$.
(2) For every y, $(\partial_t + H_x)p(x,t,y) = 0$.
(3) For any continuous section s of $\mathcal{E}$ with compact support, with respect to the supremum norm,

$$\lim_{t\to 0}\int_M p(x,t,y)s(y)\,d\mu(y) = s(x).$$

Here we may choose any metric on $\mathcal{E}$.

Remark. If $p(x,t,y)$ exists, we may define an operator

$$P_t : C_0^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$$

by $P_t s(x) := \int_M p(x,t,y)s(y)\,d\mu(y)$. Thus (2) may be rewritten as

$$(\partial_t + H)P_t = 0,$$

which is the heat equation associated with H. Also (3) is just an initial condition for the first order differential equation

$$\lim_{t\to 0} P_t = \mathrm{Id}.$$

Thus, formally, we may think of P_t as e^{-tH}.

Theorem. For any generalized Laplacian over a compact manifold M, there exists a unique heat kernel.

The existence of heat kernels for a generalized Laplacian H follows from the results in the last section. But in that approach, we need to know about the spectrum of H and the eigenvectors. Now we use an approximation to construct the heat kernels. Nevertheless, the agreement of the heat kernels obtained by these different methods is a direct consequence of the following

Uniqueness Lemma. Suppose there exist heat kernels for all generalized Laplacians, then the heat kernel for a fixed generalized Laplacian H is unique.

Proof. Let $<,>: \mathcal{E}_x \times \mathcal{E}_x^* \to \mathbf{C}$ be a natural pairing defined by

$$< s, u > := \int_M < s(x), u(x) >_x d\mu(x).$$

Then for any given H, H^* is such that $< Hs, u > =: < s, H^*u >$. By an easy computation, we know that H^* is a generalized Laplacian on $\mathcal{E}^*$ too. Suppose H (resp. H^*) has the associated operator P_t (resp. P_t^*) as above. Obviously, if $< P_t s, u > = < s, P_t^* u >$, by the duality, we have our assertion.

To prove the last relation, consider

$$f(\theta) := < P_\theta s, P_{t-\theta}^* u >$$

for $0 < \theta < t$. Differentiating with respect to θ, we see that f is a constant. Hence $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to t} f(\theta)$, which completes the proof.

We next prove the existence of heat kernels by an approximation process from the following four steps:

(1) The existence for $H = -\sum_i \partial_i^2$ on $\mathbf{R}^m$.
(2) The existence of a formal solution.
(3) The existence of an approximate solution.

(4) Construction of an exact solution from an approximate one by a perturbation process.

Proof of the theorem. (1) Over $\mathbf{R}^m$, let $\mathcal{E}$ be the trivial line bundle $\mathbf{R}^m \times \mathbf{C}$. Then $-\sum_i \partial_i^2$ is the standard Laplacian. In this case, we may precisely let

$$p(x,t,y) := (4\pi t)^{-m/2} e^{-|x-y|^2/4t}.$$

By an easy calculation, it follows that $p(x,t,y)$ is the heat kernel of $-\sum_i \partial_i^2$.

(2) The aim in this step is to construct a formal solution for H on a compact manifold M by using (1) and the so-called normal coordinate system.

Over a Riemannian manifold M, we say that a smooth path $x : [0,1] \to M$ is a **geodesic** if it minimizes the function $L(x) := \int_0^1 |\dot{x}(t)|dt$. This leads to an Euler-Lagrange equation, which is of order 2. From this, for any $y \in M$ and $\xi \in T_yM$, there exists locally a unique geodesic x with initial conditions $x(0) = y, \dot{x}(0) = \xi$, which enable us to define $\exp\xi := x(1)$. Hence for a sufficient small ε, we have a diffeomorphism

$$\exp : \{\xi \in T_yM : |\xi| < \varepsilon\} \to \text{an open neighborhood of } y \in M.$$

Identifying T_yM with $\mathbf{R}^m$, we get coordinates near y. Such coodinates are called **normal coordinates**. In the following, for any $x \in M$ near y, under the above identification, we will also denote the point in $\mathbf{R}^m$ as x.

Now, imitating the situation over Euclidean space, we let

$$q(x,t,y) := (4\pi t)^{-m/2} e^{-|x-y|^2/4t}.$$

Then by a local calculation, we know that for any C^∞-family of sections s_t in $C^\infty(M,\mathcal{E})$,

$$\begin{aligned}(\partial_t + H_x)&(q(x,t,y)s_t(x)) \\ &=((\partial_t + t^{-1}\nabla_{\sum \xi^i\partial_i} + |\det g_{ij}|^{1/2} \circ H \circ |\det g_{ij}|^{-1/2})s_t(x))\, q(x,t,y).\end{aligned}$$

From this, by a **formal solution** of the heat equation $(\partial_t + H_x)p(x,t,y) = 0$, we mean an element $q(x,t,y)\,\Phi(x,t,y)$ such that near y, the section $\Phi(x,t,y)$ of $\mathcal{E}_x \otimes \mathcal{E}_y^*$ satisfies the equation

$$(\partial_t + t^{-1}\nabla_{\sum \xi^i\partial_i} + |\det g_{ij}|^{1/2} \circ H \circ |\det g_{ij}|^{-1/2})\Phi(x,t,y) = 0.$$

Existence of Formal Solutions. There exist unique sections $\Phi_i(x,y,H)$ of $\mathcal{E}_x \otimes \mathcal{E}_y^*$ for all $i \geq 0$ such that
(a) $\Phi_0(y,y,H) = \mathrm{Id}$.
(b) $(\partial_t + t^{-1}\nabla_{\sum \xi^i\partial_i} + |\det g_{ij}|^{1/2} \circ H \circ |\det g_{ij}|^{-1/2}) \sum_{i\geq 0} \Phi_i t^i = 0$.

Proof. Suppose we have the assertion. Let $f_i(s) := s^i\Phi_i(\exp(sx), y, H)$. Then the conditions become

$$\frac{d}{ds} f_i(s) = \begin{cases} 0, & \text{if } i = 0; \\ -s^{i-1}|\det g_{ij}|^{1/2} \circ H_x \circ |\det g_{ij}|^{-1/2}\Phi_{i-1}, & \text{if } i > 0. \end{cases}$$

But this is a system of differential equations of order one, which we may solve easily. Thus by the recurrence on i, we have the existence of formal solutions.

(3) In this step, we use a cut-off function and the formal solution above to obtain the following approximations:

Existence Of Approximate Solutions. For every positive integer N, there exists a smooth family of smooth sections $K^N(x,t,y)$ of $\mathcal{E}_x \otimes \mathcal{E}_y^*$, such that for any integer d, we have

(a) $\forall T > 0$, $K^N(x,t,y)$ is uniformly bounded for $|\ |_d$ in the range $0 \leq t \leq T$.

(b) For any section s of $\mathcal{E}$,

$$\lim_{t\to 0} K^N s = s$$

with respect to $|\ |_d$.

(c) As $t \to 0$,

$$|(\partial_t + H_x)k^N|_d \leq O(t^{N-\frac{n}{2}-\frac{d}{2}}).$$

Proof. Note that since M is compact, we may find a ε such that the normal coordinates map exp is well-defined for all point $y \in M$ in a small ball $|\xi| < \varepsilon$. (In fact, we may assume that ε is smaller than the **injectivity radius**, i.e. the radius of the largest ball in $T_{x_0}M$ such that the exponential map is a diffeomorphism from this ball around zero in $T_{x_0}M$ to the neighborhood of x_0 in M: Geometrically, the injectivity radius is simply the largest ball in the normal coordinates for which geodesics do not intersect.) Now define a cut-off function $\varphi : \mathbf{R}_{\geq 0} \to [0,1]$ by

$$\varphi(s) = \begin{cases} 1, & \text{if } s < \varepsilon^2/4; \\ 0, & \text{if } s > \varepsilon^2. \end{cases}$$

Using this cut-off function, we may smooth our formal solution by defining

$$k^N(x,t,y) := \varphi(d(x,y)^2)q(x,t,y)(\sum_{i=0}^{N} t^i \Phi_i(x,y,H)).$$

Here $d(x,y)$ denotes the distance of x and y in M. Now by a local estimation, we have the assertion.

(4) The last step is to construct the exact solution from the approximate solutions. For this, we need to use the Volterra series: a perturbation process.

We illustrate the situation by an analogue for a finite dimensional space V, i.e. when M becomes a point. Suppose that there exists a function $K_t : \mathbf{R}_{\geq 0} \to \mathrm{End}(V)$ such that

$$R_t := \frac{dK_t}{dt} + HK_t = O(t^\alpha),$$

for some $\alpha \geq 0$ and $K_0 = 1$. Here H is a linear endomorphism. We introduce the following perturbation process: Let $Q_t^k : \mathbf{R}_{\geq 0} \to \mathrm{End}(V)$ be defined by

$$Q_t^k := \int_{\Delta_t^k := \{(t_1,\ldots,t_k) : 0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq t\}} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2-t_1} R_{t_1} dt_1 \ldots dt_k.$$

Then $Q_t^0 = K_t$ and the sum of the convergent series $\sum_{k\geq 0}(-1)^k Q_t^k$ is equal to $P_t = e^{-tH}$. Thus the fact that the volume of

$$\{(t_1, \ldots, t_k) : 0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq 1\}$$

is $\frac{t^k}{k!}$, which decays rapidly, we have $P_t = K_t + O(t^{1+\alpha})$.

The situation in general is very similar. We fix an $N \geq \frac{m}{2}$ and omit the N in our notation. Define $r(x,t,y) := (\partial_t + H_x)k(x,t,y)$. Let

$$q^k(x,t,y) := \int_{\Delta_t^k} \int_{M^{(k)}} k(x, t - t_k, z_k)\, r(z_k, t_r - t_{k-1}, z_{k-1}) \ldots r(z_1, t_1, y)\, dt_1 \ldots dt_k$$

and

$$r^{k+1}(x,t,y) := \int_{\Delta_t^k} \int_{M^{(k)}} r(x, t - t_k, z_k) \ldots r(z_1, t_1, y)\, dt_1 \ldots dt_k.$$

Suppose that $N > \frac{m+d}{2}$. By the existence of approximate solutions, we know that q^k, r^k are C^d with respect to x and y. Moreover,

$$|q^k|_d \leq A^k t^{k(N-m/2)-d/2} \frac{t^k}{(k-1)!}$$

and

$$|r^k|_d \leq B^k t^{k(N-m/2)-d/2} \frac{t^k}{(k-1)!}$$

for certain constants $A, B \geq 0$.

Thus, put all above together, we have the following

Existence Of Heat Kernels. Let $p(x,t,y) := \sum_{k\geq 0}(-1)^k q^k(x,t,y)$. Then
(1) The series converges absolutely.
(2) p_t is C^d with respect to x, y and satisfies

$$(\partial_t + H_x)\, p(x,t,y) = 0.$$

(3) With respect to the sup-norm,

$$\lim_{t\to 0} P_t s = s.$$

(4)

$$|p(x,t,y) - k(x,t,y)|_d \leq O(t^{N-m/2-d/2}).$$

§I.3.5. Clifford Algebras And Dirac Operators

In the previous sections, we have seen two methods to construct the heat kernels. The first is rather neat and is very convenient for us to study axiomatically. On the other hand, the approximation method is rather complicated, and does yield certain geometric properties directly. In order to discuss this later aspect, we recall in this section certain concepts and results concerning Clifford algebras and Dirac operators.

Once we have obtained a generalized Laplacian, a natural question is to ask what are the first order operators D which have the generalized Laplacians as their squares. Suppose D may be expressed locally as

$$D = \sum_\alpha a_\alpha(x)\partial_\alpha + b(x),$$

with a_α, b the sections of $\mathrm{End}(\mathcal{E})$. By an easy calculation, we see that as a section of $\mathrm{Hom}(T^*M, \mathrm{End}(\mathcal{E}))$,

$$\sigma(D)(x, \sum_\alpha a_\alpha dx^\alpha) = i \sum_\alpha a_\alpha(x)\xi_\alpha.$$

Thus the square of D is

$$D^2 = \frac{1}{2}\sum_{ij}(a^i(x)a^j(x) + a^j(x)a^i(x))\partial_i\partial_j + \text{first order operator.}$$

Thus D^2 is a generalized Laplacian if and only if for any $\xi, \eta \in T^*M$, we have

$$< a(x), \xi >< a(x), \eta > + < a(x), \eta >< a(x), \xi >= -2(\xi, \eta)_x,$$

where $(,)$ is the metric on T^*_xM.

Thus locally, we may proceed as follows. Let V be a real vector space with a quadratic from Q. The **Clifford algebra** of (V, Q), denoted by $C(V, Q)$, is the algebra over $\mathbf{R}$ generated by V with the relation

$$vw + wv = -2\,Q(v, w)$$

for all $v, w \in V$. If Q is fixed, we may write $C(V)$ for $C(V, Q)$ and (v, w) for $Q(v, w)$. Since $C(Q, V)$ is a quotient of the tensor algebra $T(V) := \oplus_i \otimes^i V$, and is a superalgebra with the generators contained in the evenly graded subalgebra of $T(V)$, we know that $C(V)$ is a superalgebra too. We say that a superspace E is a **Clifford module**, if there is a super-action of $C(V)$ on E. We will denote by $c(a)$ the action of an element a of $C(V)$ on E. Let $a \mapsto a^*$ be the anti-automorphism of $T(V)$ such that $v \in V$ is sent to $-v$. Since * leaves the quotient relations unchanged, we obtain an anti-automorphism $a \mapsto a^*$ on $C(V)$. We say a Clifford module E of $C(V)$ with an inner-product is **self-adjoint** if Q is positive definite and $c(a^*) = c(a)^*$. We also denote $\mathrm{End}_{C(V)}(E)$ the algebra of endomorphism of E supercommuting with the action of $C(V)$.

$C(V)$ has a natural increasing filtration

$$C_i(V) := \text{ the span of elements of the form } v_1 \dots v_k \text{ with } v_j \in V \text{ and } k \leq i.$$

Obviously the associated graded algebra $\operatorname{gr} C(V)$ is naturally isomorphic to the exterior algebra $\wedge V$. Thus we may define a Clifford module action of $C(V)$ on $\wedge V$ as follows:

For any $v \in V$, let $\varepsilon(v)$, $\iota(v)$ be the exterior product, the contraction via Q, respectively, of v. Then the Clifford action is defined by

$$c(v) := \varepsilon(v) - \iota(v).$$

Moreover if Q is positive-definite, ι is the adjoint of ε, so that $\wedge V$ is self-adjoint. Usually, we call the isomorphism $\sigma : C(V) \to \wedge V$ defined by $\sigma(a) := c(a)1$ the **symbol map.** Its inverse $\mathbf{c}$ is called the **quantization map.** Since $C(0, V)$ is just $\wedge V$, we may also think of $\wedge V$ as a deformation of Clifford algebras.

The most important result in the local situation is the following

Lemma. Let V be a $2m$-dimensional oriented Euclidean vector space. Then there is a unique $\mathbf{Z}_2$-graded Clifford module $S = S^+ \oplus S^-$, called the **Spinor module,** such that

$$C(V) \otimes \mathbf{C} \simeq \operatorname{End}(S).$$

Proof. Let $\{e_j\}$ be an oriented orthonormal basis of V and let P be the span of elements $e_{2j-1} - ie_{2j}$ with $1 \leq j \leq m$. Then we know that $V \otimes \mathbf{C} = P \oplus \bar{P}$. Now let $S = \wedge P$, and define a Clifford action as follows: If $w \in P$, $c(w)s := \sqrt{2}\varepsilon(w)s$; if $\bar{w} \in \bar{P} \simeq P^*$, $c(\bar{w})s := -\sqrt{2}\iota(\bar{w})s$. Since the algebra of matrices is simple, it has a unique irreducible module. Hence by comparing the dimensions, we have

$$C(V) \otimes \mathbf{C} \simeq \operatorname{End}(S).$$

Now we consider the global situation. Let M be a Riemannian manifold. The **Clifford bundle** $C(M)$ is the bundle of the Clifford algebra over M whose fiber at $x \in M$ is the Clifford algebra $C(T_x^*M)$ of the Euclidean space T_x^*M. There is a natural symbol map $\sigma : C(M) \to \wedge T^*M$ defined by the local symbol maps $\sigma_x : C(T_x^*M) \to \wedge T_x^*M$. A **Clifford module** $\mathcal{E}$ on an even-dimensional Riemannian manifold M is a $\mathbf{Z}_2$-graded bundle $\mathcal{E}$ on M with a graded action of $C(M)$ on it. If $\mathcal{E}$ is a Clifford module with metric ρ, for which $\mathcal{E}^+$ and $\mathcal{E}^-$ are orthogonal, we say that the Clifford module is **self-adjoint** if the Clifford action is self-adjoint at each point. For any vector bundle $\mathcal{E}'$, the **twisted Clifford module** obtained from $\mathcal{E}$ by twisting with $\mathcal{E}'$ is the bundle $\mathcal{E}' \otimes \mathcal{E}$, with Clifford action $1 \otimes c$. If $a \in A(M, C(M))$ is a Clifford algebra-valued differential form on M, we may define an operator $c(a)$ as follows: If α, β are differential forms on M, a is a Clifford algebra section, and s is a section of $\mathcal{E}$, all homogeneous with respect to the $\mathbf{Z}_2$-grading, then

$$(c(\alpha \otimes a))(\beta \otimes s) := (-1)^{|\alpha||\beta|}(\alpha \wedge \beta) \otimes (c(a)s).$$

Define a **Dirac operator** D on a super-vector bundle $\mathcal{E}$ as a first-order differential odd operator on $\mathcal{E}$,

$$D : C^\infty(M, \mathcal{E}^\pm) \to C^\infty(M, \mathcal{E}^\mp),$$

such that D^2 is a generalized Laplacian. By a local calculation, we know that if D is a Dirac operator, then the action of T^*M on $\mathcal{E}$ defined by $[D, f] := c(df)$ is a Clifford action. Conversely, any differential operator D which satisfies $[D, f] = c(df)$ for all $f \in C^\infty(M)$ is actually a Dirac operator. Thus the collection of all Dirac operators on a Clifford module is an affine space modelled on $C^\infty(M, \mathrm{End}^-(\mathcal{E}))$. In order to sharpen this identification, we consider a special kind of connection:

A connection $\nabla^\mathcal{E}$ on a Clifford module $\mathcal{E}$ is called a **Clifford connection** if for any $a \in C^\infty(M, C(M))$ and $X \in C^\infty(M, TM)$,

$$[\nabla^\mathcal{E}_X, c(a)] = c(\nabla_X a),$$

with ∇ the Levi-Civita connection extended to $C(M)$. A superconnection $\mathbf{A}$ on a Clifford bundle $\mathcal{E}$ is called a **Clifford superconnection** if for any $a \in C^\infty(M, C(M))$,

$$[\mathbf{A}, c(a)] = c(\nabla a),$$

with ∇_a the Levi-Civita connection at a, which is an element of $A^1(M, C(M))$. Since locally we may decompose $\mathcal{E}$ as $\mathrm{End}_C(M)(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S}$ for a certain Clifford module $\mathcal{S}$, we know that there exists a Clifford superconnection on any Clifford module by using a partition of unity.

Now let $\mathbf{A}$ be a Clifford superconnection on a Clifford module $\mathcal{E}$. We may define a first-order differential operator on $C^\infty(M, \mathcal{E})$, denoted as $D_\mathbf{A}$, by composing the superconnection with the Clifford multiplication:

$$C^\infty(M, \mathcal{E}) \xrightarrow{\mathbf{A}} A(M, \mathcal{E}) \xrightarrow{c} C^\infty(M, C(M) \otimes \mathcal{E}) \xrightarrow{c} C^\infty(M, \mathcal{E}).$$

There is an orthonormal frame $\{e_i\}$ of the tangent bundle such that with respect to this local coordinate system we have

$$\mathbf{A} = \sum_i dx^i \otimes \partial_i + \sum_{I \subset \{1, \ldots, 2m\}} c(e^I) \otimes A_I,$$

with A_I being sections of $\mathrm{End}(\mathcal{E})$. Hence

$$D_\mathbf{A} = \sum_i c(dx^i)\partial_i + \sum_{I \subset \{1, \ldots, 2m\}} c(e^I) A_I.$$

In particular, if $\nabla^\mathcal{E}$ is a Clifford connection, then the associated first-order differential operator D is a Dirac operator. Locally, we have

$$D = \sum_i c(dx^i) \nabla^\mathcal{E}_{\partial_i}.$$

Moreover we know that the map sending $\mathbf{A}$ to $D_{\mathbf{A}}$ is a one-to-one correspondence between Clifford superconnections and Dirac operators compatible with the given Clifford action on $\mathcal{E}$, i.e.

$$D_{\mathbf{A}}(fs) = c(df)s + fD_{\mathbf{A}}s.$$

Indeed, this is a consequence of the fact that the difference of two Dirac operators is a section of $\mathrm{End}^-(\mathcal{E})$.

Proposition 1. If $\mathbf{A}$ is a Clifford superconnection on $\mathcal{E}$, then the curvature of $\mathbf{A}$ decomposes under the isomorphism

$$\mathrm{End}(\mathcal{E}) \simeq C(M) \otimes \mathrm{End}_{C(M)}(\mathcal{E})$$

as follows:

$$\mathbf{A}^2 = R^{\mathcal{E}} + F^{\mathcal{E}/S}.$$

Here $R^{\mathcal{E}} \in A^2(M, C(M)) \subset A^2(M, \mathrm{End}(\mathcal{E}))$ is the action of the Riemannian curvature of M on the bundle $\mathcal{E}$ given by the formula

$$R^{\mathcal{E}}(e_i, e_j) = \frac{1}{4}\sum_{kl}(R(e_i, e_j)e_k, e_l)\, c(e^k)\, c(e^l)$$

and $F^{\mathcal{E}/S} \in A(M, \mathrm{End}_{C(M)}(\mathcal{E}))$ is an invariant of $\mathbf{A}$, called the **twisting curvature** of the Clifford module $\mathcal{E}$.

Proof. Let $F^{\mathcal{E}/S}$ be the difference $\mathbf{A}^2 - R^{\mathcal{E}}$. We have to show that $\varepsilon(F^{\mathcal{E}/S})$ commutes with the operator $c(a)$ for any $a \in C^\infty(M, T^*M) \subset C^\infty(M, C(M))$. But this is a direct consequence of the condition that $\mathbf{A}$ is a Clifford superconnection. In fact, we have

$$[\mathbf{A}^2, c(a)] = [\mathbf{A}, [\mathbf{A}, c(a)]] = [\mathbf{A}, c(\nabla(a))] = c(\nabla^2 a) = c(Ra) = [R^{\mathcal{E}}, c(a)].$$

Now, by applying the existence of the heat kernels for a generalized Laplacian, we have the following

Proposition 2. Let D be a Dirac operator on a compact manifold, then D has a finite dimensional kernel, and D^2, acting on $C^\infty(M, \mathcal{E})$, has smooth heat kernels

$$< x|e^{-tD^2}|y > \in C^\infty(M \times M, p_1^*\mathcal{E} \otimes p_2^*\mathcal{E}^*).$$

We define the **index space** of a self-adjoint Dirac operator

$$D := \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

to be its kernel

$$\mathrm{Ker}(D) := \mathrm{Ker}(D^+) \oplus \mathrm{Ker}(D^-).$$

Then the **index** of D is the dimension of the superspace $\mathrm{Ker}(D)$,

$$\mathrm{Ind}(D) := \dim(\mathrm{Ker}(D^+)) - \dim(\mathrm{Ker}(D^-)).$$

Since for a self-adjoint operator, we have $D^- = (D^+)^*$, hence

$$\mathrm{Ind}(D) = \dim(\mathrm{Ker}(D^+)) - \dim(\mathrm{Coker}(D^+)),$$

which is the classical definition.

McKean-Singer Theorem. Let D be a self-adjoint Dirac operator on a compact manifold M. Let $< x|e^{-tD^2}|y >$ be the heat kernels of the generalized Laplacian D^2. Then for any $t > 0$,

$$\mathrm{Ind}(D) = \mathrm{Tr}_s(e^{-tD^2}) = \int_M \mathrm{Tr}_s(< x|e^{-tD^2}|x >)dx.$$

It follows that the index is an invariant of M and $\mathcal{E}$.

Proof. This result has already been proved by using the spectral theorem for D^2, and this is closely related to our first method for the existence of heat kernels. Here we give another proof by using the approximation process.

Let $a(t) := \mathrm{Tr}_s(e^{-tD^2})$ and let $P_1 := 1 - P_0$ be the projection onto the orthogonal complement of $\mathrm{Ker}(D)$. Then, using the approximate solutions for heat kernels, we know that for t big enough,

$$|\mathrm{Tr}_s[e^{-tD^2}] - P_0| = |\int_M \mathrm{Tr}_s[< x|P_1 e^{-tD^2} P_1|x >]\, dx| \le C\,\mathrm{vol}(M)\, e^{-t\lambda_1/2},$$

with λ_1 the smallest non-zero eigenvalue of D^2. Thus, by the exponential decay,

$$a(\infty) := \lim_{t\to\infty} \mathrm{Tr}_s[e^{-tD^2}] = \mathrm{Ind}(D).$$

Now the assertion is a consequence of the fact that $a(t) = a(\infty)$ for all t. In fact, by differentiation with respect to t, we have

$$\frac{d}{dt}a(t) = -\mathrm{Tr}_s[D^2 e^{-tD^2}] = -\frac{1}{2}\mathrm{Tr}_s[[D, De^{-tD^2}]] = 0.$$

From this result, we see that it is possible to express the index in geometric terms which only involve M and $\mathcal{E}$: This is what the usual index theorem means.

§I.3.6. Local Index Theorem

In this section, we give a proof of a local version of the index theorem. By the McKean-Singer theorem above, we know that the usual index theorem is an integration of this local result: The local index theorem is at the level of differential forms.

Let M be a compact oriented Riemannian manifold of dimension $2m$. Let D be a Dirac operator on a Clifford module $\mathcal{E}$ on M associated with a Clifford connection $\nabla^{\mathcal{E}}$. Let $k(x,t,y) =< x|e^{-tD^2}|y >$ be the heat kernel associated with the generalized Laplacian D^2. From the McKean-Singer formula, it is enough for us to know the behavior of the restriction of the heat kernel to the diagonal, that is $k(x,t,x)$. But for a general t, there is no satisfactory expression for $k(x,t,x)$. However, if t is small enough, our system is in the situation at the very beginning. So, the change of the heat flow is rather regular. Hence we can control it. More precisely, in our situation, we may go as follows:

Think of the heat kernel $k(x,t,x)$ as a section of the bundle of filtered algebra

$$\mathrm{End}(\mathcal{E}) \simeq C(M) \otimes \mathrm{End}_{C(M)}(\mathcal{E}),$$

where the filtration is induced by the filtration of $C(M) := C(T^*M)$ and the elements of $\mathrm{End}_{C(M)}(\mathcal{E})$ are given degree 0. Denote by $C_i(M)$ the subbundle of $C(M)$ of Clifford elements of degree less than or equal to i. The associated graded algebra is the bundle $\wedge T^*M \otimes \mathrm{End}_{C(M)}(\mathcal{E})$.

Local Index Theorem. Let M be a compact oriented Riemannian manifold of dimension $2m$. Let D be a Dirac operator of a Clifford module $\mathcal{E}$ on M associated with a Clifford connection $\nabla^{\mathcal{E}}$. Also let $k(x,t,y) =< x|e^{-tD^2}|y >$ be the heat kernel associated with the generalized Laplacian D^2. Then for $t \to 0^+$, the restriction of $k(x,t,y)$ to the diagonal has the asymptotic expansion

$$k(x,t,x) \sim (4\pi t)^{-m} \sum_{i=0}^{\infty} t^i k_i(x) = \frac{k_0(x)}{(4\pi)^{-m}} \frac{1}{t^m} + \ldots + \frac{k_{m-1}(x)}{(4\pi)^{-m}} \frac{1}{t} + \frac{k_m(x)}{(4\pi)^{-m}} + \ldots.$$

Further we have
(1) The coefficients $k_i \in C^\infty(M, C_{2i}(M) \otimes \mathrm{End}_{C(M)}(\mathcal{E}))$.
(2) If $\sigma(k) := \sum_{i=0}^m \sigma_{2i}(k_i)$, then

$$\sigma(k) = \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \exp(-F^{\mathcal{E}/S}),$$

where R is the Riemannian curvature of M and $F^{\mathcal{E}/S} = (\nabla^{\mathcal{E}})^2 - R^{\mathcal{E}}$ is the twisting curvature. Usually, we call $\det^{1/2}(\frac{R/2}{\sinh(R/2)})$ the **$\hat{A}$-genus form of the manifold** M with respect to the Riemannian curvature R.

Proof. The basic idea for proving this theorem is that first by using the normal coordinates, one may reduce the problem to a local one; then by Lichnerowicz's formula, the problem becomes the one about heat kernels for the harmonic oscillator on Euclidean spaces. In this process, naturally, we may use the rescaling technique to make everything go through clearly.

Basically, the proof may be divided into the following four steps.

(a) To reduce the situation to the local one by using the normal coordinates.
(b) Lichnerowicz's formula.
(c) Heat kernels for harmonic oscillators over Euclidean spaces.
(d) The expansion of the heat equation.

(a) Fix $x_0 \in M$ and trivialize the vector bundle $\mathcal{E}$ in a neighborhood of x_0 by a parallel transport along geodesics. More precisely, let $V := T_{x_0}M, E := \mathcal{E}_{x_0}$ and $U := \{\xi \in V | |\xi| < \varepsilon\}$, where ε is smaller than the injectivity radius of M at x_0. By the exponential map $\xi \mapsto \exp_{x_0}\xi$, we identify U with a neighborhood of x_0 in M. For $x = \exp_{x_0}\xi$, the fiber $\mathcal{E}_x$ and E are identified by the parallel transport map $\tau(x_0, x) : \mathcal{E}_x \to E$ along the geodesic $x_s := \exp_{x_0} s\xi$. Thus the space $C^\infty(U, \mathcal{E})$ of sections of $\mathcal{E}$ over U is identified with the space of E-valued C^∞-functions on V, defined in the neighborhood U. We also identify $C^\infty(U, \mathrm{End}(\mathcal{E}))$ with $C^\infty(U, \mathrm{End}(E))$. Hence $D = \sum_\alpha a_\alpha(\xi)\partial_\xi^\alpha$, with $a_\alpha(\xi) \in \mathrm{End}(E)$. On the other hand, by the isomorphism σ, we have

$$\mathrm{End}_{C(M)}(E) \simeq C(V^*) \otimes \mathrm{End}_{C(V^\bullet)}(E) \simeq \wedge V^* \otimes \mathrm{End}_{C(V^\bullet)}(E).$$

In this way, for simplicity, if we introduce a rescaling on the space of functions on $\mathbf{R}_{\geq 0} \times U$ with values in $\wedge V^* \otimes \mathrm{End}_{C(V^\bullet)}(E)$ by the formula

$$(\delta_u \alpha)(t, \xi) := \sum_{i=0}^{m} u^{-i/2}\alpha(ut, u^{1/2}\xi)_{[i]}.$$

Then the local index theorem is equivalent to saying that, with

$$k(t, \xi) := \sigma(< \exp_{x_0}^\xi | e^{-tD^2} | x_0 >),$$

$$\begin{aligned} &\lim_{u \to 0}(u^m \delta_u k)|_{(t,\xi)=(1,0)} \\ &= (4\pi)^{-m} \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \exp(-F^{\mathcal{E}/S}). \end{aligned}$$

The reason we choose the rescaling operator as above is that for t, the heat equation only contains first order derivatives, but for x, it contains the second order derivatives. Historically, this technique was introduced by Getzler [Ge 86].

The rescaling operator δ_u introduces a filtration on the algebra of differential operators action on $C^\infty(\mathbf{R}_{\geq 0} \times U, \wedge V^* \otimes \mathrm{End}_{C(V^\bullet)}(E))$: An operator D has the **filtration degree** d

if $\lim_{u\to 0} u^{d/2}\delta_u D\delta_u^{-1}$ exists. Especially, since

$$\begin{aligned}
\delta_u\phi(\xi)\delta_u^{-1} &= \phi(u^{1/2}\xi),\ \forall\phi\in C^\infty(U);\\
\delta_u\partial_t\delta_u^{-1} &= u^{-1}\partial_t;\\
\delta_u\partial_i\delta_u^{-1} &= u^{-1/2}\partial_i;\\
\delta_u\varepsilon(\alpha)\delta_u^{-1} &= u^{-1/2}\varepsilon(\alpha),\ \forall\alpha\in V^*;\\
\delta_u\iota(\alpha)\delta_u^{-1} &= u^{-1/2}\iota(\alpha),
\end{aligned}$$

we know that a polynomial $P(\xi)$ has degree $-\deg(P)$, that a polynomial $P(t)$ has degree $-2\deg(P)$, that a derivative $\partial/\partial\xi^i$ has degree one, that a derivative $\partial/\partial t$ has degree two, that an exterior multiplication operator ε^i has degree one, and that an interior multiplication operator ι^i has degree -1. In particular, in (b), we will show that D^2 has degree two, i.e. up to an operator of lower order, D^2 may be identified with a harmonic oscillator with differential form coefficients.

(b) Here we want to understand the generalized Laplacian D^2 associated with the Clifford connection $\nabla^{\mathcal{E}}$. Since now we do everything locally, we may find a spin decomposition $E = S\otimes W$ with $W := \mathrm{Hom}_{C(V^*)}(S,E)$ so that

$$\mathrm{End}(E)\simeq\mathrm{End}(S)\otimes\mathrm{End}(W)\simeq C(V^*)\otimes\mathrm{End}(W).$$

Let ∂_i be the orthonormal basis of V with its dual basis $d\xi^i$ of V^*. Denote c^i as $c(d\xi^i)\in\mathrm{End}(E)$. Let e_i be the local orthonormal frame obtained by parallel transport along geodesics from the orthonormal basis ∂_i of $T_{x_0}M$, and let e^i be the dual frame of T^*M. Thus by the fact that for the **radical vector field** $\mathcal{R} := \sum_i x_i\partial_i$,

$$[\nabla^{\mathcal{E}}_{\mathcal{R}}, c(e^i)] = c(\nabla_{\mathcal{R}}e^i) = 0,$$

we know that the $\mathrm{End}(E)$-valued function $c(e^i)_\xi$ is the constant endomorphism c^i.

Theorem. (i) (**Lichnerowicz's formula**) Let $\mathbf{A}$ be a Clifford superconnection of a Clifford module $\mathcal{E}$. Denote the Laplacian with respect to $\mathbf{A}$ by $\Delta^{\mathbf{A}}$ and let r_M be the scalar curvature of M. Then

$$D_{\mathbf{A}}^2 = \Delta^{\mathbf{A}} + \frac{1}{4}r_M + \mathrm{c}(F^{\mathcal{E}/S}).$$

Here $\mathrm{c}(F^{\mathcal{E}/S}) = \sum_{i<j} F^{\mathcal{E}/S}(e_i, ej)\,c(e^i)\,c(e^j)$.

ii Let L be the differential operator on $U\subset V$, with coefficients in $C(V^*)\otimes\mathrm{End}(W)$, defined by

$$L := \Delta^{\mathcal{E}} + \mathrm{c}(F^{\mathcal{E}/S}) + \frac{1}{4}r_M.$$

Denote $L(u) := u\delta_u L\delta_u^{-1}$ as the rescaling operator of L. Then for $u\to 0^+$,

$$L(u)\sim K := -\sum_i(\partial_i - \frac{1}{4}\sum_j R_{ij}\xi_j)^2 + F.$$

Here $R_{ij} := (R_{x_0}\partial_i, \partial_j) = \sum_{k<l}(R_{x_0})_{jikl}\varepsilon^k\varepsilon^l$, and F is the element of $\wedge^2 V^* \otimes \mathrm{End}(W)$ obtained by evaluating the twisting curvature $F^{\mathcal{E}/S}$ at the point x_0.

Proof. (i) Locally, we have $D_{\mathbf{A}} = \sum c(dx^i)\mathbf{A}_i$ with $\mathbf{A}_i$ the covariant differentiation in the direction ∂/∂_i. Thus

$$\begin{aligned}
D_{\mathbf{A}}^2 =& \frac{1}{2}\sum_{ij}[c(dx^i)\mathbf{A}_i, c(dx^j)\mathbf{A}_j] \\
=& \frac{1}{2}\sum_{ij}[c(dx^i), c(dx^i)]\mathbf{A}_i\mathbf{A}_j + \sum_{ij} c(dx^i)[\mathbf{A}_i, c(dx^j)]\mathbf{A}_j \\
&+ \frac{1}{2}\sum_{ij} c(dx^i)c(dx^j)[\mathbf{A}_i, \mathbf{A}_j] \\
=& -\sum_{ij} g^{ij}(\mathbf{A}_i\mathbf{A}_j + \sum_k \Gamma_{ij}^k\mathbf{A}_k) + \sum_{i<j} c(dx^i)c(dx^j)[\mathbf{A}_i, \mathbf{A}_j] \\
=& \Delta^{\mathbf{A}} + \sum_{i<j} c(e^i)c(e^j)\mathbf{A}^2(e_i, e_j).
\end{aligned}$$

On the other hand, by Proposition 5.1, we know that

$$\sum_{i<j} c(e^i)c(e^j)\mathbf{A}^2(e_i, e_j) = \sum_{i<j} F^{\mathcal{E}/S}(e_i, ej)\, c(e^i)c(e^j) - \frac{1}{8}\sum_{ijkl} R_{ijkl}\, c(e^i)c(e^j)c(e^k)c(e^l).$$

Since the antisymmetrization of R_{ijkl} over ijk vanishes, we have

$$\begin{aligned}
\sum_{ijkl} R_{ijkl}\, c(e^i)c(e^j)c(e^k)c(e^l) =& -\sum_{ijl} c(e^i)c(e^l)\, R_{jlij} + \sum_{ijl} c(e^j)c(e^l)\, R_{ilij} \\
=& 2\sum_{ij} c(e^j)c(e^i) \sum_k R_{ikjk} \\
=& 2\sum_k\sum_{ij} c(e^j)c(e^i)\, R_{ikjk} \\
=& -2\sum_{ik} R_{ikik} = -2r_M.
\end{aligned}$$

So we have (i).

(ii) First consider the local expression of the operator $\nabla_{\partial_i}^{\mathcal{E}}$. By definition, if $F^{\mathcal{E}}$ is the curvature of $\nabla^{\mathcal{E}}$, then

$$F^{\mathcal{E}} = \frac{1}{2}\sum_{i<j;k<l}((R(\partial_i, \partial_j)e_k, e_l)c^k c^l d\xi^i \wedge d\xi^j + F^{\mathcal{E}/S}(\partial_i, \partial_j)d\xi^i \wedge d\xi^j.$$

So by a local calculation,

$$\nabla_{\partial_i}^{\mathcal{E}} = \partial_i + \frac{1}{4}\sum_{j;k<l} R_{klij}\xi^j c^k c^l + \sum_{k<l} f_{ikl}(\xi)c^k c^l + g_i(\xi),$$

where the error terms are

$$f_{ikl}(\xi) = O(|\xi|^2) \in C^\infty(U),\ g_i(\xi) = O(|\xi|) \in C^\infty(U, \mathrm{End}_{C(V^*)}(E)) = C^\infty(U, \mathrm{End}(W)).$$

Hence

$$\begin{aligned}\nabla^{\mathcal{E},u}_{\partial_i} :=& u^{1/2}\delta_u \nabla^{\mathcal{E}}_{\partial_i}\delta_u^{-1}\\ =&\partial_i + \frac{1}{4}\sum_{j;k<l} R_{klij}\xi^j(\varepsilon^k - u\iota^k)(\varepsilon^l - u\iota^l)\\ &+ u^{-1/2}\sum_{k<l} f_{ikl}(u^{1/2}\xi)(\varepsilon^k - u\iota^k)(\varepsilon^l - u\iota^l) + u^{1/2}g_i(u^{1/2}\xi).\end{aligned}$$

Therefore

$$\lim_{u\to 0}\nabla^{\mathcal{E},u}_{\partial_i} = \partial_i + \frac{1}{4}\sum_{j;k<l}(R_{x_0})_{klij}\xi^j\varepsilon^k\varepsilon^l.$$

That is,

$$\nabla^{\mathcal{E},0}_{\partial_i} = \partial_i - \frac{1}{4}\sum_j (R)_{ij}\xi^j.$$

Now

$$\begin{aligned}L(u) =& -\sum_i(\nabla^{\mathcal{E},u}_{e_i})^2 + \sum_{i<j} F^{\mathcal{E}/S}(e_i, e_j)(u^{1/2}\xi)(\varepsilon^i - u\iota^i)(\varepsilon^j - u\iota^j)\\ &+ \frac{1}{4}ur_M(u^{1/2}\xi) + u^{1/2}\sum_i \nabla^{\mathcal{E},u}_{\nabla_{e_i}e_i}.\end{aligned}$$

Take the limits for these four terms: we know that the first term has the limit

$$-\sum_i(\nabla^{\mathcal{E},0}_{\partial_i})^2 = -\sum(\partial_i - \frac{1}{4}\sum_j (R)_{ij}\xi^j)^2;$$

that the second term becomes F, while others are zero.

We end this step by the following observation: $L(0) = K$ is a generalized harmonic oscillator over the Euclidean space V. Therefore, we may prove the local index theorem by considering the situation on Euclidean spaces.

(c) Mehler formula.

Now we consider the situation for the harmonic oscillator on Euclidean space V. Let R be an $2m \times 2m$ antisymmetric matrix and let F be an $N \times N$ matrix, both with coefficients in a commutative algebra A. By definition, a differential operator H acting on $A \otimes \mathrm{End}(\mathbf{C}^N)$-valued functions on V is called a **generalized harmonic oscillator** if

$$H = -(\sum_i \nabla_i^2) + F = -\sum_i(\partial_i + \frac{1}{4}\sum_j R_{ij}x_j)^2 + F.$$

Obviously, H is the Laplacian on V associated with the non-trivial connection $\nabla = d + \frac{1}{4}\sum_{ij} R_{ij}x_j dx_i$. Thus if let $\mathcal{R} := \sum_i x_i\partial_i$ be the radical vector field on V, and s_t a $A\otimes \mathrm{End}(\mathbf{C}^N)$-valued smooth function on V, then

$$(\partial_t + H)q_t s_t = q_t(\partial_t + t^{-1}\mathcal{R} + H)s_t.$$

In this case, as in the proof for the existence of heat kernels, for a formal power series $\Phi_t(x)$ in t, whose coefficients are smooth $A\otimes \mathrm{End}(\mathbf{C}^N)$-valued functions defined in a neighborhood of 0 in V, we say $q_t\Phi_t$ is a **formal solution** of the heat equation

$$(\partial_t + H)p_t = 0$$

if

$$(\partial_t + t^{-1}\mathcal{R} + H)\Phi_t = 0.$$

Before we introduce the most important result concerning with the generalized harmonic oscillator, we need the following notation:

Let

$$j_V(R) := \det(\frac{e^{R/2} - e^{-R/2}}{R/2}).$$

Since $j_V(0) = 1$, $j_V^{-1/2}(tR)$ is well-defined for t small. Similarly, we know that the A-valued quadratic form

$$< x|\frac{tR}{2}\coth(\frac{tR}{2})|x >$$

is well-defined for small t. With this, we may state our result as

Theorem (i)(Mehler's formula) The kernel $p_t(x, R, F)$, taking values in $A\otimes \mathrm{End}(\mathbf{C}^N)$ and defined for small t by the formula

$$(4\pi t)^{-m} j_V^{-1/2}(tR)\exp(-\frac{1}{4t} < x|\frac{tR}{2}\coth(\frac{tR}{2})|x >)\exp(-tF)$$

is a solution of the heat equation

$$(\partial_t + H_x)\, p_t(x) = 0.$$

(ii) For any $a_0 \in A\otimes \mathrm{End}(\mathbf{C}^N)$, there exists a unique formal solution $p_t(x, R, F, a_0)$ of the heat equation

$$(\partial_t + H_x)\, p_t(x) = 0$$

with

$$p_t(x) = q_t(x)\sum_{k=0}^{\infty} t^k\Phi_k(x)$$

such that $\Phi_0(0) = a_0$. Furthermore, the function $p_t(x, R, F, a_0)$ is given by the formula

$$(4\pi t)^{-m} j_V^{-1/2}(tR)\exp(-\frac{1}{4t} < x|\frac{tR}{2}\coth(\frac{tR}{2})|x >)\exp(-tF)a_0.$$

Proof. (i) may be checked by a direct calculation. Classically, we only consider the harmonic oscillator $-\frac{d^2}{dx^2}+x^2$ on the real line. In that very simple situation, Mehler first offered his famous solution.

For (ii), we have to solve

$$(\partial_t + t^{-1}\mathcal{R} + H_x)\sum_{k=0}^{\infty} t^k \Phi_k(x) = 0.$$

Therefore we should have

$$\begin{aligned}\mathcal{R}\Phi_0 &= 0\\ (\mathcal{R}+k)\Phi_k &= -H_x\Phi_{k-1} \text{ if } k>0.\end{aligned}$$

From here, by recurrence, we have the proof of the theorem.

(d) The Expansion of the Heat Equation.

Form (b), we know that $L(u)$ has a limit K when u goes to 0. On the other hand, if $p(x,t,x_0)$ is the heat kernels of the operator D^2, we may let

$$k(t,\xi) := r(x_0,x)\,p(x,t,x_0).$$

Obviously, the $\wedge V^* \otimes \mathrm{End}(W)$-valued function $k(t,\xi)$ satisfies the differential equation

$$(\partial_t + L)\,k(t,\xi) = 0.$$

Also if we let

$$r(u,t,\xi) := u^m(\delta_u k)(t,\xi),$$

then the local index theorem is equivalent to saying that

$$\lim_{u\to 0} r(u,t,\xi)|_{(t,\xi)=(1,0)} = (4\pi)^{-m}\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp(-F^{\mathcal{E}/S}).$$

Thus by the fact that $r(u,t,\xi)$ satisfies the differential equation

$$(\partial_t + u\delta_u L\delta_u^{-1})r(u,t,\xi) = 0,$$

if we can prove certain results which make us expanse the heat equation, we may only need to consider the formal solutions for the harmonic oscillator on an Euclidean space to complete the proof of the local index theorem. For this, we have the following

Lemma. There exist $\wedge V^* \otimes \mathrm{End}(W)$-valued polynomials $\gamma_i(t,\xi)$ on $\mathbf{R}_{\geq 0}\times V$, such that for every integer N, the function $r^N(u,t,\xi) := q_t(\xi)\sum_{i=-2m}^{2N} u^{i/2}\gamma_i(t,\xi)$ approximates $r(u,t,\xi)$ in the following sense:

To $N > j + |\alpha|/2$, there is a constant $C(N, j, \alpha)$ such that

$$||\partial_t^j \partial_\xi^\alpha (r(u,t,\xi) - r^N(u,t,\xi))|| \leq C(N,j,\alpha) u^N,$$

for $0 < u \leq 1$ and $(t,\xi) \in (0,1) \times U$. Furthermore $\gamma_i(0,0) = 0$ if $i \neq 0$, while $\gamma_0(0,0) = 1$.

Suppose that this lemma holds, then expanding the equation

$$(\partial_t + L(u))r(u,t,\xi) = 0$$

and $r(u,t,\xi)$ in $u^{1/2}$, we have

$$r(u,t,\xi) \sim q_t(\xi) \sum_{i=-2m}^{\infty} u^{i/2} \gamma_i(t,\xi)$$

and the leading term satisfies the heat equation

$$(\partial_t + K_\xi)(q_t(\xi)\gamma_{-l}(t,\xi)) = 0.$$

Since the formal solution of the heat equation for the harmonic oscillator is uniquely determined by $\gamma_{-l}(0,0)$, and $\gamma_{-l}(0,0) = 0$ for $l > 0$, we see that $\gamma_{-l} = 0$ unless $l = 0$. In particular, we see that there is no pole in the Laurent expansion of r in $u^{1/2}$. Also we know that the leading term of the expansion of $r(u,t,\xi)$, i.e. $r(0,t,\xi) = q_t(\xi)\gamma_0(t,\xi)$ satisfies the heat equation for the operator $L(0) = K$ with the initial condition $\gamma_0(0,0) = 1$. Thus finally by the expression of K and the situation for the generalized harmonic oscillator, we have the following

Theorem. The limit $\lim_{u\to 0} r(u,t,\xi)$ exists, and is given by

$$(4\pi t)^{-m} \det{}^{1/2}\left(\frac{tR/2}{\sinh tR/2}\right) \exp\left(-\frac{1}{4t} < \xi | \frac{tR}{2} \coth\left(\frac{tR}{2}\right) | \xi >\right) \exp(-tF).$$

In particular, we have the local index theorem by letting $(t,\xi) = (1,0)$.

Thus, we only need to give the following

Proof of the lemma. By the proof of the existence of the heat kernel, we know that there exist functions $\phi_i \in C^\infty(U, \mathrm{End}(E))$, with $\phi_0(0) = 1$, such that for any $\xi \in U$,

$$||k(t,\xi) - q_t(\xi) \sum_{i=0}^{N} t^i \phi_i(\xi)|| \leq C(N) t^{N-m}.$$

Note that since

$$|x^k e^{-x^2/4t}| \leq C_k t^{k/2},$$

we may replace $\phi_i(\xi)$ by its Taylor expansion $\varphi_i(\xi)$ of order $2(N-i)$:

$$\|k(t,\xi) - q_t(\xi)\sum_{i=0}^{N} t^i\varphi_i(\xi)\| \leq C'(N)t^{N-m}.$$

Thus, we have

$$\|k(u,t,\xi) - q_t(\xi)\sum_{i=0}^{N}(ut)^i\varphi_i(u^{1/2}\xi)\| \leq C'(N)u^N$$

for (t,ξ) in $(0,1)\times U$ and $0 < u \leq 1$. We may also make a similar estimate for the derivatives of $k(u,t,\xi)$. Therefore, the function $k(u,t,\xi)$ has an asymptotic expansion in $u^{1/2}$, u small, of the form

$$k(u,t,\xi) \sim q_t(\xi)\sum_{i=0}^{\infty}\Psi_i(u,t,\xi),$$

where $\Psi_i(\xi)$ is an End(E)-valued polynomial so that $\Psi_i(\xi)$ on V and $\Psi_0(0) = 1$. Moreover this expansion is uniform for (t,ξ) lying in compact subsets of $(0,1)\times U$, and the asymptotic expansions for the derivatives $\partial_t^k\partial_\xi^\alpha(k(u,t,\xi))$ may be obtained by differentiation the above estimation. Thus considering at the p-th term, we have

$$\|k(u,t,\xi)_{[p]} - u^{-p/2}q_t(\xi)\sum_{i=0}^{N}(ut)^i\Psi_i(u^{1/2}\xi)_{[p]}\| \leq C(N)u^{N-p/2}t^{N-m}.$$

If $\gamma_j(t,\xi)_{[p]}$ is the coefficient of $u^{j/2}$ in the sum

$$u^{-p/2}\sum_{i=0}^{(j+p)/2}(ut)^i\Psi_i(u^{1/2}\xi)_{[p]},$$

then $\gamma_j(t,\xi)_{[p]}$ is a polynomial on $\mathbf{R}_{\geq 0}\times V$ with values in $\wedge^p V^*\otimes \mathrm{End}(W)$. It is clear that the sum $\gamma_j(t,\xi) := \sum_{i=0}^{2m}\gamma_j(t,\xi)_{[p]}$ satisfies $\delta_u\gamma_j = u^{j/2}\gamma_j$. And $\gamma_j(t,\xi)_{[p]} = 0$ for $j < -p$. Hence $\gamma_j(t,\xi) = 0$ for $j < -2m$. In particular,

$$\gamma_0(0,0) = \sum_{i=-m}^{\infty} u^{i/2}\gamma_i(0,0) = (\delta_u\Psi)(0,0) = 1.$$

Similarly, we have the statement for the derivates. This completes the whole proof.

§I.3.7. Applications Of Local Index Theorem

In this section, we give an application of the local index theorem itself and an application of the proof of the local index theorem.

I.3.7.a An Application Of The Local Index Theorem

By the McKean-Singer formula, for every $t > 0$, we have

$$\mathrm{Ind}(D) = \int_M \mathrm{Tr}_s k(x,t,x) d\mu.$$

The local index theorem implies that if D is associated with a Clifford connection, then the integrand itself has a limit when t tends to zero. Now as the supertrace vanishes on all elements of the Clifford filtration strictly less than $2m = \dim(M)$, the first part of the local index theorem implies that

$$\mathrm{Tr}_s\, k(x,t,x) \sim (4\pi t)^{-m} \sum_{i\geq m} t^i \mathrm{Tr}_s\, k_i(x).$$

Hence there are no poles in the asymptotic expansion of $\mathrm{Tr}_s\, k(x,t,x)$. Furthermore, as the left hand side $\mathrm{Ind}(D)$ of the Mckean-Singer formula is independent of t, we necessarily have

$$\mathrm{Ind}(D) = (4\pi)^{-m} \int_M \mathrm{Tr}_s\, k_m(x)\, d\mu,$$

while the integrals of all other terms $\int_M \mathrm{Tr}_s\, k_j(x)\, d\mu$ vanishes for $j \neq m$.

To identify the term $\mathrm{Tr}_s\, k_m(x)$ as a characteristic form on M, we need certain more notation. Let $\Gamma \in C^\infty(M, C(M))$ be the **chirality operator**, i.e. locally, if V is a Euclidean space with $\{e_i\}$ an oriented, orthonormal basis, then

$$\Gamma := i^p e_1 \dots e_n,$$

where $p = n/2$ if n is even, and $p = (n+1)/2$ if n is odd. We know that as an element in $C(V) \otimes \mathbf{C}$, Γ does not depend on the basis of V used above, and, further, Γ satisfies $\Gamma\nu = -\nu\Gamma$ if n is even, while $\Gamma\nu = \nu\Gamma$ if n is odd. Also $\Gamma^2 = 1$. Locally, if V is an even dimensional real Euclidean space, then every finite dimensional super-Clifford module E of $C(V)$ is isomorphic to $W \otimes S$ with S the spinor space and $W = \mathrm{Hom}_{C(V)}(S, E)$. In this case, by a direct calculation, we have $\mathrm{Tr}_{s,S}(\Gamma) = 2^{n/2}$, and the supertrace over W of an element

$$F \in \mathrm{End}(W) \simeq \mathrm{End}_{C(V)}(E)$$

is given by the formula

$$\mathrm{Tr}_{s,W}(F) = 2^{-n/2}\mathrm{Tr}_{s,E}(\Gamma F).$$

Motivated by this, we may define the relative supertrace of $a \in C^\infty(M, \mathrm{End}_{C(M)}(\mathcal{E}))$ by

$$\mathrm{Tr}_{s,\mathcal{E}/S}(a) := 2^{-m}\mathrm{Tr}_{s,\mathcal{E}}(\Gamma(a)).$$

Then we extend the relative supertrace to a linear map

$$\mathrm{Tr}_{s,\mathcal{E}/S} : A(M, \mathrm{End}_{C(M)}(\mathcal{E})) \to A(M).$$

Thus if $b \in C^\infty(M, C(M))$, the point-wise supertrace of the section

$$b \otimes a \in C^\infty(M, C(M) \otimes \mathrm{End}_{C(M)}(\mathcal{E})) \simeq C^\infty(M, \mathrm{End}(\mathcal{E}))$$

is equal to the Berezin integral: This may be described as follows:

$$\mathrm{Tr}_{s,\mathcal{E}}(b(x) \otimes a(x)) = (-2i)^m \sigma_{2m}(b(x))\mathrm{Tr}_{s,\mathcal{E}/S}(a(x)).$$

Hence

$$\mathrm{Tr}_{s,\mathcal{E}}\, k_m(x) = (-2i)^m \mathrm{Tr}_{s,\mathcal{E}/S}[\sigma_n(k_m(x))].$$

Thus we have the following

> **Theorem.** (Patodi, Gilkey) Let M be a compact oriented Riemannian manifold of even dimension $2m$, with Clifford module $\mathcal{E}$ and Clifford connection $\nabla^{\mathcal{E}}$, let D be the associated Dirac operator. If $k(x,t,x)$ is the restriction of the heat kernel of the generalized Laplacian D^2 to the diagonal, then $\lim_{t\to 0}\mathrm{Tr}_s(k(x,t,x))|d\mu|$ exists and is the volume form on M obtained by taking the $2m$-form piece of
>
> $$(2\pi i)^{-m}\det{}^{1/2}\Big(\frac{R/2}{\sinh(R/2)}\Big)\mathrm{Tr}_{s,\mathcal{E}/S}[\exp(-F^{\mathcal{E}/S})].$$

I.3.7.b An Application Of The Proof Of The Local Index Theorem

In this subsection, we will give another construction for the classical Bott-Chern secondary characteristic form via superconnections by using the perturbation method in the proof of the local index theorem, especially the Duhamel's formula. (Even through this formula is very important, we will not give a precise formulation, as there are too many variations for it. But still, the reader may get a good feeling from section 3, the use of the Volterra series.) Later we will use a similar method when we discuss the relative Bott-Chern secondary characteristic objects associated with Chern forms.

Let

$$\mathcal{E}_\cdot : \quad 0 \to \mathcal{E}_n \xrightarrow{v} \ldots \xrightarrow{v} \mathcal{E}_1 \xrightarrow{v} \mathcal{E}_0 \to 0$$

be an exact sequence of vector sheaves on a complex manifold M. Put hermitian metrics ρ_j on $\mathcal{E}_j$ for $j = 0, \ldots, n$. Thus we construct a supervector sheaf $\mathcal{E}$ on M as follows: Set

$$\mathcal{E}^+ := \oplus_p \mathcal{E}_{2p}, \quad \mathcal{E}^- := \oplus_p \mathcal{E}_{2p+1}.$$

The metrics ρ_j give us canonical connections $\nabla^{\pm}$ on $\mathcal{E}^{\pm}$. Also v acts as an odd endmorphism of $\mathcal{E}$. With respect to the metrics ρ_j, there is an adjoint v^* for v. Hence we have the superconnection $\mathbf{A} := \nabla^+ + \nabla^- + v + v^*$ on $\mathcal{E}$.

Next we introduce a new parameter in $\mathbf{P}^1$. First, we extend $(\mathcal{E}_\cdot, \rho_\cdot)$ naturally to a complex on $M \times \mathbf{C}$. Let

$$\mathbf{A}_z := \nabla^+ + \nabla^- + dz\frac{\partial}{\partial z} + d\bar{z}\frac{\partial}{\partial \bar{z}} + zv + \bar{z}v^*$$

be a superconnection on $\mathcal{E}$ over $M \times \mathbf{P}^1$.

Theorem. With the same notation as above, we have

$$\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_\cdot, \rho_\cdot) = [2\pi \mathrm{i}] \int_{\mathbf{C}} [\log|z|^2] \ \mathrm{Tr}_s [\exp(\mathcal{E}, \mathbf{A}_z)].$$

Proof. By the construction, it is enough to prove that axiom 1 is satisfied by

$$\eta := \int_{\mathbf{C}} [\log|z|^2] \ \mathrm{Tr}_s [\exp(\mathcal{E}, \mathbf{A}_z)].$$

But this may be proved as follows.

First, we consider the convergence of η when $|z| \to \infty$. By definition, we know that

$$\mathbf{A}_z^2 = |z|^2 \Delta + R_z,$$

where $\Delta := (vv^* + v^*v)$, $R_z = \nabla^2 + v dz + v^* d\bar{z} + z\nabla(v) + \bar{z}\nabla(v^*)$ has the degree ≥ 1 on $M \times \mathbf{P}^1$ with $\nabla := \nabla^+ + \nabla^-$, so it is nilpotent. Thus by Duhamel's formula, we have the finite sum expression:

$$\exp(-\mathbf{A}_z^2) = \sum_k (-1)^k \int_{\Delta^k} e^{-(1-t_k)|z|^2\Delta} R_z \ldots R_z e^{-t_1|z|^2\Delta} dt_1 \ldots dt_k.$$

Note that since Δ has the smallest eigenvalue $\lambda > 0$, we have

$$\|e^{-(t_j - t_{j-1})|z|^2\Delta}\| \leq 1$$

for all j, and for at least one j,

$$\|e^{-(t_j - t_{j-1})|z|^2\Delta}\| \leq e^{-\lambda \frac{1}{k}|z|^2}.$$

Therefore, we have

$$\|\exp(-\mathbf{A}_z^2)\| \leq C(1 + |z|^m) \leq e^{-\frac{\lambda}{m}|z|^2}$$

with C a uniform constant with respect to the M-coordinates. That is, we have the exponential decay.

Similar estimates hold for the derivatives of $\exp(-\mathbf{A}_z^2)$ with respect to $z, \bar{z}$.

Thus, we may extend $a(z) := \exp(-\mathbf{A}_z^2)$ in a smooth way onto $M \times \mathbf{P}^1$ by declaring that $a(z)$ is 0 on $M \times \{\infty\}$. Also it is not difficult to show that $a(z) \in \oplus_p A^{p,p}(M \times \mathbf{P}^1)$. So $a(z)$ is also $d^c_{M\times\mathbf{P}^1}$-closed. In this way, we have

$$\begin{aligned}
d_M d_M^c \eta &= d_M d_M^c \int_{\mathbf{P}^1} [\log|z|^2]\, a(z) \\
&= \int_{\mathbf{P}^1} d_{M\times\mathbf{P}^1} d_{M\times\mathbf{P}^1}([\log|z|^2]\, a(z)) \\
&= \int_{\mathbf{P}^1} dd^c[\log|z|^2]\, a(z) \\
&= i_0^* a(z) - i_\infty^* a(z).
\end{aligned}$$

So we have our theorem.

Chapter I.4
The Mellin Transform

In this chapter, we introduce the Mellin transform. The technique comes from classical mathematics, but as the existence of the Bott-Chern secondary characteristic objects depends heavily on this technique, we will give all the details of it. The references are [T 30] and [WG 89].

This chapter consists of the following sections. In section one, we recall the basic properties of Bernoulli polynomials and Bernoulli numbers. In section two, we recall certain properties of Gamma functions. In section three, we recall the properties of the Riemann zeta function. In section four, we discuss the Mellin transform in general. Finally, as an application of the Mellin transform, we give another construction for the classical Bott-Chern secondary characteristic forms following [BGS 88].

§I.4.1. Bernoulli Polynomials and Bernoulli Numbers

The n^{th} **Bernoulli polynomial**, denoted by $B_n(x)$, are defined by using a generating function as follows:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n(x). \tag{1}$$

The series is convergent for $|t|<2\pi$, since the nearest singularities of the generating function to $t=0$ are $\pm 2\pi i$. When $x=0$, we have

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n(0). \tag{2}$$

Usually, this formula is expressed as

$$t(\frac{1}{e^t-1}+\frac{1}{2})=\frac{t}{2}(\frac{e^{t/2}+e^{-t/2}}{e^{t/2}-e^{-t/2}})=1+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^{2n}}{(2n)!}B_n.$$

Thus, we see that

$$B_0(0)=1,\ B_1(0)=-\frac{1}{2},\ B_{2n}(0)=(-1)^{n-1}B_n,\ B_{2n+1}(0)=0 \tag{3}$$

for $n = 1, 2, \ldots$. As usual, we call B_n the **Bernoulli numbers**.

Since
$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{k^n}{k!} B_k(0) \sum_{l=0}^{\infty} \frac{t^l}{l!} x^l = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k(0) x^{n-k},$$
we have the following relation of Bernoulli numbers and Bernoulli polynomials
$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k(0) x^{n-k}. \tag{4}$$

Also by
$$1 = \frac{e^t - 1}{t} \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(0) = \sum_{l=1}^{\infty} \frac{t^{l-1}}{l!} \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(0) = \sum_{n=1}^{\infty} t^{n-1} \sum_{k=0}^{n-1} \frac{B_k(0)}{k!(n-k)!},$$
we get the **recurrence formula** for Bernoulli numbers:
$$B_0(0) = 1, \quad \sum_{k=0}^{n-1} \frac{B_k(0)}{k!(n-k)!} = 0 \tag{5}$$
for $n \geq 2$. Symbolically, we may write the above formula as
$$B_n(x) = (B(0) + x)^n$$
for $n = 0, 1, \ldots$ and
$$(B(0) + 1)^n - B_n(0) = 0$$
for $n = 2, 3, \ldots$. Here it is understood that, after the binomial expsnsions have been developed, the symbols $B^k(0)$ for powers are to be replaced by $B_k(0)$.

We list the properties of Bernoulli polynomials and Bernoulli numbers as follows:

Properties. 1 (Derivatives)
$$\frac{d^p}{dx^p} B_n(x) = \frac{n!}{(n-p)!} B_{n-p}(x).$$

2. (Difference Relations)
$$\begin{aligned} B_0(x+1) &= B_0(x), \\ B_1(x+1) &= B_1(x) + 1, \\ B_n(x+1) &= B_n(x) + nx^{n-1}, \quad \text{for } n \geq 2. \end{aligned}$$

3. (Functional Equation)
$$B_n(1-x) = (-1)^n B_n(x).$$

4. (Addition Formula)

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_n(y) x^{n-k}.$$

5. (Summation Formula)

$$\sum_{s=1}^{m} s^n = \frac{1}{n+1}[B_{n+1}(m+1) - B_{n+1}(0)]$$

for $n \geq 1$.

We end this section by using the Bernoulli polynomials to obtain the asymptotic expansion of an analytic function, which can then be used to deduce the famous Stirling formula for $n!$.

Let $f(z)$ be an analytic function along the straight line from a point a to z. Then for any polynomial $\phi(t)$ of degree n, and $0 \leq t \leq 1$, we have

$$\begin{aligned}
&\frac{d}{dt} \sum_{m=1}^{n} (-1)^m (z-a)^m \phi^{(n-m)}(t) f^{(m)}(a+(z-a)t) \\
&= -(z-a)\phi^{(n)}(t) f'(a+(z-a)t) + (-1)^n (z-a)^{n+1} \phi(t) f^{(n+1)}(a+(z-a)t).
\end{aligned}$$

Hence by integration, we have the **Darboux formula**:

$$\begin{aligned}
&\phi^{(n)}(0)(f(z)-f(a)) \\
&= \sum_{m=1}^{n} (-1)^{m-1} (z-a)^m [\phi^{(n-m)}(1) f^{(m)}(z) - \phi^{(n-m)}(0) f^{(m)}(a)] \\
&+ (-1)^n (z-a)^{n+1} \int_0^1 \phi(t) f^{(n+1)}(a+(z-a)t)\,dt.
\end{aligned}$$

Now let $\phi(t) = B_n(t)$ and replace n by $2n$. Note that since

$$B_{2n}^{(2n)}(0) = (2n)!,\ B_{2n}^{(2n-m)}(x) = \frac{(2n)!}{m!} B_m(x),$$

$$B_m(1) = (-1)^m B_m(0),\ B_1(0) = -\frac{1}{2},\ B_{2k+1}(0) = 0\ (k \geq 1),$$

we have

$$\begin{aligned}
&f(z)-f(a) \\
&= \frac{z-a}{2}[f'(z)+f'(a)] + \sum_{k=1}^{n} (-1)^k \frac{(z-a)^{2k}}{(2k)!} B_k\, [f^{(2k)}(z) - f^{(2k)}(a)] \\
&+ \frac{(z-a)^{2n+1}}{(2n)!} \int_0^1 \phi_{2n}(t)\, f^{(2n+1)}(a+(z-a)t)\,dt.
\end{aligned}$$

Let $F(z) = f'(z)$, write h for $z - a$, take the summation of the integration of the above formula with respect to $[a, a+h], [a+h, a+2h], \ldots, [a+(m-1)h, a+mh]$, we have

$$\begin{aligned}\int_a^{a+mh} & F(x)dx \\ =& h[\frac{F(a)}{2} + F(a+h) + \ldots + F(a+(m-1)h) + \frac{F(a+mh)}{2}] \\ &+ \sum_{k=1}^{n} \frac{(-1)^k B_k h^{2k}}{(2k)!}[F^{(2k-1)}(a+mh) - F^{(2k-1)}(a)] + R_n,\end{aligned}$$

with

$$R_n := \frac{h^{2n+1}}{(2n)!}\int_0^1 B_{2n}(t)\sum_{s=0}^{m-1} F^{(2n)}(a+hs+ht)dt.$$

We call this formula the **Euler formula**. We can further simplify the formula by introducing periodic functions $P_n(t)$ with the period 1 as follows:

$$P_n(t) := B_n(t)/n! \quad \forall t \in [0,1).$$

Since

$$\frac{d}{dt}P_n(t) = P_{n-1}(t), \quad P_{2n+1}(1) = (-1)^{2n+1}P_{2n+1}(0) = 0,$$

we have

$$\begin{aligned}R_n =& h^{2n+1}\int_0^m P_{2n}(t)F^{(2n)}(a+ht)dt \\ =& -h^{2n+2}\int_0^m P_{2n+1}(t)F^{(2n+1)}(a+ht)dt.\end{aligned}$$

A natural question is to ask how fast the error term R_n goes to 0. To answer this, let us consider the period function $P_n(t)$. Since it is a function with the period 1, and

$$P_{2n}(t) = P_{2n}(1-t) = P_{2n}(-t)$$

for $0 \le t < 1$, we see that $P_{2n}(t)$ is an even function. So

$$P_{2n}(t) = \sum_{k=0}^{\infty} a_k \cos(2k\pi t),$$

with

$$\begin{aligned}a_0 =& \int_0^1 P_{2n}(t)dt = P_{2n+1}(1) - P_{2n+1}(0) = 0, \\ a_k =& 2\int_0^1 P_{2n}(t)\cos(2k\pi t)\,dt = (-1)^{n+1}\frac{2}{(2k\pi)^{2n}},\end{aligned}$$

for $k \geq 1$. We may do the same for $P_{2n+1}(t)$. In summary, we have

$$\begin{aligned} P_{2n}(t) &=(-1)^{n+1}\sum_{k=1}^{\infty}\frac{2\cos(2k\pi t)}{(2k\pi)^{2n}} \quad \forall n \geq 1 \\ P_{2n+1}(t) &=(-1)^{n+1}\sum_{k=1}^{\infty}\frac{2\sin(2k\pi t)}{(2k\pi)^{2n+1}} \quad \forall n \geq 0. \end{aligned}$$

Especially, we have the estimates

$$|P_n(t)| \leq \frac{2}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{1}{k^n}. \tag{6}$$

Example. Let $F(x) = e^{tx}, a = 0, m = 1, h = 1$, we have

$$\begin{aligned} \frac{e^t-1}{t} = &\frac{1}{2}(e^t+1) + \sum_{k=0}^{n}\frac{(-1)^k B_k}{(2k)!}t^{2k-1}(e^t-1) \\ &- t^{2n+1}\int_0^1 P_{2n+1}(s)e^{ts}\,ds. \end{aligned}$$

Thus

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} - \sum_{k=1}^{n}\frac{(-1)^k B_k}{(2k)!}t^{2k} + \frac{t^{2n+2}}{e^t-1}\int_0^1 P_{2n+1}(s)e^{ts}\,ds, \tag{7}$$

which is the finite Taylor expansion of the function $t/(e^t-1)$. Note that there is no restriction on t now (compare this with the equation (2)).

§I.4.2. Gamma Function

Here we list the basic properties of the Gamma function which are needed for our own interests.

The **Gamma function** $\Gamma(s)$ is the function which is given for $\mathrm{Re}(z) > 0$ by the formula

$$\Gamma(z) := \int_0^{\infty} e^{-t}t^z\frac{dt}{t}. \tag{1}$$

Since

$$\begin{aligned} \Gamma(z+1) = &\int_0^{\infty} e^{-t}t^z\,dt \\ =&[-e^{-t}t^z]_{t=0}^{t=\infty} + z\int_0^{\infty} e^{-t}t^{z-1}\,dt = z\Gamma(z), \end{aligned}$$

we have that

$$\Gamma(z) = \frac{1}{(z)_n}\Gamma(z+n), \tag{2}$$

where $(z)_n := z(z+1)\dots(z+n-1)$, and

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1.$$

(Here, the reader should know that the Gamma function is a natural generalization of $n!$.) Thus $\Gamma(z)$ is a meromorphic function with the simple poles at $z = 0, -1, -2, \dots, -n, \dots$, and where the corresponding residue at $z = -n$ is

$$\mathrm{Res}_{z=-n}(\Gamma) = \lim_{z\to-n}(z+n)\Gamma(z) = \frac{\Gamma(z+n+1)}{(z)_n}|_{z=-n} = \frac{(-1)^n}{n!}. \tag{3}$$

Using the fact that

$$e^{-t} = \lim_{n\to\infty}(1-\frac{t}{n})^n,$$

we have, for large n,

$$\Gamma(z) - \int_0^n (1-\frac{t}{n})^n t^z \frac{dt}{t} = \int_0^n [e^{-t} - (1-\frac{t}{n})^n] t^z \frac{dt}{t} + \int_n^\infty e^{-t}t^z\frac{dt}{t}.$$

Obviously, the second term has the limit 0. For the first, let us look at $e^{-t} - (1-\frac{t}{n})^n$. Since for $0 \le y < 1$,

$$1 + y \le e^y \le (1-y)^{-1}.$$

Thus

$$(1+\frac{t}{n})^{-n} \ge e^{-t} \ge (1-\frac{t}{n})^n.$$

Now by $e^t \ge (1+t/n)^n$,

$$0 \le e^{-t} - (1-\frac{t}{n})^n = e^{-t}[1 - e^t(1-\frac{t}{n})^n] \le e^{-t}[1-(1-\frac{t^2}{n^2})^n].$$

Hence by the fact that $(1-\alpha)^n \ge 1 - n\alpha$ for $\alpha \in [0,1]$, we have

$$0 \le e^{-t} - (1-\frac{t}{n})^n \le \frac{t^2}{n}e^{-t}.$$

In particular, the first limit goes to

$$\lim_{n\to\infty} \int_0^n \frac{1}{n}e^{-t}t^{\mathrm{Re}(z)+1}dt = 0.$$

That is

$$\lim_{n\to\infty} A_n(z) = \Gamma(z)$$

with

$$A_n(z) := \int_0^n (1-\frac{t}{n})^n t^z \frac{dt}{t}.$$

On the other hand, if we let $t = n\tau$,

$$\begin{aligned}A_n(z) =& n^z \int_0^1 (1-\tau)^n \tau^{z-1} d\tau \\ =& n^z [\frac{\tau^z}{z}(1-\tau)^n]_0^1 + \frac{n^z n}{z}\int_0^1 (1-\tau)^{n-1}\tau^z d\tau \\ =& \dots \\ =& \frac{n^z n(n-1)\dots 2.1}{z(z+1)\dots(z+n-1)}\int_0^1 \tau^{z+n-1} d\tau \\ =& \frac{n!}{(z)_{(n+1)}} n^z.\end{aligned}$$

Hence by the fact that

$$n^z = \prod_{m=1}^{n-1}(1+\frac{1}{m})^z$$

(resp.

$$n^z = e^{z\ln n} = \exp[z(\ln n - \sum_{m=1}^{n}\frac{1}{m})]\prod_{m=1}^{n} e^{z/m},)$$

we have the following

Euler's Infinite Product Expression.

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}[(1+\frac{z}{n})^{-1}(1+\frac{1}{n})^z]$$

(resp. **Weierstrass' Infinite Product Expression**

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}[(1+\frac{z}{n})e^{-z/n}]$$

where $\gamma := \lim_{n\to\infty}\{\sum_{m=1}^{n}\frac{1}{m} - \ln n\}$ is the **Euler constant.**)

Remark. The right hand side of Euler's product formula is well-defined for all z in $\mathbf{C}$, so we may take it as the general definition for $\Gamma(z)$.

As a consequence of Weierstrass' infinite product formula, we know that

$$\Gamma(z)\Gamma(-z) = -\frac{1}{z^2}\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})^{-1} = -\frac{\pi}{z\sin(\pi z)}.$$

Hence

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

and

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

Another very important property of the Γ function is the following

Multiplication formula.

$$\begin{aligned}\Gamma(z)\Gamma(z+\frac{1}{n})\Gamma(1+\frac{2}{n})\ldots\Gamma(z+\frac{n-1}{n})\\ = (2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz).\end{aligned}$$

In particular, when $n=2$, we have

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \pi^{\frac{1}{2}}\Gamma(2z).$$

Proof. For this, let

$$\phi(z) := \frac{n^{nz}}{n\Gamma(nz)}\prod_{r=0}^{n-1}\Gamma(z+\frac{r}{n}).$$

Then, by the limit formula, we obtain

$$\begin{aligned}\phi(z) =& n^{nz-1}\lim_{m\to\infty}\frac{[(m-1)!]^n m^{nz+\frac{1}{2}(n-1)}n^{nm}}{(nm-1)!(nm)^{nz}}\\ =&\lim_{m\to\infty}\frac{[(m-1)!]^n m^{\frac{1}{2}(n-1)}n^{nm-1}}{(nm-1)!}.\end{aligned}$$

Thus $\phi(z)$ is independent of z. Now let $z=\frac{1}{2}$, we have

$$\phi = \prod_{r=0}^{n-1}\Gamma(\frac{r+1}{n}) = \prod_{r=1}^{n-1}\Gamma(\frac{r}{n}) = \prod_{r=1}^{n-1}\Gamma(1-\frac{r}{n}).$$

So

$$\phi^2 = \prod_{r=1}^{n-1}\Gamma(\frac{r}{n})\Gamma(1-\frac{r}{n}) = \pi^{n-1}\prod_{r=1}^{n-1}(\sin\frac{\pi r}{n})^{-1}.$$

But by

$$\sum_{r=0}^{n-1}z^r = \frac{z^n-1}{z-1} = \prod_{r=1}^{n-1}(z-e^{2\pi ri/n}),$$

we have

$$n = \prod_{r=1}^{n-1}(1-r^{2\pi ir/n}) = \prod_{r=1}^{n-1}r^{\pi ri/n}(-2i\sin\frac{\pi r}{n}) = 2^{n-1}\prod_{r=1}^{n-1}\sin\frac{\pi r}{n},$$

which completes the proof.

We end this section by deriving the Stirling formula.

Let

$$\varphi(z) = \frac{d}{dt}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Then, by the functional equation, we know that

$$\varphi(z+1) = \varphi(z) + \frac{1}{z}.$$

Therefore

$$\varphi(z+n) = \varphi(z) + \sum_{r=0}^{n-1}\frac{1}{z+r}.$$

On the other hand, by Weierstrass' infinite product formula, we have

$$\begin{aligned}\varphi(z) &= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)\\ &= -\frac{1}{z} + \lim_{m\to\infty}\{\ln m - \sum_{n=1}^{m}\frac{1}{z+n}\}.\end{aligned}$$

So $\Gamma'(1) = \varphi(1) = -\gamma$. Furthermore, by

$$\ln m = \int_0^{\infty}(e^{-t} - e^{-mt})\frac{dt}{t},$$

we have

$$\varphi(z) = \lim_{m\to\infty}\left[\int_0^{\infty}(e^{-t} - e^{-mt})\frac{dt}{t} - \int_0^{\infty}\frac{e^{-zt}(1-e^{-(m+1)t})}{1-e^{-t}}dt\right].$$

Now since the integrations of the terms containing e^{-mt} go to 0, we have

$$\varphi(z) = \int_0^{\infty}\left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt.$$

When $z = 1$, we have

$$\gamma = \int_0^{\infty}\left[\frac{1}{1-e^{-t}} - \frac{1}{t}\right]e^{-t}dt,$$

and

$$\varphi(z) = -\gamma + \int_0^{\infty}\frac{e^{-t} - e^{-zt}}{1-e^{-t}}dt.$$

But

$$\begin{aligned}\varphi(z) =& \ln z + \int_0^\infty (\frac{1}{t} - \frac{1}{1-e^t}) e^{-zt} dt \\ =& \ln z - \frac{1}{2z} + \int_0^\infty (\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^t}) e^{-zt} dt.\end{aligned}$$

Thus, integrating with respect to z from 1 to z, we obtain

$$\begin{aligned}\ln \Gamma(z) =& (z - \frac{1}{2}) \ln z - z + 1 \\ &+ \int_0^\infty (\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}) [e^{-t} - e^{-zt}] \frac{dt}{t}.\end{aligned}$$

Now we claim that

$$I := \int_0^\infty (\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}) e^{-t} \frac{dt}{t} = \frac{1}{2} \ln (2\pi) - 1.$$

In fact, let $z = \frac{1}{2}$ and we have

$$I - J = \frac{1}{2} \ln \pi - \frac{1}{2},$$

where

$$J := \int_0^\infty (\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}) e^{-t/2} \frac{dt}{t}.$$

Also

$$I = \int_0^\infty (\frac{1}{2} + \frac{2}{t} - \frac{1}{1-e^{-t/2}}) e^{-t/2} \frac{dt}{t},$$

so

$$I - J = \int_0^\infty (\frac{1}{t} - \frac{e^{-t/2}}{1-e^{-t}}) e^{-t/2} \frac{dt}{t}.$$

Thus

$$J = \int_0^\infty (\frac{1}{2} e^{-t} + \frac{1}{t} e^{-t} - \frac{e^{-t/2}}{t}) \frac{dt}{t}.$$

Integrating by parts with respect to the last two terms, we have

$$\begin{aligned}J =& -\frac{e^{-t} - e^{-t/2}}{t} \Big|_0^\infty - \frac{1}{2} \int_0^\infty [e^{-t} - e^{-t/2}] \frac{dt}{t} \\ =& -\frac{1}{2} - \frac{1}{2} \ln \frac{1}{2}.\end{aligned}$$

Therefore,

$$I = \frac{1}{2} \ln (2\pi) - 1$$

and

$$\begin{aligned}\ln \Gamma(z) =& [(z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln (2\pi)] \\ &- \int_0^\infty (\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}) e^{-zt} \frac{dt}{t}.\end{aligned}$$

Furthermore, by the result at the end of 4.1, we know that

$$\frac{-t}{e^{-t}-1} = 1 + \frac{t}{2} + \sum_{r=1}^{n} \frac{(-1)^{r-1}B_r}{(2r)!} t^{2r} + \frac{t^{2n+2}}{e^{-t}-1} \int_0^1 P_{2n+1}(x) e^{-tx} dx.$$

But

$$\int_0^{\infty} t^{2r-2} e^{-zt} dt = \frac{(2r-2)!}{z^{2r-1}},$$

and

$$\left| \int_0^1 P_{2n+1}(x) e^{-tx} dx \right| \leq \frac{4}{(2\pi)^{2n+1}} \int_0^1 e^{-tx} dx = \frac{4}{(2\pi)^{2n+1}} \frac{e^{-t}-1}{-t}.$$

So, we have

$$\begin{aligned} \ln \Gamma(z) =& (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln (2\pi) \\ &+ \sum_{r=1}^{n} \frac{(-1)^{r-1} B_r}{2r(2r-1)} z^{-2r+1} + O(z^{-2n-1}). \end{aligned}$$

Then, by putting $z = n$, and using $\Gamma(n+1) = n!$, we have

Stirling's formula

$$\ln (n!) \sim n \ln n - n + \frac{1}{2} \ln n + \frac{1}{2} \ln (2\pi) + O(n^{-1}).$$

§I.4.3. Riemann Zeta Function

In this section, we consider the famous Riemann zeta function. The reference here is [T 51].

The **Riemann zeta function**, $\zeta_{\mathbb{Q}}(s)$, is defined by

$$\zeta_{\mathbb{Q}}(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\mathrm{Re}(s) > 1$. Since each integer n has a unique factorization as a product of primes, we know that

$$\zeta_{\mathbb{Q}}(s) = \prod_p (1 - \frac{1}{p^s})^{-1}.$$

The restriction to $\mathrm{Re}(s) > 1$ in the definition depends on the fact that, for any $\delta > 1$, the infinite sum and product are absolutely convergent when $\mathrm{Re}(s) > \delta$. On the other hand, we have the following

Theorem. $\zeta_{\mathbf{Q}}(s)$ has a meromorphic continuation on the whole complex plane. On the complex plane, this function is regular for all s except $s = 1$, where there is a simple pole with residue 1. Furthermore, it satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

Here

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta_{\mathbf{Q}}(s).$$

Proof. (Riemann) (I) The meromorphic continuation: This is based on the following fundamental formula

$$\zeta_{\mathbf{Q}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^s}{e^x - 1}\frac{dx}{x}$$

for $\mathrm{Re}(s) > 1$. In fact, for $\mathrm{Re}(s) > 1$,

$$\int_0^\infty x^s e^{-nx}\frac{dx}{x} = \frac{1}{n^s}\int_0^\infty y^s e^{-y}\frac{dy}{y} = \frac{\Gamma(s)}{n^s}.$$

Hence, by the absolute convergence for $\mathrm{Re}(s) > 1$, we have

$$\Gamma(s)\zeta_{\mathbf{Q}}(s) = \sum_{n=1}^\infty \int_0^\infty x^s e^{-nx}\frac{dx}{x} = \int_0^\infty x^s \sum_{n=0}^\infty e^{-nx}\frac{dx}{x} = \int_0^\infty \frac{x^s}{e^x - 1}\frac{dx}{x}.$$

Now consider the integral

$$I(s) := \int_C \frac{z^{s-1}}{e^z - 1}dz,$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction, excluding the points $\pm 2\pi i, \pm 4\pi i, \ldots$ and returns to positive infinity. Here z^{s-1} is taken as $e^{(s-1)\log z}$ when the logarithm is real at the beginning of the contour. (Thus $\mathrm{Im}(\log z)$ varies from 0 to 2π round the contour.)

Take C as the real axis from ∞ to ρ, the circle $|z| = \rho$, and the real axis from ρ to ∞ with $0 < \rho < 2\pi$. On the circle,

$$|z^{s-1}| = e^{(\mathrm{Re}(s)-1)\log|z| - t\arg z} \leq |z|^{\mathrm{Re}(s)-1}e^{2\pi|t|},$$

and

$$|e^z - 1| > A|z|.$$

Hence the integral round this circle tends to 0 with ρ if $\mathrm{Re}(s) > 1$. But if $\rho \to 0$, we have

$$\begin{aligned} I(s) = & -\int_0^\infty \frac{x^s}{e^x - 1}\frac{dx}{x} + \int_0^\infty \frac{(e^{2\pi i})^{s-1}x^s}{e^x - 1}\frac{dx}{x} \\ =&(e^{2\pi i s} - 1)\Gamma(s)\zeta_{\mathbf{Q}}(s) \\ =&\frac{2\pi i e^{\pi i s}}{\Gamma(1-s)}\zeta_{\mathbf{Q}}(s). \end{aligned}$$

Hence

$$\zeta_{\mathbf{Q}}(s) = \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i} I(s)$$

for $\mathrm{Re}(s) > 1$. On the other hand, the integral $I(s)$ is uniformly convergent in any finite region of the s-plane, and so defines an integral function of s. Hence the formula provides the analytic continuation of $\zeta_{\mathbf{Q}}(s)$ over the whole s-plane. In this way, we see that the only possible singularities are the pole of $\Gamma(1-s)$, say, $s = 1, 2, 3, \ldots$. But we know that $\zeta_{\mathbf{Q}}(s)$ is regular at $s = 2, 3, \ldots$. So the only possible singularity is a simple pole at $s = 1$. Since

$$I(1) = \int_C \frac{dz}{e^z - 1} = 2\pi i$$

and

$$\Gamma(1-s) = -\frac{1}{s-1} + \text{the regular part},$$

the residue at $s = 1$ is 1.

(II) The functional equation. To deduce the functional equation, take the integral along the contour C_n consisting of the positive real axis from infinity to $(2n+1)\pi$, then round the square with corners $(2n+1)\pi(\pm 1 \pm i)$, and finally back to infinity along the positive real axis. In the region between the contours C and C_n the integrand has poles at the points $\pm 2\pi i, \ldots, \pm 2n\pi i$. The corresponding residues at $2m\pi i$ and $-2m\pi i$ are

$$(2m\pi e^{\pi i/2})^{s-1} + (2m\pi e^{3\pi i/2})^{s-1} = -2(2m\pi)^{s-1} e^{\pi i s} \sin(\frac{1}{2}\pi s).$$

So, by the residue theorem,

$$I(s) = \int_{C_n} \frac{z^{s-1}}{e^{-z} - 1} dz + 4\pi i e^{\pi i s} \sin(\frac{1}{2}\pi s) \sum_{m=1}^{n} (2m\pi)^{s-1}.$$

Now let $\mathrm{Re}(s) < 0$ and $n \to \infty$. The function $1/(e^z - 1)$ is bounded on C_n, and $z^{s-1} = O(|z|^{\mathrm{Re}(s)-1})$. Hence the integral round C_n tends to 0, and we obtain

$$I(s) = 4\pi i e^{\pi i s} \sin(\frac{1}{2}\pi s)(2\pi)^{s-1}\zeta_{\mathbf{Q}}(1-s).$$

From here, by the properties of the Gamma function, we easily have the assertion.

With above, we may also obtain the values of the Riemann zeta function on the positive integers. In fact, if n is an integer, the integrand in $I(n)$ is one valued, and $I(n)$ can be evaluated by the residue theorem. Therefore

$$\zeta_{\mathbf{Q}}(0) = -\frac{1}{2}, \ \ \zeta_{\mathbf{Q}}(-2m) = 0, \ \ \zeta_{\mathbf{Q}}(1-2m) = \frac{(-1)^m B_m}{2m},$$

for each positive integer m. Now by the functional equation of the Riemann zeta function, let $s = 1 - 2m$ and we have

$$\zeta_{\mathbf{Q}}(2m) = 2^{2m-1}\pi^{2m}\frac{B_m}{(2m)!}.$$

Another minor application is as follows: By the functional equation, we know that

$$-\frac{\zeta'_{\mathbf{Q}}(1-s)}{\zeta_{\mathbf{Q}}(1-s)} = -\log 2\pi - \frac{1}{2}\pi\tan\frac{1}{2}s\pi + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'_{\mathbf{Q}}(s)}{\zeta_{\mathbf{Q}}(s)}.$$

But in the neighborhood of $s = 1$,

$$\frac{1}{2}\pi\tan\frac{1}{2}s\pi = -\frac{1}{(s-1)} + O(|s-1|),$$

$$\frac{\Gamma'(s)}{\Gamma(s)} = \frac{\Gamma'(1)}{\Gamma(1)} + \ldots = -\gamma + \ldots,$$

and

$$\frac{\zeta'_{\mathbf{Q}}(s)}{\zeta_{\mathbf{Q}}(s)} = \frac{-1/(s-1)^2 + k + \ldots}{1/(s-1) + \gamma + k(s-1) + \ldots} = -\frac{1}{s-1} + \gamma + \ldots,$$

where k is a constant. Hence, making $s \to 1$, we have

$$\zeta'_{\mathbf{Q}}(0) = -\frac{1}{2}\log 2\pi.$$

We end this section with the following remark. The Riemann zeta function is very important in number theory, complex analysis, etc. One reason is that there are two expressions for it: One is as a sum, while the other is as a product. Especially, one may study primes by using the Riemann zeta function. There are many conjectures related to the Riemann zeta function. The most famous one is the following

Riemann Hypothesis. All the non-real complex zeros of $\zeta_{\mathbf{Q}}(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$.

§I.4.4. Mellin Transform

The Mellin transform comes originally from the Mellin inversion formula, which connects two functions $f(x)$ and $\mathcal{F}(s)$ by the relations

$$\mathcal{F}(s) = \int_0^\infty f(x)x^s\frac{dx}{x}, \quad f(x) = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\mathcal{F}(s)x^{-s}ds,$$

where a is given a real number. The simplest example of this is

$$f(x) = e^{-x}, \quad \mathcal{F}(s) = \Gamma(s).$$

From an earlier section, we also have the example with

$$f(x) = \frac{1}{e^x - 1}, \quad \mathcal{F}(s) = \Gamma(s)\zeta_{\mathbf{Q}}(s).$$

The importance of the above formula is that after such a transform, we may make a continuation of a function, which is originally only defined in a restricted region.

Let $f(t)$ be a continuous function of $t > 0$, such that

M1. For $t \to \infty$, $f(t)$ decays exponentially. That is, $f(t)$ is $O(e^{-ct})$ with $c \in \mathbf{R}_{>0}$.

M2. There is n, for $t \to 0$, $t^n f(t)$ is C^∞. That is, $f(t)$ has an asymptotic expansion

$$f(t) = \sum_{k=-n}^{0} a_k t^k + O(t).$$

Then the **Mellin transform** of $f(t)$, denoted as $M[f](s)$, is defined to be the complex function

$$\frac{1}{\Gamma(s)} \int_0^\infty f(t) t^s \frac{dt}{t}.$$

The basic properties of $M[f](s)$ are in the following

Proposition 1. (a) $M[f](s)$ converges for $\mathrm{Re}(s) > n$.
(b) There is a meromorphic continuation of $M[f](s)$ to the whole complex s-plane.
(c) $M[f](s)$ is holomorphic at 0, and hence it makes sense to talk about $M[f]'(0)$.

Proof. We divide the integration into three parts: $[0, \delta], [\delta, N]$, and $[N, \infty]$ for $0 < \delta < N < \infty$. The fact that the integrate for each part is convergent just comes from the above conditions. Hence we have (a). For the proof of (b), we use the same method as in last section by using the contour C. In this way, we get the expression that

$$M[f](s) = \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}}{f(z)} dz.$$

Hence we have (b). Now (c) is a direct consequence of the above expression. Since the only possible singularities are that $s = 1, 2, \ldots$. Therefore $s = 0$ is a regular point for $M[f](z)$.

Example. Let $f(t) = e^{-\lambda t}$, then we have $M[e^{-\lambda t}](s) = \lambda^{-s}$ and

$$\int_0^\infty e^{-\lambda t} \frac{dt}{t} = -\log\lambda,$$

which is equal to $M[e^{-\lambda t}]'(0)$. Motivated by this fact, we will also denote $M[f]'(0)$ by $\int_0^\infty f(t)\frac{dt}{t}$, even through the integrate may not really exist as what stands.

Now we give a precise expression for $M[f]'(0)$.

Proposition 2. With the same condition and notation as above, if

$$f(t) = \sum_{k\leq 0} a_k t^k + \rho_0(t),$$

then

$$M[f]'(0) = \int_0^1 \rho_0(t)\frac{dt}{t} + \int_1^\infty f(t)\frac{dt}{t} - \Gamma'(1)a_0 + \sum_{n<0}\frac{a_n}{n}.$$

Proof. In fact, by our condition, we have

$$\begin{aligned} M[f](s) =& \frac{1}{\Gamma(s)}\int_0^1 \rho_0(t)t^s\frac{dt}{t} + \sum_{n\leq 0}\frac{a_n}{\Gamma(s)}\int_0^1 t^{n+s}\frac{dt}{t} \\ &+\frac{1}{\Gamma(s)}\int_1^\infty f(t)t^s\frac{dt}{t}. \end{aligned}$$

Hence,

$$\begin{aligned} M[f]'(0) =& \int_0^1 \rho_0(t)(\frac{t^s}{\Gamma(s)})'_{s=0}\frac{dt}{t} + \sum_{n\leq 0}(\frac{a_n}{\Gamma(s)(n+s)})'_{s=0} \\ &+\int_1^\infty f(t)(\frac{t^s}{\Gamma(s)})_{s=0}\frac{dt}{t}. \end{aligned}$$

Now the conclusion comes from the following facts:

$$\begin{aligned} (\frac{t^s}{\Gamma(s)})'_{s=0} &= 1, \\ (\frac{1}{\Gamma(s)s})'_{s=0} &= (\frac{1}{\Gamma(s+1)})'_{s=0} = -\frac{\Gamma'(1)}{\Gamma(1)^2} = -\Gamma'(1), \\ (\frac{1}{\Gamma(s)(n+s)})'_{s=0} &= \frac{1}{n}. \end{aligned}$$

Now we turn to applications of the above idea. The final aim is to deal with Laplacians. By Lemma 3.2, we can show that the eigenvalues of a Laplacian over a compact manifold are non-negative numbers and they are discrete.

Recall the following

Fact. Let A be an endomorphism of an n-dimensional Euclidean vector space V with real strictly positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

Then

$$\det A = \prod_{j=1}^n \lambda_j = \exp(-\frac{d}{ds}\sum_{j=1}^n \lambda_j^{-s}|_{s=0}).$$

But for infinite many positive numbers λ_j, as the product may not be convergent, how we can offer a reasonable definition for $\prod \lambda_j$? To do so, let us look at Stirling's formula. So we may take the finite part of the expression of $n!$ as the definition for $\infty!$. That is

$$\infty! := \sqrt{2}\pi = \exp(-\zeta_{\mathbf{Q}}'(0)).$$

With this in mind, we introduce the follows:

For an increasing sequence of positive real numbers

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots,$$

the **zeta function** associated with this sequence is defined by

$$\zeta_\lambda(s) := \sum_{n\geq 1} \lambda_n^{-s}.$$

Suppose we have the following conditions:
Z1. $\zeta_\lambda(s)$ converges for $\mathrm{Re}\, s \gg 0$.
Z2. $\zeta_\lambda(s)$ has a meromorphic continuation to the whole s-plane.
Z3. $\zeta_\lambda(s)$ has no pole at $s = 0$.

By definition, we set

$$\prod_{n=1}^{\infty} \lambda_n := \exp(-\zeta_\lambda'(0)).$$

With an increasing sequence as above, we define the associated **theta function** as

$$\Theta_\lambda(t) := \sum_{n=1}^{\infty} e^{-\lambda_n t}.$$

Lemma. Suppose λ is such that
Θ1. $\Theta_\lambda(t)$ converges for $t > 0$;
Θ2. For $t \to \infty$, $\Theta_\lambda(t)$ decays exponentially.
Θ3. For $t \to 0$, $t^n \Theta_\lambda(t)$ is C^∞.
Then we have

$$\zeta_\lambda(s) = M[\Theta_\lambda](s)$$

satisfies **Z1**, **Z2**, and **Z3**. In particular,

$$\zeta_\lambda'(0) = \int_0^\infty \Theta_\lambda(t) \frac{dt}{t}.$$

§I.1.4.5. Another Construction of Classical Bott-Chern Secondary Characteristic Forms

I.4.5.a The Double Transgression Formula

Let B be a connected complex manifold, and let $J \in \mathrm{End}\,(TB)$ be the complex structure of B. Let

$$0 \to E_0 \xrightarrow{v} \dots \xrightarrow{v} E_m \to 0$$

be a holomorphic chain complex of finite dimensional complex holomorphic vector bundles on B with hermitian metrics ρ_j on E_j for $0 \le j \le m$. Set

$$E^+ := \oplus_{j \text{ even}} E_j, \quad E^- := \oplus_{j \text{ odd}} E_j, \quad E = E^+ \oplus E^-.$$

Let N be the number operator on E which defines the $\mathbf{Z}$-grading of E, i.e. N is the multiplication by j on E_j. Similarly, let τ be the number operator defining the $\mathbf{Z}_2$-grading of E, i.e. $\tau = \pm 1$ on $E^{\pm}$. Also let v^* be the adjoint of v. For $a \in \mathbf{C}$, set $V^a := av + \bar{a}v^*$ and $V := V^1$. Then, if $\nabla = \nabla' + \nabla''$ is the canonical connection of $(E, \rho := \oplus\rho_j)$ with ∇', ∇'' the holomorphic and antiholomorphic parts of ∇, $\nabla + V^a$ is a superconnection on E.

In the proof later, we have to use the following properties of the number operator N, which may be easily checked:

$$[\nabla, N] = 0, \quad [v, N] = -v, \quad [v^*, N] = v^*, \quad [\bar{\partial}, N] = -\bar{\partial}, \quad [\bar{\partial}^*, N] = -\bar{\partial}^*.$$

Let P be the subspace of the smooth sections of $\wedge T^*_{\mathbf{C}}B$, which are sums of the differential forms of complex type (p,p). Let P' be the set of ∂, $\bar{\partial}$-exact smooth forms in P, i.e. these that can be written as $\partial^B\eta + \bar{\partial}^B\eta'$ with η, η' smooth forms on B. Then if A denotes the vector subspace of $\wedge T^*_{\mathbf{C}}B \hat{\otimes} \mathrm{End} E$ generated by smooth sections of $\wedge^{p,q}T^*_{\mathbf{C}}B \hat{\otimes} \mathrm{Hom}(E_j, E_{j+p-q})$ for all $p, q, j \ge 0$, A is an algebra and $\mathrm{Tr}_s \eta \in P$ for any $\eta \in A$.

Theorem. (1) For any $a \in \mathbf{C}$, the differential forms

$$\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2)], \quad \mathrm{Tr}_s[N \exp(-(\nabla + V^a)^2)]$$

are in P and only depend on $|a|$.
(2) $\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2)]$ is closed.
(3) **(Double Transgression Formula.)**

$$\begin{aligned}
\frac{\partial}{\partial a}\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2)] &= -\,\partial^B \mathrm{Tr}_s[v \exp(-(\nabla + V^a)^2)];\\
\frac{\partial}{\partial \bar{a}}\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2)] &= -\,\bar{\partial}^B \mathrm{Tr}_s[v^* \exp(-(\nabla + V^a)^2)];\\
\mathrm{Tr}_s[av \exp(-(\nabla + V^a)^2)] &= -\,\partial^B \mathrm{Tr}_s[N \exp(-(\nabla + V^a)^2)];\\
\mathrm{Tr}_s[\bar{a}v^* \exp(-(\nabla + V^a)^2)] &= \partial^B \mathrm{Tr}_s[N \exp(-(\nabla + V^a)^2)].
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial a}\mathrm{Tr}_s[\exp(-(\nabla+V^a)^2)] = -\frac{1}{a}\bar\partial^B\partial^B\mathrm{Tr}_s[N\exp(-(\nabla+V^a)^2)];$$
$$\frac{\partial}{\partial \bar a}\mathrm{Tr}_s[\exp(-(\nabla+V^a)^2)] = -\frac{1}{\bar a}\bar\partial^B\partial^B\mathrm{Tr}_s[N\exp(-(\nabla+V^a)^2)].$$

Proof. 1. From the definitions, we have

$$e^{i\theta N}(\nabla+V^a)e^{-i\theta N} = \nabla + V^{ae^{i\theta}}$$

for $\theta\in\mathbf{R}$. Thus $\mathrm{Tr}_s[\exp(-(\nabla+V^a)^2)]$ and $\mathrm{Tr}_s[N\exp(-(\nabla+V^a)^2)]$ are radical functions of a. On the other hand, by a direct calculation,

$$(\nabla+V^a)^2 = \nabla^2 + |a|^2(vv^*+v^*v) + a\nabla'v + \bar a\nabla''v^*.$$

So, by definition, we have (1).

(2) That $\mathrm{Tr}_s[\exp(-(\nabla+V^a)^2)]$ is closed is a standard result in the superconnection formalism, say Prop. 1.3.2.

(3) The last two relations are direct consequences of the formal relations. For the first two equalities, we proceed as follows: On $B\times\mathrm{C}$, the form $\mathrm{Tr}_s[\exp(-(\nabla+da\frac{\partial}{\partial a}+d\bar a\frac{\partial}{\partial \bar a}+V^a)^2)]$ is closed. It is equal to $\mathrm{Tr}_s[\exp(-(\nabla+V^a)^2 - da\,v - d\,\bar a\,v^*)]$. Thus by Duhamel's formula, it becomes

$$\begin{aligned}&\mathrm{Tr}_s[\exp(-(\nabla+V^a)^2)] - \mathrm{Tr}_s[v\exp(-(\nabla+V^a)^2)]\,da\\ &\quad-\mathrm{Tr}_s[v^*\exp(-(\nabla+V^a)^2)]\,d\bar a + \alpha\,da\,d\bar a,\end{aligned}$$

with α a differential form on B, since Tr_s vanishes on supercommutators. Now our first two relations may be deduced from the fact that this last combination of form is closed under $\partial^B + da\frac{\partial}{\partial a}$ and $\bar\partial^B + d\bar a\frac{\partial}{\partial \bar a}$. Finally, a simple degree counting argument of the following relations will complete the proof:

$$\begin{aligned}&d^B\mathrm{Tr}_s[N\exp(-(\nabla+V^a)^2)]\\ &=\mathrm{Tr}_s[\nabla+V^a, N\exp(-(\nabla+V^a)^2)]\\ &=\mathrm{Tr}_s([V^a,N]\exp(-(\nabla+V^a)^2)]\\ &=\mathrm{Tr}_s[(-av+\bar a v^*)\exp(-(\nabla+V^a)^2)].\end{aligned}$$

Here we use the properties of number operators listed before the theorem.

I.4.5.b. **Asymptotic Expansion**

For any $u \geq 0$, let $\mathbf{A}_u := \nabla + \sqrt{u}V$. Then, as in the Mellin transform, *if* we have the right asymptotic expansion, then

$$\zeta_{E.,\rho.}(s) := \frac{1}{\Gamma(s)} \int_0^{+\infty} u^s \mathrm{Tr}_s[N \exp(-\mathbf{A}_u^2)] \frac{du}{u}$$

is a well-defined element in P for $s \in \mathbf{C}$, $\mathrm{Re}(s) > 0$, and $\zeta_{E.,\rho.}(s)$ has a unique meromorphic continuation on the whole complex plane such that it is holomorphic at 0. Thus by the above theorem, we know that

$$\int_0^{+\infty} \mathrm{Tr}_s[V \exp(-\mathbf{A}_u^2)] u^{1/2} \frac{du}{u} = -(\partial^B - \bar{\partial}^B)\zeta'_{E.,\rho.}(0)$$
$$\mathrm{Tr}_s[\exp(-\nabla^2)] = -\bar{\partial}^B \partial^B \zeta'_{E.,\rho.}(0).$$

Hence, if we define

$$\mathrm{ch}_{\mathrm{BC}}(E.,\rho.) := [2\pi \mathbf{i}]\zeta'_{E.,\rho.}(0),$$

axiom 1 is valid. Axiom 2 is trivial. And axiom 3 is a consequence of the fact that, in the splitting case, v commutes with ∇ and $vv^* + v^*v = \mathrm{Id}$.

Now we have to show that $\zeta_{E.,\rho.}(s)$ makes sense as in the Mellin transform. For this, we check the condition for the Mellin transform. First, look at the asymptotic expansion of $\mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)]$ when $u \to 0^+$. Even through at this stage, we may offer a simple method to deal with it, but in order to explain the basic idea of the proof for the infinite dimensional case later, we prove it by a certain concrete calculation. Also, we will go widely. More precisely, we have the following

Theorem. When $u \to 0^+$,

$$\mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)] = \sum_{j=-1}^{k} \mu_j \, u^j + o(u^k)$$

with

$$\mu_0 = \mathrm{Tr}_s(N\exp(-\nabla^2).$$

Here o is uniform for any compact subset of B.

We do this by introduce a new parameter b. The key point for this is the following easy generalization of the theorem in subsection a:

Proposition 1. For any $a, b \in \mathbf{C}$, we have

$$\begin{aligned}&\partial^B \mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + bN)] \\ &\quad = b\bar{a}\, \mathrm{Tr}_s[v^* \exp(-(\nabla + V^a)^2 + bN)]; \\ &\bar{\partial}^B \mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + bN)] \\ &\quad = -\, ba\, \mathrm{Tr}_s[v \exp(-(\nabla + V^a)^2 + bN)].\end{aligned}$$

Proof. Indeed, we may deduce the result from the degree counting of the following obviouse relation:

$$\begin{aligned}
&d^B\mathrm{Tr}_s(\exp(-(\nabla+V^a)^2+bN))\\
&\quad=\mathrm{Tr}_s([\nabla+V^a,\exp(-(\nabla+V^a)^2+bN)])\\
&\quad=b\mathrm{Tr}_s([V^a,N]\exp(-(\nabla+V^a)^2+bN)])\\
&\quad=b\mathrm{Tr}_s((-av+\bar{a}v^*)\exp(-(\nabla+V^a)^2+bN)).
\end{aligned}$$

Now let $\alpha_1(a), \alpha_2(a)$, and $\alpha_3(a)$ denote the differential forms on B defined by

$$\begin{aligned}
&\mathrm{Tr}_s[N\exp(-(\nabla+da\frac{\partial}{\partial a}+d\bar{a}\frac{\partial}{\partial \bar{a}}+V^a)^2)]\\
&\quad=\mathrm{Tr}_s[N\exp(-(\nabla+V^a)^2)]+\alpha_1\,da+\alpha_2\,d\bar{a}+\alpha_3\,da\,d\bar{a}
\end{aligned}$$

according the Grassmannian degree on $\mathbf{C}$. (In general, if $\eta=\eta_0+\eta_1\,da+\eta_2\,d\bar{a}+\eta_3\,da\,d\bar{a}$ is a decomposition of η according to the Grassmannian degree of $\mathbf{C}$, we may also denote η_3 by $[\eta]^{da\ d\bar{a}}$.) On the other hand, since

$$\mathrm{Tr}_s[\exp(-(\nabla+V^a)^2+bN)]$$

is a smooth function of $|a|^2$, then by Duhamel's formula, there exists a smooth form $\beta_1(x,a,b)$ for $(x,a,b)\in B\times\mathbf{C}\times\mathbf{C}$, which depends smoothly on $|a|^2$ and is such that

$$\mathrm{Tr}_s[\exp(-(\nabla+V^a)^2+bN)]=\mathrm{Tr}_s[\exp(-\nabla^2+bN)]+|a|^2\beta_1(x,a,b).$$

Proposition 2. With the same notation as above, we have

$$\begin{aligned}
\alpha_1(a)&=\bar{a}\,\bar{\partial}^B[\frac{1}{2}\frac{\partial^2}{\partial b^2}\beta_1]_{b=0};\\
\alpha_2(a)&=-\,a\,\partial^B[\frac{1}{2}\frac{\partial^2}{\partial b^2}\beta_1]_{b=0}.
\end{aligned}$$

Proof. By Duhamel's formula, we know that in the expression of the Grassmannian degree on $\mathbf{C}$,

$$\mathrm{Tr}_s[\exp(-(\nabla+da\frac{\partial}{\partial a}+d\bar{a}\frac{\partial}{\partial \bar{a}}+V^a)^2+bN)],$$

the form which appears as the coefficient of da is given by

$$-\mathrm{Tr}_s[v\exp(-(\nabla+V^a)^2+bN)],$$

i.e.

$$\frac{1}{ba}\bar{\partial}^B\mathrm{Tr}_s[\exp-(\nabla+V^a)^2]$$

by the above proposition for $ab \neq 0$. On the other hand, formally, by the fact that

$$\frac{\partial}{\partial x}e^{sx+t}|_{x=0} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(xe^{sx+t})|_{x=0} = se^t,$$

we have

$$\begin{aligned} &\frac{\partial}{\partial b}\mathrm{Tr}_s[\exp(-(\nabla + da\frac{\partial}{\partial a} + d\bar{a}\frac{\partial}{\partial \bar{a}} + V^a)^2 + bN)]_{b=0} \\ = &\frac{1}{2}\frac{\partial^2}{\partial b^2}b\mathrm{Tr}_s[\exp(-(\nabla + da\frac{\partial}{\partial a} + d\bar{a}\frac{\partial}{\partial \bar{a}} + V^a)^2 + bN)]_{b=0} \\ = &\mathrm{Tr}_s[N\exp(-(\nabla + da\frac{\partial}{\partial a} + d\bar{a}\frac{\partial}{\partial \bar{a}} + V^a)^2)]. \end{aligned}$$

Hence,

$$a\alpha_1(a) = \bar{\partial}^B\frac{1}{2}\frac{\partial^2}{\partial^2 b}\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + bN)]_{b=0}.$$

Let $a = 0$ and we have

$$\bar{\partial}^B\frac{1}{2}\frac{\partial^2}{\partial^2 b}\mathrm{Tr}_s[\exp(-\nabla^2 + bN)]_{b=0} = 0.$$

So

$$a\alpha_1(a) = \bar{\partial}^B\frac{1}{2}\frac{\partial^2}{\partial^2 b}(\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + bN) - \exp(-\nabla^2 + bN)])_{b=0}.$$

Now by the definition of $\beta_1(x, a, b)$, we have the first relation. The proof for $\alpha_2(a)$ is similar.

Proposition 3. For any $(a, b) \in \mathbf{C} \times \mathbf{C}$,

$$\begin{aligned} \frac{\partial}{\partial a}(\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + b|a|^2N)]) = -d^B(\mathrm{Tr}_s[v\exp(-(\nabla + V^a)^2 + b|a|^2N)]) \\ + b\bar{a}\,\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + (b|a|^2 + da\,d\bar{a})N) - a\,da\,v - \bar{a}\,d\bar{a}\,v^*)]^{da d\bar{a}}. \end{aligned}$$

In particular,

$$\begin{aligned} \frac{\partial}{\partial a}|a|^2(\mathrm{Tr}_s[N\exp(-(\nabla + V^a)^2)]) = -d^B\frac{\partial}{\partial b}(\mathrm{Tr}_s[v\exp(-(\nabla + V^a)^2 + b|a|^2N)])_{b=0} \\ + \bar{a}\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + da\,d\bar{a}\,N - a\,da\,v - \bar{a}\,d\bar{a}\,v^*)]^{da d\bar{a}}. \end{aligned}$$

Proof. Differentiating the first relation at $b = 0$, we easily have the second relation. For the first, we know that the LHS is the coefficient of the da in

$$[da\frac{\partial}{\partial a}, \mathrm{Tr}_s[\exp(-(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N)]].$$

But, by the properties of the number operator N, we know that the last expression is

$$\begin{aligned}
&\mathrm{Tr}_s([da\frac{\partial}{\partial a}, -(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N]\exp(-(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N)) \\
&= \mathrm{Tr}_s((([\nabla + V^a, (\nabla + da\frac{\partial}{\partial a} + V^a)^2] + b|a|^2N)\exp(-(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N)) \\
&= \mathrm{Tr}_s((([\nabla + V^a, (\nabla + da\frac{\partial}{\partial a} + V^a)^2 - b|a|^2N] + b|a|^2[V^a, N] + b\bar{a}\, N\, da) \\
&\qquad \exp(-(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N)] \\
&= -d^B\mathrm{Tr}_s(\exp(-(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N)) \\
&\qquad + \mathrm{Tr}_s(b|a|^2(-av + \bar{a}v^*) + b\bar{a}\, N\, da)\exp(-(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N)).
\end{aligned}$$

Here, as before, we also use the properties of the number operator N. On the other hand, taking the factor of da in

$$\mathrm{Tr}_s([da\frac{\partial}{\partial a}, -(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N]\exp(-(\nabla + da\frac{\partial}{\partial a} + V^a)^2 + b|a|^2N)),$$

we have

$$\begin{aligned}
&\frac{\partial}{\partial a}\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + b|a|^2N)] = \\
&\qquad - d^B(\mathrm{Tr}_s[v\exp(-(\nabla + V^a)^2 + b|a|^2N)]) \\
&\qquad + b\bar{a}\,\mathrm{Tr}_s[\exp(-(\nabla + V^a)^2 + (b|a|^2 + da\, d\bar{a})N) \\
&\qquad - av\, da - (-av - \bar{a}v^*)\, d\bar{a} + N\, da\, d\bar{a})]^{da d\bar{a}}.
\end{aligned}$$

Thus the assertion comes from the facts that

$$\begin{aligned}
av\, da + (-av + \bar{a}V^*)\, d\bar{a} &= av\,(da - d\bar{a}) + \bar{a}\, v^*\, d\bar{a}; \\
(da - d\bar{a})d\bar{a} &= da\, d\bar{a}.
\end{aligned}$$

With above propositions, if we let $a = u^2$ for $u \geq 0$, we easily see the following

Corollary. With the same notation as above,

$$\frac{\partial}{\partial u}[u\,\mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)]]_{u=0} = \mathrm{Tr}_s[N\exp(-\nabla^2)].$$

From this we easily see that as $u \to 0^+$,

$$\mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)]$$

has an asymptotic expansion starting from u^{-1}. Hence, we have the theorem stated at the beginning of this subsection. Moreover, in the definition of ζ_E, we do have the same situation as in the Mellin transform. Indeed, for the purpose here, we only need to know the asymototic expansion for the associated trace class. We will see later that for infinite dimensional cases the analogue still holds, because we will carefully choose the superconnection and the number operator, which provide the right cancellation.

I.4.5.c. A Construction

In this subsection, we use the results above to give another construction for the classical Bott-Chern secondary characteristic forms.

In order to use the Mellin transform, we still need to discuss the behavior of

$$\mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)]$$

when $u \to \infty$. For this, we introduce the **basic assumption** that (E, v) is acyclic. With this assumption, $V = (v+v^*)^2$ is self-adjoint and positive definite. So we may use Duhamel's formula to deduce the fact that as $u \to +\infty$, $\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]$ and $\mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)]$ decay exponentially and uniformly on compact subsets of B. Therefore, we have the following

Proposition and Definition. Let $(\mathcal{E}., \rho.)$ be a complex of hermitian vector bundles on B.
(1) For $s \in \mathbf{C}, \mathrm{Re}(s) > 0$, let

$$\zeta_{(\mathcal{E}.,\rho.)}(s) =: \frac{1}{\Gamma(s)} \int_0^{+\infty} u^s \mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)] \frac{du}{u}.$$

Then $\zeta_{(\mathcal{E}.,\rho.)}(s) \in P$ is well-defined.
(2) There exists a meromorphic extension of $\zeta_{(\mathcal{E}.,\rho.)}(s)$ on the whole complex plane which is holomorphic at 0.

Obviously, we know that

$$\begin{aligned}
\zeta_{(\mathcal{E}.,\rho.)}(0) =&\mathrm{Tr}_s[N\exp(-\nabla^2)]\\
\zeta'_{(\mathcal{E}.,\rho.)}(0) =& \int_0^1 [\mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)] + \mathrm{Tr}_s[N\exp(-\nabla^2)] \frac{du}{u}\\
&\int_1^{+\infty} \mathrm{Tr}_s[N\exp(-\mathbf{A}_u^2)] \frac{du}{u} - \Gamma'(1)\mathrm{Tr}_s[N\exp(-\nabla^2)].
\end{aligned}$$

Theorem. Suppose $\mathcal{E}.$ is acyclic, then

$$\begin{aligned}
\int_0^{+\infty} \mathrm{Tr}_s[(\nu+\nu^*)\exp(-\mathbf{A}_u^2)]u^{1/2}\frac{du}{u} =&(\partial^B - \bar\partial^B)\zeta'_{(\mathcal{E}.,\rho.)}(0);\\
\mathrm{Tr}_s[\exp(-\nabla^2)] = \bar\partial_M \partial_M \zeta'_{(\mathcal{E}.,\rho.)}(0).&
\end{aligned}$$

In particular, we get

$$\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}., \rho.) = [2\pi \mathrm{i}]\zeta'_{(\mathcal{E}.,\rho.)}(0)$$

Later we will give a similar construction for the relative Bott-Chern secondary characteristic objects. At that moment, since the situation is infinite dimensional, we have to choose the correct superconnection and number operator in order to have certain cancellation, and hence, a suitable asymptotic expansion.

Chapter I.5
Local Family Index Theorem

In order to prove the existence of relative Bott-Chern secondary characteristic forms with respect to a smooth morphism, we need to study the local family index theorem: Roughly speaking, the construction of relative Bott-Chern secondary characteristic forms for smooth morphisms is very similar to the one in the final section of the last chapter. There we used the local index theorem in the absolute situation by a discussion about the associated heat kernels. Here we will use the heat kernel associated with a suitable generalized Laplacian.

To do so, we now meet certain problems. The most important one is about the convergence: For the local index theorem in the absolute situation, we know that the heat kernels associated with a generalized Laplacian has an asymptotic expansion when time goes to small; in the relative situation, if we choose a 'natural' connection, the associated second order differential operator has a kernel, and usually, there is no good asymptotic expansion for it when the time is small. So we need to modify this connection. Now, by the proof of the local index theorem in the absolute situation, we see that the Lichnerowicz formula is a crucial point. Therefore, we know basically how to make this modification. (In practice, we introduce the Bismut superconnection.) The reference here is [BGV 92].

I.5.1. The Bismut Superconnection

Let $\pi : M \to B$ be a family of oriented Riemannian manifolds $(M_z | z \in B)$ with a Riemannian metric $g_{M/B}$ on each fiber M_z, and let $\mathcal{E}$ be a vector sheaf on M such that $\mathcal{E}_z = \mathcal{E}|_{M_z}$ is a Clifford module for each $z \in B$. We denote by $T(M/B)$ the bundle of vertical tangent vectors. We assume that the bundle M/B possesses the following additional structure: a splitting $TM = T_H M \oplus T(M/B)$, so that the subbundle $T_H M$ is isomorphic to the vector bundle $\pi^* TB$; and a connection $\nabla^{M/B}$ on $T(M/B)$. Let P be the projection operator $P : TM \to T(M/B)$ with kernel the chosen horizontal tangent space $T_H M$. For X a vector field on B, denote by X_H the horizontal lift on M. Choose a Riemannian metric g_B on the base B and pull it up to $T_H M$ by means of the identification $T_H M \simeq \pi^* TB$, we then obtain an inner product on the bundle $T_H M$, which we call a **horizontal metric**. (We usually make use of a local frame e_i of the vertical tangent bundle, and a local frame f_α of TB, with dual frames e^i and f^α.) We form the total metric $g = g_B \oplus g_{M/B}$ on the tangent bundle TM of M. Let ∇^g be the Levi-Civita connection on TM with respect to this metric, and define a connection $\nabla^{M/B}$ on the bundle $T(M/B)$ by projecting this connection

$$\nabla^{M/B} := P\nabla^g P.$$

The first result is the following

Proposition 1. The connection $\nabla^{M/B}$ on $T(M/B)$ is independent of the metric g_B on TB used in the definition.

Proof. We first recall the following basic formula for the Levi-Civita connection

$$\begin{aligned}2(\nabla^g_X Y, Z) &= ([X,Y],Z) - ([Y,Z],X) + ([Z,X],Y)\\ &\quad + X(Y,Z) + Y(Z,X) - Z(X,Y).\end{aligned}$$

Thus, if X, Y and Z are all vertical, the right hand side reduces to the Levi-Civita connection on the fibers for the vertical metric $g_{M/B}$. On the other hand, if X is horizontal, but Y and Z are vertical, then $[Y,Z]$ is vertical, so that $([Y,Z],X)$ vanishes, and we see that

$$2(\nabla^{M/B}_X Y, Z) = (P[X,Y],Z) + (P[Z,X],Y) + X(Y,Z).$$

From this formula, it is clear that only the vertical metric $g_{M/B}$ and the vertical projection P are used to define $\nabla^{M/B}_X$ for X horizontal. This completes the proof.

Next we construct a new connection on TM, that is

$$\nabla^{\oplus} := \nabla^B \oplus \nabla^{M/B}.$$

Note that if we replace g by the rescaled metric $ug_B \oplus g_{M/B}$, with $u > 0$, then ∇^B does not change, and neither does $\nabla^{\oplus}$. Usually, the connection $\nabla^{\oplus}$ has a non-vanishing torsion, even through it preserves the metric. The next proposition shows us the relation between $\nabla^{\oplus}$ and ∇^g.

Proposition 2. There exists a three-tensor ω on M such that, for all $X, Y, Z \in C^\infty(M, TM)$,

$$(\nabla^g_X Y, Z)_g = (\nabla^{\oplus}_X Y, Z)_g + \omega(X)(Y,Z).$$

Furthermore $\omega \in A^1(M, \wedge^2 T^*M)$ is defined by the formula

$$\begin{aligned}\omega(X)(Y,Z) :=& S(X,Z)(Y) - S(X,Y)(Z)\\ &+ \frac{1}{2}(\Omega(X,Z),Y) - \frac{1}{2}(\Omega(X,Y),Z) + \frac{1}{2}(\Omega(Y,Z),X).\end{aligned}$$

Here the tensor S, which is called the second fundamental form, is the section of the bundle

$$\mathrm{End}(T(M/B)) \otimes T^*_H M \simeq T^*(M/B) \otimes T(M/B) \otimes T^*_H M$$

defined by

$$(S(X,\theta),Z) := < \nabla_Z^{M/B} X - P[Z,X], \theta >$$

for $Z \in C^\infty(M,T_HM), X \in C^\infty(M,T(M/B))$ and $\theta \in C^\infty(M,T^*(M/B))$, while the tensor Ω is the section of the bundle $\mathrm{Hom}(\wedge^2 T_H M, T(M/B))$ over M defined by

$$\Omega(X,Y) := -P[X,Y]$$

for $X, Y \in C^\infty(M,T_HM)$, i.e., the negative vertical projection of $[X,Y]$.

Proof. First, since for any $\phi \in C^\infty(B)$, if Y is the horizontal lift of a vector field on B, $Y(\pi^*\phi) = \pi^*(\pi_* Y\phi)$. Hence, for any vertical vector X, $[X,Y]$ is also vertical.

Now observe that $(\nabla_X^g Y, Z) - (\nabla_X^\oplus Y, Z)$ is antisymmetric in Y and Z, because each of the connections ∇^g and $\nabla^\oplus$ preserves the total metric g on M. Hence we may check the proposition case by case for the different situation in which X, Y and Z are horizontal or vertical. As an example, if X, Y and Z are all horizontal lifts from the base, it is easy to see that the difference is 0, while $\omega(X)(Y,Z)$ vanishes. Other cases which are very similar, are left to the reader.

Let $u \in (0,1]$, and let

$$g^u := ug^B \oplus g^{M/B}$$

be the metric on T^*M. Then $g^0 = \lim_{u\to 0} g^u$ is the degenerate metric, which induces a metric on T^*M and vanishes in the horizontal cotangent direction $T(M/B)^\perp \subset T^*M$. The family of metrics g^u is a powerful tool to investigate the geometry of g^0. Let g_u be the dual metric on TM so that

$$g_u = u^{-1} g_B \oplus g_{M/B}$$

which explodes in the horizontal direction as $u \to 0$. We call this process the **blowing-up of the base metric**.

Let $C_u(M) = C(T^*M, g^u)$ be the Clifford algebra bundle, and denote the canonical quantization map from $\wedge T^*M$ to $C_u(M)$ by c_u. Thus the Clifford bundle $C_0(M)$ is the limit of the one parameter family of algebras bundle $C_u(M)$. Let

$$\tau^u : \wedge^2 T^*M \to \mathrm{End}(T^*M)$$

be defined by

$$\mathrm{c}_u(\tau^u(\alpha)\xi) := [\mathrm{c}_u(\alpha), \mathrm{c}_u(\xi)],$$

where $\alpha \in \wedge^2 T_x^* M$ and $\xi \in T_x^* M$. Then

$$\tau^u(\nu_1 \wedge \nu_2)\xi = 2((\nu_1,\xi)_{g^u}\nu_2 - g^u(\nu_2,\xi)_{g^u}\nu_1).$$

Therefore, for the orthogonal frame of T^*M of the form $\{e^i\} \cup \{f^\alpha\}$, we have

$$\begin{aligned}
\frac{1}{2}\tau^u(e^ie^j)e^k &= \delta^{ik}e^j - \delta^{jk}e^i, \\
\frac{1}{2}\tau^u(e^if^\alpha)e^j &= \delta^{ij}f^\alpha, \\
\tau^u(f^\alpha f^\beta)e^i &= 0, \\
\frac{1}{2}\tau^u(e^if^\alpha)f^\beta &= -u\delta^{\alpha\beta}e^i, \\
\frac{1}{2}\tau^u(f^\alpha f^\beta)f^\gamma &= u(\delta^{\alpha\gamma}f^\beta - \delta^{\beta\gamma}f^\alpha), \\
\tau^u(e^ie^j)f^\alpha &= 0.
\end{aligned}$$

Taking the limit as $u \to 0^+$, we have

$$\begin{aligned}
\frac{1}{2}\tau^0(e^ie^j)e^k &= \delta^{ik}e^j - \delta^{jk}e^i, \\
\frac{1}{2}\tau^0(e^if^\alpha)e^j &= \delta^{ij}f^\alpha, \\
\tau^0(f^\alpha f^\beta)e^i &= 0, \\
\tau^0(a)f^\alpha &= 0,
\end{aligned}$$

for all $a \in \wedge^2T^*M$. Hence, τ^0 vanishes on T^*B.

Thus, if we denote the negative of the adjoint of $\tau^u(a) \in \mathrm{End}(T^*M)$ by $\tau_u(a)$ for $u \in [0,1]$, then

$$\frac{1}{2}(\tau_u(\alpha)X, Y)_{g_u} = <\alpha, X \wedge Y>,$$

for $X, Y \in C^\infty(M, TM)$ and $\alpha \in \wedge^2T^*M$. Hence if $\nabla^{M,u}$, $u > 0$, is the Levi-Civita connection on TM corresponding to the metric g_u, then we may restate the proposition above as

Proposition 2′: $\nabla^{M,u} = \nabla^{\oplus} + \frac{1}{2}\tau_u(\omega)$.

It follows that the family of connections $\nabla^{M,u}$ has a well-defined limit as $u \to 0$, which we will denote by $\nabla^{M,0}$. In particular,

$$\nabla^{M,0} := \nabla^{\oplus} + \frac{1}{2}\tau_0(\omega).$$

Obviously, we have the following facts: The connection $\nabla^{M,u}$ is torsion free. The projection of $\nabla^{M,0}$ to the bundle $T(M/B)$ equals $\nabla^{M/B}$. The restriction of $\nabla^{M,0}$ to each fiber depends only on the vertical metric $g_{M/B}$ and the connection on the fiber bundle M/B. We let $\nabla^{T^*M,u}$ be the dual connection on T^*M for $u \in [0,1]$. Motivated by this, we introduce the following discussion:

Now let $\mathcal{E}$ be a vector sheaf on M such that $\mathcal{E}_z = \mathcal{E}|_{M_z}$ is a Clifford module for each z and suppose that there is a connection $\nabla^{\mathcal{E}}$ given on $\mathcal{E}$ whose restriction to each bundle $\mathcal{E}_z$ is a Clifford connection. Denote by $\pi_*\mathcal{E}$ the infinite dimensional bundle over B whose fiber at $z \in B$ is the space $C^\infty(M_z, \mathcal{E}_z)$. Let $D = (D^z | z \in B)$ be a family of Dirac operators acting on the fibers of $\pi_*\mathcal{E}$, constructed from the Clifford module structure and Clifford connection on $\mathcal{E}$. That is, on the fiber M_z, D^z is the composition of the Clifford connection followed by the Clifford action. Introduce the vector bundle $\mathbf{E}$ over M by

$$\mathbf{E} := \pi^*(\wedge T^*B) \otimes \mathcal{E}.$$

This bundle carries a natural action m_0 of the degenerate Clifford algebra $C_0(M)$: the Clifford action of a horizontal cotangent vector $\alpha \in C^\infty(M, T_H^*M)$ is given by exterior multiplication $m_0(\alpha) = \varepsilon(\alpha)$ acting on the first factor $\wedge T_H^*M$ in $\mathbf{E}$, while the Clifford action of a vertical cotangent vector simply equals its Clifford action on $\mathcal{E}$. This Clifford module will be the main tool in calculating the index of the family operator D.

In order to study $\mathbf{E}$, we write it as the limit of a family of Clifford modules for the bundles of Clifford algebras $C_u(M)$, all constructed on the same underlying vector bundle $\mathbf{E}$: The Clifford action

$$m_u : C_u(M) \to \mathrm{End}(\mathbf{E})$$

is defined as follows: For a horizontal cotangent vector $\alpha \in C^\infty(M, T_H^*M)$,

$$m_u(\alpha) := \varepsilon(\alpha) - u\iota(\alpha),$$

acting on the first factor $\wedge T_H^*M$ in $\mathbf{E}$; while the Clifford action of a vertical cotangent vector is simply its Clifford action on $\mathcal{E}$.

There are connections $\nabla^{\mathbf{E},u}$ and $\nabla^{\mathbf{E},\oplus}$ on the Clifford module $\mathcal{E}$ analogous to those in T^*M: The connection $\nabla^{\mathbf{E},\oplus}$ on $\mathbf{E} \simeq \pi^* \wedge T^*B \otimes \mathcal{E}$ is defined by

$$\nabla^{\mathbf{E},\oplus} := \pi^*\nabla^B \otimes 1 + 1 \otimes \nabla^{\mathcal{E}}.$$

The connection $\nabla^{\mathbf{E},u}$ is defined by the following formula (inspired by the discussion for $\nabla^{M,u}$),

$$\nabla^{\mathbf{E},u} := \nabla^{\mathbf{E},\oplus} + \frac{1}{2} m_u(\omega).$$

Proposition 3. For all $u \in [0,1]$, the connection $\nabla^{\mathbf{E},u}$ is a Clifford connection for the Clifford action m_u of $C_u(M)$ on $\mathbf{E}$. In particular, the connection

$$\nabla^{\mathbf{E},0} := \lim_{u\to 0} \nabla^{\mathbf{E},u} = \nabla^{\mathbf{E},\oplus} + \frac{1}{2} m_0(\omega)$$

is a Clifford connection for the Clifford action m_0 of the Clifford algebra bundle $C_0(M)$ on $\mathbf{E}$. The restriction of those connections to each fiber of the bundle M/B is independent of the choice of the horizontal metric g_B used in the definition.

Proof. Note that since

$$[\nabla_X^{\mathbf{E},\oplus}, m_u(\alpha)] = m_u(\nabla_X^{\oplus}\alpha),$$

we know that the connection $\nabla^{\mathbf{E},\oplus}$ is a Clifford connection with respect to the Clifford action m_u. Now the first two statements are the consequences of the fact that

$$[m_u(\omega(X)), m_u(\alpha)] = m_u(\tau^u(\omega(X))\alpha).$$

Others are trivial.

Next we define the Bismut superconnection on $\pi_*\mathcal{E}$. We consider that the space $A(B,\pi_*\mathcal{E})$ of differential forms on B with coefficients in $\pi_*\mathcal{E}$ as the space of smooth sections of the Clifford module $\mathbf{E}$ over M. The **Bismut superconnection B**,

$$\mathbf{B} : A(B,\pi_*\mathcal{E}) \to A(B,\pi_*\mathcal{E})$$

is the following Dirac operator for the Clifford module $\mathbf{E} \to M$:

$$\mathbf{B} := \sum_a m_0^a \nabla_a^{\mathbf{E},0}.$$

Here m_u^a denotes $m_u(e^i)$ or $m_u(f^\alpha)$, $\nabla_a^{\mathbf{E},u}$ denotes $\nabla_{e_i}^{\mathbf{E},u}$ or $\nabla_{f_\alpha}^{\mathbf{E},u}$, and the summation is taken over all the orthonormal frames e_i and f_α.

Remark. Remember that the Dirac operator for the absolute situation is given by

$$D = \sum_i f_i \nabla_{e_i}.$$

Let $\mathbf{B} = \mathbf{B}_{[0]} + \mathbf{B}_{[1]} + \mathbf{B}_{[2]} + \dots$, be the decomposition of the Bismut superconnection according to its degree. Then we have the following

Proposition 4. With the same notation as above, the restriction of $\mathbf{B}$ to $C^\infty(M,\mathcal{E})$ has the expression

$$\sum_i c^i\nabla_i^{\mathcal{E}} + \sum_\alpha \varepsilon^\alpha(\nabla_\alpha^{\mathcal{E}} + \frac{1}{2}\sum_i (S(e_i,e^i),f_\alpha)) - \frac{1}{4}\sum_{\alpha<\beta}\sum_i \varepsilon^\alpha\varepsilon^\beta c^i(\Omega(f_\alpha,f_\beta),e_i).$$

In particular, $\mathbf{B}_{[0]} = D$ and $\mathbf{B}_{[1]} = \nabla^{\pi_*\mathcal{E}}$. (For the precise definition of $\nabla^{\pi_*\mathcal{E}}$, see I.5.3.)

Proof. From the definition, we see that

$$\mathbf{B} = \sum_i c^i\nabla_i^{\mathbf{E},\oplus} + \sum_\alpha \varepsilon^\alpha\nabla_\alpha^{\mathbf{E},\oplus} + \frac{1}{4}\sum_{abc}\omega(e_a)(e_b,e_c)m_0^a m_0^b m_0^c.$$

On the other hand, by a direct calculation, we have

$$\sum_{abc}\omega(e_a)(e_b,e_c)m_u^a m_u^b m_u^c = 2\sum_{\alpha}\sum_{i}(S(e_i,e^i),f_\alpha)m_u^\alpha - \frac{1}{2}\sum_{\alpha\beta i}(\Omega(f_\alpha,f_\beta),e_i)m_u^\alpha m_u^\beta m_u^i.$$

Thus we have the first formula. Then, by comparing the degree, we have the other two.

We saw that the Lichnerowicz formula is very important in the proof of the local index theorem, which makes us to use the Mehler formula possible. Next we discuss the corresponding formula for the Bismut superconnection.

Theorem. (The super-Lichnerowicz formula) With the notation as above, we have

$$\mathbf{B}^2 = \Delta^{M/B} + \frac{1}{4}r_{M/B} + \sum_{a<b} m_0^a m_0^b F^{\mathcal{E}/S}(e_a,e_b).$$

Here $r_{M/B}$ denotes the scalar curvature, and $F^{\mathcal{E}/S} \in A(M, \mathrm{End}_{C(M/B)}(\mathcal{E}))$ denotes the twisted curvature of the Clifford module $\mathcal{E}$.

Proof. By definition, we know that

$$\begin{aligned}\mathbf{B}^2 =& \frac{1}{2}\sum_{ab}[m_0^a\nabla_a^{\mathbf{E},0}, m_0^b\nabla_b^{\mathbf{E},0}]\\ =& \frac{1}{2}\sum_{ab}[m_0^a, m_0^b]\nabla_a^{\mathbf{E},0}\nabla_b^{\mathbf{E},0}\\ &+\sum_{ab} m_0^a[\nabla_a^{\mathbf{E},0}, m_0^b]\nabla_b^{\mathbf{E},0} + \frac{1}{2}\sum_{ab} m_0^a m_0^b[\nabla_a^{\mathbf{E},0},\nabla_b^{\mathbf{E},0}].\end{aligned}$$

Now we compute each term in the last equation. Since for any $\alpha \in C^\infty(M, T^*M)$,

$$m_u(\alpha) = \varepsilon(\alpha) - u\iota(\alpha), \quad [m_u(\alpha), m_u(\alpha)] = -g^u(\alpha,\alpha),$$

we know that for the orthonormal basis e^i, f^α, the first term is equal to $-\sum_i(\nabla_i^{\mathbf{E},0})^2$.

By the fact that $\nabla^{\mathbf{E},0}$ is a Clifford connection, we know that the second term is

$$\begin{aligned}\sum_{ab} m_0^a m_0(\nabla_a^{T^*M,0}e^b)\nabla_b^{\mathbf{E},0} =& \sum_{abc} m_0^a m_0^c < \nabla_a^{T^*M,0}e^b, e_c > \nabla_b^{\mathbf{E},0}\\ =& -\frac{1}{2}\sum_{ac} m_0^a m_0^c \nabla_{[e_a,e_c]}^{\mathbf{E},0} + \sum_i \nabla_{\nabla_i^{M/B}e^i}^{\mathbf{E},0}.\end{aligned}$$

Here we have used the facts that $\nabla^{M,0}$ agrees with $\nabla^{M/B}$ when restricted to a fiber M_z, that the connection $\nabla^{M,0}$ is torsion-free, and the adjunction formula

$$\nabla_a^{T^*M,0}e^b = \sum_c < \nabla_a^{T^*M,0}e^b, e_c > e^c = -\sum_c < e^b, \nabla_a^{M,0}e_c > e^c.$$

On the other hand, by definition, we have

$$\Delta^{M/B} = \sum_i \nabla^{\mathbf{E},0}_{\nabla_i^{M/B} e^i} - \sum_i (\nabla_i^{\mathbf{E},0})^2.$$

Therefore,

$$\mathbf{B}^2 = \Delta^{M/B} + \frac{1}{2} \sum_{ab} m_0^a m_0^b ([\nabla_a^{\mathbf{E},0}, \nabla_b^{\mathbf{E},0}] - \nabla^{\mathbf{E},0}_{[e^a, e^b]}).$$

Furthermore, we know that

$$(\nabla^{\mathcal{E}})^2 = -\frac{1}{2} \sum_{i<j;a<b} R^{M/B}_{ijab} c^i c^j \varepsilon^a \varepsilon^b + \sum_{a<b} F^{\mathcal{E}/S}_{ab} \varepsilon^a \varepsilon^b;$$

$$[\nabla^{\mathbf{E},\oplus}, m_0(\omega)] = m_0(\nabla^{\oplus}\omega);$$

$$[m_0(\omega), m_0(\omega)] = m_0([\omega,\omega]_0);$$

Also, if, locally, for any $A \in \mathrm{End}(V)$, we define

$$\lambda(A) := \sum_{j,k} < e^j, Ae_k > \varepsilon_j \iota^k,$$

globally, we get

$$(\nabla^{\mathbf{E},\oplus})^2 = \lambda((\nabla^B)^2) + (\nabla^{\mathcal{E}})^2 = \lambda(R^B) + \frac{1}{2} m_0(R^{M/B}) + F^{\mathcal{E}/S}.$$

So, by the fact that

$$(\nabla^{\mathbf{E},0})^2 = (\nabla^{\mathbf{E},\oplus})^2 + \frac{1}{2}[\nabla^{\mathbf{E},\oplus}, m_0(\omega)] + \frac{1}{8}[m_0(\omega), m_0(\omega)],$$

we get

$$\begin{aligned}&\sum_{ab} m_0^a m_0^b (\nabla^{\mathbf{E},0})^2(e_a, e_b)\\ &= \sum_{ab} m_0^a m_0^b \lambda(R^B(e_a, e_b)) - \frac{1}{4}\sum_{abcd} m_0^a m_0^b m_0^c m_0^d R_{abcd} + \sum_{ab} m_0^a m_0^b F^{\mathcal{E}/S}(e_a, e_b).\end{aligned}$$

From the fact that the antisymmetrization of R^B over any three indices is zero, we know that the first term is zero. In particular, we see that

$$\mathbf{B}^2 = \Delta^{M/B} + \sum_{a<b} m_0^a m_0^b F^{\mathcal{E}/S}(e_a, e_b) - \frac{1}{8} \sum_{abcd} m_0^a m_0^b m_0^c m_0^d R_{abcd}.$$

Now the original formula is a consequence of the following standard calculation:

$$\begin{aligned}\sum_{abcd} m_0^a m_0^b m_0^c m_0^d R_{abcd} =& \sum_{abd} m_0^a m_0^b m_0^a m_0^d R_{abad} + \sum_{abd} m_0^a m_0^b m_0^b m_0^d R_{abbd} \\ =& 2\sum_{adi} m_0^a m_0^d R_{aidi} = -2\sum_{ij} R_{ijij} = -2r_{M/B}.\end{aligned}$$

§I.5.2. Existence Of Heat Kernels In Relative Version

In this section, we prove that, associated with the Bismut superconnection, there exist heat kernels. For doing so, we put the problem in a relatively large content.

With the same situation as in the previous section, we have a family of manifolds, together with the associated structures. Let $\mathcal{D}(\mathcal{E})$ be the bundle of algebras over B whose fiber at z is the algebra of differential operators, and whose smooth sections are families of differential operators D^z, with coefficients (in a local trivialization of M and $\mathcal{E}$) depending smoothly on the coordinates in B. Let $\mathcal{K}(\mathcal{E})$ be the bundle of algebras whose fiber at z is the algebra of smoothing families of smoothing operators K^z. Since $\mathcal{K}(\mathcal{E})$ is a bundle of modules for $\mathcal{D}(\mathcal{E})$, we may form an algebra from the sums of operators in $\mathcal{D}(\mathcal{E})$ and $\mathcal{K}(\mathcal{E})$. We refer to the smooth sections of the bundle $\mathcal{D}(\mathcal{E}) + \mathcal{K}(\mathcal{E})$ as the **$\mathcal{P}$-endomorphisms** of the infinite-dimensional bundle $\pi_*\mathcal{E}$, and denote by $\mathrm{End}_{\mathcal{P}}(\pi_*\mathcal{E})$ the space of its smooth sections.

As an illustration, first, we consider the situation when B is a point. Let $A = \sum_{i=0}^n A^i$ be a finite-dimensional graded algebra with identity. Let $\mathcal{M}$ be the algebra $\mathcal{P} \otimes A$. There is a natural decreasing filtration of the algebra $\mathcal{M}$ with $\mathcal{M}_i := \sum_{j \geq i} \mathcal{P} \otimes A^i$. Let

$$\mathcal{F} := H_0 + K + \mathcal{F}_{[+]}$$

be in $\mathcal{M}$ with H_0 a generalized Laplacian, $K \in \mathcal{K}$ and $\mathcal{F}_{[+]} \in \mathcal{M}_1$. We define a **heat kernel** for $\mathcal{F}$ to be a continuous map $(t, x, y) \mapsto p_t(x, y) \in \mathcal{E}_x \otimes \mathcal{E}_y \otimes A$ which is C^1 in t, C^2 in x and satisfies the equation

$$\left(\frac{\partial}{\partial t} + \mathcal{F}_x\right) p_t(x, y) = 0,$$

with the boundary condition that for every $s \in C^\infty(M, \mathcal{E}) \otimes A$,

$$\lim_{t \to 0} \int_{y \in M} p_t(x, y)\, s(y) = s(x)$$

uniformly in $x \in M$.

Theorem 1. Let
$$\mathcal{F} := H_0 + K + \mathcal{F}_{[+]}$$
be an element of $\mathcal{M}$ with H_0 a generalized Laplacian, $K \in \mathcal{K}$ and $\mathcal{F}_{[+]} \in \mathcal{M}_1$. (In the sequel, we call $\mathcal{F}$ an essential generalized Laplacian.) Then there exists a unique heat kernel $p_t(x, y)$ for $\mathcal{F}$.

Proof. First, we assume that $A = \mathbf{C}$. Then $\mathcal{F}_{[+]} = 0$. In this case, since we know that H_0 has a smooth heat kernel satisfying certain strong estimates, it is not surprising that we can prove the same things for $H = H_0 + K$ by Duhamel's formula. To construct the heat kernel for H, we use a generalization of the Volterra series as follows:

Lemma 1. With respect to any C^l-norm, $l \geq 0$, the series
$$Q_t := \sum_{k=0}^{\infty}(-t)^k \int_{\Delta_k} e^{-\sigma_0 t H_0} K e^{-\sigma_1 t H_0} \dots K e^{-\sigma_k t H_0} d\sigma$$
converges to a kernel
$$q_t \in C^{\infty}(M \times M, p_1^*\mathcal{E} \otimes p_2^*\mathcal{E}^*).$$
The sum is C^{∞} with respect to t and is a solution of the heat equation
$$(\partial_t + (H_0)_x + K_x)\, q_t(x, y) = 0$$
with the following boundary condition at $t = 0$: If ϕ is a section of $\mathcal{E}$, then with the uniform norm,
$$\lim_{t \to 0} Q_t s = s.$$
Hence $q_t(x, y)$ is a heat kernel for H. Furthermore, the difference
$$e^{-tH} - e^{-tH_0} = \sum_{k=1}^{\infty}(-t)^k \int_{\Delta_k} e^{-\sigma_0 t H_0} K e^{-\sigma_1 t H_0} \dots K e^{-\sigma_k t H_0} d\sigma$$
tends to 0 when $t \to 0$.

Proof. Since K is a smoothing operator, the operator $e^{-tH_0}K$ has a smooth kernel for all $t \geq 0$, and
$$\|e^{tH_0} K\|_l \leq C(l) \|K\|_l,$$
for some constant $C(l)$ depending on l. It follows that, for $k \geq 1$,
$$\| \int_{\Delta_k} e^{-\sigma_0 t H_0} K e^{-\sigma_1 t H_0} \dots K e^{-\sigma_k t H_0} d\sigma \| \leq \frac{C(l)^{k+1} \|K\|_l^k}{k!}.$$
Thus the series $\sum_{k \geq 1}$ converges with respect to the C^l-norm, uniformly for $t \geq 0$, with similar estimates for the derivative with respect to t. Others are easily to check.

In general, the operators $\mathcal{F}$ and H differ by an operator $\mathcal{F}_{[+]}$ of positive degree in the finite-dimensional graded algebra A. Thus by the heat kernels of H, the Volterra series once more gives a candidate for the heat kernel of $\mathcal{F}$: For fixed $t > 0$, define the operator $e^{-t\mathcal{F}}$ by

$$e^{-t\mathcal{F}} := e^{-tH} + \sum_{k>0}(-t)^k I_k,$$

where

$$I_k = \int_{\Delta_k} e^{-\sigma_0 tH_0}\mathcal{F}_{[+]}e^{\sigma_1 tH_0}\dots\mathcal{F}_{[+]}e^{-\sigma_k tH_0}d\sigma.$$

The sum is finite, since $I_k \in \mathcal{M}_k$, and for k large, $A^k = 0$, hence $\mathcal{M}_k = 0$. Thus it is sufficient to make sense of each term in this finite sum. For this, we need the following

Lemma 2. Let D be a differential operator of order k. There exists a constant $C > 0$ such that if K is a smoothing operator, for $t \in [0,T]$, with T being a positive real number, we have

$$\|De^{-tH}K\|_l \leq C\|K\|_{k+l}$$
$$\|Ke^{-tH}D\|_l \leq C\|K\|_{k+l}.$$

Proof. By the fact that there exists a constant $C(l)$ such that for $\phi \in \Gamma^l(M,\mathcal{E})$, one has for $t \in [0,T]$,

$$\|e^{-tH}\phi\|_l \leq C(l)\|\phi\|_l,$$

the bound $\|De^{-tH}K\|_l \leq C(l)\|K\|_{k+l}$ follows easily. Using the adjoint, we have the other inequality.

Now we can complete the proof of our theorem. Obviously, it is enough to show that each term I_k has a smooth kernel. On the simplex Δ_k, one of the σ_i must be greater than $(k+1)^{-1}$. Since for a fixed σ with $(k+1)^{-1} < \sigma \leq 1$ and a fixed t, the operator $e^{-\sigma tH}$ has a uniformly smooth kernel, it follows by iterated applications of the above lemma that the operator

$$e^{-\sigma_0 tH_0}\mathcal{F}_{[+]}e^{-\sigma_1 tH_0}\dots\mathcal{F}_{[+]}e^{-\sigma_k tH_0}$$

has a smooth kernel which depends continuously on $(\sigma_0,\dots,\sigma_k) \in \Delta_k$. Thus the integral makes sense as an operator with smooth kernel. The rest is trivial.

We now come back to the relative situation. Let $\pi : M \to B$ be a family of manifolds over a base B. Denote by $M \times_\pi M$ the fiber product which is a fiber bundle over B with fiber at $z \in B$ being $M_z \times M_z$. Let $\mathcal{E} \to M$ be a family of vector bundles. By definition, a **smooth family of smoothing operator** acting on the bundles $\mathcal{E}_z \to M_z$ along the fibers is a family of operators with kernel

$$k \in C^\infty(M \times_\pi M, p_1^*\mathcal{E} \otimes p_2^*\mathcal{E}^*).$$

When restricted to the fiber $M_z \times M_z$, the kernel k may be viewed as a kernel k^z in $C^\infty(M_z \times M_z, p_1^*\mathcal{E}_z \otimes p_2^*\mathcal{E}_z^*)$. Let $\mathcal{K}(\mathcal{E})$ be a bundle over B, whose smooth sections are given by

$$C^\infty(B,\mathcal{K}(\mathcal{E})) = C^\infty(M \times_\pi M, p_1^*\mathcal{E} \otimes p_2^*\mathcal{E}^*).$$

As explained above, $\mathcal{K}(\mathcal{E})$ is a sub-bundle of $\mathrm{End}_{\mathcal{P}}(\mathcal{E})$.

Theorem 2. Suppose we are given a smooth family of generalized Laplacians H^z along the fibers of $M \to B$, then the corresponding heat kernel $p_t(x, y, z)$ defines a smooth family of smoothing operators, that is, a section in $C^\infty(B, \mathcal{K}(\mathcal{E}))$.

Proof. As usual, around any point $z_0 \in B$, we can find a neighborhood on which the families M and $\mathcal{E}$ are trivialized. Thus, we may replace B by a ball $U \subset \mathbf{R}^p$ centered at 0; M by the trivial bundle $M_0 \times U$; and $\mathcal{E}$ by the bundle $\mathcal{E}_0 \times U$, where $\mathcal{E}_0$ is a bundle over M_0. Since the change of coordinates, and its inverse, used to obtain this trivialization are smooth, we see that the data used to define the family of generalized Laplacians H^z gives a smooth family of one for defining generalized Laplacian on the bundle $\mathcal{E}_0$, parameterized by the ball U. Hence by the result above, with respect to the parameter, we have our assertion.

Similarly, by localization, we have the following theorem, which shows that the heat kernel exists for the Bismut superconnection $\mathbf{B}_u$.

Theorem 3. Let A be a bundle of finite-dimensional graded algebras with identity over B and let $\mathcal{M}$ be the bundle of filtered algebras $\mathcal{M} = A \otimes \mathrm{End}_{\mathcal{P}}(\mathcal{E})$. Let $\mathcal{F} \in C^\infty(B, A \otimes \mathrm{End}_{\mathcal{P}}(\mathcal{E}))$ be a smooth family of $\mathcal{P}$-endomorphism with coefficients in $\mathcal{A}$, of the form

$$\mathcal{F} = H_0 + K + \mathcal{F}_{[+]},$$

where $H_0 \in C^\infty(B, \mathcal{D}(\mathcal{E}))$ is a smooth family of generalized Laplacians, $K \in C^\infty(B, \mathcal{K}(\mathcal{E}))$ is a smooth family of smoothing operators, and $\mathcal{F}_{[+]}$ is an element of $C^\infty(B, A^i \otimes \mathrm{End}_{\mathcal{P}}(\mathcal{E}))$. Then for $t > 0$, the kernel of the operator $e^{-t\mathcal{F}}$ is a smooth family of smoothing operators with coefficients in A, that is, a smooth section in $C^\infty(B, A \otimes \mathcal{K}(\mathcal{E}))$.

§I.5.3. Chern Characteristic Forms In The Relative Situation

In this section, following Bismut, we will extend Quillen's theory of superconnection to the infinite dimensional bundle $\pi_*\mathcal{E} \to B$, thereby obtaining a formula in terms of heat kernels for Chern characteristic forms in the relative situation. As the space of sections of $\pi_*\mathcal{E}$ is $C^\infty(M, \mathcal{E})$, it is natural to define the space of differential forms on B with values in $\pi_*\mathcal{E}$ by $\mathcal{A}(B, \pi_*\mathcal{E}) = C^\infty(M, \pi^*(\wedge T^*B) \otimes \mathcal{E})$.

By definition, a differential operator on $\mathcal{A}(B, \pi_*\mathcal{E})$ is a differential operator on the space $C^\infty(M, \pi^*(\wedge T^*B) \otimes \mathcal{E})$. Let

$$\mathcal{A}(B, \mathcal{D}(\mathcal{E})) := C^\infty(B, \wedge T^*B \otimes \mathcal{D}(\mathcal{E}))$$

be the space of vertical differential operators with differential coefficients. If a differential operator D on $\mathcal{A}(B, \pi_*\mathcal{E})$ is supercommutative with the action of $\mathcal{A}(B)$, then this operator is given by the action of an element of $\mathcal{A}(B, \mathcal{D}(\mathcal{E}))$. Similarly, we write

$$\mathcal{A}(B, \mathcal{K}(\mathcal{E}))$$

for the space of smooth families of smoothing operators K^t with differential form coefficients. Denote by d_B the exterior differential on B.

Let D be a smooth family of Dirac operators on $\mathcal{E}$. A **superconnection** associated to D is a differential operator $\mathbf{A}$ on $\mathcal{A}(B,\pi_*\mathcal{E})$ of odd parity such that
(1) (Leibniz's rule) For all $\nu \in \mathcal{A}(B)$ and $\phi \in \mathcal{A}(B,\pi_*\mathcal{E})$,

$$\mathbf{A}(\nu\phi) = (d_B\nu)\phi + (-1)^{|\nu|}\nu\mathbf{A}(\phi).$$

(2) $\mathbf{A} = D + \sum_{i=1}^{\dim(B)} \mathbf{A}_{[i]}$, where $\mathbf{A}_{[i]} : \mathcal{A}^{\cdot}(B,\pi_*\mathcal{E}) \to \mathcal{A}^{\cdot+i}(B,\pi_*\mathcal{E})$.

It is easy to see that $\mathbf{A}_{[i]}$ supercommutes with $\mathcal{A}(B)$ if $i \neq 1$, and hence belongs to $\mathcal{A}^i(B,\mathcal{D}(\mathcal{E}))$.

Next we construct a superconnection associated with a family of Dirac operators D: It is sufficient to define a connection $\nabla^{\pi_*\mathcal{E}}$ on the bundle $\pi_*\mathcal{E}$, i.e., a differential operator from $C^\infty(B,\pi_*\mathcal{E}^\pm)$ to $\mathcal{A}^1(B,\pi_*\mathcal{E}^\mp)$ such that

$$\nabla^{\pi_*\mathcal{E}}(f\phi) = df \wedge \phi + f\nabla^{\pi_*\mathcal{E}}\phi$$

for all $f \in C^\infty(B)$ and $\phi \in C^\infty(B,\pi_*\mathcal{E})$.

For doing so, assume that the bundle M/B possesses a splitting $TM = T_HM \oplus T(M/B)$, so that the subbundle T_HM is isomorphic to the vector bundle π^*TB. If X is a vector field on the base B, denote by X_H its horizontal lift on M, i.e. the vector field on M which is a section of T_HM and which projects to X under the pushforward $\pi_* : (T_HM)_x \to T_{\pi(x)}B$. Furthermore, let us suppose that the bundle $\mathcal{E}$ over M is provided with a connection $\nabla^{\mathcal{E}}$, which is compatible with its hermitian structure. We can now define a canonical linear connection on the vertical tangent space $T(M/B)$ using the projection operator

$$P : TM \to T(M/B)$$

which has the chosen horizontal tangent space T_HM as its kernel.

Proposition and Definition. Let $s \in C^\infty(M,\mathcal{E})$. For X a vector field on B, define the action of $\nabla_X^{\pi_*\mathcal{E}}$ on s by the formula

$$\nabla_X^{\pi_*\mathcal{E}} s = \nabla_{X_H}^{\mathcal{E}} s.$$

Then, we have
(1) This formula defines a connection on $\pi_*\mathcal{E}$ over B.
(2) The connection $\nabla^{\pi_*\mathcal{E}}$ is compatible with the inner product on $\pi_*\mathcal{E}$.

Proof. To show that $\nabla^{\pi_*\mathcal{E}}$ is a connection, we must show that $\nabla_{fX}^{\pi_*\mathcal{E}} = \pi^*f\nabla_X^{\pi_*\mathcal{E}}$ for $f \in C^\infty(B)$. But that is a direct consequence of the definition. On the other hand, by the fact that $\nabla^{\mathcal{E}}$ is compatible with the hermitian metric, we also have (2).

Thus associated with a connection on the fiber bundle M/B and a connection on the bundle $\mathcal{E}$, there is a natural superconnection $\mathbf{A} := D+\nabla^{\pi_*\mathcal{E}}$ for the family of Dirac operators $(D^z|z \in B)$. The curvature

$$\mathcal{F} = \mathbf{A}^2 = D^2 + \mathcal{F}_{[+]} \in \mathcal{A}(B, \mathcal{D}(\mathcal{E}))$$

of the superconnection $\mathbf{A}$ is a vertical differential operator with differential form coefficients. Also D^2 is a smooth family of generalized Laplacians and

$$\mathcal{F}_{[+]} \in C^\infty(B, \wedge T^*B \otimes \mathcal{D}(\mathcal{E}))$$

is a smooth family of differential operators with differential form coefficients which raises exterior degree in $\wedge T^*_z B \otimes C^\infty(M_z, \mathcal{E}_z)$. Here, by the results in section 5.2, we obtain the existence of a smooth family of heat kernels for $\mathcal{F}$, which we denote by $e^{-t\mathcal{F}} \in \mathcal{A}(B, \mathcal{K}(\mathcal{E}))$:

$$e^{-t\mathcal{F}} = e^{-tD^2} + \sum_{k>0} (-t)^k I_k,$$

with

$$I_k = \int_{\Delta_k} e^{-\sigma_0 t D^2} \mathcal{F}_{[+]} e^{-\sigma_1 t D^2} \dots \mathcal{F}_{[+]} e^{-\sigma_k t H_0} d\sigma.$$

Since I_k vanishes for $k > \dim(B)$, the sum above is finite.

On the other hand, for $K = (K^z|z \in B) \in \mathcal{A}(B, \mathcal{K}(\mathcal{E}))$ a smooth family of smoothing operators with coefficients in $\mathcal{A}(B)$, given by a kernel

$$< x|K|y > \in C^\infty(M \times_\pi M, \pi^*(\wedge T^*B) \otimes p_1^*\mathcal{E} \otimes p_2^*\mathcal{E}^*),$$

there is a supertrace on $\mathcal{K}(\mathcal{E}_z)$ over each fiber M_z of M/B, which gives a supertrace

$$\mathrm{Tr}_s : C^\infty(B, \mathcal{K}(\mathcal{E})) \to C^\infty(B).$$

When restricted to the diagonal, the kernel $< x|K^z|x >$ is a smooth section of the bundle $\pi^* \wedge T^*B \otimes \mathrm{End}(\mathcal{E})$ over M, and its pointwise supertrace $\mathrm{Tr}_{s,\mathcal{E}} < x|K^z|x >$ is a section in $C^\infty(M, \pi^* \wedge T^*B)$. Such a section can be integrated over the fibers, and hence gives a differential form on B. Thus the $\mathcal{A}(B)$-valued supertrace $\mathrm{Tr}_s : \mathcal{A}(B, \mathcal{K}(\mathcal{E})) \to \mathcal{A}(B)$ of the family of operators K is the differential form on B

$$z \mapsto \int_{M_z} \mathrm{Tr}_{s,\mathcal{E}} < x|K^z|x > .$$

A local calculation shows that

$$d_B \mathrm{Tr}_s(K) = \mathrm{Tr}_s([\mathbf{A}, K]) \in \mathcal{A}(B).$$

With this, we may define the **Chern characteristic forms** in the relative situation for a superconnection $\mathbf{A}$ on the bundle $\pi_*\mathcal{E}$, associated with a family of Dirac operators D, to be the differential form on B given by the formula

$$\mathrm{ch}(\mathbf{A}) = [2\pi i]\mathrm{Tr}_s(e^{-\mathbf{A}^2}).$$

Since $e^{-\mathbf{A}^2} = e^{-\mathcal{F}} \in \mathcal{A}(B, \mathcal{K}(\mathcal{E}))$, it is well defined. As an immediate consequence, we have the following

Theorem. Let $\mathbf{A}$ be a superconnection on the bundle $\pi_*\mathcal{E}$ for the family of Dirac operators D.

(1) The differential form $\mathrm{ch}(\mathbf{A})$ is closed.

(2) If $\mathbf{A}_\sigma$ is a one parameter family of superconnections on the bundle $\pi_*\mathcal{E}$ for a family of Dirac operators D_σ, then

$$\frac{d}{d\sigma}\mathrm{Tr}_s(e^{-\mathbf{A}_\sigma^2}) = -d_B\mathrm{Tr}_s(\frac{d\mathbf{A}_\sigma}{d\sigma}e^{-\mathcal{F}_\sigma}).$$

Thus, the class of $\mathrm{ch}(\mathbf{A}_\sigma)$ in de Rham cohomology is a homotopy invariant of the superconnection $\mathbf{A}$.

As we have seen above, one may study the Chern characteristic forms by using the heat kernels associated with a superconnection on $\pi_*\mathcal{E}$. In the absolute situation, by rescaling the superconnection, we show that the associated heat kernels give us the information we need. Unfortunately, in the relative situation, the problem is rather complicated. The main reason is that when the time goes to small, the associated heat kernel is not convergent. Thus, the first thing we have to do is to modify the superconnection so that the limit of the supertrace of the heat kernel on the diagonal exists whenever the time goes to infinity or small; this is why we need to use the Bismut superconnection.

§I.5.4. Local Family Index Theorem

In this section, as in the absolute situation, we prove the local family index theorem by studying the behavior of the heat kernel associated with the square of Bismut superconnection $\mathbf{B}$ when the time goes to zero. (By the result of §I.5.1, we know that these heat kernels do exist.)

Let us summarize the data that we are working with:

(1) A relative dimension $2m$ fiber bundle $\pi : M \to B$ with a vertical metric $g_{M/B}$ and a splitting $TM = T_HM \oplus T(M/B)$ with $T_HM \simeq \pi^*T_B$. From this, we obtain a connection $\nabla^{M/B}$ on the vertical tangent bundle $T(M/B)$.

(2) A Clifford module $\mathcal{E}$ for the vertical Clifford bundle $C(M/B)$ with a Clifford connection $\nabla^{\mathcal{E}}$ which is compatible with $\nabla^{M/B}$.

Using this data, we construct a family of twisted Dirac operators $D = (D^z | z \in B)$, and the Bismut superconnection $\mathbf{B}$, with $\mathbf{B}_{[0]} = D$. Hence the curvature $\mathcal{F} := \mathbf{B}^2$ of $\mathbf{B}$ acts on the space $\mathcal{A}_z \otimes (\pi_*\mathcal{E})_z$ of sections of the bundle $\mathbf{E} = \wedge\pi^*(T^*B)\otimes\mathcal{E}$ along the fiber M_z. Here $\mathcal{A}_z$ denotes the finite dimensional algebra $\wedge T_z^*B$. With this, we may state the following

Local Family Index Theorem. (1) For each $t > 0$, the heat operator $e^{-t\mathcal{F}}$ acting on $C^\infty(M, \mathbf{E})$ has a kernel

$$< x|e^{-t\mathcal{F}}|y > \in C^\infty(M \times_\pi M, \pi^*\mathcal{A} \otimes p_1^*\mathcal{E} \otimes p_2^*\mathcal{E}^*),$$

in the sense that if $\phi \in C^\infty(M, \mathbf{E})$, we have

$$(e^{-t\mathcal{F}}\phi)(x) = \int_{M_z} < x|e^{-t\mathcal{F}}|y > \phi(y)dy,$$

where dy is the Riemannian volume form of the fiber M_z and $z = \pi(x)$.

(2.) When $t \to 0^+$, we have the asymptotic expansion

$$k(x,t,x) \sim (4\pi t)^{-m} \sum_{i=0}^{\infty} t^i k_i(x)$$

such that

(a) The coefficient k_i lies in $\sum_{j\leq i} \mathcal{A}^{2j}(M, \mathrm{End}_{C(M/B)}(\mathcal{E}))$.

(b) The full symbol of $k(x,t,x)$, defined by $\sigma(k) := \sum_{i=0}^{\dim(M)/2} \sigma_{2i}(k_i)$, is given by the formula

$$\sigma(k) = \hat{A}(R^{M/B})\exp(-F^{\mathcal{E}/S}) \in \mathcal{A}(M, \mathrm{End}_{C(M/B)}(\mathcal{E})).$$

Proof. By the result of section 1 and section 2, we easily have (1). The proof of (2) is similar to that in the absolute situation:

For $z \in B$ and $x_0 \in M_z$, let $V = T_{x_0}(M/B)$ and $H = T_zB$ be the vertical and horizontal tangent spaces at x_0. Then $T := T_{x_0}M = V \oplus H$. Let $U := \{\xi \in V : ||\xi|| < \varepsilon\}$, where ε is a positive number which is smaller than the injectivity radius of the fiber M_z. So we may identify U with a neighborhood of x_0 in M_z by the exponential map $\xi \mapsto \exp_{x_0}\xi$. Let $\tau^{M/B}(x_0, x)$ be the parallel transport map in the bundle $T(M/B)$ along the geodesic from x to x_0, defined with respect to the connection $\nabla^{M/B}$. Since we are working on a single fiber M_z, this connection is nothing but the Levi-Civita connection of M_z. Using this map, we identify the fiber $T_x(M/B)$ with the space V, so that the space of differential forms $\mathcal{A}(U)$ is identified with $C^\infty(U, \wedge V^*)$. Choose an orthonormal basis $d\xi_i$ of V^*, and let $e^i \in C^\infty(U, T^*(M/B)) = C^\infty(U, V^*)$ be the orthonormal frame of $T^*(M/B)$ over U obtained by parallel transport of $d\xi_i$ along geodesics by the Levi-Civita connection on M_z. We denote by e^a a local frame of T^*M on U consisting of the union of the cotangent frame e^i and of a fixed basis f^α of T_z^*B.

Let $E := \mathcal{E}_{x_0}$ be the fiber of the Clifford module $\mathcal{E}$ at x_0, let S_V be the spinor space of V^*, and let $W = \mathrm{Hom}_{C(V^*)}(S_V, E)$, so that E is naturally isomorphic to $S_V \otimes W$. Let $\tau^{\mathbf{E}}(x_0, x)$ be the parallel transport map in the bundle $\mathcal{A}_z \otimes \mathcal{E}$ along the geodesic from x to

x_0, defined with respect to the Clifford connection $\nabla^{\mathbf{E},0}$. Using this map, we identify the fiber $\mathcal{A}_z \otimes \mathcal{E}_x$ of $\mathbf{E}$ at x with the space $\wedge H^* \otimes S_V \otimes W$, and the space $C^\infty(U, \mathcal{A}_z \otimes \mathcal{E}_x)$ with $C^\infty(U, \wedge H^* \otimes S_V \otimes W)$.

If we let Δ^z be the Laplacian on M_z associated with the connection $\nabla^{\mathbf{E},0}$, then by the super-Lichnerowicz formula, we have

$$\mathcal{F}^z = \Delta^z + \frac{1}{4} r_{M_z} + \sum_{a<b} m_0^a m_0^b F^{\mathcal{E}/S}(e_a, e_b).$$

Hence we may transform this operator to the one on $C^\infty(U, \wedge T^* \otimes \mathrm{End}(W))$ by using the quantization map; that is, replace the Clifford action $m_0(e^a)$ at $\xi = 0$ by the action $m^i = \varepsilon^i - \iota^i$, $m^\alpha = \varepsilon^\alpha$. In this way, we get the corresponding operator

$$L = -\sum_i ((\nabla_i^{\mathbf{E},0})^2 - \nabla_{\nabla_i e_i}^{\mathbf{E},0}) + \frac{1}{4} r_{M_z} + \sum_{a<b} F_{ab}^{\mathcal{E}/S} m^a m^b.$$

Next we introduce the rescaling operator δ_u on the space $C^\infty(U, \wedge T^* \otimes \mathrm{End}(W))$: If $a \in C^\infty(U, \wedge^i T^* \otimes \mathrm{End}(W))$, then

$$\delta_u(a)(\xi) := u^{-i/2} a(u^{1/2}\xi).$$

In the same way, if $a \in C^\infty(U \times \mathbf{R}_{>0}, \wedge^i T^* \otimes \mathrm{End}(W))$, we define

$$\delta_u(a)(t, \xi) := u^{-i/2} a(ut, u^{1/2}\xi).$$

Thus if we let

$$k(t, \xi) := \tau^{\mathbf{E}}(x_0, x) < x | e^{-t\mathcal{F}} | x_0 >,$$

where $x = \exp_{x_0} \xi$, we know that $k(t, \xi)$ is in $C^\infty(U, \wedge H^* \otimes \mathrm{End}(S_V) \otimes \mathrm{End}(W))$, which, by the symbol map, may be thought of as a map from u to $\wedge T^* \otimes \mathrm{End}(W)$ and satisfies the heat equation

$$(\partial_t + L)\, k(t, \xi) = 0$$

with the initial condition

$$\lim_{t \to 0} k(t, \xi) = \delta(\xi).$$

Hence first as in the absolute situation, if we rescale $k(t, \xi)$ as

$$r(u, t, \xi) := \sum_{i=0}^{n} u^{(2m-i)/2} k(ut, u^{1/2}\xi)_{[i]},$$

we have

$$(\partial_t + u\delta_u L \delta_u^{-1}) r(u, t, \xi) = 0.$$

In this way, we see that the local family index theorem means that

$$\lim_{u\to 0} r(u,t,\xi)|_{(t,\xi)=(1,0)} = (4\pi)^{-m}\det{}^{1/2}\left(\frac{R^{M/B}/2}{\sinh R^{M/B}/2}\right)\exp(-F^{\mathcal{E}/S}).$$

Now we expand the equation

$$(\partial_t + u\delta_u L\delta_u^{-1})\, r(u,t,\xi) = 0$$

with respect to $u^{1/2}$. For this, we need the following easy

Proposition. When $u \to 0^+$, the differential operator $u\delta_u L\delta_u^{-1}$ on $C^\infty(U, \wedge T^* \otimes \mathrm{End}(W))$ has a limit

$$K := -\sum_i(\partial_i - \frac{1}{4}\sum_j a_{ij}\xi_j)^2 + F.$$

Here F is the element of $\wedge^2 V^* \otimes \mathrm{End}(W)$, obtained by evaluting the twisted curvature $F^{\mathcal{E}/S}$ at x_0.

The proof of it is quite similar to the one in the absolute situation.

On the other hand, we also have the $\wedge T^* \otimes \mathrm{End}(W)$-valued polynomials $\gamma_i(t,\xi)$ on $\mathbf{R}_{>0} \times V$ such that for $N > j + |\alpha|/2$, $u \in (0,1], (t,\xi) \in (0,1) \times U$,

$$\|\partial_t^j \partial_\xi^\alpha (r(u,t,\xi) - q_t(\xi)\sum_{i=-2m}^{2N} u^{i/2}\gamma_i(t,\xi)\| \le C(N,j,\alpha)u^N.$$

Therefore, we have

$$r(u,t,\xi) \sim q_t(\xi)\sum_{i=0}^{\infty} u^{i/2}\gamma_i(t,\xi).$$

Hence, by using the Mehler formula, we complete the proof.

We end this section with the following application of the local family index theorem.

Theorem. (1) Let $\mathbf{B}_t := t^{1/2}\delta_t^B \mathbf{B}(\delta_t^B)^{-1}$ be the rescaled Bismut superconnection, then

$$\mathrm{ch}(\mathbf{B}_t) = \int_{M_s} \delta_t^B(\mathrm{Tr}_{s,\mathcal{E}_x}[k(x,t,x)])dx.$$

(2) When $t \to 0^+$, the section $\delta_t^B(\mathrm{Tr}_{s,\mathcal{E}_x}[k(x,t,x)]) \in C^\infty(M, \pi^* \wedge T^*B)$ has a limit, which is equal to

$$(2\pi i)^{-m} T_{M/B}(\hat{A}(M/B)\mathrm{ch}(\mathcal{E}/S)).$$

Hence, we have

$$\mathrm{ch}(\mathbf{B}_t) = (2\pi i)^{-m}\int_{M/B} \hat{A}(R^{M/B})\mathrm{ch}(\mathcal{E}/S).$$

Here $T_{M/B} : \mathcal{A} \to C^\infty(M, \pi^*(\wedge T^*B))$ is the map given by decomposing the bundle $\wedge T^*M$ as a tensor product $\wedge T^*(M/B) \otimes \pi^*(\wedge T^*B)$ and applying the Berezin integral to the first factor; that is, projecting onto $\wedge^n T^*(M/B) \otimes \pi^*(\wedge T^*B)$ and then dividing by the vertical Riemannian volume form.

Proof. (1) Note that since $\mathcal{F}_t = t\delta_t^B \mathcal{F}(\delta_t^B)^{-1}$, we have

$$\begin{aligned}\mathrm{ch}(\mathbf{B}_t) =&\mathrm{Tr}_s[e^{-\mathcal{F}_t}]\\ =&\mathrm{Tr}_s[\delta_t^B(e^{-t\mathcal{F}})]\\ =&\delta_t^B(\mathrm{Tr}_s[e^{-t\mathcal{F}}]).\end{aligned}$$

Thus by

$$\mathrm{Tr}_s(e^{-t\mathcal{F}_s}) = \int_{M_s} \mathrm{Tr}_{s,\mathcal{E}_x}[k_t(x,x)]\,dx,$$

we have (1).

For (2), we define the bigrading on

$$\wedge^{p,q}T_x^*M := \sum_{p,q} \mathcal{A}_s^p \otimes \wedge^q T_x^*(M/B).$$

Thus for any $a \in \mathcal{A}_s \otimes \mathrm{End}(\mathcal{E}_x)$,

$$\mathrm{Tr}_{s,\mathcal{E}_x}(a) = (-2\pi i)^m \sum_p \mathrm{Tr}_{s,\mathcal{E}/S}[\sigma_{[p,n]}(a)].$$

Therefore, we have

$$\delta_t^B(\mathrm{Tr}_{s,\mathcal{E}_x}[k(x,t,x)]) \sim (2\pi i)^{-m} \sum_{j,p} t^{j-m-p/2}\mathrm{Tr}_{s,\mathcal{E}/S}[\sigma_{[p,n]}(k_j(x))].$$

Since for $2j \leq n+p$, $\mathrm{Tr}_s[\sigma_{[p,n]}(k_j)] = 0$, we see that there is no singular term in the asymptotic expansion of $\delta_t^B(\mathrm{Tr}_{\mathcal{E}_x}[k(x,t,x)])$ as $t \to 0$. Hence, we have the result.

§I.5.5. The Situation At Infinity

In this section, we will study the behavior of the heat kernels associated with any superconnection $\mathbf{A}$ when $t \to +\infty$. This is a result of Berline and Vergne. Since the proof for this result has the same structure as the one for the finite dimensional case, we will first explain such a finite dimensional result.

I.5.5.a. The Situation In Finite Dimension

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \to M$ be a hermitian super-vector bundle over a manifold M. Let D be an odd endomorphism of $\mathcal{E}$ with components $D^\pm : \mathcal{E}^\pm \to \mathcal{E}^\mp$, such that $\ker D$ has constant rank. Then the family of superspaces $\{\mathrm{Ker}(D_z) : z \in M\}$ forms a superbundle, the **index bundle** of D, over M. Suppose that $\mathcal{E}$ has a hermitian structure so that the adjoint of D^- is D^+. Then D is self-adjoint. So, if let $\mathcal{E}_0 \subset \mathcal{E}$ be the superbundle $\mathrm{Ker} D$ graded by $E^\pm$, let P_0 be the orthogonal projection of $\mathcal{E}$ on $\mathcal{E}_0$, and let $P_1 = 1 - P_0$ be the orthogonal projection of $\mathcal{E}$ to $\mathcal{E}_1$, where $\mathcal{E}_1 = \mathrm{Im}(D) \subset \mathcal{E}$ is the image of the operator D, then there are decompositions

$$\mathcal{E}^\pm = \mathcal{E}_0^\pm \oplus \mathcal{E}_1^\pm$$

with $\mathcal{E}_0^\pm := \mathrm{Ker}(D^\pm)$ and the endomorphism D^+ gives an isomorphism between the bundles $\mathcal{E}_1^+$ and $\mathcal{E}_1^-$.

Let $\mathbf{A} := \mathbf{A}_{[0]} + \mathbf{A}_{[1]} + \mathbf{A}_{[2]} + \dots$ be a superconnection of $\mathcal{E}$, with curvature $\mathcal{F} := \mathbf{A}^2 \in \mathcal{A}(M, \mathrm{End}(\mathcal{E}))$. Then,

$$\mathrm{ch}(\mathbf{A}) = [2\pi \mathbf{i}]\mathrm{Tr}_s(e^{-\mathbf{A}^2}),$$

in de Rham cohomology, is equal to the difference of the Chern characters of the bundles $\mathcal{E}^+$ and $\mathcal{E}^-$. Furthermore, if $\mathbf{A}_{[0]} = D$, then

$$\begin{aligned}
\mathrm{ch}(\mathbf{A}) =& \mathrm{ch}(\mathcal{E}^+) - \mathrm{ch}(\mathcal{E}^-) \\
=& \mathrm{ch}(\mathcal{E}_0^+) + \mathrm{ch}(\mathcal{E}_1^+) - \mathrm{ch}(\mathcal{E}_0^-) - \mathrm{ch}(\mathcal{E}_1^-) \\
=& \mathrm{ch}(\mathcal{E}_0^+) - \mathrm{ch}(\mathcal{E}_0^-) = \mathrm{ch}(\mathrm{Ker}\,(D)).
\end{aligned}$$

Now we may say that our final result in this section is a refined version of the similar result at the level of differential forms.

Let $\tilde{\mathbf{A}}$ be the superconnection

$$\tilde{\mathbf{A}} := P_0 \mathbf{A} P_0 + P_1 \mathbf{A} P_1$$

which preserves the spaces $A(M, \mathcal{E}_0)$ and $A(M, \mathcal{E}_1) \subset A(M, \mathcal{E})$. We need to use the following notation: If $K \in \mathcal{A}(M, \mathrm{End}(\mathcal{E}))$, we write

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

which simply means that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} P_0 K P_0 & P_0 K P_1 \\ P_1 K P_0 & P_1 K P_1 \end{pmatrix}$$

with $\alpha \in \Gamma(M, \mathrm{End}(\mathcal{E}_0))$, etc.

Since $\tilde{\mathbf{A}}$ commutes with P_0, we see that its curvature has the form

$$\tilde{\mathcal{F}} = \tilde{\mathbf{A}}^2 = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$$

with R being obtained as follows: Denote by ∇_0 the connection on the bundle $\mathcal{E}_0$ given by the projection of the connection $\mathbf{A}_{[1]}$ onto the bundle $\mathcal{E}_0$:

$$\nabla_0 = P_0 \mathbf{A}_{[1]} P_0.$$

We filter the algebra $\mathcal{M} := A(M, \mathrm{End}(\mathcal{E}))$ by the subspaces

$$\mathcal{M}_i := \sum_{j \geq i} A^j(M, \mathrm{End}(\mathcal{E})).$$

Then we have

Lemma 1. The differential form R lies in $\mathcal{M}_2$, and the curvature of the connection ∇_0 equals $R_{[2]}$.

Proof. This comes from the fact that the superconnection $\mathbf{A}_{[0]} = P_0 \mathbf{A} P_0$ on the bundle $\mathcal{E}_0$ has its curvature $\mathbf{A}_0^2 = R$, and

$$\mathbf{A}_0 = \nabla_0 + \sum_{i \geq 2} P_0 \mathbf{A}_{[i]} P_0$$

as $P_0 \mathbf{A}_{[0]} P_0 = P_0 D P_0 = 0$.

Now, for $t > 0$, let δ_t be the automorphism of $A(M, \mathcal{E})$ which acts on $A^i(M, \mathcal{E})$ by multiplication by $t^{-i/2}$. Then

$$\mathbf{A}_t := t^{1/2} \delta_t \mathbf{A} \delta_t^{-1}$$

is again a superconnection on $\mathcal{E}$ and the decomposition of $\mathbf{A}_t$ into homogeneous components with respect to the exterior degree is given by the formula

$$\mathbf{A}_t = t^{1/2} \mathbf{A}_{[0]} + \mathbf{A}_{[1]} + t^{-1/2} \mathbf{A}_{[2]} + \dots.$$

The curvature $\mathcal{F}_t := \mathbf{A}_t^2$ of $\mathbf{A}_t$ is the operator $t \delta_t \mathcal{F} \delta_t^{-1}$, and the cohomology class of $\mathrm{ch}(\mathbf{A}_t) = [2\pi \mathbf{i}] \operatorname{Tr}_s[e^{-\mathcal{F}_t}]$ is independent of t: It is equal to the difference of the Chern characters $\mathrm{ch}(\mathcal{E}_0^+) - \mathrm{ch}(\mathcal{E}_0^-)$ for all $t > 0$. Next, we study the limit of $\mathrm{ch}(\mathbf{A}_t)$ as $t \to +\infty$. It is remarkable that the following stronger result holds.

Theorem. Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be an hermitian super-vector sheaf and let D be an odd endomorphism of $\mathcal{E}$ whose kernel has constant rank. Let $\mathbf{A}$ be a superconnection of $\mathcal{E}$ with zero-degree term being D. For $t > 0$, let

$$\mathbf{A}_t = t^{1/2} \delta_t \mathbf{A} \delta_t^{-1} = t^{1/2} D + \mathbf{A}_{[1]} + t^{-1/2} \mathbf{A}_{[2]} + \dots$$

be the rescaled superconnection, with curvature $\mathcal{F}_t$. Then for t large enough,

$$\|e^{-\mathcal{F}_t} - e^{-R_{[2]}}\|_l = O(t^{-1/2})$$

uniformly on compact subsets of M.

Proof. We begin with the following lemma.

Lemma 2. (a) Under the decomposition $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, the curvature $\mathcal{F}$ may be written as

$$\mathcal{F} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \in \begin{pmatrix} \mathcal{M}_2 & \mathcal{M}_1 \\ \mathcal{M}_1 & \mathcal{M}_0 \end{pmatrix}.$$

(b) The endomorphism $T_{[0]} \in \Gamma(M, \mathrm{End}(\mathcal{E}_1))$ is equal to $P_1 D^2 P_1$ and is positive definite.

(c) Denote the inverse of $T_{[0]}$ on $\mathcal{E}_1$ by G. The curvature $R_{[2]}$ of the connection ∇_0 on $\mathcal{E}_0$ is given by

$$R_{[2]} = X_{[2]} - Y_{[1]} G Z_{[1]}.$$

Proof. Let $\mathbf{A} = \tilde{\mathbf{A}} + \omega$ with

$$\omega = P_0 \mathbf{A} P_1 + P_1 \mathbf{A} P_0 = \begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix} \in \mathcal{M}_1.$$

Then

$$\mathcal{F} = \tilde{\mathcal{F}} + [\tilde{\mathbf{A}}, \omega] + \omega \wedge \omega,$$

so

$$\begin{aligned} \mathcal{F} &= \begin{pmatrix} R + \mu\nu & P_0[\tilde{\mathbf{A}}, \mu] P_1 \\ P_1[\tilde{\mathbf{A}}, \nu] P_0 & S + \nu\mu \end{pmatrix} \\ &\equiv \begin{pmatrix} R_{[2]} + \mu_{[1]}\nu_{[1]} & \mu_{[1]} D \\ D\nu_{[1]} & D^2 \end{pmatrix} \left(\mathrm{mod} \begin{pmatrix} \mathcal{M}_3 & \mathcal{M}_2 \\ \mathcal{M}_2 & \mathcal{M}_1 \end{pmatrix}\right). \end{aligned}$$

Thus if we write

$$\mathcal{F} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix},$$

we see that

$$X_{[2]} - Y_{[1]} G Z_{[1]} = (R_{[2]} + \mu_{[1]}\nu_{[1]}) - (\mu_{[1]} D) G (D\nu_{[1]}) = R_{[2]}.$$

Next we give a key technical lemma.

Lemma 3. There exists an invertible matrix g with $g - 1 \in \mathcal{M}_1$ such that

$$g \mathcal{F} g^{-1} = g \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} g^{-1} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

Furthermore

$$U \equiv X - YGZ \ (\mathrm{mod}\, \mathcal{M}_3),$$
$$V \equiv T \ (\mathrm{mod}\, \mathcal{M}_1).$$

Hence

$$t\delta_t(U) = R_{[2]} + O(t^{-1/2}).$$

Proof. Obviously the set of matrices of the form $1 + K$ with $K \in \mathcal{M}_1$ forms a group. So it makes sense for us to find an invertible matrix g so that $g - 1 \in \mathcal{M}_1$. To construct such a matrix g which puts $\mathcal{F}$ into a diagonal form, we use the induction on $\dim M - i$.

Assume that there exists g_i such that

$$g_i \mathcal{F} g_i^{-1} = \begin{pmatrix} X_i & Y_i \\ Z_i & T_i \end{pmatrix} \in \begin{pmatrix} \mathcal{M}_2 & \mathcal{M}_i \\ \mathcal{M}_i & \mathcal{M}_0 \end{pmatrix},$$

with $T_i \equiv D^2$ (mod$\mathcal{M}_1$). In particular

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 - GT_i \end{pmatrix} \in \mathcal{M}_1.$$

But

$$\begin{pmatrix} 0 & -Y_iG \\ GZ_i & 0 \end{pmatrix} \in \mathcal{M}_i,$$

so

$$\begin{pmatrix} 1 & -Y_iG \\ GZ_i & 1 \end{pmatrix}^{-1} - \begin{pmatrix} 1 & Y_iG \\ -GZ_i & 1 \end{pmatrix} \in \mathcal{M}_{2i}.$$

Hence, if we define $\tilde{X}_i$, etc., by

$$\begin{pmatrix} 1 & -Y_iG \\ GZ_i & 1 \end{pmatrix} \begin{pmatrix} X_i & Y_i \\ Z_i & T_i \end{pmatrix} \begin{pmatrix} 1 & Y_iG \\ GZ_i & 1 \end{pmatrix}^{-1} =: \begin{pmatrix} \tilde{X}_i & \tilde{Y}_i \\ \tilde{Z}_i & \tilde{T}_i \end{pmatrix},$$

then

$$\begin{aligned}
\tilde{X}_i \equiv & X_i - 2(Y_iG)Z_i + (Y_iG)T_i(GZ_i) \\
\equiv & X_i \ (\mathrm{mod}\ \mathcal{M}_{2i}); \\
\tilde{Y}_i \equiv & Y_i(1 - GT_i) + (X_i - (Y_iG)Z_i)(Y_iG) \in \mathcal{M}_{i+1}; \\
\tilde{Z}_i \equiv & (1 - T_iG)Z_i + (GZ_i)X_i - (GZ_i)Y_i(GZ_i) \in \mathcal{M}_{i+1}; \\
\tilde{T}_i \equiv & T_i + (GZ_i)X_i(Y_iG) + Z_i(Y_iG) + (GZ_i)Y_i \\
\equiv & T_i \ (\mathrm{mod}\mathcal{M}_1).
\end{aligned}$$

Thus by looking at the sub-index, we may continue the induction.

Now suppose that we have a matrix g of the required form which diagonalizes $\mathcal{F}$. Then

$$\begin{pmatrix} 1+K & M \\ N & 1+L \end{pmatrix} \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} 1+K & M \\ N & 1+L \end{pmatrix},$$

for some

$$\begin{pmatrix} K & M \\ N & L \end{pmatrix} \in \mathcal{M}_1.$$

So

$$\begin{pmatrix} X+KX+MZ & Y+KY+MT \\ NX+Z+LZ & NY+T+LT \end{pmatrix} = \begin{pmatrix} U(1+K) & UM \\ VN & V(1+L) \end{pmatrix}.$$

Since $X \in \mathcal{M}_2$ and $K, L, M, N, Y, Z \in \mathcal{M}_1$, we have

(a) $V = (T+LT+NY)(1+L)^{-1} \equiv T \ (\mathrm{mod}\mathcal{M}_1)$, hence $GV \equiv 1 \ (\mathrm{mod}\mathcal{M}_1)$.
(b) $U = (X+KX+MZ)(1+K)^{-1} \equiv X+MZ \ (\mathrm{mod}\mathcal{M}_3)$.
(c) $Y+MT = UM - KY \in \mathcal{M}_2$. Hence multiplying on the right by G, we have $M \equiv -YG \ (\mathrm{mod}\mathcal{M}_2)$. But this is what we want for this lemma.

As for the theorem, we may write

$$e^{-t\delta_t(\mathcal{F})} = \delta_t(g)^{-1} \begin{pmatrix} e^{-t\delta_t(U)} & 0 \\ 0 & e^{-t\delta_t(V)} \end{pmatrix} \delta_t(g).$$

Now by the fact that $V_{[0]} = T_{[0]} = D^2$ is positive definite on $\mathcal{E}_1$, using the Volterra series, we have the following

Lemma 4. There exist constants $\varepsilon, C > 0$, such that $|e^{-t\delta_t(V)}| \leq Ce^{-\varepsilon t}$.

Hence, we have

$$e^{-t\delta_t(\mathcal{F})} = \delta_t(g)^{-1} \begin{pmatrix} e^{-R_{[2]}} + O(t^{-1/2}) & 0 \\ 0 & 0 \end{pmatrix} \delta_t(g) + O(e^{-\varepsilon t}).$$

But $\delta_t(g)$ and $\delta_t(g^{-1}) = \delta_t(g)^{-1}$ have the form

$$\begin{pmatrix} 1+O(t^{-1/2}) & O(t^{-1/2}) \\ O(t^{-1/2}) & 1+O(t^{-1/2}) \end{pmatrix}.$$

It follows that

$$e^{-t\delta_t(\mathcal{F})} = \begin{pmatrix} e^{-R_{[2]}} + O(t^{-1/2}) & O(t^{-1/2}) \\ O(t^{-1/2}) & O(t^{-1}) \end{pmatrix}.$$

Similarly, we can deal with the situation for derivatives with respect to the base. Hence, we have the assertion.

Corollary. The limit $\lim_{t\to+\infty} \mathrm{ch}(\mathbf{A}_t)$ exists, and equals the Chern character of the connection ∇_0 on the superbundle $\mathcal{E} = \mathrm{Ker}(D)$.

I.5.5.b. **The Infinite Dimensional Situation.**

We use the same notation as in §I.5.1. What we discuss in this section is a fundamental theorem of Berline-Vergne, which generalizes the result of last subsection to the case of a family of Dirac operators. Assume now that D is a family of Dirac operators such that $\mathrm{Ker}D^z$ has a constant dimension, so that $\mathrm{Ker}D$ is a superbundle over M. If P_0^Z is the orthogonal projection from $f_*\mathcal{E}_z$ to $\mathrm{Ker}D^z$, then $P_0 \in \Gamma(M, \mathcal{K}(\mathcal{E}))$ is a smooth family of smoothing operators. Also we have the following easy

Lemma 1. The operator ∇_0 defined by the formula $\nabla_0 := P_0 \mathbf{A}_{[1]} P_0$ is a connection on the superbundle $\mathrm{Ker}D$.

For $t > 0$, let δ_t be the automorphism of $A(M, \pi_*\mathcal{E})$ which multiplies $A^i(M, \pi_*\mathcal{E})$ by $t^{-i/2}$. Then $\mathbf{A}_t := t^{1/2}\delta_t \mathbf{A} \delta_t^{-1}$ is a superconnection for the family of Dirac operators $t^{1/2}D$.

Theorem. For $t > 0$, let

$$\mathbf{A}_t := t^{1/2}\delta_t \mathbf{A}\delta_t^{-1} = t^{1/2}D + \mathbf{A}_{[1]} + t^{-1/2}\mathbf{A}_{[2]} + \dots$$

be the rescaled superconnection with curvature $\mathcal{F}_t = t\delta_t(\mathcal{F})$. Then for t large enough,

$$||e^{-\mathcal{F}_t} - e^{-\nabla_0^2}||_l \le C(l)t^{-1/2}$$

uniformly on compact subsets of $M \times_\pi M$.

Proof. The structure of the proof for this theorem is the same as the one for the finite dimensional case. First we filter the algebra

$$\mathcal{M} := \mathcal{A}(M, \mathrm{End}_{\mathcal{P}}(\mathcal{E})) = \Gamma(M, \pi^* \wedge T^*M \otimes \mathrm{End}_{\mathcal{P}}(\mathcal{E}))$$

by the subspaces

$$\mathcal{M}_i := \sum_{j \ge i} A^j(M, \mathrm{End}_{\mathcal{P}}(\mathcal{E})).$$

In the same way, we also get a filtration for the algebra $\mathcal{N} := A(B, \mathcal{K}(\mathcal{E}))$.

Let $G := (G^z : z \in M)$ be the family of Green's operators G^z of $(D^z)^2$. Thus locally

$$G = \int_0^\infty e^{-t(D^2+P_0)}dt - P_0.$$

Then G preserves $\mathcal{N}$. In fact, for every $K \in \mathcal{N}$, we may decompose $\partial^\alpha((G + P_0)K)$ into terms proportional to

$$\int_0^\infty (\partial_z^{\alpha_1} e^{-t(D^2+P_0)})(\partial_z^{\alpha_2} K)dt$$

with $\alpha_1 + \alpha_2 = \alpha$. Thus by Duhamel's formula, we see that, in general, this integration has the form

$$\int_0^\infty \dots \int_0^\infty e^{-t_1(D^2+P_0)}D_1 e^{-t_2(D^2+P_0)})D_2 \dots e^{-t_k(D^2+P_0)}D_k dt_1 \dots dt_k,$$

where $k = |\alpha_1|+1$ and $D_i \in \Gamma(M, \mathrm{End}_{\mathcal{P}}(\pi_*\mathcal{E}))$. So the fact that $< x|e^{-t(D^2+P_0)})|x >$ decaies exponentially implies that the above integratal is bounded. Now by the formula

$$\partial^\alpha(GK) = \partial^\alpha((G+P_0)K) - \partial^\alpha(P_0K),$$

we see that $GK \in \mathcal{N}$. Similarly, we know that $KG \in \mathcal{N}$.

Let $P_1 = 1 - P_0$ be the projection onto $\mathrm{Im}\, D$. If $K \in \mathcal{M}$, we let

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} P_0KP_0 & P_0KP_1 \\ P_1KP_0 & P_1KP_1 \end{pmatrix} \in \begin{pmatrix} \mathcal{N} & \mathcal{N} \\ \mathcal{N} & \mathcal{M} \end{pmatrix}.$$

Lemma 2. (a) Let $R_{[2]}$ be the curvature of the connection ∇_0 on the bundle Ker D, and let

$$\mathcal{F} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$$

be the curvature of the superconnection $\mathbf{A}$. Then $X, Y, Z \in \mathcal{N}$ and

$$R_{[2]} = X_{[2]} - Y_{[1]}GZ_{[1]}.$$

(b) There exists $g \in \mathcal{M}$ with $g - 1 \in \mathcal{N}_1$, such that

$$g\begin{pmatrix} Z & Y \\ Z & T \end{pmatrix} g^{-1} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

Furthermore

$$U \equiv X - YGZ \ (\mathrm{mod}\, \mathcal{N}_3)$$
$$V \equiv T \ (\mathrm{mod}\, \mathcal{N}_1).$$

Proof. If G is the Green operator of D^2, and $Y, Z \in \mathcal{N}_i$, then the operators YG and GZ are also in $\mathcal{N}_i$, but $Y(1-GT)$ and $(1-TG)Z$ are in $\mathcal{N}_{i+1}$. So we may construct g as in last section as a product of matrices of the form

$$\begin{pmatrix} 1 & -YG \\ GZ & 1 \end{pmatrix} \in 1 + \begin{pmatrix} 0 & \mathcal{N}_1 \\ \mathcal{N}_1 & 0 \end{pmatrix}.$$

Now the proof of this lemma is obvious from the one in the last section.

Since $U \equiv R_{[2]} \ (\mathrm{mod}\, \mathcal{N}_3)$ and $V \equiv D^2 \ (\mathrm{mod}\, \mathcal{N}_1)$. So for each $t > 0$, the family of operators $\delta_t(V)$ is the sum of a family of generalized Laplacians D^2 and an element of $P_1\mathcal{M}_1P_1$. Hence, $e^{-t\delta_t(V)}$ is a section in $\mathcal{N}$ for each $t > 0$. Thus, by the uniqueness of the heat kernels, we know that

$$e^{-t\delta_t(\mathcal{F})} = \delta_t(g)^{-1} \begin{pmatrix} e^{-t\delta_t(U)} & 0 \\ 0 & e^{-t\delta_t(V)} \end{pmatrix} \delta_t(g).$$

With this, if U is a relatively compact open subset of M, and λ is the infinitum over U of the lowest non-zero eigenvalue of the operators D^2, then, by the Volterra series, we easily see that over U,

$$P_1 e^{-t\delta_t(V)} P_1 = O(e^{-t\lambda/2}).$$

(Here, we use the following convention: If $A(t) : \mathbf{R}_{>0} \to \Gamma(B, \mathcal{K}(\mathcal{E}))$, we write $A(t) = O(f(t))$ if for all $\varepsilon > 0$, $l \in \mathbf{N}$ and each function $\phi \in C_c^\infty(M)$ of compact support, there is a constant $C(l, \varepsilon, \phi)$ such that

$$\|\pi^*(\phi)(x) < x|A(t)|y > \|_l \leq C(l, \varepsilon, \phi) f(t)$$

for all $t > \varepsilon$.) Therefore, we have

$$e^{-t\delta_t(\mathcal{F})} = \delta_t(g)^{-1} \begin{pmatrix} e^{-R_{[2]}} & 0 \\ 0 & 0 \end{pmatrix} \delta_t(g) + \delta_t(g)^{-1} \begin{pmatrix} O(t^{-1/2}) & 0 \\ 0 & O(e^{-t\lambda/2}) \end{pmatrix} \delta_t(g).$$

But $\delta_t(g) - 1 = O(t^{-1/2})$, so we have

$$e^{-t\delta_t(\mathcal{F})} = \begin{pmatrix} e^{-R_{[2]}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} O(t^{-1/2}) & O(t^{-1/2}) \\ O(t^{-1/2}) & O(t^{-1}) \end{pmatrix}.$$

From here, the theorem follows for $l = 1$. For derivations, we may proceed in the same way. The details are left to the reader.

Corollary. The limit

$$\lim_{t\to\infty} \mathrm{ch}(\mathbf{A}_t) = \mathrm{ch}(P_0 \mathbf{A}_{[1]} P_0) \in \mathcal{A}(M)$$

holds with respect to each C^l-norm on compact subsets of M.

§I.5.6. From The Real Situation To The Complex Situation

In this section, we present the results discussed in the last few sections in the sense of Kähler geometry, by comparing them with those for Riemannian geometry.

I.5.6.a. Absolute Situation: Dirac Operators

First we present the complex theory of dirac operators. This may be treated by the following

Theorem. (1) Over a Riemannian manifold, the de Rham complex is given by

$$0 \to A^0(M) \xrightarrow{d_0} A^1(M) \xrightarrow{d_1} A^2(M) \xrightarrow{d_2} \ldots.$$

Furthermore, we have

(a) The bundle $\wedge T^*M$ is a Clifford module defined by

$$c(\alpha)\beta = \varepsilon(\alpha)\beta - \iota(\alpha)\beta$$

for $\alpha \in \Gamma(M, T^*M)$, $\beta \in A(M)$.

(b) The Levi-Civita connection on the bundle $\wedge T^*M$ is a Clifford connection.

(c) The Dirac operator associated with the above data is the operator $d + d^*$, with

$$d^* : A^{\cdot}(M) \to A^{\cdot -1}(M)$$

the adjoint of the exterior differential d.

(d) (**Weizenböck's formula**):

$$(d + d^*)^2 = \Delta^{\wedge T^*M} - \sum_{ijkl} R_{ijkl} \varepsilon^k \iota^l \varepsilon^i \iota^j.$$

(2) Over a Kähler manifold M, the Dolbeault complex is defined by

$$0 \to A^{p,0} \xrightarrow{\bar\partial} A^{p,1} \xrightarrow{\bar\partial} A^{p,2} \xrightarrow{\bar\partial} \dots,$$

with $\bar\partial := \sum_{i=1}^n \varepsilon(d\bar z^i) \frac{\partial}{\partial \bar z^i}$ locally. Furthermore, we have

(a) The vector bundle of anti-holomorphic differential forms $\wedge(T^{0,1}M)^*$ is a Clifford module defined by

$$c(f)\mu = \sqrt{2}(\varepsilon(f^{0,1}) - \iota(f^{1,0}))\mu,$$

where $f = f^{0,1} + f^{1,0}$ and $f^{a,b} \in (T^{b,a}M)^*$.

(b) The Levi-Civita connection is a Clifford connection.

(c) The associated Dirac operator is $\sqrt{2}(\bar\partial + \bar\partial^*)$.

(d) (**Bochner-Kodaira's formula**):

$$(\bar\partial + \bar\partial^*)^2 = \Delta^{0,\cdot} + \sum_{i,j} \varepsilon(d\bar z^i)\iota(dz^j) F^{K^*}(\partial_{z^j}, \partial_{\bar z^i}).$$

Here $K = \wedge^n (T^{1,0}M)^*$ is the canonical bundle of M.

Proof. (1) By a local calculation in this case, since ∇ is torsion-free, we know that $d = \varepsilon \circ \nabla, d^* = -\iota \circ \nabla$. So $d + d^* = c \circ \nabla$. Now by the Lichnerowicz formula, we have

$$(d + d^*)^2 = \Delta^{\wedge T^*M} + \frac{1}{2} \sum_{ijkl} R_{ijkl} (\varepsilon^k - \iota^k)(\varepsilon^l - \iota^l) \varepsilon^i \iota^j.$$

But R_{ijkl} vanishes over the antisymmetrization of three indices, so

$$\sum \varepsilon^i \varepsilon^j \varepsilon^k \iota^l R_{ijkl} = 0, \quad \sum \iota^i \iota^j \varepsilon^k \iota^l R_{ijkl} = 0.$$

Hence we have the assertion.

(2) We know that without the Kähler condition, the vector bundle $\wedge(T^{0,1})^*$ is still a Clifford module and the Clifford action is self-adjoint. In general, the corresponding canonical connection is not a Clifford connection. But once we have the Kähler condition, the situation changes dramatically. In fact, by the Kähler condition, we know that the Levi-Civita connection of the underlying Riemannian structure preserves the bundles $T^{1,0}M$ and $T^{0,1}M$ (which may be thought of as the definition of the Kähler condition.) Hence we have the assertion that the corresponding canonical connection is a Clifford connection. Next, we have to prove that the associated Dirac operator is $\sqrt{2}(\bar{\partial}+\bar{\partial}^*)$.

Let Z_i be a local orthonormal frame of $T^{1,0}M$ with dual frame $Z^i \in (T^{1,0}M)^*$. Then

$$d = \sum_i (\varepsilon(Z^i)\nabla_{Z_i} + \varepsilon(\bar{Z}^i)\nabla_{\bar{Z}^i}), \quad \bar{\partial} = \sum_i \varepsilon(\bar{Z}^i)\nabla_{\bar{Z}^i}.$$

Hence it is enough to show that

$$\bar{\partial}^* = -\sum_i \iota(\bar{Z}^i)\nabla_{\bar{Z}^i}.$$

Let α be the one-form on M such that for $\beta_q \in A^{p,q}(M)$ and $\beta_{q+1} \in A^{p,q+1}(M)$,

$$\alpha(X) = (\beta_q, \iota(X^{0,1})\beta_{q+1}).$$

Since ∇ preserves the splitting $TM \otimes_{\mathbf{R}} \mathbf{C} = T^{1,0}M \oplus T^{0,1}M$,

$$\mathrm{Tr}(\nabla\alpha) = \sum_i (\bar{Z}_i\alpha(\bar{Z}_i) - \alpha(\nabla_{\bar{Z}_i}\bar{Z}_i)).$$

Thus

$$((\sum_i \varepsilon(\bar{Z}^i)\nabla_{\bar{Z}_i})\beta_q, \beta_{q+1})_x = -(\beta_q, (\sum_i \iota(\bar{Z}^i)\nabla_{\bar{Z}^i}\beta_{q+1})_x + \mathrm{Tr}(\nabla\alpha)_x.$$

Now the assertion comes from the fact that the integration of the last term over M vanishes.

Finally, for $(\bar{\partial}+\bar{\partial}^*)^2$, do the same thing as for Lichnerowicz's formula. We have

$$(\bar{\partial}+\bar{\partial}^*)^2 = \nabla^{0,\cdot} + \sum_{ij} \varepsilon(d\bar{z}^i)\iota(dz^j)R^+(\partial_{z^j}, \partial_{\bar{z}^i})$$

with R^+ the curvature of $\wedge(T^{0,1}M)^*$. Thus, it is enough to show that

$$\sum_{ij} \varepsilon(d\bar{z}^i)\iota(dz^j)R^+(\partial_{z^j}, \partial_{\bar{z}^i}) = \sum_{ij} \varepsilon(d\bar{z}^i)\iota(dz^j)F^{K^*}(\partial_{z^j}, \partial_{\bar{z}^i}).$$

By definition, we know that the left hand side is equal to

$$\begin{aligned}
&\sum_{ijkl} \varepsilon(\bar{Z}_i^*)\iota(Z_j^*)\varepsilon(\bar{Z}_l^*)\iota(Z_k^*)(R(Z_j,\bar{Z}_i)Z_k, \bar{Z}_l) \\
&= \sum_{ijk} \varepsilon(\bar{Z}_i^*)\iota(Z_k^*)(R(Z_j,\bar{Z}_i)Z_k, \bar{Z}_j).
\end{aligned}$$

Now by the facts that

$$R(Z_j,\bar{Z}_i)Z_k + R(Z_k,Z_j)\bar{Z}_i + R(\bar{Z}_i,Z_k)Z_j = 0$$

and that $R(Z_k,Z_j)=0$, we have

$$R(Z_j,\bar{Z}_i)Z_k = R(Z_k,\bar{Z}_i)Z_j.$$

Hence we have the final assertion.

I.5.6.b. The Absolute Situation: Index Theorem

Theorem. (1) (**Atiyah-Singer Index Theorem**) The index of a Dirac operator on a Clifford module $\mathcal{E}$ over a compact oriented even-dimensional manifold is given by the cohomological formula

$$\operatorname{Ind}(D) = \int_M \hat{A}(M)\mathrm{ch}(\mathcal{E}/S).$$

(2) (**Hirzebruch-Riemann-Roch Theorem**) The Euler number of the holomorphic vector bundle $\mathcal{E}$ over a Kähler manifold M is given by the cohomological formula:

$$\chi(M,\mathcal{E}) = \int_M \mathrm{td}(M)\,\mathrm{ch}(\mathcal{E}).$$

Proof. (1) This is a consequence of the result of Patodi and Gilkey stated in subsection 3.7.a.

(2) We may deduce this formula from (1). In fact, if we consider the Riemannian curvature R to be the matrix with two-form coefficients, then the curvature operator $(\nabla^{\wedge(T^{0,1}M)^*})^2$ is $\sum_{ij}(RZ_i,\bar{Z}_j)\varepsilon(\bar{Z}^j)\iota(Z^k)$. But the $\mathrm{End}(\wedge(T^{0,1}M)^*)$-valued two-form $R^{\wedge(T^{0,1}M)^*}$ equals

$$\frac{1}{4}\sum_{ij}(RZ_i,\bar{Z}_j)c(Z^j)c(\bar{Z}^j) + \frac{1}{4}\sum_{ij}(R\bar{Z}_i,Z_j)c(\bar{Z}^i)c(Z^j).$$

So

$$(\nabla^{\wedge(T^{0,1}M)^*})^2 = R^{\wedge(T^{0,1}M)^*} + \frac{1}{2}\sum_i R(Z_i,\bar{Z}_i).$$

We know that, by definition,

$$F^{\wedge(T^{0,1}M)^*\otimes\mathcal{E}/S} = \frac{1}{2}\mathrm{Tr}_{T^{1,0}M}(R^+) + F^{\mathcal{E}}.$$

Here R^+ denotes the curvature of the bundle $T^{1,0}M$ and $F^{\mathcal{E}}$ denotes the curvature of $\mathcal{E}$. Now by the splitting $TM\otimes_{\mathbf{R}}\mathbf{C} = T^{1,0}M\oplus T^{0,1}M$, we see that

$$\hat{A}(M) = \det\left(\frac{R^+}{e^{R^+/2}-e^{-R^+/2}}\right).$$

Hence

$$\hat{A}(M)\mathrm{Tr}_{\wedge(T^{0,1}M)^*\otimes\mathcal{E}/S}(\exp(-F^{\wedge(T^{0,1}M)^*\otimes\mathcal{E}/S})) = \mathrm{td}(M)\,\mathrm{Tr}(\exp(-F^{\mathcal{E}})),$$

since the Todd genus

$$\mathrm{td}(M) = \det(\frac{R^+}{e^{R^+}-1}) = \det(\frac{R^+}{e^{R^+/2}-e^{-R^+/2}})\exp(-\mathrm{Tr}(R^+/2)).$$

Therefore, we have our assertion.

I.5.6.c. **The Relative Situation: Smooth Fibrations**

Let $\pi : M \to B$ be a smooth family as in the previous chapter. We assume that M and B are complex manifolds of dimension n and m respectively. Then we have the following exact sequence of holomorphic tangent bundles over M:

$$0 \to T^{1,0}Z \to T^{1,0}M \to \pi^*T^{1,0}B \to 0.$$

Also, as C^∞-bundles, we know that $T^{1,0}_H M \simeq \pi^*T^{1,0}B$. However, in general, we do not have this isomorphism as holomorphic bundles. Hence $T^{1,0}_H M$ is not a holomorphic subbundle of $T^{1,0}M$. Now we state the complex situation for the smooth family as a triple $(\pi, g_Z, T_H M)$ with a smooth 2-form ω on M of complex type $(1,1)$, which has the following properties:
(1) ω is closed;
(2) $T_H M$ and TZ are orthogonal with respect to ω;
(3) If $X, Y \in TZ$, then $\omega(X,Y) = < X, JY >$.

Usually, we call such a family a **Kähler fibration** with associated $(1,1)$ form ω. In this case, we know that (M,ω) and (Z, g^Z) are Kähler and B is locally Kähler; i.e. there is an open covering $\mathcal{U}$ of B, such that there is a closed $(1,1)$ form η^U on U, which induces a Kähler metric on TB. We also know that on $\pi^{-1}(U)$, one may replace ω by $\omega + \lambda\pi^*\eta^U$ for any $\lambda > 0$. Since the fiber Z is compact, if λ is large enough, $\omega + \lambda\pi^*\eta^U$ is a Kähler form on $\pi^{-1}(U)$, which induces the metric g^Z on Z and is such that $T_H M = (TZ)^\perp$. Denote by ω_H, ω_Z the restrictions of ω to $T_H M, TZ$, respectively. Thus, on TM, we have the relation $\omega = \omega_H + \omega_Z$. We know that the pair $(g_Z, T_H M)$ is entirely determined by ω. In fact, we easily have the following

Proposition 1. Let ω be a smooth 2-form on M of complex type $(1,1)$, which has the following properties:
(a) ω is closed;
(b) If $X, Y \in TZ, (X,Y) \mapsto \omega(JX, Y)$ defines a hermitian product g_Z on TZ.
For any $x \in M$, let

$$T_{H\,x}M := \{Y \in T_x M : \omega(X,Y) = 0, \forall X \in T_x Z\}.$$

Then, $T_H M$ is a smooth subbundle of TM such that $(\pi, g_Z, T_H M)$ is a Kähler fibration with associated $(1,1)$-form ω.

The bundle $T^{*0,1}Z$ is identified to $T^{1,0}Z$ by the metric g_Z. Therefore $T^{*0,1}Z$ inherits the holomorphic structure of $T^{1,0}Z$. ∇^Z induces the corresponding canonical connections on $T^{*0,1}Z$. Hence we know that $\wedge T^{*0,1}Z$ is also a holomorphic hermitian vector bundle on M. If ξ is a holomorphic hermitian vector bundle on M of complex rank k, for $0 \leq p \leq l$, we let E^p denote the set of C^∞ sections over M of $\wedge^p T^{*0,1}Z \otimes \mathcal{E}$. We also regard E^p as the set of C^∞ sections over B of an infinite dimensional bundle: For any $y \in B$, the corresponding fiber E^p_y is the set of C^∞ sections over Z_y of $\wedge^p T^{*0,1}Z \otimes \mathcal{E}$. Set

$$E^+ := \oplus_{p \text{ even}} E^p, \quad E^- := \oplus_{p \text{ odd}} E^p, \quad E = E^+ \oplus E^-.$$

Let dx be the Riemannian volume element for the fiber Z. Then for any $y \in B$, we have an L^2 metric

$$\int_{Z_y} < e, e' > (x) dx$$

on E_y. Let $(z^1 = x^1 + iy^1, \ldots, z^l = x^l + iy^l)$ be a complex system of coordinates in one given fiber Z, and let TZ be oriented by the base $(\partial/\partial x^1, \partial/\partial y^1, \ldots, \partial/\partial x^l, \partial/\partial y^l)$. (So we have $\partial/\partial y^j = J\partial/\partial x^j$.) Let

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right),$$

and

$$dz^j = dx^j + idy^j, \quad d\bar{z}^j = dx^j - idy^j.$$

Locally, let

$$\bar{\partial}^{Z_y} := \sum_{j=1}^{l} d\bar{z}^j \wedge \frac{\partial}{\partial \bar{z}^j}.$$

Then for every $y \in B$, the operator $\bar{\partial}^{Z_y}$ acts naturally on E_y. Let $\bar{\partial}^{Z_y *}$ be the formal adjoint of $\bar{\partial}^{Z_y}$ with respect to the hermitian metrics on E_y. Motivated by the result of subsection 6.a, we let

$$\bar{\partial}_y = \sqrt{2}\bar{\partial}^{Z_y}, \quad \bar{\partial}^*_y = \sqrt{2}\bar{\partial}^{Z_y *}, \quad D_y = \bar{\partial}_y + \bar{\partial}^*_y.$$

Then the Dirac operator D_y interchanges E^+_y and E^-_y. Let $D_{\pm,y}$ be the restriction of D_y to $E^\pm_y$. Also, by a local trivialization of the fibration π, we know that $\bar{\partial}_y, \bar{\partial}^*_y, D_y$ are first order differential operators whose coefficients depend smoothly on $x \in M$, and D_y is formally self-adjoint on E_y.

Next we define a Clifford module structure on $\wedge T^{*0,1}Z \otimes \mathcal{E}$: If $X \in T^{1,0}Z$, denote $X^* \in T^{*0,1}Z$ the 1-form $Y \in T_{\mathbb{C}}Z \mapsto < X, Y >$, we define $c(X) \in \mathrm{End}(\wedge T^{*0,1}Z \otimes \mathcal{E})$ by $c(X) := \sqrt{2}X^* \wedge$. If $X' \in T^{0,1}Z$, we let $c(X') := -\sqrt{2}\iota_{X'}$.

With this, we can give another description of D_y. In fact, if we let $e_1, \ldots, e_n$ be an orthonormal basis of TZ, $w_1, \ldots w_l$ an orthonormal basis of $T^{1,0}Z$, with the corresponding basis $\bar{w}_j, w^j, \bar{w}^j$ on $T^{0,1}Z, T^{*1,0}Z, T^{*0,1}Z$ respectively. Then by the fact that Z_y is Kähler, we have

$$\bar{\partial}^{Z_y} = d\bar{z}^j \wedge \nabla_{\partial/\partial\bar{z}^j} = \bar{w}^j \wedge \nabla_{\bar{w}_j}, \quad \bar{\partial}^{Z_y *} = -\iota_{\bar{w}_j}\nabla_{w_j}.$$

Therefore we have

Proposition 2. For any $y \in B$, $D_y = \sum_{k=1}^n c(e_k)\nabla_{e_k}$.

Just as in the real situation, we introduce a connection $\nabla^{\pi_*\mathcal{E}}$ on E as follows: for any C^∞ section of E, if $Y \in TB$, then

$$\nabla_Y^{\pi_*\mathcal{E}} h := \nabla_{Y^H} h,$$

where Y^H is the lifting of Y in $T_H M$.

Theorem. (1) The connection $\nabla^{\pi_*\mathcal{E}}$ does not depend on the metric on B, and preserves the hermitian metric on E.

(2) As a 2-form $(\nabla^{\pi_*\mathcal{E}})^2$ is of complex type $(1,1)$. Furthermore, for any $U \in T^{1,0}B, V \in T^{0,1}B$,

$$\nabla_V^{\pi_*\mathcal{E}}\bar{\partial} = 0, \quad \nabla_U^{\pi_*\mathcal{E}}\bar{\partial}^* = 0.$$

Proof. (1) may be proved as in the real situation.

(2) We know, by a local calculation, that the curvature of $\nabla^{\pi_*\mathcal{E}}$ is given by

$$(\nabla^{\pi_*\mathcal{E}})^2(Y,Y') = R^Z(Y^H, Y'^H) \otimes 1 + 1 \otimes R^{\mathcal{E}}(Y^H, Y'^H) - \nabla_{T(Y^H, Y'^H)},$$

for $Y, Y' \in TB$ with $R^Z, R^{\mathcal{E}}$ the curvature of TZ and $\mathcal{E}$ respectively, and T the torsion of $\nabla^{\oplus}$. Thus by the condition for a Kähler fibration, we have that ∇^Z on TZ preserves the complex structure of TZ and induces on $T^{1,0}Z$ its canonical connection, and T is of type (1,1). So we know that $(\nabla^{\pi_*\mathcal{E}})^2$ is of complex type (1,1). On the other hand, if $(y^1, \ldots, y^{l'})$ is a complex coordinate system of B with $(\partial/\partial y^\alpha)$ the corresponding basis of $T^{1,0}B$, etc, then by a local calculation, since $R^Z, R^{\mathcal{E}}, T$ are of type (1,1), we also have that

$$\begin{aligned}
\nabla^{\pi_*\mathcal{E}} D =& \nabla^{\pi_*\mathcal{E}}\bar{\partial} + \nabla^{\pi_*\mathcal{E}}\bar{\partial}^* \\
=& dy^\alpha c(w_j)[R^Z((\partial/\partial y^\alpha), \bar{w}_j) \otimes 1 \\
&+ 1 \otimes R^{\mathcal{E}}((\partial/\partial y^\alpha), \bar{w}_j) - \nabla_{T((\partial/\partial y^\alpha), \bar{w}_j)}] \\
&+ d\bar{y}^\alpha c(\bar{w}_j)[R^Z((\partial/\partial \bar{y}^\alpha), w_j) \otimes 1 \\
&+ 1 \otimes R^{\mathcal{E}}((\partial/\partial \bar{y}^\alpha), w_j) - \nabla_{T((\partial/\partial \bar{y}^\alpha), w_j)}].
\end{aligned}$$

Since $\nabla^{\pi_*\mathcal{E}}$ preserves the grading in E, and so $\nabla^{\pi_*\mathcal{E}}\bar{\partial}$ (resp. $\nabla^{\pi_*\mathcal{E}}\bar{\partial}^*$) increases (resp. decreases) the degree in E by 1. But R^Z, $R^{\mathcal{E}}$, ∇_T do not change the grading in E, hence

$$\begin{aligned}
\nabla^{\pi_*\mathcal{E}}\bar{\partial} =& dy^\alpha c(w_j)[R^Z((\partial/\partial y^\alpha), \bar{w}_j) \otimes 1 \\
&+ 1 \otimes R^{\mathcal{E}}((\partial/\partial y^\alpha), \bar{w}_j) - \nabla_{T((\partial/\partial y^\alpha), \bar{w}_j)}]; \\
\nabla^{\pi_*\mathcal{E}}\bar{\partial}^* =& d\bar{y}^\alpha c(\bar{w}_j)[R^Z((\partial/\partial \bar{y}^\alpha), w_j) \otimes 1 \\
&+ 1 \otimes R^{\mathcal{E}}((\partial/\partial \bar{y}^\alpha), w_j) - \nabla_{T((\partial/\partial \bar{y}^\alpha), w_j)}].
\end{aligned}$$

Therefore, by counting the degrees of both sides, we have the assertion.

Remark. If E is a finite dimensional complex hermitian vector bundle on B, endowed with a hermitian connection ∇ whose curvature is of complex type (1,1), then by Newlander-Nirenberg theorem we know that there is a unique holomorphic structure on E such that ∇ is the corresponding canonical connection. For the infinite dimensional situation, as stated above, we can still attach a unitary connection whose curvature is of complex type (1,1). Also $\nabla^{\pi_*\mathcal{E}}\bar{\partial} = 0$, so formally, we have a kind of 'holomorphic' on E. In this sense we usually call such a connection a **holomorphic connection**. We do not use the notation of canonical connection here as by the result in the real situation, this connection is not the right one for us to study the problem at the level of differential forms; it does not give us the nice cancellation. For this reason, we have to use the Bismut superconnection.

We end this section by the following explicit formula for the Bismut superconnection in the complex situation.

Let X, Y be two vector fields on B. Then we have the horizontal lifting X_H, Y_H to $T_H M$. Let $[X, Y]$ be the commutator of X_H, Y_H and $T(X,Y) \in T_{M/B}$ be the projection of $-[X,Y]$ along $T_H M$. The map T defines a tensor in $C^\infty(M, T_{M/B} \otimes \wedge^2 T_H^* M)$. Let $T := T^{1,0} + T^{0,1}$ be the decomposition of T according to its type in $T_{M/B}$ and $c(T) = c(T^{1,0}) + c(T^{0,1})$ the corresponding decomposition of Clifford action $c(T)$. Then, we know that the Bismut superconnection in our case is just given as follows:

$$\mathbf{B} := \nabla^{\pi_*\mathcal{E}} + \bar{\partial} + \bar{\partial}^* - \frac{1}{4}c(T^{1,0}) - \frac{1}{4}c(T^{0,1}).$$

Moreover, if we scale the metric for the fiber by a factor $\frac{1}{u}$, we know that the associated rescaled Bismut superconnection associated with this new metric is given by

$$\mathbf{B}_u := \nabla^{\pi_*\mathcal{E}} + u(\bar{\partial} + \bar{\partial}^*) - \frac{1}{4u}c(T).$$

Chapter I.6
Relative Bott-Chern Secondary Characteristic Forms With Respect To Smooth Morphisms II: Existence

From the construction for the classical Bott-Chern secondary characteristic forms at the end of Chapter 4 by the Mellin trasform, we know that such a theory depends on the existence of ceratin trace classes which have the right asymptotic behaviors when the time goes to zero or goes to infinity. In this chapter, we will give a construction for relative Bott-Chern secondary characteristic forms with respect to smooth morphisms. Similarly, the basic idea is to use the Mellin transform. In general, this process is rather complicated. So we first deal with a special situation for the infinite dimensional case, in order to get an easy statement for the exponential decay of our objects when the parameter goes to infinity. Then, we study the most general situation, by using the key observation which comes from both the finite dimensional situation and the special infinite dimensional situation mentioned above. The references for this chapter are [BGS 88] and [Fa 92].

§I.6.1. A Special Case In The Infinite Dimensional Situation

In this section, by imitating the process in 4.5 for the finite dimensional situation, we give a construction of relative Bott-Chern secondary characteristic forms for certain special cases following [BGS 88], from which we may get a good feeling for the construction in general.

I.6.1.a. **Bismut's superconnection**

We use the same notation as in the previous chapter: $(\pi : M \to B, g_Z, T_H M)$ is a Kähler fibration with the associated (1,1) form ω, etc. Let

$$0 \to \xi_0 \xrightarrow{v} \xi_1 \xrightarrow{v} \dots \xrightarrow{v} \xi_m \to 0$$

be a holomorphic chain complex of finite dimensional holomorphic vector bundles on M with hermitian metrics ρ_j on ξ_j. Set

$$\xi^+ := \oplus_{j \text{ even}} \xi_j, \quad \xi^- := \oplus_{j \text{ odd}} \xi_j, \quad \xi := \xi^+ \oplus \xi^- .$$

Let ∇^j be the canonical connection of (ξ_j, ρ_j) for $j = 0, \ldots, m$, $\nabla^\xi =: \oplus \nabla^j$, $L^\xi := (\nabla^\xi)^2$ and $R^\xi := -\frac{1}{2\pi i}(\nabla^\xi)^2$. Let v^* be the formal adjoint of v. Set $V := v + v^*$.

For $0 \leq j \leq m$, we make the various constructions as in 5.6.c for ξ_j. Denote by $E_j^p, E_j^\pm, E_j$ the corresponding infinite dimensional hermitian vector bundles on B which we endow with the (unlabelled) 'holomorphic' hermitian connection ∇^{π_*}. Also we have the unlabelled families of operators $\bar\partial, \bar\partial^*, D$ on E_j as well as the vertical Clifford multiplication operators $c(e_i)$.

Let τ be the involution defining the grading on E_j, i.e. $\tau = \pm 1$ on $E_j^\pm$. We also make the convention that v, v^*, V act on E_j like $\tau(1 \otimes v)$; therefore they anticommute with $c(e_i)$. Hence we have a 'holomorphic' double chain complex of infinite dimensional vector bundles on B:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \to & E_0^0 & \xrightarrow{v} & E_1^0 & \xrightarrow{v} & \cdots & \xrightarrow{v} & E_m^0 & \to & 0 \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & & & \downarrow \bar\partial & & \\
\cdots & & & \cdots & & \cdots & \cdots & \cdots & & & \cdots \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & & & \downarrow \bar\partial & & \\
0 & \to & E_0^l & \xrightarrow{v} & E_1^l & \xrightarrow{v} & \cdots & \xrightarrow{v} & E_m^l & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & .
\end{array}$$

Taking the grading naturally, we also set

$$E := \oplus_{j,p} E_j^p, \quad E^+ := \oplus_{j+p \text{ even}} E_j^p, \quad E^- := \oplus_{j+p \text{ odd}} E_j^p.$$

The operators $\bar\partial, \bar\partial^*, D, v, v^*, V$ are odd in $\operatorname{End} E$. Hence for $u \geq 0$, we have the superconnection

$$\nabla^{\pi_*} + \sqrt{u}(D + V)$$

on E. But as we have already seen in the real situation, this superconnection is not the right one for our purpose. More precisely, by a complex realization, we have the following Bismut superconnection: For $u \geq 0$,

$$\mathbf{A}_u := \nabla^{\pi_*} + \sqrt{u}(D + V) - \frac{c(T)}{4\sqrt{u}}$$

where

$$c(T) = dy^\alpha d\bar y^\beta c(T(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar y^\beta})).$$

I.6.1.b. **Local Family Index Theorem**

With the same notation as above, from the local family index theorem in the real situation which was proved in Chapter 5, by taking the correspondence at the end of the last chapter, we have the following

Theorem. 1. Let P be the vector space of smooth forms on B with complex type (p,p), $0 \leq p \leq \dim B$. For any $u > 0$, the smooth differential forms on B

$$\mathrm{Tr}_s[\exp(-(\nabla^{\pi_*} + \sqrt{u}(D+V))^2)], \quad \mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]$$

are in P; and they are closed.
(2) (**Local Family Ihdex Theorem.**) Uniformly on compact subsets in B,

$$[2\pi\mathbf{i}]\lim_{u\to 0}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] = \int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-R^{\xi})].$$

Remark. Even through $\mathrm{Tr}_s[\exp(-(\nabla^{\pi_*} + \sqrt{u}(D+V))^2]$, $\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]$ are in the same cohomology class, in general, $\mathrm{Tr}_s[\exp(-(\nabla^{\pi_*} + \sqrt{u}(D+V))^2)]$ does not converge as $u \to 0$. It is at this part that we must use the Bismut superconnection.

I.6.1.c. Number Operators

The double complex E has a horizontal and a vertical grading. Let N_H, N_V be the number operators corresponding to these two gradings: N_H and N_V act on E_j^k by the multiplication with j and k respectively. Thus $N = N_H + N_V$ is the total grading number operator. We know that this number operator is the right choice of the number operator if we use the superconnection $\nabla^{\pi_*} + \sqrt{u}(D+V)$, when we do everything as in the finite dimensional case. For example, we may have the double transgression formula associated with $\mathrm{Tr}_s[\exp(-(\nabla^{\pi_*} + \sqrt{u}(D+V))^2)]$. On the other hand, since finally we use the Bismut superconnection, so we have to change number operators, in order to make our theory go through. Now suppose N_u is the right number operator with respect to the Bismut superconnection. For getting the right cancellation, similarly to the finite dimensional case, we need the following basic relations

$$[\nabla^{\pi_*}, N_u] = 0, \ [\bar{\partial}, N_u] = -\bar{\partial}, \ [\bar{\partial}^*, N_u] = \bar{\partial}^*,$$

$$[v, N_u] = -v, \ [v^*, N_u] = v^*, \ [c(T^{1,0}), N_u] = -c(T^{1,0}), \ [c(T^{0,1}), N_u] = -c(T^{0,1}).$$

Therefore, by a direct calculation, we have a natural choice for N_u. To explain it, we make the following observation.

First note that we do not change the horizontal data, so it is enough to modify N_V. For this purpose, we first evaluate N_V in a more geometric way, via the vertical Kähler form ω_Z. In fact, by a local calculation, we know that, as an element of the Clifford algebra $C(TZ)$, ω_Z is given by

$$\omega_Z^c := -\frac{1}{4}\omega_Z(w_j, \bar{w}_j)[c(\bar{w}_j), c(w_j)],$$

i.e.

$$\omega_Z^c = \frac{i}{2}(-\iota_{\bar{w}_j}\bar{w}_j \wedge + \bar{w}_j \wedge \iota_{\bar{w}_j}) = i(\bar{w}_j \wedge \iota_{\bar{w}_j} - \frac{l}{2}).$$

Here, l is the relative dimension. So from the fact that $\sum_{j=1}^{l} \bar{w}_j \wedge \iota_{\bar{w}_j}$ on E_j^k is given by the multiplication by k, we know that

$$N_V = -i\omega_Z^c + \frac{l}{2}.$$

From here, with the above basic relations, we let

$$N_{V,u} := -i\omega_Z^c + \frac{i}{2u}\omega_H \wedge + \frac{l}{2}, \quad N_u := N_{V,u} + N_H.$$

I.6.1.d. Double Transgression Formula

With above notation, we have the following

Theorem. (1) For any $u > 0$, the smooth differential form $\mathrm{Tr}_s[N_u \exp(-\mathbf{A}_u^2)]$ is in P.
(2) **(Double Transgression Formula)**

$$\begin{aligned}
&\frac{\partial}{\partial u}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] \\
&\quad = -\frac{1}{2u}(\partial^B + \bar{\partial}^B)\mathrm{Tr}_s[(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2)]; \\
&\mathrm{Tr}_s[(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2)] \\
&\quad = (\partial - \bar{\partial}^B)\mathrm{Tr}_s[N_u \exp(-\mathbf{A}_u^2)].
\end{aligned}$$

In particular,

$$\frac{\partial}{\partial u}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] = -\frac{1}{u}\bar{\partial}^B\partial^B\mathrm{Tr}_s[N_u \exp(-\mathbf{A}_u^2)].$$

Proof. By the construction, (1) is trivial. For (2), by the choices of the Bismut superconnection and the properties of the number operator listed in the last subsection, one may exactly imitate the proof of the corresponding assertion for the finite dimensional situation, to give a complete proof for this part. The details of this translation are left to the reader.

I.6.1.e. Asymptotic Behaviors Of Certain Forms

By the result in the last subsection, in order to imitate the definition of the classical Bott-Chern secondary characteristic forms for finite dimensional case, via the Mellin transform, we now need to consider the asymptotic behaviors of

$$\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)],$$

$$\mathrm{Tr}_s[(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2)],$$

and

$$\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)],$$

when $u \to 0^+$ and $u \to +\infty$ respectively. In this respect, we have the following

Theorem. (1) There exist C^∞ even differential forms $A_0, A_1, \ldots$ in P such that for any $k \in \mathbf{N}$,

$$\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] = \sum_{j=0}^{k} A_j u^j + o(u^k)$$

where

$$A_0 = \frac{1}{(2\pi i)^l}\int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-L_\xi)].$$

(2) There exist C^∞ odd differential forms $B_0, B_1, \ldots$ such that for any $k \in \mathbf{N}$,

$$\mathrm{Tr}_s[(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2)] = \sum_{j=0}^{k} B_j u^j + o(u^k).$$

Moreover, $B_0 = 0$.
(3) There exist smooth differential forms $C_{-1}, C_0, \ldots$ in P such that as $u \to 0$,

$$\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)] = \sum_{j=-1}^{k} C_j u^j + o(u^k).$$

(4) The various $o(u^k)$ are uniform on compact subsets in B.

Proof. By the local family index theorem I.5.4, we have the asymptotic expansion in (1), since the super-trace vanishes on all elements of Clifford degree strictly less than $2l$, and the fact that the corresponding o is uniform on the compact subsets in B. Furthermore, by the discussion for the local family index theorem in section 5.6 for the compelx geometry, we have the expression for the term A_0.

Now, we consider (2). By Duhamel's formula,

$$\begin{aligned}&\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2+(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\,du)]\\ =&\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] + \mathrm{Tr}_s[(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2)]\,du.\end{aligned}$$

Therefore, it suffices to give the asymptotic expansion for

$$\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2+(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\,du)].$$

For doing so, we may use exactly the same procedure for the proof of the local family index theorem to deduce the asymptotic expansion, since, as an essential generalized Laplacian the structure of the super-Lichnerowicz formula is just the same. With this, $B_0 = 0$ is a direct consequence of (1) and (3), provided that we use the double transgression formula in the previous subsection.

So, to complete the proof of the theorem, we need to prove (3). From the above proof of the asymptotic expansions in (1) and (2), we see that if we could give a formula, in which only the terms $\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2 + B_u)$ occur, where, as essential generalized Laplacians, $-\mathbf{A}_u^2 + B_u$ has the same structure as what is for the super-Lichnerowicz formula, we then can use the same procedure as what we did for the proof of the local family index theorem to give the asymptotic expansion. In fact, this is the general method for the proof of the existence of asymptotic expansion in the sequel. Here we demonstrate it by the example with

$$\frac{\partial}{\partial u}(u\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)]).$$

We are supposed to show that its values at $u = 0$ is a smooth form C_{-1} on B.

We first recall the situation for the finite dimensional case. At that place, we got the assertion

$$\frac{\partial}{\partial u}[u\mathrm{Tr}_s(N\exp-(\nabla+\sqrt{u}V)^2)]_{u=0} = \mathrm{Tr}_s[N\exp(-\nabla^2)].$$

Therefore, the assertion (3) is not really surprising. But now we get a certain trouble, as the higher Grassmannian degree terms in $\mathbf{A}_u^2$, N_u scale with negative powers of $u^{1/2}$. Fortunately, finally when we count the Clifford degrees as we did before, we will find that the terms with a low u-power also have low Clifford degree by the properties of the Bismut superconnection and the number operator listed at section 5.6 and subsection 6.1.c. Thus by the fact that the super-trace vanishes on elements with low Clifford degrees, we find the correct cancellation.

In practice, we do as follows, which is the same as for the finite dimension situation: First, we have a generalization of the double transgression formula: For $u > 0$, $b \geq 0$

$$\begin{aligned} &bu\mathrm{Tr}_s[(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2+buN_u)] \\ =&(\partial^B-\bar{\partial}^B)\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2+buN_u)]. \end{aligned}$$

Hence, by the fact that

$$\begin{aligned} &\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2+(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\,du)]^{du} \\ =&\frac{\partial}{\partial b}\mathrm{Tr}_s[(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2+buN_u)]_{b=0}, \end{aligned}$$

we have, if we replace $\frac{\partial}{\partial b}$ by $\frac{1}{2}\frac{\partial^2}{\partial b^2}b$,

$$\begin{aligned}&\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2+(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\,du)]^{du}\\&=(\partial^B-\bar\partial^B)\frac{1}{2}\frac{\partial^2}{\partial b^2}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2+bN_u)]_{b=0},\end{aligned}$$

On the other hand, by the fact that

$$\frac{\partial}{\partial u}(uN_u)=-i\omega_Z^c+N_H+\frac{l}{2},$$

we get

$$\begin{aligned}&\frac{\partial}{\partial u}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2+buN_u)]\\&=-\frac{1}{2u}(\partial^B+\bar\partial^B)\mathrm{Tr}_s[(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2+buN_u)]\\&\quad+b(\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2-(\sqrt{u}(\bar\partial+v)+\frac{c(T^{(1,0)})}{4\sqrt{u}})\,da\\&\qquad-(\sqrt{u}(\bar\partial^*+v^*)+\frac{c(T^{(0,1)})}{4\sqrt{u}})\,d\bar a-i\omega_Z^c\,da\,d\bar a+buN_u)]^{da\,d\bar a}\\&\quad+\mathrm{Tr}_s[(N_H+\frac{l}{2})\exp(-\mathbf{A}_u^2+buN_u)]).\end{aligned}$$

Differentiating with respect to b and evaluating at $b=0$, we then get

$$\begin{aligned}&\frac{\partial}{\partial u}(u\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)])\\&=\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2-(\sqrt{u}(\bar\partial+v)+\frac{c(T^{(1,0)})}{4\sqrt{u}})\,da\\&\qquad-(\sqrt{u}(\bar\partial^*+v^*)+\frac{c(T^{(0,1)})}{4\sqrt{u}})\,d\bar a-i\omega_Z^c\,da\,d\bar a)]^{da\,d\bar a}\\&\quad+\mathrm{Tr}_s[(N_H+\frac{l}{2})\exp(-\mathbf{A}_u^2)])\\&\quad-(\partial^B+\bar\partial^B)(\frac{1}{2}\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2+(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\,du)]^{du}.\end{aligned}$$

Therefore, put all this together, we get

$$\begin{aligned}&\frac{\partial}{\partial u}(u\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)])\\&=\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2-(\sqrt{u}(\bar\partial+v)+\frac{c(T^{(1,0)})}{4\sqrt{u}})\,da\\&\qquad-(\sqrt{u}(\bar\partial^*+v^*)+\frac{c(T^{(0,1)})}{4\sqrt{u}})\,d\bar a-i\omega_Z^c\,da\,d\bar a)]^{da\,d\bar a}\\&\quad+\mathrm{Tr}_s[(N_H+\frac{l}{2})\exp(-\mathbf{A}_u^2)])\\&\quad-\bar\partial^B\partial^B\frac{1}{2}\frac{\partial^2}{\partial b^2}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2+bN_u)]_{b=0}.\end{aligned}$$

Hence, we expressed the term

$$\frac{\partial}{\partial u}(u\mathrm{Tr}_s[N_u \exp(-\mathbf{A}_u^2)])$$

as a combination of the forms of

$$\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2 + B_u)],$$

as stated above. Thus, by the method used in the proof of the local family index theorem, i.e., first, to use the normal coordinate to localize the problem, then, as an essential generalized Laplacian, to use the structure for $-\mathbf{A}_u^2 + B_u$ in the sense of the super-Lichnerowicz formula to give the discussion over the Euclidean spaces via the generalized oscillators to give the asymptotic expansion. More precisely, we have

$$\mathrm{Tr}_s[(N_H + \frac{l}{2})\exp(-\mathbf{A}_u^2)]) = \sum_{j=0}^{k} F_j u^j + o(u^k)$$

with

$$F_0 = \frac{1}{(2\pi i)^l}\int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[(N_H + \frac{l}{2})\exp(-L_{\zeta_\cdot,\rho_\cdot})],$$

$$\begin{aligned}\mathrm{Tr}_s[\exp(&-\mathbf{A}_u^2 - (\sqrt{u}(\bar\partial + v) + \frac{c(T^{(1,0)})}{4\sqrt{u}})\,da \\ &-(\sqrt{u}(\bar\partial^* + v^*) + \frac{c(T^{(0,1)})}{4\sqrt{u}})\,d\bar a - i\omega_Z^c\,da\,d\bar a)]^{da\,d\bar a} \\ &= \sum_{j=0}^{k} E_j u^j + o(u^k),\end{aligned}$$

and

$$\frac{\partial^2}{\partial b^2}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2 + bN_u)]_{b=0} = \sum_{j=-2}^{k} D_j u^j + o(u^k)$$

with D_{-1}, D_{-2} closed forms. Thus, we complete the proof of the theorem.

From the proof above, if we write all terms down in a precise form from the local discussion over Euclidean spaces as what we did for the proof of the local family index theorem, we also can have the following

Corollary. With the same notation as above,

$$\begin{aligned}
C_{-1} =& \frac{1}{(2\pi i)^l}\int_Z i\frac{\omega}{2}\mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-L_{\zeta_\cdot,\rho_\cdot})],\\
C_0 =& \frac{1}{(2\pi i)^l}\int_Z \frac{\partial}{\partial b}(\mathrm{td}(-R_Z-bI))_{b=0}\mathrm{Tr}_s[\exp(-L_{\zeta_\cdot,\rho_\cdot})]\\
&+l\frac{1}{(2\pi i)^l}\int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-L_{\zeta_\cdot,\rho_\cdot})]\\
&+\frac{1}{(2\pi i)^l}\int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[N_H\exp(-L_{\zeta_\cdot,\rho_\cdot})]\\
&-\frac{1}{2}d^B E_0.
\end{aligned}$$

Indeed,

$$C_{-1} = \lim_{u=0}(u\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)]).$$

So, by the fact that

$$uN_u = -iu\omega_Z^c + \frac{i}{2}\omega^H + u(N_H+\frac{l}{2})$$

contains no nagative powers of u, we may get above expression of C_{-1} by imitating the process in the proof of the local family index theorem. For C_0, we see that

$$\lim_{u\to 0}\frac{\partial}{\partial u}(u\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)]) = C_0.$$

Thus, by the last part of the proof of the theorem above, we see that only the part for

$$\begin{aligned}
\mathrm{Tr}_s[\exp(&-\mathbf{A}_u^2-(\sqrt{u}(\bar\partial+v)+\frac{c(T^{(1,0)})}{4\sqrt{u}})\,da\\
&-(\sqrt{u}(\bar\partial^*+v^*)+\frac{c(T^{(0,1)})}{4\sqrt{u}})\,d\bar a - i\omega_Z^c\,da\,d\bar a)]^{da\,d\bar a}
\end{aligned}$$

matters. But then, we may first assume that $v=0$ to deduce the result. In general, we need to know that fact that the 0 order operator $[\bar\partial^*,v]+[\bar\partial,v^*]$ has the weight u, so it does not contribute to the limit. In this way, we get

$$\begin{aligned}
\lim_{u=0}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2&-(\sqrt{u}(\bar\partial+v)+\frac{c(T^{(1,0)})}{4\sqrt{u}})\,da\\
&-(\sqrt{u}(\bar\partial^*+v^*)+\frac{c(T^{(0,1)})}{4\sqrt{u}})\,d\bar a - i\omega_Z^c\,da\,d\bar a)]^{da\,d\bar a}\\
=\frac{1}{(2\pi i)^l}\int_Z \frac{\partial}{\partial b}&[A(R_Z-ibJ_Z)]_{b=0}\exp(-\frac{1}{2}\mathrm{Tr}[R_Z])\mathrm{Tr}_s[\exp(-L_{\zeta_\cdot,\rho_\cdot})],
\end{aligned}$$

where A denote the Hirzebruch A genus. Hence, we get

$$\begin{aligned}C_0 =&\frac{1}{(2\pi i)^l}\int_Z \frac{\partial}{\partial b}[A(R_Z+bI)]_{b=0}\exp(-\frac{1}{2}\mathrm{Tr}[R_Z])\mathrm{Tr}_s[\exp(-L_{\zeta.,\rho.})]\\ &+\frac{1}{(2\pi i)^l}\int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[(N_H+\frac{l}{2})\exp(-L_{\zeta.,\rho.})]-\frac{1}{2}d^B E_0.\end{aligned}$$

So, by using the relation between complex Kähler geometry and Riemannian geometry listed in the final section of the last chapter, we get the corollary.

I.6.1.f. The Construction In A Special Case

We have already described the asymptotic behavior of

$$\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)]$$

as $u \to 0^+$. Therefore, if we can also prove that it decays exponentially when $u \to +\infty$, then we may use the Mellin transform to construct a good object. In general, however, this is not the case, say, we do not always have the condition that the double complex $(E,\bar\partial+v)$ is acyclic. So in order to go further, we make the basic **additional assumption** that $(E,\bar\partial+v)$ is acyclic. Then, as in the finite dimensional case, the eigenvalues of the corresponding Laplacian are strictly positive. So by using Volterra's series, it not difficult to show that when $u\to+\infty$,

$$\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)],$$

$$\mathrm{Tr}_s[(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}})\exp(-\mathbf{A}_u^2)]$$

and

$$\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)]$$

all decay exponentially and uniformly on compact subsets in B. Therefore, we can use the Mellin transform to give the following

Theorem. With the same notation as above, assume that $(E,\bar\partial+v)$ is acyclic. Then, (1) For $s\in\mathbf{C},\mathrm{Re}(s)>1$, let

$$\zeta_{\xi.,\rho.,\pi}(s):=\frac{1}{\Gamma(s)}\int_0^{+\infty}u^s\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)]\frac{du}{u}.$$

As a notation, usually, if there is no confusion, we may also denote $\zeta_{\xi.,\rho.,\pi}$ as ζ_E. Then ζ_E is well-defined, and is an element in P.
(2) There exists a meromorphic continuation of $\zeta_{\xi.,\rho.,\pi}(s)$ to the whole complex plane such that this extension is holomorphic at $s=0$. In particular, it makes sense for us to talk about $\zeta'_{\xi.,\rho.,\pi}(0)$.

I.6.1.g. **Certain Properties Of The Construction**

Here, we prove that the object constructed above satisfies the corresponding modification of the axioms for relative Bott-Chern secondary characteristic forms with respect to smooth morphisms, i.e. the axioms subtracting the term associated with $\pi_*\xi$, since we here assume that $(E, \bar\partial + v)$ is acyclic.

Theorem. If (ξ, v) is acyclic, then
(1) We have the modified axiom 1, i.e. the double transgression formula:

$$\bar\partial_B\,\partial\,\zeta_E'(0) = \frac{1}{(2\pi i)^l}\int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-L_{\xi.,\rho.})].$$

(2) We have the modified axiom 3, i.e.

$$\zeta_E{}'(0) = \frac{1}{(2\pi i)^l}\int_Z \mathrm{td}(-R_Z)\,\zeta_{\xi.,\rho.}{}'(0).$$

Proof. (1) is nothing but the integrated form of the double transgression formula: Since we have the correct decay at infinity and the right asymptotic expansion at zero, the integrating process works well.

The proof of (2) is based on a deformation process. More precisely, it comes from the following two statements:

(a) For $t > 0$, let $\zeta_{E,t}(s)$ be the zeta function associated with the chain complex $(E, \sqrt{t}\bar\partial + v)$, then as an element in P/P',

$$\zeta_{E,t}'(0) = \zeta_E'(0) + A\,\mathrm{Log}|t|.$$

(b) When $t \to 0^+$, we have

$$[2\pi \mathbf{i}]\zeta_{E,t}'(0) + \frac{A'}{t} \to \int_Z \mathrm{td}(-R_Z)\,\mathrm{ch}_{\mathrm{BC}}(\xi.,\rho.).$$

Here A and A' are smooth forms of B.

Suppose we have (a) and (b), formally, after we consider the constant terms, we complete the proof of the theorem. But, in practice, it is not so simple: The difficult is that in general P' is not closed in P. Hence, in the convergence arguments, we have to be more careful.

We next give a proof of (a). The basic idea for proving this is to use a deformation process. That is, we prove the result by studying the relations between the zeta functions associated with the following complexes:

$$(E, \bar\partial + \nu),\ \ (E, a\,\bar\partial + \nu),\ \ (E, \bar\partial + b\,\nu),\ \ (E, a\bar\partial + b\,\nu).$$

(i) The Bismut superconnections:

First, for $(E, \bar\partial + v)$, from the previous discussion, we get the associated Bismut superconnection

$$\mathbf{A}_u := \nabla^{\pi_*} + \sqrt{u}(D+V) - \frac{c(T)}{4\sqrt{u}}.$$

Thus, if we consider the chain complex $(E, \bar\partial + av)$ for $(y,a) \in B \times \mathbf{C}$, which is acyclic for $a \neq 0$, then the corresponding Bismut superconnection is

$$\mathbf{A}_{\mathbf{C},u} := \nabla^{\pi_*} + da\frac{\partial}{\partial a} + d\bar a\frac{\partial}{\partial \bar a} + \sqrt{u}(D+V^a) - \frac{c(T)}{4\sqrt{u}}.$$

Here we set $V^a := av + \bar a v^*$. Similarly, for $t \geq 0$, we scale $\bar\partial, \bar\partial^*$ by the factor $\sqrt{t}$, then we may get the associated Bismut superconnections as follows

$$\begin{aligned}\mathbf{A}_u^t :=& \nabla + \sqrt{u}(\sqrt{t}D + V) - \frac{c(T)}{4\sqrt{u}};\\ \mathbf{A}_{\mathbf{C},u}^t :=& \nabla + da\frac{\partial}{\partial a} + d\bar a\frac{\partial}{\partial \bar a} + \sqrt{u}(\sqrt{t}D + V^a) - \frac{c(T)}{4\sqrt{u}}.\end{aligned}$$

With above, if we look at the dependence of those elements on v, i.e., we let

$$\mathbf{A}_u =: \mathbf{A}_u(v), \quad \mathbf{A}_{\mathbf{C},u} =: \mathbf{A}_{\mathbf{C},u}(v),$$

then we have the following relations among the above superconnections.

$$\mathbf{A}_u^t = \mathbf{A}_{ut}(\frac{v}{\sqrt{t}}), \quad \mathbf{A}_{\mathbf{C},u}^t = \mathbf{A}_{\mathbf{C},ut}(\frac{v}{\sqrt{t}}),$$

(ii) The number operators.

With the Bismut superconnections as above, we easily know that the corresponding number operators are N_u, N_u, N_{ut} and N_{ut}.

(iii) The zeta functions.

Hence, we also know that the associated zeta functions are respectively as follows:

$$\zeta_E(s) = -\frac{1}{\Gamma(s)}\int_0^\infty u^s \mathrm{Tr}_s[N_u \exp(-(\mathbf{A}_u)^2)]du,$$

$$\zeta_{E,\mathbf{C}}(s) = -\frac{1}{\Gamma(s)}\int_0^\infty u^s \mathrm{Tr}_s[N_u \exp(-(\mathbf{A}_{\mathbf{C},u})^2)]du,$$

$$\zeta_E^t(s) = -\frac{1}{\Gamma(s)}\int_0^\infty u^s \mathrm{Tr}_s[N_{ut} \exp(-(\mathbf{A}_u^t)^2)]du,$$

and

$$\zeta^t_{E,\mathbf{C}}(s) = -\frac{1}{\Gamma(s)}\int_0^\infty u^s \mathrm{Tr}_s[N_{ut}\exp(-(\mathbf{A}^t_{\mathbf{C},u})^2)]du.$$

Also, if we consider them as a function of v, then, we get

$$\zeta^t_{E,v}(s) = t^{-s}\zeta_{E,\frac{v}{\sqrt{t}}}(s),\ \zeta^t_{E,\mathbf{C},v}(s) = t^{-s}\zeta_{E,\mathbf{C},\frac{v}{\sqrt{t}}}(s).$$

As a corollary, we see that

$$\zeta^t_{E,v}{}'(0) = \zeta_{E,\frac{v}{\sqrt{t}}}{}'(0) - \zeta_{E,\frac{v}{\sqrt{t}}}(0)\,\mathrm{Log}t,$$

and

$$\zeta^t_{E,\mathbf{C},v}{}'(0) = \zeta_{E,\mathbf{C},\frac{v}{\sqrt{t}}}{}'(0) - \zeta_{E,\mathbf{C},\frac{v}{\sqrt{t}}}(0)\log t.$$

(iv) The local family index theorem.

On the other hand, by the local family index theorem for $\pi \times \mathrm{Id}_{\mathbf{C}}$, we see that

$$\lim_{u\to 0}[2\pi \mathbf{i}]\mathrm{Tr}_s[\exp(-\mathbf{A}^2_{\mathbf{C},u})] = \int_Z \mathrm{td}(-R_Z)\,\mathrm{Tr}_s[\exp(-R_{\xi_\cdot,\rho_\cdot})].$$

So, we see that there is no da or $d\bar{a}$ term in the right hand side. Thus, later on, we may use the trick of counting the Grassmannian degree in $\mathbf{C}$ to deduce the result.

(v) The expansion of $\zeta_{\mathbf{C}}'(0)$ with respect to $\mathbf{C}$.

First, we consider the acyclic chain complex $(E, a\bar{\partial} + v)$ on $B \times \mathbf{C}^*$ for $a \neq 0$. For $\zeta_{E,\mathbf{C}}$, the corresponding zeta function, there exist differential forms $\theta_0, \theta_1, \theta_{-1}$, and θ_2 on B depending smoothly on $(y, a) \in B \times \mathbf{C}^*$, such that

$$\zeta_{E,\mathbf{C}}'(0) = \theta_0 + \theta_1\,da + \theta_{-1}\,d\bar{a} + \theta_2\,dad\bar{a}.$$

Similarly, by considering the Grassmannian degree with respect to $\mathbf{C}$, for $a \in \mathbf{C}^*$, there exist smooth forms θ^t_j on B, depending smoothly on $a \in \mathbf{C}^*$ such that

$$\zeta^t_{\mathbf{C},E}{}'(0) = \theta^t_0 + \theta^t_1\,da + \theta^t_{-1}\,d\bar{a} + \theta^t_2\,dad\bar{a}.$$

Thus, if $i_a : B \to B \times \mathbf{C}^*$ is the embedding $y \mapsto (y, a)$, then

$$\theta_0 = i^*_a\zeta_{E,\mathbf{C}}'(0), \quad \zeta_E'(0) = i^*_1\zeta_{E,\mathbf{C}}'(0),$$

while

$$\theta^t_0(y,1) = i^*_1\zeta^t_{\mathbf{C},E}{}'(0),$$

and

$$\theta^t_0(y,1) = i^*_1\zeta^t_{\mathbf{C},E}{}'(0) = \zeta^t_E{}'(0).$$

Therefore, to study the relation of $\zeta_E{}'$ and $\zeta_{E,\mathbf{C}}{}'$, we need to study θ and θ^t.

(vi) θ_0 and θ_0^t.

First, we study θ_0. Since, by the double transgression formula, we have

$$d_B d_B^c\,[2\pi\mathbf{i}]\zeta_E{}'(0) = \int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-R_{\xi.,\rho.})];$$
$$d_{B\times\mathbf{C}} d_{B\times\mathbf{C}}^c\,[2\pi\mathbf{i}]\zeta_{E,\mathbf{C}}{}'(0) = \int_Z \mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-R_{\xi.,\rho.})].$$

In particular, $d_{B\times\mathbf{C}} d^c_{B\times\mathbf{C}}\zeta_{E,\mathbf{C}}{}'(0)$ does not contain da or $d\bar a$ terms. Hence, by the relation of $\partial_{B\times\mathbf{C}}, \bar\partial_{B\times\mathbf{C}}$ with those for B and $\mathbf{C}$, we have

$$\frac{\partial^2\theta_0}{\partial a\partial\bar a} - \partial^B\frac{\partial\theta_1}{\partial\bar a} - \bar\partial^B\frac{\partial\theta_{-1}}{\partial a} - \bar\partial^B\partial^B\theta_2 = 0.$$

So, by the facts that θ_0 is a radical function of $|a|$, that $\frac{\partial^2}{\partial a\partial\bar a}$, acting on the radical function of $|a| = r$, coincides with

$$\frac{1}{4}[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}],$$

we have

$$\frac{\partial\theta_0}{\partial r}(r) = \frac{1}{r}[\frac{\partial\theta_0}{\partial r}(1) + 4\int_1^r [\partial^B(\frac{\partial\theta_1}{\partial\bar a}) + \bar\partial^B(\frac{\partial\theta_{-1}}{\partial a}) + \bar\partial^B\partial^B(\theta_2)](b)b\,db].$$

Thus, by integration, if I is the linear operator $C^\infty(\mathbf{R}_+^*)$ into itself such that

$$f \mapsto I(f) : r \mapsto I(f)(r) := 4\int_1^r f(b)b\,\mathrm{Log}\frac{r}{b}\,db,$$

we have

Lemma. With the same notation as above, for $a \in \mathbf{C}^*$,

$$\theta_0 = \zeta_E'(0) + \frac{\partial\theta_0}{\partial r}(1)\mathrm{Log}|a| + \partial^B I(\frac{\partial\theta_1}{\partial\bar a}) + \bar\partial^B I(\frac{\partial\theta_{-1}}{\partial a}) + \bar\partial^B\partial^B I(\theta_2).$$

On the other hand, if we let $r_t : \mathbf{C} \to \mathbf{C}$ be the map defined by $a \mapsto \frac{a}{\sqrt t}$, Then

$$\zeta_{E,\mathbf{C},\frac{1}{\sqrt t},(y,a)}{}'(0) = r_t^*\zeta_{E,\mathbf{C},v,(y,a/\sqrt t)}{}'(0).$$

In this manner, we get

$$\begin{aligned}\theta_0(y,\frac{1}{\sqrt t}) =& \zeta_E{}'(0) - \frac{1}{2}\frac{\partial\theta_0}{\partial r}(1)\mathrm{Log}t\\ &+ \partial^B I(\frac{\partial\theta_1}{\partial\bar a})(y,\frac{1}{\sqrt t}) + \bar\partial^B I(\frac{\partial\theta_{-1}}{\partial a})(y,\frac{1}{\sqrt t})\\ &+ \bar\partial^B\partial^B I(\theta_2)(y,\frac{1}{\sqrt t}).\end{aligned}$$

Now we are ready to consider θ_0^t. For this, we use that fact that

$$\zeta^t_{E,\mathbf{C},(y,a)}{}'(0)=r_t^*\zeta_{E,\mathbf{C},v,(y,\frac{a}{\sqrt{t}})}{}'(0)-\zeta_{E,\mathbf{C},\frac{a}{\sqrt{t}}}(0)\,\mathrm{Log}\,t.$$

In particular, we get

$$\theta_0^t(y,a)+\theta_1^t(y,a)\,da+\theta_{-1}^t(y,a)\,d\bar a+\theta_2^t(y,a)\,dad\bar a=r_t^*\zeta_{E,\mathbf{C},v,(y,\frac{a}{\sqrt{t}})}{}'(0)-\zeta_{E,\mathbf{C},\frac{a}{\sqrt{t}}}(0)\,\mathrm{Log}\,t.$$

But, by definition, we get

$$\begin{aligned}r_t^*\zeta_{E,\mathbf{C},v,(y,\frac{a}{\sqrt{t}})}{}'(0)=&\theta_0^t(y,\frac{a}{\sqrt{t}})\\&+\frac{1}{\sqrt{t}}\theta_1^t(y,\frac{a}{\sqrt{t}})\,da+\frac{1}{\sqrt{t}}\theta_{-1}^t(y,\frac{a}{\sqrt{t}})\,d\bar a\\&+\frac{1}{t}\theta_2^t(y,\frac{a}{\sqrt{t}})\,dad\bar a,\end{aligned}$$

so we get

$$\begin{aligned}&\theta_0^t(y,a)+\theta_1^t(y,a)\,da+\theta_{-1}^t(y,a)\,d\bar a+\theta_2^t(y,a)\,dad\bar a\\=&\theta_0^t(y,\frac{a}{\sqrt{t}})+\frac{1}{\sqrt{t}}\theta_1^t(y,\frac{a}{\sqrt{t}})\,da+\frac{1}{\sqrt{t}}\theta_{-1}^t(y,\frac{a}{\sqrt{t}})\,d\bar a\\&+\frac{1}{t}\theta_2^t(y,\frac{a}{\sqrt{t}})\,dad\bar a-\zeta_{E,\mathbf{C},\frac{a}{\sqrt{t}}}(0)\,\mathrm{Log}\,t.\end{aligned}$$

To go futher, we need an expression for $\zeta_{E,\mathbf{C},\frac{a}{\sqrt{t}}}(0)$.

(vii) $\zeta_{E,\mathbf{C},\frac{a}{\sqrt{t}}}(0)$

To give the value of $\zeta_{E,\mathbf{C},\frac{a}{\sqrt{t}}}(0)$, we start with $\zeta_E(0)$. With the same notation as in the asymptotic expansion theorem and its corollary in subsection e, by definition, we see that

$$\zeta_E(0)=-C_0$$

and

$$\begin{aligned}\zeta_E{}'(0)=&-\int_0^1(\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)]-\frac{C_1}{u}-C_0)\frac{du}{u}\\&-\int_1^\infty\mathrm{Tr}_s[N_u\exp(-\mathbf{A}_u^2)]\frac{du}{u}+C_{-1}+\Gamma'(1)C_0.\end{aligned}$$

We could do the same thing for $\zeta_{E,\mathbf{C}}(s)$. For that, we may introduce the following family of closed differential forms on B:

$$\begin{aligned}C_{-1}(u):=&\frac{1}{(2\pi i)^l}\int_Z i\frac{\omega}{2}\mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-(\nabla_{\xi_\cdot,\rho_\cdot}+\sqrt{u}V)^2)],\\C_0(u)=&\frac{1}{(2\pi i)^l}\int_Z\frac{\partial}{\partial b}(\mathrm{td}(-R_Z-bI))_{b=0}\mathrm{Tr}_s[\exp(-(\nabla_{\xi_\cdot,\rho_\cdot}+\sqrt{u}V)^2)]\\&+\frac{l}{(2\pi i)^l}\int_Z\mathrm{td}(-R_Z)\mathrm{Tr}_s[\exp(-(\nabla_{\xi_\cdot,\rho_\cdot}+\sqrt{u}V)^2)]\\&+\frac{1}{(2\pi i)^l}\int_Z\mathrm{td}(-R_Z)\mathrm{Tr}_s[N_H\exp(-(\nabla_{\xi_\cdot,\rho_\cdot}+\sqrt{u}V)^2)]\\&-\frac{1}{2}d^B\tilde E_0\mathrm{Tr}_s[\exp(-(\nabla_{\xi_\cdot,\rho_\cdot}+\sqrt{u}V)^2)].\end{aligned}$$

Here $\bar{E}_0$ is E_0 calculated with $\xi = \mathbf{C}$ endowed with its canonical hermitian metric.

Directly from the definition, note that

$$\begin{aligned} E_0 =& \bar{E}_0 \mathrm{Tr}_s[\exp(-\nabla^2_{\xi_\cdot,\rho_\cdot})] \\ &+ \frac{1}{(2\pi i)^l}\int_Z i\frac{\omega}{2}\mathrm{td}(-R_Z)(\frac{1}{\sqrt{u}}\mathrm{Tr}_s[V\exp(-(\nabla_{\xi_\cdot,\rho_\cdot}+\sqrt{u}V)^2)]_{u=0}, \end{aligned}$$

and hence

$$\begin{aligned} d^B E_0 =& d^B \bar{E}_0 \mathrm{Tr}_s[\exp(-\nabla^2_{\xi_\cdot,\rho_\cdot})] \\ &- \frac{1}{(2\pi i)^l}\int_Z i\omega\mathrm{td}(-R_Z)(\frac{\partial}{\partial u}\mathrm{Tr}_s[\exp(-(\nabla_{\xi_\cdot,\rho_\cdot}+\sqrt{u}V)^2)]_{u=0}, \end{aligned}$$

we immediately get

$$C_{-1} = C_{-1}(0), \quad C_0 = C_0(0) + C_{-1}'(0).$$

Similarly, with the definition from above, replacing $\nabla_{\xi_\cdot,\rho_\cdot}$ by

$$\nabla_{\zeta_\cdot} + da\frac{\partial}{\partial a} + d\bar{a}\frac{\partial}{\partial \bar{a}}$$

and V by V^a, we may also define $C_{\mathbf{C},0}(u)$ and $C_{\mathbf{C},-1}(u)$. In this manner, using the same process as above, we have the following

Lemma. With the same notation as above,

$$\zeta_{E,\mathbf{C},\frac{v}{\sqrt{t}}}(0) = -C_0(0) - \frac{C_{\mathbf{C},-1}'(0)}{t}.$$

Indeed, here only v is changed to $\frac{v}{\sqrt{t}}$, but v does not appear in $C_0(0)$, so we have the assertion.

Now we return back to (vi). Write out the closed form $C_{\mathbf{C},-1}'(0)$ with respect to $\mathbf{C}$, we find that there exist smooth forms $K_1,\ K_{-1},\ K_2$ on B, such that with respect to the Grassmannian degree in $\mathbf{C}$

$$C_{\mathbf{C},-1}'(0) = |a|^2 C_{-1}'(0) + \bar{a}K_1\,da + aK_{-1}\,d\bar{a} + K_2\,dad\bar{a}.$$

Thus, by the d-closed property of $C_{\mathbf{C},-1}'(0)$, we have

$$\partial^B K_1 = C_{-1}'(0),\ \bar{\partial}^B K_{-1} = C_{-1}'(0),\ \bar{\partial}^B K_2 = K_1,\ \partial^B K_2 = -K_{-1}.$$

So, by comparing the Grassmannian degree with respect to $\mathbf{C}$, from the last part of (vi) above, we get

$$\begin{aligned} \theta_0^t(y,a) =& \theta_0(y,\frac{a}{\sqrt{t}}) + (C_0(0) + |a|^2\frac{C_{-1}'(0)}{t})\mathrm{Log}\,t; \\ \theta_1^t(y,a) =& \frac{1}{\sqrt{t}}\theta_1(y,\frac{a}{\sqrt{t}}) + \frac{\bar{a}K_1}{t}\mathrm{Log}\,t; \\ \theta_{-1}^t(y,a) =& \frac{1}{\sqrt{t}}\theta_{-1}(y,\frac{a}{\sqrt{t}}) + \frac{aK_{-1}}{t}\mathrm{Log}\,t; \\ \theta_2^t(y,a) =& \frac{1}{t}\theta_2(y,\frac{a}{\sqrt{t}}) + \frac{K_2}{t}\mathrm{Log}\,t. \end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial\bar a}\theta_1(y,a) =&t\frac{\partial}{\partial\bar a}\theta_1^t(y,a\sqrt t)-K_1\operatorname{Log}t;\\ \frac{\partial}{\partial a}\theta_{-1}(y,a) =&t\frac{\partial}{\partial a}\theta_{-1}^t(y,a\sqrt t)-K_{-1}\operatorname{Log}t;\\ \theta_3(y,a) =&t\theta_3^t(y,a\sqrt t)-K_2\operatorname{Log}t.\end{aligned}$$

So

$$\begin{aligned}I(\frac{\partial}{\partial\bar a}\theta_1(y,\frac{1}{\sqrt t})) =&4\int_1^{1/\sqrt t}\frac{\partial}{\partial\bar a}\theta_1(y,b)\operatorname{Log}(\frac{1}{\sqrt t b})b\,db\\ =&\int_t^1\frac{\partial}{\partial\bar a}\theta_1(y,\frac{1}{\sqrt b})\operatorname{Log}\frac{b}{t}\frac{db}{b^2}\\ =&\int_t^1\frac{\partial}{\partial\bar a}\theta_1^b(y,1)\operatorname{Log}\frac{b}{t}\frac{db}{b}-K_1\int_t^1\operatorname{Log}(b)\operatorname{Log}\frac{b}{t}\frac{db}{b^2}.\end{aligned}$$

Thus, if J is the operator acting on $C^\infty(\mathbf{R}_+^*)$ such that

$$f\mapsto J(f):t\mapsto\int_t^1 f(b)\operatorname{Log}\frac{b}{t}\frac{db}{b},$$

then

$$I(\frac{\partial}{\partial\bar a}\theta_1(y,\frac{1}{\sqrt t}))=J(\frac{\partial}{\partial\bar a}\theta_1^{\cdot}|_{a=1})-K_1(\frac{\operatorname{Log}t}{t}+\operatorname{Log}t+\frac{2}{t}-2).$$

In the same way, we have expansions for

$$I(\frac{\partial}{\partial\bar a}\theta_{-1}(y,\frac{1}{\sqrt t})),\quad I(\theta_2(y,\frac{1}{\sqrt t})).$$

Putting them together, by the fact that

$$\theta_0^t(y,1)\doteq{\zeta_E^t}'(0),$$

we have

$$\begin{aligned}{\zeta_E^t}'(0) =&\zeta_E'(0)+(C_0(0)-\frac{1}{2}\frac{\partial\theta_0}{\partial r}(1))\operatorname{Log}t+C_{-1}'(0)\frac{\operatorname{Log}t}{t}\\ &+\partial^B(J(\frac{\partial}{\partial\bar a}\theta_1^{\cdot}|_{a=1}))(t)+\bar\partial^B(J(\frac{\partial}{\partial a}\theta_{-1}^{\cdot}|_{a=1}))(t)\\ &+\bar\partial^B\partial^B(J(\theta_2^{\cdot}|_{a=1}))(t)\\ &-(\partial^B K_1+\bar\partial^B K_{-1}+\bar\partial^B\partial^B K_2)(\frac{\operatorname{log}t}{t}+\operatorname{Log}t+\frac{2}{t}-2).\end{aligned}$$

Thus, by the relation that

$$\partial^B K_1=\bar\partial^B K_{-1}=-\bar\partial^B\partial^B K_2=C_{-1}'(0),$$

we have the the following

Proposition. With the same notation as above, we have

$$\begin{aligned}\zeta_E^{t}{}'(0) =&\zeta_E{}'(0) + (C_0(0) - \frac{1}{2}\frac{\partial\theta_0}{\partial r}(1) - C_{-1}{}'(0))\mathrm{Log}t\\ &+ 2C_{-1}{}'(0)(1-\frac{1}{t})\\ &+ \partial^B(J(\frac{\partial}{\partial\bar{a}}\theta_1|_{a=1}))(t) + \bar{\partial}^B(J(\frac{\partial}{\partial a}\theta_2|_{a=1}))(t)\\ &+ \bar{\partial}^B\partial^B(J(\theta_3|_{a=1}))(t).\end{aligned}$$

In particular, we get (a).

(b) We now give the asymptotic expression for $\zeta'_{E,t}(0)$ as $t \to 0^+$. For this, by the definition, or, better, by the expression of $\zeta'(0)$ used in the proof of (a), we need to know the behavior of the corresponding integrant $\mathrm{Tr}_s[N_{u^t}\exp(-(\mathbf{A}_u^t)^2)]$, when the parameter u goes to zero and infinity respectively.

(i) We start with the situation for $u \to 0^+$. By definition, we know that this is equivalent to studying the asymototic expansion of

$$\mathrm{Tr}_s[N_{u'}\exp(-(\nabla + \sqrt{u'}D + \sqrt{u}V - \frac{c(T)}{4\sqrt{u'}})^2)],$$

when $u' \to 0^+$. Here, by definition, $N_{u'} = N_H + N_{V,u'}$.

We separate the above into two parts according those for $N_{u'}$. For the part with N_H, as $u' \to 0^+$, we get

$$\begin{aligned}&\mathrm{Tr}_s[N_H\exp(-(\nabla + \sqrt{u'}D + \sqrt{u}V - \frac{c(T)}{4\sqrt{u'}})^2)]\\ \to&\frac{1}{(2\pi i)^l}\int_Z \mathrm{Td}(-R_Z)\mathrm{Tr}_s[N_H\exp(-(\nabla + \sqrt{u}V)^2)].\end{aligned}$$

So it is sufficient to study the behavior of

$$\mathrm{Tr}_s[N_{V,u'}\exp(-(\nabla + \sqrt{u'}D + \sqrt{u}V - \frac{c(T)}{4\sqrt{u'}})^2)],$$

as $u' \to 0^+$. Now we use the same method as what we did in the proof of the theorem in subsection e: That is, to express

$$\mathrm{Tr}_s[N_{V,u'}\exp(-(\nabla + \sqrt{u'}D + \sqrt{u}V - \frac{c(T)}{4\sqrt{u'}})^2)]$$

as a combination of the super-trace for the heat kernel of certain essential generalized Laplacians. For doing so, by a local discussion, we see that

$$\begin{aligned}
&\frac{\partial}{\partial u'}(u'\mathrm{Tr}_s[N_{V,u'}\exp(-(\mathbf{A}_u^{u'/u})^2)]) \\
&=\mathrm{Tr}_s[\exp(-(\mathbf{A}_u^{u'/u})^2-(\sqrt{u'}\bar{\partial}+\frac{c(T^{1,0})}{4\sqrt{u'}})\,da \\
&\qquad -(\sqrt{u'}\bar{\partial}^*+\frac{c(T^{0,1})}{4\sqrt{u'}})\,d\bar{a}-i\omega_Z^c\,da\,d\bar{a}))]^{da\,d\bar{a}} \\
&\quad+\frac{l}{2}\mathrm{Tr}_s[\exp(-(\mathbf{A}_u^{u'/u})^2)] \\
&\quad-d^B\mathrm{Tr}_s[N_{V,u'}\exp(-(\mathbf{A}_u^{u'/u})^2+(\sqrt{u'}D+\frac{c(T)}{4\sqrt{u'}})\,du')]^{du'}.
\end{aligned}$$

In particular, after following what we did for the corollary of subsection e, we find that

$$\begin{aligned}
&\mathrm{Tr}_s[N_{u'}\exp(-(\nabla+\sqrt{u'}D+\sqrt{u}V-\frac{c(T)}{4\sqrt{u'}})^2)] \\
&=\frac{C_{-1}(u)}{u'}+C_0(u)+O_u(u').
\end{aligned}$$

That is, we have the following

Lemma 1. For $u>0$, as $t\to 0^+$,

$$\mathrm{Tr}_s[N_{ut}\exp(-(\mathbf{A}_u^t)^2)]=\frac{C_{-1}(u)}{ut}+C_0(u)+O_u(ut).$$

(ii) Now we consider the uniform estimates as $u\to+\infty$. For this, we may use the method of Berline and Vergne in section 5.5. Thus, by the fact that the complex E is acyclic, we have

Lemma 2. For any compact subset K of B, there exist $c_K>0$, $d_K>0$ such that for $u\geq 1,\ t>0,\ y\in K$, we have

$$|\mathrm{Tr}_s[N_{ut}\exp(-(\mathbf{A}_u^t)^2)]-\frac{C_{-1}(u)}{ut}|\leq c_K\exp(-d_K u).$$

(iii) Now we are ready to obtain the asymptotic expansion of $\zeta'_{E,t}$ with respect to t, as $t\to 0^+$. To state the result, we make the following definition: For $\mathrm{Re}(s)>1$, set

$$\begin{aligned}
\lambda_0(s)=&\frac{1}{\Gamma(s)}\int_0^\infty u^s C_0(u)\frac{du}{u}; \\
\lambda_1(s)=&\frac{1}{\Gamma(s)}\int_0^\infty u^s C_{-1}(u)\frac{du}{u}.
\end{aligned}$$

Then λ_0 and λ_1 may be extended to meromorphic functions on the whole complex plane, which are holomorphic at $s=0$.

Proposition. There exists an element $\beta \in P$ such that as $t \to 0^+$

$$\zeta'_{E,t}(0) = \frac{\lambda'_{-1}(0)}{t} + \lambda'_0(0) + \beta t + o(t),$$

and $o(t)$ is uniform over compact subsets in B.

Proof. First, by definition, we know that

$$\begin{aligned}\zeta_E^t{}'(0) =& \int_0^1 (\mathrm{Tr}_s[N_{ut}\exp(-(\mathbf{A}_u^t)^2)] - \frac{C_{-1}(0)+C_{-1}{}'(0)u}{ut} - C_0(0))\frac{du}{u} \\ &+ \int_1^\infty \mathrm{Tr}_s[N_{ut}\exp(-(\mathbf{A}_u^t)^2)]\frac{du}{u} + \frac{C_{-1}(0)}{t} \\ &- \Gamma'(1)(C_0(0) + \frac{C_{-1}{}'(0)}{t}).\end{aligned}$$

Thus, by Lemma 1, for $u \le 1$, we have $|O_u(ut)| \le C\,ut$. Thus if $t \to 0^+$, we have

$$|\int_0^1 O_u(ut)\frac{du}{u}| \le Ct.$$

So

$$\begin{aligned}&\int_0^1 (\mathrm{Tr}_s[N_{ut}\exp(-(\mathbf{A}_u^t)^2)] - \frac{C_{-1}(0)+C_{-1}{}'(0)u}{ut} - C_0(0))\frac{du}{u} \\ &= \frac{1}{t}\int_0^1 \frac{C_{-1}(u) - C_{-1}(0) - C_{-1}{}'(0)u}{u}\frac{du}{u} \\ &\quad + \int_0^1 (C_0(u) - C_0(0))\frac{du}{u} + o(t).\end{aligned}$$

On the other hand, by Lemma 2, we have

$$\begin{aligned}&\int_1^{+\infty} \mathrm{Tr}_s[N_{ut}\exp(-(\mathbf{A}_u^t)^2)]du \\ &= \frac{1}{t}\int_1^{+\infty} \frac{C_{-1}(u)}{u}\frac{du}{u} + \int_1^{+\infty} C_0(u)\frac{du}{u} + \varepsilon(t).\end{aligned}$$

Here $\varepsilon(t)$ is such that $\lim_{t\to 0}\varepsilon(t) = 0$. Hence, if we let $\bar\varepsilon(t) = \varepsilon(t) + O(t)$, then

$$\begin{aligned}\zeta_E^t{}'(0) =& \frac{1}{t}[\int_0^1 \frac{C_{-1}(u) - C_{-1}(0) - C_{-1}{}'(0)u}{u}\frac{du}{u} \\ &\int_1^{+\infty} \frac{C_{-1}(u)}{u}\frac{du}{u} + C_{-1}(0) + \Gamma'(1)C_{-1}{}'(0)] \\ &\int_0^1 (C_0(u) - C_0(0))\frac{du}{u} - \int_1^{+\infty} C_0(u)\frac{du}{u} + \Gamma'(1)C_0(0) + \bar\varepsilon(t).\end{aligned}$$

Therefore,

$$\zeta_E^t{}'(0) = \frac{\lambda'_{-1}(0)}{t} + \lambda'_0(0) + \bar{\epsilon}(t).$$

This completes the proof of the proposition, and hence (b).

Now we may finish the proof of the theorem. First, we know that $\zeta_{E,\mathbf{C}}^t{}'(0)$ has an asymptotic expansion similar to that in the above lemmas, simply replacing B by $B \times \mathbf{C}^*$. Moreover, we easily see that

$$\frac{\partial}{\partial \bar{a}}\theta_1^t|_{a=1}, \quad \frac{\partial}{\partial a}\theta_{-1}^t|_{a=1}, \quad \theta_2^t|_{a=1}$$

have similar expansions. Thus, by the fact that

$$(J(\frac{1}{b}))(t) = \int_t^1 \mathrm{Log}\frac{b}{t}\frac{db}{b^2} = \mathrm{Log}t - 1 - \frac{1}{t};$$

$$(J(1))(t) = \int_t^1 \mathrm{Log}\frac{b}{t}\frac{db}{b} = \frac{1}{2}\mathrm{Log}^2 t,$$

we know that as $t \to 0^+$,

$$J(\frac{\partial}{\partial \bar{a}}\theta_1^\cdot|_{a=1})(t) = \frac{\alpha_1^1}{t} + \alpha_1^2 \mathrm{Log}^2 t + \alpha_1^3 \mathrm{Log} t + \alpha_1^4 + \kappa(t).$$

Here $\alpha_1^j (1 \leq j \leq 4)$ are C^∞ forms on B and $\kappa(t) \to 0$ as $t \to 0^+$. Furthermore, one may verify that such expansions may be differentiated and we can apply the operator ∂^B on both sides of the the above equation and $\partial^B \kappa(t) \to 0$ as $t \to 0^+$. Similarly, we may do all of these for

$$J(\frac{\partial}{\partial a}\theta_{-1}^\cdot|_{a=1})(t), \quad J(\theta_2^\cdot|_{a=1})(t).$$

Hence, using (a) and (b), if we identify the constant term in the expansion of $\zeta_E^t{}'(0)$ as $t \to 0^+$, we find

$$\lambda_0'(0) - \zeta_E'(0) - 2C_{-1}'(0) \in P'.$$

But we know that $C_{-1}'(0) \in P'$, so

$$\lambda_0'(0) - \zeta_E'(0) \in P'.$$

On the other hand, by the fact that

$$C_0(u) \equiv \frac{1}{(2\pi i)^l}\int_Z \mathrm{Td}(-R_Z)\mathrm{Tr}_s[N_H \exp(-(\nabla + \sqrt{u}V)^2)],$$

as in the finite dimensional case, we have that

$$\lambda_0'(0) - \frac{1}{(2\pi i)^l}\int_Z \mathrm{Td}(-R_Z)\zeta_{\xi_\cdot,\rho_\cdot}'(0).$$

Here, $\zeta_{\xi.,\rho.}(s)$ is the zeta function associated with the exact sequence $(\xi., \rho.)$ on M, which is a finite dimensional version. Therefore, by the degree counting,

$$\zeta_E{}'(0) - \frac{1}{(2\pi i)^l}\int_Z \mathrm{td}(-R_Z)\zeta_{\xi.,\rho.}{}'(0) \in P'.$$

This completes the proof of the theorem.

Remark. We may also discuss the dependence of $\zeta_E{}'(0)$ on the choice of the metric g_Z on the fibre. For doing that, we also should use a deformation process as in the above proof of (2) of the theorem. More precisely, we may first consider the variation of the metric by a rational family $tg_Z + (1-t)g'_Z$. In this case, we then deformation the double copmlex $(E, \bar{\partial} + v)$ as stated in subsection a vertically. Thus the similar discussion as above could offers us a corresponding result for this change: We should first express the associated terms as a combination of the supre-trace of certain trace classes, say $\mathrm{Tr}_s\exp(-\mathbf{A}_u^2 + B_u)$. Then, one may use a local discussion for the essential generalized Laplacians $-\mathbf{A}_u^2 + B_u$, which have similar structures as what are for the super-Lechnerowicz formula, so that we could wirte the associated asymptotic expansions down. For the gerenal situation, one may use a partition of unity to deduce the final answer. We will not give the full details here, as the principle behind this now becomes quite clear, and the discussion is rather dull and tedious.

§I.6.2. The Construction In General

From above, we know that if we put the acyclic condition on $(E, \bar{\partial} + v)$ then the corresponding

$$\mathrm{Tr}_s[N_u \exp(-\mathbf{A}_u^2)]$$

decays exponentially when $u \to \infty$: Since in this case, the corresponding Laplacian is positive and self-adjoint, hence the eigenvalues are strictly positive. But in general, this is not always the case. In this section, following Faltings, we give a method which removes this technical assumption.

In fact, it is not very difficult. Recall that from sheaf theory, for any two complexes $E.$ and $F.$ of coherent sheaves with a quasi-isomorphism $\phi. : E. \to F.$, then there is the **cone construction**, $\mathrm{cone}(\phi.)$, such that we have the following exact sequence:

$$0 \to F. \to \mathrm{cone}(\phi.) \to E.[1] \to 0,$$

which is defined as follows:

$$E.[1]^p := E^{p+1}, \quad d_{E.[1]} = -d_E.$$

and

$$\mathrm{cone}(\phi.) := E.[1] \oplus F.,$$

where the differential operator is given by

$$\begin{pmatrix} d_{E.[1]} & \phi.[1] \\ 0 & d_{F.} \end{pmatrix}.$$

In this sense, at the level of cohomology classes, cone(ϕ) may be thought of formally as the summation of $F.$ and $E._{[1]}$, i.e. the difference of $F.$ and $E.$. In this section, we prove that this can go through even when we study everything at the level of differential forms. In order to do so, we have to find out the key rule behind the very complicated constructions in the previous section.

I.6.2.a. **Key Observations**

We start with the finite dimensional situation. When necessary, we suggest that the reader refers to section 4.5.

Suppose M is a complex manifold and $(\mathcal{E}., \nu.)$ is a finite complex of holomorphic vector sheaves on M. Let ρ_i be the hermitian metrics on $\mathcal{E}_j$. Then $\mathcal{E} := \oplus \mathcal{E}_j$ has a hermitian metric ρ and a canonical connection ∇. Let ν^* be the formal adjoint of ν with respect to this metric. Then we have the superconnection $\nabla_\nu := \nabla + \nu + \nu^*$ on $\mathcal{E}$, and $(\nabla + \nu + \nu^*)^2$ is of type $(1,1)$. We know that

$$[2\pi i]\mathrm{Tr}_s[\exp(-(\nabla + \nu + \nu^*)^2)]$$

represents the Chern character

$$\mathrm{ch}(\mathcal{E}.) = \sum_j (-1)^j \mathrm{ch}(\mathcal{E}_j)$$

in cohomology. Also, the classical Bott-Chern secondary characteristic forms measure this construction under the change of metrics at the level of differential forms.

By the construction in sections 4.5 and 6.1, we know that the number operator plays a very important role. Previously, we gave a precise description of the number operator. But here we look at this in an essential way, i.e. we consider the number operator as the one being the measure of the change of metrics. There are two ways to change the metrics: One is for the vector sheaf $\mathcal{E}$, another is for the Kähler metric g_M. We discuss them separately.

First we discuss the change of the metric ρ on $\mathcal{E}$, but instead of discussing it in general, we only deal with the infinitesimal variations: Suppose we have a one-parameter family of metrics on $\mathcal{E}.$ defined by

$$< e_1, e_2 >_\varepsilon := < e_1, e_2 > + \varepsilon < e_1, N(e_2) > + O(\varepsilon^2),$$

where N is a self-adjoint endomorphism of $\mathcal{E}.$. (For example, if we assume that $(\mathcal{E}., \nu.)$ is acyclic, and we multiply the metric on $\mathcal{E}_j$ by a factor t^j with $t > 0$, then we know that the ordinary number operator N, which is defined by the multiplication by j on $\mathcal{E}_j$, is given by the above definition.) With respect to this change of metrics, we know that $\nabla''_v = \nabla'' + \nu$ remains unchanged, while $\nabla'_v = \nabla' + \nu^*$ changes by $-\varepsilon[N, \nabla_v]$. Since the conjugation does not change the trace, so we may conjugate with $1 + \frac{\varepsilon}{2}N$ to get a more symmetric situation, i.e. the variation in connections is given by

$$\delta(\nabla_v) = \frac{\varepsilon}{2}[N, \nabla''_\nu - \nabla'_\nu].$$

Thus, we have

$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon}\mathrm{Tr}_s[\exp(-\nabla^2_{\nu,\varepsilon})]_{\varepsilon=0}\\
&=\frac{\partial}{\partial\varepsilon}\mathrm{Tr}_s[\exp(-\nabla^2_\nu - \frac{\varepsilon}{2}[\nabla_\nu, [N, \nabla''_\nu - \nabla'_\nu]])]_{\varepsilon=0}\\
&=\frac{1}{2}\mathrm{Tr}_s[[\nabla_\nu, [\nabla''_\nu - \nabla'_\nu, N]]\exp(-\nabla^2_\nu)]\\
&=\frac{1}{2}d\,\mathrm{Tr}_s[[\nabla''_\nu - \nabla'_\nu, N]\exp(-\nabla^2_\nu)].
\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}
&d\,\mathrm{Tr}_s[N\exp(-\nabla^2_\nu)] = \mathrm{Tr}_s[[\nabla_\nu, N]\exp(-\nabla^2_\nu)]\\
&= \mathrm{Tr}_s[[\nabla'_\nu, N]\exp(-\nabla^2_\nu)] + \mathrm{Tr}_s[[\nabla''_\nu, N]\exp(-\nabla^2_\nu)].
\end{aligned}$$

By comparing the holomorphic and antiholomorphic parts, we see that the two terms are just ∂ and $\bar\partial$, respectively, applied to $\mathrm{Tr}_s[N\exp(-\nabla^2_\nu)]$. So finally, we have

$$\frac{\partial}{\partial\varepsilon}\mathrm{Tr}_s[\exp(-\nabla^2_{\nu,\varepsilon})]_{\varepsilon=0} = \partial\bar\partial\mathrm{Tr}_s[N\exp(-\nabla^2_\nu)].$$

So locally, we may let

$$\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_\cdot, N) := [2\pi\mathrm{i}]\mathrm{Tr}_s[N\exp(-\nabla^2_\nu)],$$

we will have

$$dd^c\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_\cdot, N) = \frac{\partial}{\partial\varepsilon}\mathrm{ch}(\mathcal{E}_\cdot, \rho_{\cdot\varepsilon}).$$

In particular, if $\mathcal{E}_\cdot$ is acyclic, we may have the integral version, which is just our construction of classical Bott-Chern secondary characteristic forms in section 4.5, since, in this case, at infinity, we have the exponential decay.

Now we consider the change of g_M. In this case, the Kähler form $\omega_M = -\frac{1}{2\pi i}\sum dz_j \wedge d\bar z_j$ is changed by $\varepsilon\delta(\omega_M)$. Here dz_j denotes a local orthonormal basis. (The reason for us to consider this change comes from the definition of the Bismut superconnection.) This change effects the metric on $\mathcal{E} \otimes \wedge\Omega^{0,1}_M$ in two ways:

First the volume on M changes, and then so does the hermitian metric on $\Omega^{0,1}_M$. For the first case, if we let $Q \in \mathrm{End}(T^{1,0}_M)$ denote the hermitian operator, which infinitesimally generates the change of metrics, then this first effect is measured by the scalar number operator $\mathrm{Tr}(Q)$. For the second one, the number operator is given by

$$-\frac{1}{2}\sum d\bar{z}_j\,(-Q^t\,dz_j) = -\frac{1}{2}\mathrm{Tr}(Q) + \pi i\delta(\omega^c).$$

Thus the total number operator is given by

$$N = \frac{1}{2}\mathrm{Tr}(Q) + \pi i\delta(\omega^c).$$

Thus, for in the relative situation, we may have the follows:

Let $\pi : M \to B$ be a Kähler fibration. Then we study the metrics $g_{M,s} := g_M + s^{-1}g_B$ on M. By the definition of the Bismut superconnection, we know that the Bismut superconnection may be thought of as the limit of the corresponding ordinary Clifford superconnection as $s \to 0^+$. (For this, see the definition in section 5.1.) Therefore, we may need to consider the corresponding limit for the total number operator N_s as introduced above. We change the notation and denote this limit as $\mathbf{N}$. The formulas at the beginning of this subsection for the finite dimensional case now turn to be the one which describes the dependence of the Bismut superconnection $\mathbf{B}$ on g_X. For example, we then could have

$$\frac{\partial}{\partial\epsilon}\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]_{\epsilon=0} = \partial\bar{\partial}\mathrm{Tr}_s[\mathbf{N}\exp(-\mathbf{B}^2)],$$

or, formally, by the local family index theorem,

$$\frac{\partial}{\partial\epsilon}[\mathrm{ch}(\pi_*(\mathcal{E}\otimes\Omega^{0,1}_{M/B}))]_{\epsilon=0} = [2\pi\mathrm{i}]\,d_B d_B^c\mathrm{Tr}_s(\mathbf{N}\exp(-\mathbf{B}^2)].$$

To demonstrate how the above relation works, we go as follows: Introduce a new parameter t on replacing the original metric g_M by $t^{-1}g_M$, which could introduce a family of Bismut superconnections $\mathbf{B}_t$. Then, for $t \to 0^+$, the asymptotic expansions of the above relation go as follows: (For this, consult 6.1.d.)

On the left hand side, we obtain the derivative of $\int_Z \mathrm{td}(M/B)\mathrm{ch}(\mathcal{E})$, which is $d_B d_B^c$ of

$$\int_Z \mathrm{td}_{\mathrm{BC}}(M/B,Q)\mathrm{ch}(\mathcal{E}).$$

On the right hand side, we know that the t^{-1} term in $\mathrm{Tr}_s(\mathbf{N}_t\exp(-\mathbf{B}_t^2)]$ is equal to a closed form $\int_Z \frac{\omega_M}{2}\mathrm{td}(M/B)\mathrm{ch}(\mathcal{E})$, as what we gave in the proof of Theorem 1.g. Furthermore, the corresponding t^0 term is the sum of

$$\frac{1}{2}\int_Z \mathrm{Tr}(Q|_{T_\pi})\mathrm{td}(M/B)\mathrm{ch}(\mathcal{E}) \quad\text{and}\quad \int_Z \hat{A}_{\mathrm{BC}}(T_\pi,Q)\mathrm{ch}(\mathcal{E}).$$

So, we may get the coincidence. In fact, for t^0, the first term is the contribution of the factor $\exp(\frac{1}{2}c_1(T_\pi))$ to the secondary characteristic class $\mathrm{td}_{\mathrm{BC}}(T_\pi,Q)$ associated to the Todd genus

$\mathrm{td}(T_\pi) = \hat{A}(T_\pi)\exp(\frac{1}{2}c_1(T_\pi))$, so the sum of these two terms is just $\int_Z \mathrm{td}_{\mathrm{BC}}(M/B, Q)\mathrm{ch}(\mathcal{E})$. This proves the claim above.

As we said above, the Bismut superconnection may be thought of as the limit of the associated Clifford superconnection with the blow-up of the metrics $g_{M,s} = g_M + s^{-1}\pi^* g_B$ when $s \to 0^+$. We introduce a parameter t by rescaling the metric g_M as $t^{-1}g_M$, which affects the fibre part, and then we have the super-Lichnerowicz formula for them, and so on. In the next subsection, we use all these observations to give the corresponding concepts for the cone construction.

I.6.2.b. The Situation For The Cone Construction

Let $\pi : M \to B$ be a Kähler fibration and $(\mathcal{E}, \rho)$ be a π-acyclic vector sheaf on M. There is a natural morphism

$$\nu : \pi_*\mathcal{E} \to \pi_*(\mathcal{E} \otimes \wedge\Omega_\pi^{0,1}).$$

Here $\pi_*\mathcal{E}$ is with the natural push-out hermitian metric $\pi_*\rho$. Hence we have the associated mapping cone, $\mathrm{cone}(v)$. On this mapping cone, we consider the super-Dirac operator

$$\mathbf{A}_{\mathrm{cone}} := \nabla_{\pi_*(\mathcal{E},\rho)} + \mathbf{B} + \nu + \tilde{\nu}^*$$

with $\tilde{\nu}^*$ the limit of the formal adjoints ν_s^* when $s \to 0^+$. This contains terms of positive Grassmannian degree because the pull-back metric on $\pi^*\Omega_B$ may not coincide with its subspace metric induced from Ω_M. Such a difference in metrics, in the limit, is given by the number operator N, and $\tilde{\nu}^*$ differs from the naive ν^* by a factor $\exp(\mathrm{N})$. Furthermore, we rescale the metric g_M as $t^{-1}g_M$, by introducing the parameter t. At the same time, the metric on $\pi_*\mathcal{E} \otimes \wedge\Omega_B^{0,1}$ is scaled by an additional factor t^l with l the relative dimension of π, to account for the volume forms. (This makes sections of $\pi_*\mathcal{E}$ uniformly square integrable on the fibers of π.) In this way, we have $\mathbf{A}_{\mathrm{cone,t}}$, and we denote the associated number operator by $\mathbf{N}_t$. Then by section 5.2, we know that there exist heat kernels $\exp(-\mathbf{A}^2_{\mathrm{cone,t}})$ for the essential generalized Laplacian $\mathbf{A}^2_{\mathrm{cone,t}}$. As one may imagine, in order to give the relative Bott-Chern secondary characteristic forms associated with the π-acyclic hermitian vector sheaf $(\mathcal{E}, \rho)$, we use the Mellin transform for

$$\mathrm{Tr}_s[\mathbf{N}_t \exp(-\mathbf{A}^2_{\mathrm{cone,t}})].$$

Therefore, we need to consider the behavior of this super-trace when $t \to 0^+$ and $t \to +\infty$ respectively.

I.6.2.c. Conditions For The Mellin Transform

Let $\pi : M \to B$ be a Kähler fibration, $(\mathcal{E}, \rho)$ a π-acyclic hermitian vector sheaf on M. For the metric $g_{s,t} := t^{-1}(g_M + s^{-1}\pi^* g_B)$, consider the Laplacian $\Delta_{s,t} = t\,\Delta_{s,1}$ on the mapping cone of

$$\nu : C^\infty(Y, \pi_*\mathcal{E} \otimes \wedge\Omega_B^{0,1}) \to C^\infty(M, \mathcal{E} \otimes \wedge\Omega_M^{0,1}).$$

On $C^\infty(Y, \pi_*\mathcal{E} \otimes \wedge\Omega_B^{0,1})$ the metric is scaled by an additional factor t^l, to account for the volume forms.

By the result in subsection 6.1.d, we know that it is enough to consider the effect of the terms associated with ν. As an illustration, we treat first the absolute case, i.e. when Y is a point. In this case, ν and all its derivatives are uniformly trace class. By Duhamel's formula, the corresponding super-traces for Bismut superconnections have an asymptotic expansion when $t \to 0^+$. Furthermore, by the fact that ν only influences terms with positive t-power, we have the same formula as in subsection 6.1.e. On the other hand, the exponential decay is a direct consequence of the fact that the complex is acyclic and thus all eigenvalues are positive.

The basic idea to treat the relative situation in general is that we first consider the object for the ordinary Laplacians, then consider the limits for t, and finally let $s \to 0^+$ to get the result. If we proceed in this way, one may ask that why the final results are just the same as when we first let $s \to 0^+$ and then take the limit for t. The answer is simple: the constructions do not depend on the paths we choose. As an example, we consider the heat kernels $\exp(-\Delta_{s,t})$. We want to prove that when $t \to +\infty$, $\exp(-\Delta_{s,t})$ decays exponentially together with all its s-derivatives: By a local discussion as in the proof of the local family index theorem, it is enough to show that the associated Laplacian have strictly positive eigenvalues. Hence, we need to show the following

Proposition. With the same notation as above, the eigenvalues of $\Delta_{s,1}$ are uniformly bounded below by a fixed $\varepsilon > 0$, which is independent of s.

Proof. Let $y \in B$ and $Z = \pi^{-1}(y)$. Near Z, we identify M with $Z \times T_{B,y}$. Scaling the coordinates on the second factor by $\sqrt{s}$, we know that $\Delta_{0,1}$ is a direct sum of the Laplacian of $T_{B,y}$ and the relative Laplacian on the fiber Z. On the other hand, by our construction, we know that the cone is acyclic along the fiber, as we assume that $\mathcal{E}$ is π acyclic, so the relative Laplacian has positive eigenvalues. So by the perturbation process, it follows that for each y, there exists a small neighborhood U_y in B and an $\varepsilon_y > 0$ such that for any C^∞-section with support in U_y, and for s small enough, we have

$$< \Delta_{s,t}(f), f > \geq 2\varepsilon_y < f, f > .$$

Moreover, by the compactness of B, we may choose finitely many U_y and hence a $\varepsilon > 0$, which is independent of y. Therefore we may use the Sobolev estimation uniformly in s with the metric $g_{s,1}$ by the finiteness stated above. In particular, we have a uniform Garding-inequality for $\Delta_{s,1}$. From this, the assertion may be deduced easily: In fact, we may choose a small open cover U_i of B such that if f has support in U_i, then

$$< \Delta_{s,t}(f), f > \geq 2\varepsilon < f, f > .$$

Now choose C^∞ functions ϕ_i with support in U_i such that $\sum_i \phi_i^2 = 1$ is a partition of unity. Then the Sobolev norm of $[\Delta_{s,1}, \phi_i]$ is $O(\sqrt{s})$. So, for a C^∞ section f on M, we have

$$\begin{aligned}<\Delta_{s,1}(f), f > &= \sum_i < \phi_i \Delta_{s,1}(f), \phi_i f > \\ &= \sum_i < \Delta_{s,1}(\phi_i f), \phi_i f > - \sum_i < [\Delta_{s,1}, \phi_i](f), \phi_i f > .\end{aligned}$$

Since $\phi_i f$ has support in U_i, the first term is $\geq 2\varepsilon < f, f >$, while the second one is bounded by a fixed multiple of $\sqrt{s}(< \Delta_{s,1}(f), f > + < f, f >)$. Thus for s small enough, we have

$$< \Delta_{s,1}(f), f > \geq \varepsilon < f, f > .$$

This completes the proof.

With this lemma, we know that as $t \to +\infty$, $s^{\dim B}\exp(-t\Delta_{s,1})$ decays exponentially together with all its s-derivatives. In other words, the asymptotic s-expansion of $\exp(-t\Delta_{s,1})$ as well as all remainder terms, decay exponentially at infinity. From the fact that the s-expansion starts with $s^{-\dim B}$, and the $s^{p-\dim B}$-terms have Clifford filtration degrees in B at most $2p$, we know that when $t \to +\infty$, the super-objects for the cone construction decay expansionally, since the super-trace vanishes on the low Clifford degree terms.

Finally, we consider the asymptotic expansion for $t \to 0^+$. As we stated above, we first consider the asymptotic expansions as $t \to 0^+$ for a fixed s. Hence, we use a cut-off in B and scale the B-coordinates by $\sqrt{t}$. Thus $\Delta_{s,t}$ becomes a perturbation $t\Delta_Z + \Delta_{T_{B,s}}$. We know that the coefficient of $t^{\dim B}\exp(-t\Delta_{s,1})$ is C^∞ in $\sqrt{t}$. Furthermore, the terms with low t-power also have low Clifford degree in B. On the other hand, the terms involving ν have at least a $\sqrt{t}$ in front of them, and have Clifford degree zero. Hence, they do not contribute to the leading terms. So, taking the super-trace and letting $s \to 0^+$, we get the formula similar to that in subsection 6.1.e. That is, when restricted to the diagonal, there is no negative t-power, and the constant term is independent of ν. Similarly, we may also do this for the objects by twisting the number operator, as it could be represented as a combination of certain trace classes. In summary, we have the following

Theorem. (1) There exist C^∞ even differential forms $A_0, A_1, \ldots, C_{-1}, C_0, \ldots$ in P such that for any $k \in \mathbf{N}$,

$$\mathrm{Tr}_s[\exp(-\mathbf{A}^2_{\mathrm{cone},t})] = \sum_0^k A_j u^j + o(u^k),$$

$$\mathrm{Tr}_s[\mathbf{N}_t \exp(-\mathbf{A}^2_{\mathrm{cone},t})] = \sum_{j=-1}^k C_j u^j + o(u^k),$$

where o is uniform on compact subsets of B. Furthermore A_0, C_{-1} and C_0 are independent of ν.

(2) When $t \to +\infty$, $\mathrm{Tr}_s[\exp(-\mathbf{A}^2_{\mathrm{cone},t})]$, $\mathrm{Tr}_s[\mathbf{N}_t \exp(-\mathbf{A}^2_{\mathrm{cone},t})]$ decay exponentially.

I.6.2.d. The Construction In General

With the results in the previous subsections, we may make the following

Proposition and Definition. (1) For $s \in \mathbf{C}, \mathrm{Re}(s) > 1$, let

$$\zeta_{\xi_\cdot,\rho_\cdot,\pi}(s) := \frac{1}{\Gamma(s)} \int_0^{+\infty} u^s \mathrm{Tr}_s[\mathbf{N}_u \exp(-\mathbf{A}^2_{\mathrm{cone,u}})] \frac{du}{u}.$$

Then $\zeta_{\xi_\cdot,\rho_\cdot,\pi}(s)$ is well-defined and is an element in P.
(2) There exists a meromorphic continuation of $\zeta_{\xi_\cdot,\rho_\cdot,\pi}(s)$ to the whole complex plane such that this extension is holomorphic at $s=0$. Hence it makes sense for us to talk about $\zeta'_{\xi_\cdot,\rho_\cdot,\pi}(0)$.

Now we have the relative Bott-Chern secondary characteristic forms by the following

Main Theorem. $\dot{\mathrm{ch}}_{\mathrm{BC}}(\mathcal{E},\rho;\pi,\rho_\pi)=[2\pi\mathrm{i}]\zeta'_{\xi_\cdot,\rho_\cdot,\pi}(0)$.

The proof of this theorem is given in the next section.

§I.6.3. Checking The Axioms

To prove the theorem at the end of last section, we have to check the axioms for relative Bott-Chern secondary characteristic forms with respect to smooth morphisms.

I.6.3.a. **Downstairs Rule**

We start with the proof of the following equality:

$$dd^c\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;\pi,\rho_\pi)=\pi_*(\mathrm{ch}(\mathcal{E},\rho)\mathrm{td}(\mathcal{T}_\pi,\rho_\pi))-\mathrm{ch}(\pi_*\mathcal{E},\pi_*\rho).$$

This is a direct consequence of the double transgression formula. In fact, by the double transgression formula, it is enough to show that the limits of the corresponding heat kernels associated with the Bismut superconnection for the cone construction behave as follows:

As $t\to+\infty$, they decay exponentially; and as $t\to 0^+$, the limit is

$$f_*(\mathrm{ch}(\mathcal{E},\rho)\mathrm{td}(\mathcal{T}_\pi,\rho_\pi))-\mathrm{ch}(\pi_*\mathcal{E},\pi_*\rho).$$

For $t\to+\infty$, the assertion is obvious by Theorem c. On the other hand, as $t\to 0^+$, the limit has two parts: one comes from the local family index theorem associated with $\pi_*(\mathcal{E}\otimes\wedge\Omega_\pi^{0,1})$, which gives the term

$$\pi_*(\mathrm{ch}(\mathcal{E},\rho)\mathrm{td}(\mathcal{T}_\pi,\rho_\pi)).$$

The other part comes from $\pi_*(\mathcal{E},\rho)$, which gives the term

$$\mathrm{ch}(\pi_*\mathcal{E},\pi_*\rho),$$

since the cone construction now will just give us the difference between them. So we have axiom 1.

I.6.3.b. **Functorial Rule**

This axiom simply states that the relative Bott-Chern secondary characteristic forms are compatible with flat base changes. But this is easily checked by the fact that everything in our definition for relative Bott-Chern secondary characteristic forms is compatible with the flat base change. Therefore, we have axiom 2.

I.6.3.c. **Triangle Rule For Vector Sheaves**

There are two different ways to check axiom 3: One is obtained by using Theorem 1.g and Theorem 5.5. The other is obtained by using the $\mathbf{P}^1$-deformation.

We first give a proof using Theorem 1.g. With this theorem, the conclusion is easy, as we know that the cone construction may be thought of as the difference of terms associated with $\pi_*(\mathcal{E}\otimes\wedge\Omega_\pi^{0,1})$ and $\pi_*(\mathcal{E},\rho)$. Thus, write each term down, in which only the push-out matters, we have axiom 3. Next, we use the $\mathbf{P}^1$-deformation to check the axiom.

The $\mathbf{P}^1$-deformation method relies on axioms 1 and 2. In fact, by the construction of classical Bott-Chern secondary characteristic forms in section 1.2, we know that

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.;\rho_1,\rho_2,\rho_3)\\&\quad=\int_{\mathbf{P}^1}[\log|z|^2]\mathrm{ch}(D\mathcal{E}_2,D\rho_2).\end{aligned}$$

Then by axiom 1, we have

$$\begin{aligned}&(\pi\times\mathrm{Id}_{\mathbf{P}^1})_*(\mathrm{ch}(D\mathcal{E}_2,D\rho_2)\,\mathrm{td}(\mathcal{T}_{M\times\mathbf{P}^1/B\times\mathbf{P}^1},g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1}))\\&\quad=dd^c\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}_2,D\rho_2,\pi\times\mathrm{Id}_{\mathbf{P}^1},g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})\\&\qquad+\mathrm{ch}((\pi\times\mathrm{Id}_{\mathbf{P}^1})_*D\mathcal{E}_2,(\pi\times\mathrm{Id}_{\mathbf{P}^1})_*D\rho_2).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\pi_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho_1,\rho_2,\rho_3)\,\mathrm{td}(\mathcal{T}_{M/B},g_{M/B}))\\&\quad=\int_{\mathbf{P}^1}[\log|z|^2]((\pi\times\mathrm{Id}_{\mathbf{P}^1})_*(\mathrm{ch}(D\mathcal{E}_2,D\rho_2)\mathrm{td}(\mathcal{T}_{M\times\mathbf{P}^1/B\times\mathbf{P}^1},g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})).\end{aligned}$$

Therefore,

$$\begin{aligned}&\pi_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}.,\rho_1,\rho_2,\rho_3)\mathrm{td}(\mathcal{T}_{M/B},g_{M/B}))\\&\quad=\int_{\mathbf{P}^1}[\log|z|^2]\,dd^c\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}_2,D\rho_2,\pi\times\mathrm{Id}_{\mathbf{P}^1},g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})\\&\qquad+\int_{\mathbf{P}^1}[\log|z|^2]\mathrm{ch}((\pi\times\mathrm{Id}_{\mathbf{P}^1})_*D\mathcal{E}_2,(\pi\times\mathrm{Id}_{\mathbf{P}^1})_*D\rho_2)\\&\quad=\int_{\mathbf{P}^1}dd^c[\log|z|^2]\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}_2,D\rho_2,\pi\times\mathrm{Id}_{\mathbf{P}^1},g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})\\&\qquad+\int_{\mathbf{P}^1}\log|z|^2\mathrm{ch}((\pi\times\mathrm{Id}_{\mathbf{P}^1})_*D\mathcal{E}_2,(\pi\times\mathrm{Id}_{\mathbf{P}^1})_*D\rho_2).\end{aligned}$$

Then by the following equation of currents,

$$dd^c[\log|z|^2] = \delta_0 - \delta_\infty,$$

we have

$$\begin{aligned}\pi_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_\cdot, \rho_1, \rho_2, \rho_3)\,\mathrm{td}(\mathcal{T}_{M/B}, g_{M/B}))
&= i_0^*\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}_2, D\rho_2; \pi \times \mathrm{Id}_{\mathbf{P}^1}, g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})\\
&\quad - i_\infty^*\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}_2, D\rho_2; \pi \times \mathrm{Id}_{\mathbf{P}^1}, g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})\\
&\quad + \int_{\mathbf{P}^1}[\log|z|^2]\mathrm{ch}((\pi \times \mathrm{Id}_{\mathbf{P}^1})_* D\mathcal{E}_2, (\pi \times \mathrm{Id}_{\mathbf{P}^1})_* D\rho_2).\end{aligned}$$

Now, by axiom 2 or better by the constructions, we know that

$$\begin{aligned}i_0^*\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}_2, D\rho_2; \pi \times \mathrm{Id}_{\mathbf{P}^1}, g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})
&= \mathrm{ch}_{\mathrm{BC}}(i_0^* D\mathcal{E}_2, i_0^* D\rho_2; \pi, g_{M/B})\\
&= \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_2, \rho_2; \pi, g_{M/B}),\end{aligned}$$

and

$$\begin{aligned}i_\infty^*\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}_2, D\rho_2, \pi \times \mathrm{Id}_{\mathbf{P}^1}, g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})
&= \mathrm{ch}_{\mathrm{BC}}(i_\infty^* D\mathcal{E}_2, i_\infty^* D\rho_2; \pi, g_{M/B})\\
&= \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_1 \oplus \mathcal{E}_3, \rho_1 \oplus \rho_3; \pi, g_{M/B}).\end{aligned}$$

Obviously, our construction is compatible with the direct sum, so we have

$$\begin{aligned}i_\infty^*\mathrm{ch}_{\mathrm{BC}}(D\mathcal{E}_2, D\rho_2; \pi \times \mathrm{Id}_{\mathbf{P}^1}, g_{M\times\mathbf{P}^1/B\times\mathbf{P}^1})
&= \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_1, \rho_1; \pi, g_{M/B}) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_3, \rho_3; \pi, g_{M/B}).\end{aligned}$$

Then, by the fact that

$$\begin{aligned}\mathrm{ch}_{\mathrm{BC}}(\pi_*\mathcal{E}_\cdot, \pi_*\rho_1, \pi_*\rho_2, \pi_*\rho_3)
&= \int_{\mathbf{P}^1}\log|z|^2\mathrm{ch}((\pi \times \mathrm{Id}_{\mathbf{P}^1})_* D\mathcal{E}_2, (\pi \times \mathrm{Id}_{\mathbf{P}^1})_* D\rho_2),\end{aligned}$$

we have

$$\begin{aligned}\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_2, \rho_2; \pi, g_\pi) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_1, \rho_1; \pi, g_\pi) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_3, \rho_3; \pi, g_\pi)
&= \pi_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_\cdot, \rho_\cdot)\,\mathrm{td}(\mathcal{T}_\pi, g_\pi)) - \mathrm{ch}_{\mathrm{BC}}(\pi_*\mathcal{E}_\cdot, \pi_*\rho_\cdot).\end{aligned}$$

This is just axiom 3.

Remark. From the proof above, we know that we may use the degenerate triangles in axiom 3. That is,

Axiom 3′. (Degenerate Triangle Rule For Hermitian Vector Sheaves) For any f-acyclic vector sheaf $\mathcal{E}$ with hermitian metrics ρ and ρ'. We have

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,g_{X/Y}) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho';f,g_{X/Y})\\&\quad = f_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E};\rho,\rho')\mathrm{td}(\mathcal{T}_{X/Y},g_{X/Y})) - \mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E};f_*\rho,f_*\rho').\end{aligned}$$

In addition, let $(\mathcal{E},\rho)$ and $(\mathcal{F},\tau)$ be two f-acyclic hermitian vector sheaves on X. Then

$$\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}\oplus\mathcal{F},\rho\oplus\tau;f,g_{X/Y}) = \mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,g_{X/Y}) + \mathrm{ch}_{\mathrm{BC}}(\mathcal{F},\tau;f,g_{X/Y}).$$

I.6.3.d. Triangle Rule For Morphisms

After the proof in the last subsection for hermitian vector sheaves, one may naturally use $\mathbf{P}^1$-deformation to deduce axiom 4 for smooth morphisms from the following

Axiom 4′. (Degenerate Triangle Rule For Morphisms) Let $f: X \to Y$ be a smooth morphism of complex compact manifolds. Let $(\mathcal{E},\rho)$ be an f-acyclic hermitian vector sheaf. Let $g_{X/Y}$, $g'_{X/Y}$ be two hermitian metrics on $\mathcal{T}_{X/Y}$. Then

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,g_{X/Y}) - \mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,g'_{X/Y})\\&\quad = f_*(\mathrm{ch}(\mathcal{E},\rho)\mathrm{td}_{\mathrm{BC}}(\mathcal{T}_{X/Y},g_{X/Y},g'_{X/Y})).\end{aligned}$$

Here $\mathrm{td}_{\mathrm{BC}}(\mathcal{T}_{X/Y},g_{X/Y},g'_{X/Y})$ is the classical Bott-Chern secondary characteristic form associated with the Todd class with respect to the metrics $g_{X/Y}$, $g'_{X/Y}$ and the relative hermitian tangent sheaf. In addition, if $g: Y \to Z$ is another smooth morphism with $f_*\mathcal{E}$ being g-acyclic. Suppose that the associated short exact sequence of relative hermitian tangent sheaves is split, then

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;g\circ f,g_{X/Z})\\&\quad=\mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{E},f_*\rho;g,g_{Y/Z}) + g_*(\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f,g_{X/Y})\mathrm{td}(\mathcal{T}_{Y/Z},h_{Y/Z})).\end{aligned}$$

In practice, we proceed as follows. Recall the data: We have three Kähler fibrations $f: X \to Y$, $g: Y \to Z$, $g\circ f: X \to Z$ and an f-acyclic vector sheaf $\mathcal{E}$ on X such that $g_*\mathcal{E}$ is g-acyclic. Thus we have the following natural morphisms:

$$\begin{aligned}f_*\mathcal{E} &\xrightarrow{\tilde f} f_*(\mathcal{E}\otimes\wedge\Omega_f^{0,1});\\ g_*(f_*\mathcal{E}) &\xrightarrow{\tilde g} g_*(f_*\mathcal{E}\otimes\wedge\Omega_g^{0,1});\\ (g\circ f)_*\mathcal{E} &\xrightarrow{\widetilde{g\circ f}} (g\circ f)_*(\mathcal{E}\otimes\wedge\Omega_{g\circ f}^{0,1}).\end{aligned}$$

From them, we have associated cone constructions, and the relative Bott-Chern secondary characteristic forms with respect to f, g and $g\circ f$, respectively. So, in order to discuss the relations of the relative Bott-Chern secondary characteristic forms for those morphisms, we consider the following diagram:

$$\begin{array}{ccccc}
g_*(f_*\mathcal{E}) & \xrightarrow{\tilde g} & g_*(f_*\mathcal{E}\otimes\wedge\Omega_g^{0,1}) & \xrightarrow{\tilde f} & g_*(f_*(\mathcal{E}\otimes\wedge\Omega_f^{0,1})\otimes\wedge\Omega_g^{0,1}) \\
\| & & \| & & \| \\
g_*(f_*\mathcal{E}) & \xrightarrow{\tilde g} & g_*(f_*\mathcal{E}\otimes\wedge\Omega_g^{0,1}) & \xrightarrow{\tilde f} & g_*(f_*(\mathcal{E}\otimes\wedge\Omega_f^{0,1}\otimes f^*\wedge\Omega_g^{0,1})) \\
\| & & \| & & \| \\
g_*(f_*\mathcal{E}) & \xrightarrow{\tilde g} & g_*(f_*\mathcal{E}\otimes\wedge\Omega_g^{0,1}) & \xrightarrow{\hat f} & g_*(f_*(\mathcal{E}\otimes\wedge\Omega_{g\circ f}^{0,1})) \\
\| & & \| & & \| \\
(g\circ f)_*\mathcal{E} & \xrightarrow{\tilde g} & g_*(f_*\mathcal{E}\otimes\wedge\Omega_g^{0,1}) & \xrightarrow{\hat f} & (g\circ f)_*(\mathcal{E}\otimes\wedge\Omega_{g\circ f}^{0,1}) \\
\| & & & & \| \\
(g\circ f)_*\mathcal{E} & & \xrightarrow{g\circ f} & & (g\circ f)_*(\mathcal{E}\otimes\wedge\Omega_{g\circ f}^{0,1}).
\end{array}$$

Thus, we do as before to formulate the cone constructions with respect to the morphism $\tilde g$, $\tilde f$ and $\hat f$. We also introduce the associated superconnections, number operators and then their heat kernels respectively. Finally, one has similar elements by using the Mellin transforms. Denote those final elements by

$$\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\tilde g)),\ \mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\tilde f)),\ \mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\hat f)).$$

Then, the above diagram and the fact that $\mathrm{ch}_{\mathrm{BC}}$ corresponds to the difference of the complex give

$$\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\tilde g))+\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\hat f))=\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(g\circ f)).$$

Also the difference of $\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\tilde f))$ and $\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\hat f))$ is given by

$$\int_g \mathrm{ch}(\mathcal{E},\rho)\mathrm{td}_{\mathrm{BC}}(X,Y,Z).$$

(This is a consequence of the discussion of the remark at the end of subsection 1.g.) So, in order to check axiom 4, it is enough to show that

$$\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\tilde f))=g_*(\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(f))\,\mathrm{td}(\mathcal{T}_g,g_g)).$$

But the proof of this equality is not difficult. We have

$$f_*\mathcal{E}\xrightarrow{f}f_*(\mathcal{E}\otimes\wedge\Omega_f^{0,1});$$

$$g_*(f_*\mathcal{E}\otimes\wedge\Omega_g^{0,1})\xrightarrow{\tilde f}g_*(f_*(\mathcal{E}\otimes\wedge\Omega_f^{0,1})\otimes\wedge\Omega_g^{0,1}).$$

Thus, at least, formally by the results in the previous sections, we know that the additional term $\wedge\Omega_g^{0,1}$ will give the $\mathrm{td}(\mathcal{T}_g,g_g)$. Hence we have the assertion. Even in practice, this statement works well, since when we take the integration with respect to the parameters, our construction does not depend on the path we choose. Therefore by changing the order of taking the limits with respect to the parameters, we have the above relation as we did in section 6.2.

Up to now, we have already proved the existence of the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms. There is another statement for the uniqueness of our object. We will postpone the proof of this uniqueness after we have the arithmetic Riemann-Roch theorem in part II.

Chapter I.7
Relative Bott-Chern Secondary Characteristic Currents With Respect To Closed Immersions I: Axioms

In the previous chapters, we proved the existence of classical Bott-Chern secondary characteristic forms and the existence of relative Bott-Chern secondary characteristic forms with respect to smooth morphisms. In this chapter, we obtain the axioms for the relative Bott-Chern secondary characteristic currents with respect to closed immersions.

We first consider the classical Riemann-Roch theorem for closed immersions. Let $i : M' \hookrightarrow M$ be a closed immersion of complex manifolds. There exists an exact sequence:

$$0 \to \mathcal{T}_{M'} \to i^*\mathcal{T}_M \to \mathcal{N}_i \to 0.$$

On the other hand, for any vector sheaf η on M', the direct image $i_*\eta$ is usually not a vector sheaf on M: It is only a coherent sheaf on M. By classical sheaf theory, there exists a vector sheaf resolution of $i_*\eta$ on M:

$$0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to i_*\eta \to 0,$$

or

$$\mathcal{E}. \to i_*\eta \to 0.$$

Then the classical Riemann-Roch theorem for closed immersion says that we have the following equality at the level of cohomology classes, i.e. in $\mathrm{CH}(M)_{\mathbf{Q}}$,

$$\mathrm{ch}(i_*\eta) = i_*(\mathrm{td}(\mathcal{N})^{-1}\,\mathrm{ch}(\eta)).$$

Thus by the fact that $i_*\eta = \sum_j (-1)^j \mathcal{E}_j$, we have

$$\mathrm{ch}(\mathcal{E}.) = i_*(\mathrm{td}(\mathcal{N})^{-1}\,\mathrm{ch}(\eta)).$$

(For all of this, see II.1.)

Now we consider the problem at the level of differential forms. Then we may put metrics on the exact sequence of normal sheaves. Also, even through $i_*\eta$ is only a coherent sheaf, we may still put the metrics on $\mathcal{E}_j$. Just as for smooth morphisms, a natural question is how we can measure the change of (η, g_η), after the action of the closed immersion i, at the level of differential forms. It is for this reason that we introduce the relative Bott-Chern secondary characteristic currents with respect to closed immersions, $\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; i, g_i)$. (Here we have to use the language of currents, as, at least formally, the i_*-image of a form may be written as the product of this form with the Dirac symbol δ'_M of M' in M.) In practice, we have the downstairs rule as follows:

$$dd^c \mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; i, g_i) = \mathrm{td}(\mathcal{N}, g_{\mathcal{N}})^{-1}\, \mathrm{ch}(\eta, g_\eta)\, \delta'_M - \mathrm{ch}(\mathcal{E}_., \rho_.).$$

Nevertheless, the situation is not so simple. We know that the metrics on $\mathcal{E}.$ are not unique and, in general, we cannot control them very well. In order to introduce the relative Bott-Chern secondary characteristic currents with respect to closed immersions, we need a technical assumption on the metrics, which is nothing but the so-called Bismut condition (A), which gives certain compatibility condition for the associated metrics. The references here are [B 90] and [We 91].

§I.7.1 Basic Facts Associated With Closed Immersions and Resolutions

I.7.1.a. **Assumptions and Notations.**

In this subsection, we recall some basic facts associated with closed immersions and resolutions of the direct image of a vector sheaf. Even through we only deal with the closed immersions, since there is no further difficulty or complexity, we work with the following data:

(1) A closed immersion $i : M' \hookrightarrow M$ with $M' = \cup_1^n M'_j$. Each M_j is a compact connected complex submanifolds of dimension $l_j + l'$ of a dimension $l + l'$ complex manifold M, such that if $j \neq j'$, $M'_j \cap M'_{j'} = \emptyset$.

(2) A submersion $\pi : M \to B$ which restricts to a submersion $\pi : M' \to B$, where B is a compact connected complex manifold of dimension l'. The fibers Z of π are compact connected complex submanifolds of dimension l, and the fibers Y_j for M_j are compact connected submanifolds of dimension l_j. We let $Y = \cup_1^n Y_j$, and denote $i : Y \hookrightarrow Z$ the induced closed immersion.

3. A hermitian vector sheaf (η, g_η) on M' with a vector sheaf resolution of the coherent sheaf $i_*\eta$ by a chain complex of vector sheaves on M:

$$(\xi_., v_.) : 0 \to \xi_n \xrightarrow{\nu} \dots \xrightarrow{\nu} \xi_1 \xrightarrow{\nu} \xi_0 \to 0.$$

We also use the following notation: $T_{\mathbf{R}}$ is the real tangent bundle and T the $(1,0)$ part of the bundle $T_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. For $1 \leq j \leq n$, $N_{\mathbf{R},j}$ is the real normal bundle of M'_j in M, and N_j is the $(1,0)$ part of $N_{\mathbf{R},j} \otimes_{\mathbf{R}} \mathbf{C}$. We often write $N_{\mathbf{R}}$, N instead of $N_{\mathbf{R},j}$, N_j, etc.. For any vector bundle, the dual is denoted by a symbol *; for example, $T^*_{\mathbf{R}}$ is the dual of $T_{\mathbf{R}}$.

We have the following exact sequences

$$\begin{aligned} &0 \to TZ \to TM \to \pi^*TB \to 0,\\ &0 \to TY \to TM' \to \pi^*TB \to 0,\\ &0 \to TM' \to TM|_{M'} \to N \to 0,\\ &0 \to TY \to TZ|_{M'} \to N \to 0. \end{aligned}$$

Also there exists an exact sequence

$$0 \to \xi_m \xrightarrow{\nu} \dots \xrightarrow{\nu} \xi_1 \xrightarrow{\nu} \xi_0 \xrightarrow{r} i_*\eta \to 0.$$

Moreover, the complex $(\xi_\cdot, v_\cdot)$ is acyclic on $M - M'$.

I.7.1.b. The Homology of $(\mathcal{E}_\cdot, v_\cdot)|_{M'}$.

For $1 \le j \le n$, let $\pi_j : N_j \to M_j'$ be the natural projection. If E is a vector bundle on M_j', we will also denote by E the vector bundle $\pi_j^*(E)$ on the total space of N_j. Consider M_j' as the zero section of N_j, via k_j. Then on N_j, we have the following exact sequence:

$$0 \to \mathcal{O}_{N_j}(\wedge^{l-l_j}(N_j^*) \otimes \eta_j) \xrightarrow{i_y} \dots \xrightarrow{i_y} \mathcal{O}_{N_j}(\eta_j) \to k_{j*}\mathcal{O}_{M_j'}(\eta_j) \to 0$$

the **Koszul complex**, with i_y the interior multiplication for $y \in N_j$.

For any $x \in M_j'$, there exist holomorphic coordinates $(z^1, \dots, z^l)$ on an open neighborhood U of x in M such that $M_j' \cap U$ is represented by $z^1 = 0, \dots, z^{l-l_j} = 0$. On $M_j' \cap U$, N_j^* is spanned by the forms $dz^1, \dots, dz^{l-l_j}$, which extend to the whole open set U. Hence N_j^* on $M_j' \cap U$ extends into a holomorphic vector bundle on U, say $\tilde{N}_j^*$. In this way, $y = \sum_1^{l-l_j} z^k \frac{\partial}{\partial z^k}$ is a holomorphic section of $\tilde{N}_j$ (the dual of $\tilde{N}_j^*$ on U), which exactly vanishes on $M_j' \cap U$.

Now we choose U small enough, so that $\eta_j|_{M_j' \cap U}$ extends to a holomorphic vector bundle $\tilde{\eta}_j$ on U. Hence, we get a Koszul complex $(\wedge(\tilde{N}_j^*) \otimes \tilde{\eta}_j, i_y)$ on U. By the local uniqueness of resolutions, there exists a holomorphic acyclic chain complex (A, a) on U such that on U,

$$(\xi, \nu) \simeq (\wedge(\tilde{N}_j^*) \otimes \tilde{\eta}_j, i_y) \oplus (A, a).$$

Next, we use the above isomorphism to study the homology of the complex $(\xi, \nu)|_{M'}$: For $x \in M'$, denote by $F_{0,x}, \dots, F_{m,x}$ the homology groups of the chain complex $(\xi, \nu)_x$. Let $F_x = \oplus_0^m F_{k,x}$. For any holomorphic trivialization U of (ξ, ν), and any $X \in (T_{\mathbf{R}}M)_x$, we may define the derivative $\partial_X \nu(x)$ of the map $x \mapsto \nu(x)$ in the direction X. By the fact that $\nu^2 = 0$, we know that $\partial_X \nu$ acts naturally on the vector space F_x. This action does not depend on the local trivialization. (In fact, if $\partial_X' \nu(x)$ is the derivative of ν in the direction

X with respect to another holomorphic trivialization of ξ, there exists $A_x(X)$ acting linearly on $\xi_{0,x},\dots,\xi_{m,x}$ such that

$$\partial'_X\nu(x)=\partial_X\nu(x)+[A_x(X),\nu(x)].)$$

Obviously, $\partial_X\nu(x)$ decreases by 1 the grading in F_x. Since ν is a holomorphic section of $\mathrm{End}(\xi)$, if $X\in T_x^{(0,1)}M$, then $\partial_X\nu(x)=0$.

With the above notation, we have the following

Proposition. (1) The vector spaces $F_{0,x},\dots,F_{m,x}$ are the fibers of smooth vector bundles $F_0,\dots,F_m$ on M_j, which inherit a canonical holomorphic structure from the holomorphic vector bundle $\xi_0,\dots,\xi_m$.
(2) For any $x\in M_j'$, $X\in(T_{\mathbf{R}}M_j')_x$, $\partial_X\nu(x)=0$. Hence the linear map

$$y\in N_{j,x}\mapsto\partial_y\nu(x)\in\mathrm{End}(F_x)$$

is well defined, depends smoothly on x,y, and $(\partial_y\nu(x))^2=0$.
(3) On the normal bundle N_j, the complex

$$(F,\partial_y\nu):0\to F_m\xrightarrow{\partial_y\nu}F_{m-1}\xrightarrow{\partial_y\nu}\dots\xrightarrow{\partial_y\nu}F_0\to0$$

is a holomorphic $\mathbf{Z}$-graded chain complex, which is canonically isomorphic to the holomorphic $\mathbf{Z}$-graded Koszul complex $(\wedge(N_j^*)\otimes\eta_j,i_y)$.

Proof. (1) Since for any $x\in M_j'\cap U$, there is an isomorphism

$$F_{k,x}\simeq(\wedge^kN_j^*\otimes\eta_j)_x,$$

we know that $F_{k,x}$ has constant dimension on M_j. But the F_k's are the homology groups of the holomorphic chain complex $(\xi,\nu)|_{M_j'}$, so they are the fibers of the smooth holomorphic vector bundles on M_j'.

(2) By the fact that $\partial_X\nu$ corresponds to $\iota_{N(X)}$, where $N(X)$ is the component of X in N_j, we easily have 2.

(3) From above, for any $y\in N_j$, there exists an isomorphism

$$(F,\partial_y\nu)\simeq(\wedge N_j^*\otimes\eta_j,i_y)$$

on $M_j'\cap U$. Therefore, it is enough to prove that this isomorphism is canonical. This last statement is a direct consequence of the fact that the restriction map is canonical.

I.7.1.c. Bismut Condition (A)

We see from the above discussion that there is a natural algebraic isomorphism on N,

$$(F, \partial_y \nu) \simeq (\wedge(N)^* \otimes \eta, i_y).$$

On the other hand, if we put metrics on the vector bundles in question, there are induced metrics on each side of the above algebraic isomorphism. Now a natural question is: when is this algebraic isomorphism an isometry? It is to achieve this that Bismut condition (A) is needed.

Put hermitian metrics g_j on ξ_j for $j = 0, \ldots, m$. By the Hodge theorem in Chapter 3, we know that if ν^* is the adjoint of ν with respect to the metrics, we have

$$F_{k,x} \simeq F'_{k,x} =: \{f \in \xi_{k,x} : \nu(x)f = 0; \nu^*(x)f = 0\}.$$

Hence there is an induced metric on F_k from ξ_k. We denote this metric by g_{F_k}. We say that the hermitian metric g_k satisfies **Bismut condition (A)** with respect to the hermitian metrics g_N and g_η if the identifications of holomorphic chain complexes on N by

$$(F, \partial_y \nu) \simeq (\wedge(N^*) \otimes \eta, i_y)$$

also identifies the metrics above.

Proposition. Given hermitian metrics g_N and g_η on N and η, there exist hermitian metrics g_k on ξ_k which satisfy the Bismut condition (A) with respect to g_N and g_η.

Proof. We start with any hermitian metrics $g'_k|_{M'}$ on $\xi_k|_{M'}$. Let v^* denote the adjoint of v on M' with respect to these metrics. Therefore, for $0 \leq k \leq m$, by the Hodge theorem, there exists a decomposition

$$\xi_{k,x} = v(\xi_{k+1})_x \oplus v^*(\xi_{k-1})_x \oplus F'_{k,x},$$

and this splitting is orthogonal with respect to $g'_k|_{M'}$. Now we may modify these metrics: Obviously, there is a hermitian metric $g_k|_{M'}$ on $\xi_k|_{M'}$ so that

(1) The above splitting for ξ_k is still orthogonal;
(2) The restrictions of $g_k|_{M'}$ and $g'_k|_{M'}$ to $v(\xi_{k+1}) \oplus v^*(\xi_{k-1})$ coincide;
(3) When F'^k is equipped with the metric induced by $g'_k|_{M'}$, the canonical identification $F'^k \simeq \wedge^k(N^*) \otimes \eta$ also identifies the metrics.

Now by a partition of the unity, we may extend the metrics on ξ_k, which satisfy Bismut condition (A). This completes the proof.

Next, we give some consequences of Bismut assumption (A). This kind of result is usually proved by the so-called Mathai-Quillen's method, developed in the paper [MQ 86].

First we need some more notation. There exist hermitian metrics on the homology bundles $F_0, \ldots, F_m$. For any $y \in N$, we have $\bar{y} \in \bar{N}$ and $Y =: y + \bar{y} \in N_{\mathbf{R}}$. We know that

$\partial_{\bar{y}}\nu^*$ is the adjoint of $\partial_y\nu$. Hence $\partial_Y V =: \partial_y\nu + \partial_{\bar{y}}\nu^*$ is a self-adjoint operator acting on F. Let $\nabla^F =: \oplus_0^m \nabla^{F_k}$ be the canonical connection on F, then F lifts naturally to a vector bundle on N, which we also denote by F. Also

$$\mathbf{B} = \nabla^F + \partial_Y V$$

is a superconnection on the supervector bundle F on N. Let N_H be the number operator in $\mathrm{End}(\xi)$ which maps $f \in \xi_k$ to $kf \in \xi_k$.

Theorem. With the notation as above, if the metrics $g_0, \ldots, g_m$ on the vector bundles $\xi_0, \ldots, \xi_m$ satisfy the Bismut assumption (A) with respect to the metrics g^N and g_η on N and η, then we have the following equalities of differential forms on M':

$$\int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)] = (2\pi i)^{\dim N}\mathrm{td}^{-1}(-(\nabla^N)^2)\,\mathrm{Tr}_s[\exp(-(\nabla^\eta)^2)],$$

$$\int_N \mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)] = -(2\pi i)^{\dim N}(\mathrm{td}^{-1})'(-(\nabla^N)^2)\,\mathrm{Tr}_s[\exp(-(\nabla^\eta)^2)].$$

Proof. Under Bismut assumption (A), we have the identity of holomorphic hermitian chain complexes on N, $(F, \partial_y\nu) \simeq (\wedge N^* \otimes \eta, i_y)$. In particular $\partial_y\nu^* = \bar{y}\wedge$, and

$$\begin{aligned}&\int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)] \\ &= \mathrm{Tr}[\exp(-(\nabla^N)^2)] \int_N \mathrm{Tr}_s[\exp(-(\nabla^{\wedge N^*} + i_y + \bar{y}\wedge)^2)].\end{aligned}$$

We identify $\bar{N}$ with N^* by the metric of N. The algebra $\wedge(N^*)$ is an $N_{\mathbf{R}}$-Clifford module. Namely, if $U \in N$, $V \in \bar{N}$, set

$$c(U) = -i\sqrt{2}\iota_U, \; c(V) = -i\sqrt{2}V\wedge.$$

So if $Y = y + \bar{y}$, then

$$\partial_y\nu + \partial_{\bar{y}}\nu^* = \frac{ic(Y)}{\sqrt{2}}.$$

The connection ∇^N splits the tangent bundle $T_{\mathbf{R}}N$ into a horizontal part, and into a vertical part which may be identified with $N_{\mathbf{R}}$. If $Y \in N_{\mathbf{R}}$, $X \in TN_{\mathbf{R}}$, let $D_X Y$ be the vertical component of X. If (f_α) is a basis of $T_{\mathbf{R}}N$ with dual basis (f^α) in $T^*_{\mathbf{R}}N$, we let

$$c(DY) = -\sum_\alpha f^\alpha c(D_{f_\alpha}Y).$$

Then we have

$$(\nabla^F + \frac{ic(Y)}{\sqrt{2}})^2 = (\nabla^F)^2 - \frac{ic(DY)}{\sqrt{2}} + \frac{|Y|^2}{2}.$$

Under Bismut assumption (A), $(\nabla^F)^2$ is the action on $\wedge(N^*)$ of the curvature tensor $(\nabla^N)^2$ of the connection ∇^N on N. Furthermore, if $e_1, \ldots, e_{\dim N}$ is a complex orthonormal basis of N, let $\bar{e}_1, \ldots, \bar{e}_{\dim N}$ be the conjugate base of $\bar{N}$, then

$$(\nabla^F)^2 = \frac{1}{2} < (\nabla^N)^2 e_i, \bar{e}_j > c(\bar{e}_i)c(e_j).$$

As in [MQ 86], we may replace $(\nabla^N)^2$ by a skew-adjoint endomorphism A of N with A invertible. (Later, by the continuity, we can drop the condition that A is invertible.) The co-action A' induced by A on $\wedge(N^*)$ is given by the analogue of above relation:

$$A' := \frac{1}{2} < Ae_i, \bar{e}_j > c(\bar{e}_i)c(e_j).$$

Observe that if $X \in N_{\mathbf{R}}$,

$$[c(A^{-1}DY), c(X)] = 2 < A^{-1}DY, X > \in T^*_{\mathbf{R}}N.$$

Then, by the above expression for $\partial_y \nu + \partial_{\bar{y}} \nu^*$, we have

$$[c(A^{-1}DY), [c(A^{-1}DY), c(X)]] = 0,$$

and so

$$\exp(\frac{ic(A^{-1}DY)}{\sqrt{2}})\, c(X) \exp(-\frac{ic(A^{-1}DY)}{\sqrt{2}}) = c(X) + \frac{2i}{\sqrt{2}} < A^{-1}DY, X > .$$

Therefore

$$A' - \frac{ic(DY)}{\sqrt{2}} = \exp(\frac{ic(A^{-1}DY)}{\sqrt{2}})\, A' \exp(-\frac{ic(A^{-1}DY)}{\sqrt{2}}) - \frac{1}{2} < A^{-1}DY, DY > .$$

Since the supertrace vanishes on supercommutators, we have

$$\mathrm{Tr}_s[\exp(-A' + \frac{ic(DY)}{\sqrt{2}})] = \mathrm{Tr}_s[\exp(-A')] \exp(\frac{1}{2} < A^{-1}DY, DY >).$$

But classically

$$\mathrm{Tr}_s[\exp(-A')] = \det(I - \exp(A)).$$

Hence, we get

$$\begin{aligned}&\int_N \mathrm{Tr}_s[\exp(-A' + \frac{ic(DY)}{\sqrt{2}} - \frac{|Y|^2}{2})]\\ =&\det(I - \exp(A)) \int_N \exp(-\frac{|Y|^2}{2} - \frac{1}{2} < A^{-1}DY, DY >).\end{aligned}$$

The Pfaffian of $-A^{-1}$, with the canonical orientation of $N_{\mathbf{R}}$, is given by $\frac{i^{\dim N}}{\det(-A)}$, so that

$$\int_N \mathrm{Tr}_s[\exp(-A' + \frac{ic(DY)}{\sqrt{2}} - \frac{|Y|^2}{2})] = (2\pi i)^{\dim N}\mathrm{td}^{-1}(-A),$$

which completes the proof of the first relation.

After the above discussion, we easily have the second relation. A local calculation gives

$$N_H = -\frac{1}{2} < I_N e_i, \bar{e}_j > c(\bar{e}_i)c(e_j).$$

On the other hand, we have

$$\mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)] = \frac{\partial}{\partial b}\mathrm{Tr}_s[\exp(-((\nabla^F)^2 - bN_H - \frac{ic(DY)}{\sqrt{2}} + \frac{|Y|^2}{2}))]_{b=0}.$$

By integration over N, with a similar discussion as above, we get

$$\begin{aligned}&\int_N \mathrm{Tr}_s[\exp(-((\nabla^F)^2 - bN_H - \frac{ic(DY)}{\sqrt{2}} + \frac{|Y|^2}{2}))]\\ =&(2\pi i)^{\dim N}\mathrm{td}^{-1}(-R^N - bI_N)\mathrm{Tr}[\exp(-(\nabla^\eta)^2)].\end{aligned}$$

So, we have the second assertion.

I.7.1.d. Wave Front Sets

Let γ be a current on M and denote by $\mathrm{WF}(\gamma)$ the wave front set of γ. For the definition and the basic properties of wave front sets, see [Hö 83]. We know that $\mathrm{WF}(\gamma)$ is a closed conic subset of $T^*_{\mathbf{R}}M - \{0\}$. If $p : T^*_{\mathbf{R}}M \to M$ is the natural projection, $p(\mathrm{WF}(\gamma))$ is exactly the singular support of γ, whose complement in M is the set of points x such that γ is C^∞ on a neighborhood of x. We let

$$\mathcal{D}_{\mathcal{N}^*_{\mathbf{R}}} =: \{\gamma \in \mathcal{D} : \mathrm{WF}(\gamma) \subset \mathcal{N}^*_{\mathbf{R}}\},$$

then the elements in $\mathcal{D}_{\mathcal{N}^*_{\mathbf{R}}}$ are smooth on $M - M'$. Furthermore, there exists a natural topology on $\mathcal{D}_{\mathcal{N}^*_{\mathbf{R}}}$ given as follows:

> Let U be a small open set in M, which we identify with an open ball in $\mathbf{R}^{2l}$. Over U, we identify $T^*_{\mathbf{R}}M$ with $U \times \mathbf{R}^{2l}$. Let Γ be a closed conic set in $\mathbf{R}^{2l}$ such that if $x \in U$, $\Gamma \cap N^*_{\mathbf{R}x} = \emptyset$. Let φ be a smooth current on $\mathbf{R}^{2l}$ with a compact support included in U and let m be an integer. If γ is a current, denote by $\hat{\varphi\gamma}$ the Fourier transform of $\varphi\gamma$ (which is considered as a current on $\mathbf{R}^{2l}$). For any $\gamma \in \mathcal{D}_{\mathcal{N}^*_{\mathbf{R}}}$, let
>
> $$p_{\nu,\Gamma,\varphi,m}(\gamma) =: \sup_{\xi\in\Gamma}|\xi|^m|\hat{\varphi\gamma}(\xi)|.$$

We say that a sequence of currents $\{\gamma_n\}\in\mathcal{D}_{\mathcal{N}_{\mathbb{R}}^\bullet}$ converges to $\gamma\in\mathcal{D}_{\mathcal{N}_{\mathbb{R}}^\bullet}$ if

1. $\lim_{n\to\infty}\gamma_n=\gamma$ in the sense of distributions; and
2. $\lim_{n\to\infty}p_{u,\Gamma,\varphi,m}(\gamma_n-\gamma)=0$.

As in the case of differential forms with P and P', we also let

$$P^M_{M'}:=\{\omega\in\mathcal{D}_{\mathcal{N}_{\mathbb{R}}^\bullet}:\omega \text{ is a sum of currents of type (p, p)}\}.$$

And

$$P^{M,0}_{M'}:=\{\omega\in P^M_{M'}:\omega=\partial\alpha+\bar\partial\beta \text{ with } \alpha,\beta\in\mathcal{D}_{\mathcal{N}_{\mathbb{R}}^\bullet}\}.$$

If $M'=\emptyset$, we write P^M, $P^{M,0}$ instead of $P^M_{M'}$, $P^{M,0}_{M'}$ respectively.

§I.7.2 Axioms For Relative Bott-Chern Secondary Characteristic Currents With Respect To Closed Immersions

Suppose we have the following data: A closed immersion $i:M'\hookrightarrow M$ with hermitian metrics on N, T_M and $T_{M'}$. Let (η,g_η) be a hermitian vector sheaf on M' with a vector sheaf resolution of $i_*\eta$ on M:

$$0\to\mathcal{E}_m\xrightarrow{\nu}\dots\xrightarrow{\nu}\mathcal{E}_1\xrightarrow{\nu}\mathcal{E}_0\xrightarrow{r}i_*\eta\to 0.$$

Suppose that ρ_k are hermitian metrics on $\mathcal{E}_k$ which satisfy Bismut condition (A) with respect to g_N, g_η. In the same spirit as in Chapter 2, we introduce axioms for the relative Bott-Chern secondary characteristic current with respect to closed immersions, $\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;i,\rho_i)$, on M, as follows:

Axiom 1. (Downstairs Rule) In there is an element $\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;i,\rho_i)\in P^M_{M'}/P^{M,0}_{M'}$, such that

$$dd^c\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;i,\rho_i)=\mathrm{td}^{-1}(\mathcal{N},g_{\mathcal{N}})\mathrm{ch}(\eta,g_\eta)\delta_M-\mathrm{ch}(\mathcal{E}_\cdot,\rho_\cdot).$$

Axiom 2. (Base Change Rule) Let $f:\tilde M\to M$ be a holomorphic morphism. Assume that f is transversal to M', i.e. for any $x\in f^{-1}(M')$,

$$\mathrm{Im}\,df(x)+T_{f(x)}M'=T_xM.$$

Then we have

$$\mathrm{ch}_{\mathrm{BC}}(f^*\eta,f^*g_\eta;i_f,\rho_{i_f})=f^*\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;i,\rho_i).$$

Axiom 3. (Triangle Rule For Hermitian Vector Sheaves) Let

$$0\to\eta_1\to\eta_2\to\eta_3\to 0$$

be a short exact sequences of vector sheaves on M'. Then we may find the resolution $\mathcal{E}_{k\cdot}$ for $i_*\eta_k$ in the above sense, with the condition that, for all j,

$$0 \to \mathcal{E}_{1,j} \to \mathcal{E}_{2,j} \to \mathcal{E}_{3,j} \to 0$$

is a short exact sequence. Put metrics satisfying Bismut condition (A) on them. Then we have

$$\begin{aligned}&\sum_{k=1}^{3}(-1)^k \mathrm{ch}_{\mathrm{BC}}(\eta_k, g_{\eta_k}; i, \rho_i)\\ &\qquad = i_*(\mathrm{td}^{-1}(\mathcal{N}, \rho_{\mathcal{N}})\mathrm{ch}_{\mathrm{BC}}(\eta_\cdot, g_{\eta_\cdot})) - \sum_{j=0}^{m}(-1)^j \mathrm{ch}_{\mathrm{BC}}(\mathcal{E}_{\cdot,j}, \rho_{\cdot,j})\end{aligned}$$

in $P_{M'}^{M}/P_{M'}^{M,0}$.

Axiom 4. (Triangle Rule For Closed Immersions) Let $i' : M'' \hookrightarrow M$ be another closed immersion such that M' and M'' intersect transversely, i.e. if $x \in M' \cap M''$, then

$$T_xM' + T_xM'' = T_xM.$$

Let $i'' : M' \cap M'' \hookrightarrow M$ be the induced closed immersion. For any vector sheaves η (resp. η') on M' (resp. M''), let $\eta'' =: \eta'|_{M'} \otimes \eta'|_{M''}$. Then, in $P_{M'\cup M''}^{M}/P_{M'\cup M''}^{M}$, we have

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\eta'', g_{\eta''}; i'', \rho_{i''})\\ &\quad = \mathrm{ch}(\mathcal{E}'_\cdot, \rho'_\cdot)\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; i, \rho_i) + i_*(\mathrm{td}^{-1}(\mathcal{N}, g_{\mathcal{N}})\mathrm{ch}(\eta, g_\eta)\, i^*\mathrm{ch}_{\mathrm{BC}}(\eta', g_{\eta'}; i', \rho_{i'}))\end{aligned}$$

and

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\eta'', g_{\eta''}; i'', \rho_{i''})\\ &\quad = \mathrm{ch}(\mathcal{E}_\cdot, \rho_\cdot)\,\mathrm{ch}_{\mathrm{BC}}(\eta', g_{\eta'}; i', \rho_{i'}) + i'_*(\mathrm{td}^{-1}(\mathcal{N}', g_{\mathcal{N}'})\,\mathrm{ch}(\eta', g_{\eta'})\, i^*\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; i, \rho_i))\end{aligned}$$

§I.7.3 Existence Theorem For Relative Bott-Chern Secondary Characteristic Currents With Respect To Closed Immersions

We have the following

Existence Theorem Of Relative Bott-Chern Secondary Characteristic Currents With Respect To Closed Immersions.
With the same notation as above, let $i : M' \to M$ be a closed immersion of compact Kähler manifolds. Let (η, g_η) be a hermitian vector sheaf on X. Then there exists a unique current in $P_{M'}^{M}$, $\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; i, \rho_i)$, such that the axioms stated in the previous subsection hold.

We will prove this theorem in the next chapter, which are taking from [BGS 91]. As in the case of smooth morphisms, this theorem is proved by the following steps:

(1) The introduction of a family of superconnections with a parameter $t > 0$.
(2) Investigation of the convergence of the associated heat kernels when $t \to 0^+$, and $t \to +\infty$.
(3) The use of the Mellin transform to construct the relative Bott-Chern secondary characteristic currents.
(4) Proof of the axioms step by step.

It is clear that in this process, the number operator plays a very important role.

Chapter I.8
Relative Bott-Chern Secondary Characteristic Currents With Respect To Closed Immersions II: Existence

We now give the proof of the existence theorem for relative Bott-Chern secondary characteristic currents with respect to closed immersions as stated in the last chapter. The basic ideal for doing so is as follows: First, we need to find a family of superconnections with one parameter $t > 0$; then investigate the convergence of the associated heat kernels when $t \to 0^+$ and $t \to \infty$; thus with the right convergence, we may construct the relative Bott-Chern secondary characteristic currents by using the Mellin transform; finally, we check the axioms case by case.

We study the above aspects in different sections of this chapter. The basic strategy is that we first discuss the general formula without Bismut assumption (A), and see how far we can go. Then we adopt Bismut assumption (A) so as to deduce the final axioms. Essentially, this chapter comes from [B 90], [BGS 90], [BGS 91] and [We 91].

§I.8.1 Convergence Of Heat Kernels Associated To Certain Superconnections

We use the same notation as in the previous chapter: $i : M' \hookrightarrow M$ denotes a closed immersion of complex manifolds, etc.. Let $\nabla^\xi =: \oplus_0^m \nabla^{\xi_k}$ be the canonical connection on $\xi =: \oplus_0^m \xi_k$. Set $V = \nu + \nu^*$. For $u \geq 0$, let $\mathbf{A}_u = \nabla^\xi + \sqrt{u}V$ be a superconnection on the supervector bundle ξ, with $\mathbf{A} := \mathbf{A}_1$. There are natural hermitian metrics on the homology bundles $F_0, \ldots, F_m$. For any $y \in N$, we have $\bar{y} \in \bar{N}$ and $Y =: y + \bar{y} \in N_{\mathbf{R}}$. Since $\partial_{\bar{y}}\nu^*$ is the adjoint of $\partial_y \nu$, hence $\partial_Y V =: \partial_y \nu + \partial_{\bar{y}}\nu^*$ is a self-adjoint operator acting on F. Let $\nabla^F =: \oplus_0^m \nabla^{F_k}$ be the canonical connection on F. F lifts naturally to a vector bundle on N, which we also denote by F. In particular,

$$\mathbf{B} = \nabla^F + \partial_Y V$$

is a superconnection on the supervector bundle F on N.

Using the above superconnections and the result of Chapter 3 and Chapter 5 about the existence of heat kernels for generalized Laplacians and their beyond, we know that there exist heat kernels for $\mathbf{A}_u^2$ and $\mathbf{B}^2$. Hence it makes sense to talk about $\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]$ and $\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]$. Our first result for convergence is the following micro-local estimations.

Theorem. (1) As $u \to \infty$, in $P^M_{M'}$, we have the following convergence of currents on M

$$\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] \to [\int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)]]\delta_{M'};$$

(2) For any natural number k, there exists a constant $C_k > 0$ such that for any smooth differential form μ on M and $u \geq 1$,

$$\| \int_M \mu\{\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] - [\int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)]]\delta_{M'}\}\|_{C^k(B)} \leq \frac{C_k}{\sqrt{u}}\|\mu\|_{C^{k+1}(M)}.$$

(3) If U, Γ, φ, m are taken as 7.1.d, there exists a constant $C > 0$ such that for $u \geq 1$,

$$p_{U,\Gamma,\varphi,m}(\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] - [\int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)]\delta_{M'}) \leq \frac{C}{\sqrt{u}}.$$

Proof. By definition, (1) is a direct consequence of (2) and (3).

(2) We first consider the case with $k = 0$. Later we will see that the proof for this special case is also valid in general.

Assume first that the compact support of μ is inclused in $M - M'$. Since the linear self-adjoint map $V \in \mathrm{End}(\xi)$ is invertible on $M - M'$, by Duhamel's formula, we know that

$$\int_Z \mu \mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] \to 0$$

uniformly together with its derivatives.

Take now $x_0 \in M'$, let $z = (z^1, \ldots, z^{l+l'})$ be holomorphic coordinates on an open neighborhood U of x_0 in M, such that locally, M' is the vector subspace $(z^1, \ldots, z^e) = 0$. There exists an open neighborhood $\mathcal{V}$ of x_0 in M' and $\varepsilon > 0$ such that if D_ε is the open ball, center 0 and radius ε in $\mathbf{C}^e$, then $U = \mathcal{V} \times D_\varepsilon$. We then need to prove the statement for the case that μ has compact support included in U.

Let $\sigma_u : \mathcal{V} \times D_{\varepsilon\sqrt{u}} \to \mathcal{V} \times D_\varepsilon$ be defined by $\sigma_u(x, y) := (x, \frac{y}{\sqrt{u}})$. By a direct calculation, we have

$$\int_Z \mu \mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] = \int_{(Y \cap \mathcal{V}) \times D_{\varepsilon\sqrt{u}}} (\sigma_u^* \mu)(\sigma_u^* \mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]).$$

In this way, we reduce the problem to a local discussion. Obviously, the vector bundle $\sigma_u^*\xi$ on $\mathcal{V} \times D_{\varepsilon\sqrt{u}}$ is equipped with the metric $\sigma_u^* g_\xi$, and the connection $\sigma_u^* \nabla^\xi =: \nabla_u^\xi$. Thus, by $\sigma_u^* V = V(x, \frac{y}{\sqrt{u}})$, we have

$$\int_Z \mu \mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] = \int_{(Y \cap \mathcal{V}) \times D_{\varepsilon\sqrt{u}}} (\sigma_u^* \mu)\, \mathrm{Tr}_s[\exp(-(\nabla_u^\xi + \sqrt{u} V(x, \frac{y}{\sqrt{u}}))^2].$$

But for $u \geq \varepsilon^{-2}$, we have

$$\int_{(Y\cap\mathcal{V})\times D_{\epsilon\sqrt{u}}}(\sigma_u^*\mu)\mathrm{Tr}_s[\exp(-(\nabla_u^\xi+\sqrt{u}V(x,\frac{y}{\sqrt{u}})^2)]$$
$$=\int_{(Y\cap\mathcal{V})\times D_1}(\sigma_u^*\mu)\mathrm{Tr}_s[\exp(-(\nabla_u^\xi+\sqrt{u}V(x,\frac{y}{\sqrt{u}})^2)]$$
$$+\int_{(Y\cap\mathcal{V}\times(D_{\epsilon\sqrt{u}}-D_1)}(\sigma_u^*\mu)\mathrm{Tr}_s[\exp(-(\nabla_u^\xi+\sqrt{u}V(x,\frac{y}{\sqrt{u}})^2)].$$

Now we let P be the projection from ξ to F defined locally by the orthogonal projection operation ξ_x to F_x for any $x\in M'$. Hence for any smooth section f of F on M', we have

$$\nabla^F f = P\nabla^\xi f.$$

The first result in the theorem is then a direct consequence of the following

Lemma. For $u\to+\infty$, we have the following estimates:
(1)

$$|\int_{\mathcal{V}\times D_1}(\sigma_u^*\mu)\mathrm{Tr}_s[\exp(-(\nabla_u^\xi+\sqrt{u}V(x,\frac{y}{\sqrt{u}})^2)]$$
$$-\int_{\mathcal{V}}(i^*\mu)\int_{|y|\leq 1}\mathrm{Tr}_s[\exp(-(\nabla^F+P\nabla_Y^\xi VP)^2)]|\leq\frac{C}{\sqrt{u}}\|\mu\|_{C^1(M)};$$

(2)

$$|\int_{\mathcal{V}\times D_{\epsilon\sqrt{u}}-D_1}(\sigma_u^*\mu)\mathrm{Tr}_s[\exp(-(\nabla_u^\xi+\sqrt{u}V(x,\frac{y}{\sqrt{u}})^2)]$$
$$-\int_{\mathcal{V}}(i^*\mu)\int_{1\leq|y|\leq\epsilon\sqrt{u}}\mathrm{Tr}_s[\exp(-(\nabla^F+P\nabla_Y^\xi VP)^2)]|\leq\frac{C}{\sqrt{u}}\|\mu\|_{C^1(M)};$$

(3)

$$|\int_{\mathcal{V}}(i^*\mu)\int_{\epsilon\sqrt{u}\leq|y|<+\infty}\mathrm{Tr}_s[\exp(-(\nabla^F+P\nabla_Y^\xi VP)^2)]|\leq\frac{C}{\sqrt{u}}\|\mu\|_{C^1(M)}.$$

Proof of the lemma. First, we give the proof of (3). By Duhamel's formula, it is sufficient to prove the following

Sublemma 1. There exists a constant $c>0$ such that for any $x\in M'$, $Y\in N_{\mathbf{R},x}$, $f\in F_x$,

$$|\partial_Y V(x)f|^2\geq c|Y|^2_{N_{\mathbf{R}}}|f|^2_F.$$

In particular, on N, $\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]$ decays faster than $\exp(-C|y|^2_N)$ when $|y|_N\to\infty$.

Proof of Sublemma 1. We only need to prove the assertion for $|Y|=1$. In this case, the complex $(F,\partial_y\nu)\simeq(\wedge N^*\otimes\eta,i_y)$ is acyclic. Thus by Hodge theory, $\partial_Y V$ is a self-adjoint invertible operator. Hence by the compactness of the sphere bundle $S_{N_{\mathbf{R}}}$ on $N_{\mathbf{R}}$, we deduces that there exists $c > 0$ which is a lower bound for the smallest eigenvalues of $(\partial_Y V)^2_{Y \in S_{N_{\mathbf{R}}}}$. So we have Sublemma 1.

Next we prove (1) and (2) in the lemma. For doing so, we need some more notation.

By Prop. 7.1.b, since M' is compact, we know that there exists a constant $b > 0$ such that if $x \in M'$, $V^2(x)$ has no eigenvalue in the interval $]0, 2b]$. Therefore, if $\varepsilon > 0$ is small enough, $x \in \mathcal{V}$ and $|y| \leq \varepsilon$, we may choose $b > 0$ so that b is not an eigenvalue of the operator $V^2(x,y)$. Hence if for $0 \leq k \leq m$, we let $\xi^+_{k,(x,y)}$ (resp. $\xi^-_{k,(x,y)}$) be the direct sum of the eigenspaces of the restriction of $V^2(x,y)$ to $\xi_{k,(x,y)}$ corresponding to eigenvalues which are strictly larger (resp. smaller) than b, then

$$\xi^{\pm,+}_{\cdot,(x,y)} := \oplus_{k \text{ even}} \xi^{\pm}_{k,(x,y)}, \quad \xi^{\pm,-}_{\cdot,(x,y)} := \oplus_{k \text{ odd}} \xi^{\pm}_{k,(x,y)}, \quad \xi^{\pm}_{(x,y)} := \xi^{\pm,+}_{\cdot,(x,y)} \oplus \xi^{\pm,-}_{\cdot,(x,y)}$$

are smooth vector bundles on $\mathcal{V} \times D_\varepsilon$ with

$$\xi_k = \xi_k^+ \oplus \xi_k^-, \quad \xi^{\pm} = \xi^{\pm,+} \oplus \xi^{\pm,-}.$$

Furthermore, we know that as $\mathbf{Z}$-graded vector bundles, $\xi^-|_{\mathcal{V}} = F$. Hence $\xi^+|_{\mathcal{V}}$ is exactly the subbundle $F^\perp$ of $\xi|_{\mathcal{V}}$ orthogonal to F. Let $P^{\pm}$ be the orthogonal projection operator from ξ to $\xi^{\pm}$. Obviously on $\mathcal{V} \simeq \mathcal{V} \times \{0\}$, P^- is just P defined before the lemma. Thus if we let I_ξ, I_F be the identify maps on ξ and F, the orthogonal projection operation Q from $\xi|_{M'}$ to $F^\perp$ is given by $Q = I_\xi - P$. And on $\mathcal{V}$, $Q = P^+$.

Let

$$\nabla^+ := P^+ \nabla^\xi, \quad \nabla^- = P^- \nabla^\xi, \quad \nabla^\oplus := \nabla^+ \oplus \nabla^-,$$

which are connections on ξ^+, ξ^- and ξ respectively. As in the smooth morphism case, we consider the difference of these two connections, and denote it as S, i.e., $S := \nabla^\xi - \nabla^\oplus$. If we identify the fiber $\xi_{(x,y)}$ with the fiber $\xi_{(x,0)} = \xi_x$ by the parallel transport along the line $s \in [0,1] \mapsto x + sy$ with respect to the unitary connection $\nabla^\oplus$, the linear map $V(x,y)$ acts as a self-adjoint operator on the fiber ξ_x and preserves the splitting $\xi_x = \xi_x^+ \oplus \xi_x^-$. In this way, we introduce the operators $V^{\pm}$ as the restrictions of V on $\xi^{\pm}$.

Since a 1-form on $\mathcal{V} \times D_\varepsilon$ is the sum of a 1-form on $\mathcal{V}$ and a 1-form on D_ε, we may denote by H the set of one forms of the first kind, i.e. on V, and by $H^\perp$ the 1-forms of the second kind, i.e. on D_ε. Let $(z^1, \ldots, z^{l+l'})$ be holomorphic coordinates on $\mathcal{V}$. Then we define

$$\nabla^\xi_H V(x,y) = \sum (dz^\alpha \nabla^\xi_{\frac{\partial}{\partial z^\alpha}} V(x,y) + d\bar{z}^\alpha \nabla^\xi_{\frac{\partial}{\partial \bar{z}^\alpha}} V(x,y)).$$

Using the identification $\xi_{(x,y)} \simeq \xi_x$, we find that $\nabla^\xi_H V(x,y)$ lies in $\wedge^1(T^*_{\mathbf{R}} M' \hat{\otimes} \mathrm{End}\xi)_x$. Hence if we identify $Y \in \mathbf{R}^{2e_j}$ with the vector field $(x,Y) \to (0,Y)$ on $\mathcal{V} \times \mathbf{R}^{2e_j}$, where we let $e_j := l - l_j$, then $[Y, \frac{\partial}{\partial z^\alpha}] = 0$, $[Y, \frac{\partial}{\partial \bar{z}^\alpha}] = 0$ for any α. In particular, $\nabla^F_H(P \nabla^\xi_Y V(x) P)$ is a well-defined 1-form on $\mathcal{V}$ which takes values in $\mathrm{End}(F)$. Now we need the following technical

Sublemma 2. (a) If $x \in M'$, $X \in (T_{\mathbf{R}}M')_x$, then $\nabla^\xi_X V(x)$ maps F into $F^\perp$. With respect to the splitting $\xi|_{M'} = F \oplus F^\perp$, if $x \in \mathcal{V}$, $U \in (T_{\mathbf{R}}M)_x$, we have

$$S_x(U) = \begin{pmatrix} 0 & P(\nabla^\xi_U V)Q(V^+)^{-1} \\ -(V^+)^{-1}Q(\nabla^\xi_U V)P & 0 \end{pmatrix}.$$

In particular,

$$(\nabla^F)^2 = P(\nabla^\xi)^2 P - P(\nabla^\xi V(V^+)^{-2}\nabla^\xi V)P.$$

(b) For any $x \in \mathcal{V}$, $Y \in \mathbf{R}^{2e_j}$,

$$P\nabla^\oplus_Y(\nabla^\xi_H V)(x)P = \nabla^F_H(P\nabla^\xi_Y V(x)P).$$

Suppose that Sublemma 2 is proved, we also need the following

Sublemma 3. (a) If $\mathcal{V}$ and $\varepsilon > 0$ are small enough, there exists a constant $C > 0$ such that if $(x, y) \in \mathcal{V} \times D_\varepsilon$, then

$$V(x,y)^2 \geq C|y|^2.$$

In particular, for $\lambda \in \Gamma, x \in \mathcal{V}, y \in \mathbf{C}^e$ with $y \neq 0, |y| \leq \varepsilon\sqrt{u}$, $\lambda I_\xi - \sqrt{u}V(x, \frac{y}{\sqrt{u}}) \in$ Endξ_x is invertible.
(b) If $x \in \mathcal{V}, y \in N_x, Y = y + \bar{y}$, we have

$$|P\nabla^\xi_Y V(x)P| \geq C_0|y|.$$

In particular, for $\lambda \in \Gamma, x \in \mathcal{V}$, $\lambda I_{F_x} - P\nabla^\xi_Y V(x)P \in \mathrm{End}\, F_x$ is invertible.
(c) For $\lambda \in \Gamma, x \in \mathcal{V}, y \in \mathbf{C}^e$ with $y \neq 0, |y| \leq \varepsilon\sqrt{u}$, let $Y = y + \bar{y}$ and define $A(u, x, y, \lambda) \in \mathrm{End}(\xi_x)$ by

$$\begin{aligned}(\lambda I_\xi - \sqrt{u}V(x, \frac{y}{\sqrt{u}}))^{-1} =& P(\lambda I_F - P\nabla^\xi_Y V(x)P)^{-1}P \\ &+ \frac{1}{\sqrt{u}}\{\frac{1}{2}P(\lambda I_F - P\nabla^\xi_Y V(x)P)^{-1}P\nabla^\oplus_Y\nabla^\oplus_Y V(x)P \\ &(\lambda I_F - P\nabla^\xi_Y V(x)P)^{-1}P - Q(V^+)^{-1}(x)Q\} \\ &+ A(u, x, y, \lambda).\end{aligned}$$

Then for $\varepsilon > 0$ small enough, there exists a constant $C > 0$ such that

$$||A(u, x, y, \lambda)|| \leq \frac{C}{u}(|y| + |y|^4 + |\lambda| + |\lambda|^3).$$

Suppose we have the above two sublemmas, for the first estimation in the lemma, we then need the following

Proposition. For any $\lambda \in \mathbf{C}$ with $|\mathrm{Im}\lambda| = 1$, let $I(u, \lambda)$ be

$$\{(I_\xi - (\lambda^2 I_\xi - uV^2(x, \frac{y}{\sqrt{u}})))^{-1}((\nabla^\xi_u)^2 + \sqrt{u}\nabla^\xi_u V(x, \frac{y}{\sqrt{u}}))\}^{-1}(\lambda I_\xi - \sqrt{u}V(x, \frac{y}{\sqrt{u}}))^{-1}.$$

Then for $u \to +\infty$, $I(u,\lambda) \to I(\infty,\lambda)$, where

$$\begin{aligned} I(\infty,\lambda) :=& P\{I_F - (\lambda^2 I_F - (P\nabla^\xi_Y VP)^2)^{-1}((\nabla_F)^2 + \nabla^F P\nabla^\xi_Y VP))\}^{-1} \\ & ((\lambda I_F - P\nabla^\xi_Y VP)^{-1} - (\lambda^2 I_F - (P\nabla^\xi_Y VP)^2)(P\nabla^\xi_H V(V^+)^{-1}Q). \end{aligned}$$

Proof of the proposition. Let

$$\begin{aligned} & J(\lambda) \\ =:& P\{I_F - (\lambda^2 I_F - (P\nabla^\xi_Y VP)^2)^{-1}((\nabla_F)^2 + \nabla^F P\nabla^\xi_Y VP))\}^{-1}(\lambda I_F - P\nabla^\xi_Y VP)^{-1}, \end{aligned}$$

then it easily follows that

$$\frac{1}{2\pi i}\int_\Gamma \exp(-\lambda^2) I(\infty,\lambda)d\lambda = \frac{1}{2\pi i}\int_\Gamma \exp(-\lambda^2)J(\lambda)d\lambda,$$

where Γ is the oriented contour in $\mathbf{C}$ defined by $|\mathrm{Im}\lambda| = 1$ and taken clockwise, since the second term in $I(\infty,\lambda)$ is an even function of λ. On the other hand, we may also easily have

$$\exp(-(\nabla^\xi + \sqrt{u}V)^2) = \frac{1}{2\pi i}\int_\Gamma \frac{\exp(-\lambda^2)}{\lambda I_\xi - \nabla^\xi - \sqrt{u}V}d\lambda.$$

A priori, the right hand side of this equality is a differential operator of degree 1, while the left hand side is of degree 0: There is some cancellation here, which may be explained as follows: Since $[\nabla^\xi, V]$ belongs to $\wedge^1(T^*_{\mathbf{R}}M)\hat{\otimes}\mathrm{End}\xi$, $(\lambda I_\xi - \nabla^\xi - \sqrt{u}V)^{-1}$ can be expressed as the sum of two differential operators of degrees 0 and 1 respective:

$$\begin{aligned} & (\lambda I_\xi - \nabla^\xi - \sqrt{u}V)^{-1} \\ =& \{I_\xi - (\lambda^2 I_\xi - uV^2)^{-1}((\nabla^\xi)^2 + \sqrt{u}[\nabla^\xi, V])\}^{-1}((\lambda I_\xi - \sqrt{u}V)^{-1} + (\lambda^2 I_\xi - uV^2)^{-1}\nabla_\xi). \end{aligned}$$

By the fact that the λ-function

$$\exp(-\lambda^2)\{I_\xi - (\lambda^2 I_\xi - uV^2)^{-1}((\nabla^\xi)^2 + \sqrt{u}[\nabla^\xi, V])\}^{-1}(\lambda^2 I_\xi - uV^2)^{-1}$$

is even, we know that its integral on Γ vanishes. From this,

$$\begin{aligned} & \exp(-(\nabla^\xi + \sqrt{u}V)^2) \\ =& \frac{1}{2\pi i}\int_\Gamma \exp(-\lambda^2)\{I_\xi - (\lambda^2 I_\xi - uV^2)^{-1}((\nabla^\xi)^2 + \sqrt{u}[\nabla^\xi, V])\}^{-1}(\lambda I_\xi - \sqrt{u}V)^{-1}d\lambda. \end{aligned}$$

Hence, we get

$$\exp(-(\nabla^\xi + \sqrt{u}V(x, \frac{y}{\sqrt{u}}))^2) = \frac{1}{2\pi i}\int_\Gamma \exp(-\lambda^2)I(u,\lambda)d\lambda.$$

Similarly, we have

$$\exp(-(\nabla_F + P\nabla_Y^\xi VP)^2) = \frac{1}{2\pi i}\int_\Gamma \exp(-\lambda^2)J(\lambda)d\lambda.$$

The connection ∇_u^ξ on ξ converges to the connection ∇_∞^ξ, which is the pull-back of the connection $\nabla^\xi|_V$ on the vector bundle $\xi|_V$ by σ_∞. This means that in a given trivialization of ξ, the connection-forms of ∇_u^ξ converge uniformly to the connection-form of ∇_∞^ξ together with their derivatives. Hence if we define $E(u,x,y)$ by

$$(\nabla_u^\xi)^2(x,y) := (\nabla_\infty^\xi)^2(x) + E(u,x,y),$$

then

$$\|E(u,x,y)\| \le \frac{C}{\sqrt{u}}(1+|y|).$$

Moreover, we know that if $F(u,x,y)$ is defined by

$$\sqrt{u}\nabla_u^\xi V(x,\frac{y}{\sqrt{u}}) = \sqrt{u}\nabla_H^\xi V(x) + \nabla_Y^\oplus \nabla_H^\xi V(x) + \nabla_{H^\perp}^\xi V(x) + F(u,x,y),$$

then

$$\|F(u,x,y)\| \le \frac{C}{\sqrt{u}}(|y|+|y|^2).$$

Note that since $\nabla_H^\xi V(x)$ maps F_x into $F_x^\perp$ by sublemma 2 (a), we know that if $\lambda \in \Gamma$, as $u \to \infty$,

$$\begin{aligned}
(\lambda I_\xi + &\sqrt{u}V(x,\frac{y}{\sqrt{u}}))^{-1}\sqrt{u}\nabla_H^\xi V(x)(\lambda I_\xi - \sqrt{u}V(x,\frac{y}{\sqrt{u}}))^{-1}\\
=&(V^+)^{-1}(x)(\nabla_H^\xi V(x))P(\lambda I_F - P\nabla_Y^\xi V(x)P)^{-1}P\\
&- P(\lambda I_F - P\nabla_Y^\xi V(x)P)^{-1}P(\nabla_H^\xi V(x))(V^+)^{-1} + O(\frac{1}{\sqrt{u}}).
\end{aligned}$$

But by Sublemma 2 (b), we know that

$$P\nabla_Y^\oplus(\nabla_H^\xi V)(x)P = \nabla_H^F(P\nabla_Y^\xi V(x)P),$$

so, together with Sublemma 3 (c)

$$\begin{aligned}
(\lambda I_\xi + &\sqrt{u}V(x,\frac{y}{\sqrt{u}}))^{-1}\nabla_Y^\oplus(\nabla_H^\xi V)(x)(\lambda I_\xi - \sqrt{u}V(x,\frac{y}{\sqrt{u}}))^{-1}\\
=&P(\lambda I_F + P\nabla_Y^\xi V(x)P)^{-1}P\nabla_H^F(P\nabla_Y^\xi V(x)P)(\lambda I_F - P\nabla_Y^\xi V(x)P)^{-1}P\\
&+ O(\frac{1}{\sqrt{u}}).
\end{aligned}$$

On the other hand, by definition,

$$P(\nabla_{H^\perp}^\xi V)(x)P = P\nabla_{H^\perp}^F(P\nabla_Y^\xi V(x)P),$$

so similarly, by using Sublemma 3 (c), we have

$$\begin{aligned}(\lambda I_\xi+&\sqrt{u}V(x,\frac{y}{\sqrt{u}}))^{-1}\nabla^{\xi}_{H^\perp}V(x)(\lambda I_\xi-\sqrt{u}V(x,\frac{y}{\sqrt{u}}))^{-1}\\ =&P(\lambda I_F+P\nabla^{\xi}_{Y}V(x)P)^{-1}\nabla^{F}_{H^\perp}(P\nabla^{\xi}_{Y}V(x)P)(\lambda I_F-P\nabla^{\xi}_{Y}V(x)P)^{-1}P\\ &+O(\frac{1}{\sqrt{u}}).\end{aligned}$$

Therefore, for $\lambda\in\Gamma$, as $u\to+\infty$, the form $I(u,\lambda)$ converges pointwise to the form $I(\infty,\lambda)$. Also, for $\lambda\in\Gamma$, we have

$$\|(\lambda I_{\xi_x}-\sqrt{u}V(x,\frac{y}{\sqrt{u}}))^{-1}\|\leq 1,\quad \|(\lambda I_{F_x}-P\nabla^{\xi}_{Y}V(x)P)^{-1}\|\leq 1.$$

Therefore, in the above expression, the norms of the various $O(\frac{1}{\sqrt{u}})$ can be dominated by $\frac{C}{\sqrt{u}}(1+|\lambda|^3)$. This proves the proposition.

With this proposition, the proof of the first estimation in the lemma becomes quite easy.

Indeed, if $G(u,x,y)$ is the form defined by

$$(\sigma_u^*\mu)(x,y):=(\sigma_\infty^* i^*\mu)(x,y)+G(x,y,u),$$

then

$$|G(x,y,u)|\leq\frac{C}{\sqrt{u}}\|\mu\|_{C^1(M)}(1+|y|).$$

Finally, for any $\lambda\in\Gamma$,

$$|\exp(-\lambda^2)|=\exp(-|\mathrm{Re}\lambda|^2+1)$$

and so for any $p\in\mathbf{N}, |\lambda|^p\exp(-\lambda^2)|$ is integrable on Γ. Thus, by (*), let $u\to\infty$, we get the first estimate in the lemma. The second estimate is proved similarly: In this case, instead of using Γ, we have to use the contour

$$\Gamma_y=:\{z\in\mathbf{C}:\mathrm{Re}(z)\geq\frac{C_0|y|}{2}\text{ with }|\mathrm{Im}|(z)=1\text{ or }\mathrm{Re}(z)=\frac{C_0|y|}{2}\}.$$

The details are left to the reader.

Next we give the proof of Sublemma 3 and Sublemma 2 that were stated above.

Proof of Sublemma 3. (a) By the uniqueness of the local resolution, we know that if $\mathcal{V}$ and ε are small enough, then

$$(\xi,\nu)\simeq(\wedge\tilde{N}^*\otimes\tilde{\eta},i_y)\oplus(A,a).$$

Further, there is a metric $\tilde{h}$ on the right hand such that

(1) The $(A_k)_0^m$ are mutually orthogonal.
(2) The splitting is orthogonal.
(3) The metric on the complex $(\wedge \tilde{N}^* \otimes \tilde{\eta}, i_y)$ comes from metrics $g^{\tilde{N}}$ and $g^{\tilde{\eta}}$ on $\tilde{N}$ and $\tilde{\eta}$ respectively.

Let $\tilde{\iota}_y^*$ be the adjoint of ι_y with respect to the metric $g^{\tilde{N}}$. If $\bar{y} \in \bar{\tilde{N}}$ is identified with an element of $\tilde{N}^*$ by the metric $g^{\tilde{N}}$, then $\tilde{\iota}_y^* = \bar{y} \wedge$. Therefore,

$$(\iota_y + \tilde{\iota}_y^*)^2 = |y|^2_{g^{\tilde{N}}}.$$

Let $\tilde{\nu}^*$ be the adjoint of ν with respect to $\tilde{h}$. Set $\tilde{V} =: \nu + \tilde{\nu}^*$. From $(\iota_y + \tilde{\iota}_y^*)^2 = |y|^2_{g^{\tilde{N}}}$, we know that there exists a constant $C > 0$ such that if $(x, y) \in \mathcal{V} \times D_\epsilon$, then

$$\tilde{V}^2(x, y) \geq C|y|^2.$$

We use this estimate to deduce the assertion.

Now fix $(x, y) \in \mathcal{V} \times (\mathcal{D}_\epsilon - \{0\})$. As all the estimates will be done at the point (x, y) in the sequel, we omit (x, y). Alos, both V^2 and $\tilde{V}^2$ preserve $\mathrm{Ker}(\nu)$, by Hodge theory, the lowest eigenvalues can be calculated by considering those on $\mathrm{Ker}(\nu)$, so let $f \in \mathrm{Ker}(\nu) \cap \xi_k$ and set $g := \tilde{\nu}^*(\tilde{V}^2)^{-1} f$, we have $f = \nu g$. But Hodge theory tells us that $g = \nu\alpha + \nu^*\beta$ for certain α, β. Hence $f = \nu\nu^*\beta$. So $\nu^*\beta = \nu^*(V^2)^{-1} f$.

Taking the metrics, we have

$$\|g\|^2_{\tilde{h}} = < (\tilde{V}^2)^{-1} f, f >_{\tilde{h}}$$

and

$$\|g\|^2_h \geq \|\nu^*\beta\|^2_h = < (V^2)^{-1} f, f >_h .$$

But for $\mathcal{V}$ and $\varepsilon > 0$ small enough, there exist constants $C > 0, C' > 0$ such that on $\mathcal{V} \times D_\epsilon$,

$$C' \| \ \|^2_{\tilde{h}} \leq \| \ \|^2_h \leq C \| \ \|^2_{\tilde{h}}.$$

Hence,

$$< (V^2)^{-1} f, f >_h \geq C < (\tilde{V}^2)^{-1} f, f >_{\tilde{h}} .$$

Therefore, if $\lambda, \tilde{\lambda}$ are the lowest eigenvalues of $V^2, \tilde{V}^2$, there is a $c > 0$, which is uniform on $\mathcal{V} \times D_\epsilon$, such that $\lambda \geq c\tilde{\lambda}$. Hence, we have the estimate (a) by using

$$\tilde{V}^2(x, y) \geq C|y|^2.$$

(b) Note that if $x \in \mathcal{V}$, $y \in N_x, Y = y + \bar{y} \in N_{\mathbf{R},x}$, then

$$\partial_Y V(x) = P\nabla^\xi_Y V(x) P = P\tilde{\nabla}^\xi_Y V(x) P.$$

Thus, by Sublemma, we find that there is a constant C_0 such that

$$P\nabla_Y^\xi V(x)P| \geq C_0|y|.$$

(c) Let $I_{\xi^\pm}$ be the identity map of $\xi^\pm$ and define $A^+(u,x,y,\lambda)$ by

$$(\lambda I_{\xi^+} - \sqrt{u}V^+(x,\frac{y}{\sqrt{u}}))^{-1} =: -\frac{1}{\sqrt{u}}(V^+)^{-1}(x,0) + A^+(u,x,y,\lambda).$$

Similarly, note that since $V^-(x)=0$, we may define $A^-(u,x,y,\lambda)$ by

$$\begin{aligned}
&(\lambda I_{\xi^-} - \sqrt{u}V^-(x,\frac{y}{\sqrt{u}}))^{-1}\\
&\quad=(\lambda I_{\xi^-} - \nabla_Y^\oplus V^-(x))^{-1}\\
&\quad\quad+\frac{1}{2\sqrt{u}}(\lambda I_{\xi^-} - \nabla_Y^\oplus V^-(x))^{-1}\nabla_Y^\oplus\nabla_Y^\oplus V^-(x)(\lambda I_{\xi^-} - \nabla_Y^\oplus V^-(x))^{-1}\\
&\quad\quad+A^-(u,x,y,\lambda).
\end{aligned}$$

Obviously, $A = A^+ + A^-$. We next deduce the estimate for A from those for $A^\pm$.

We begin with an estimate for A^+. First, we have

$$(\lambda I_{\xi^+} - \sqrt{u}V^+(x,\frac{y}{\sqrt{u}}))^{-1} = -\frac{(V^+)^{-1}(x,\frac{y}{\sqrt{u}})}{\sqrt{u}}(I_{\xi^+} - \frac{\lambda}{\sqrt{u}}(V^+)^{-1}(x,\frac{y}{\sqrt{u}}))^{-1}.$$

Thus by finite increments, we have

$$\begin{aligned}
&\|(I_{\xi^+} - \frac{\lambda}{\sqrt{u}}(V^+)^{-1}(x,\frac{y}{\sqrt{u}}))^{-1} - I_{\xi^+}\|\\
&\quad\leq\frac{\lambda}{\sqrt{u}}\|(V^+)^{-1}(x,\frac{y}{\sqrt{u}})\|\sup_{c\in[0,1]}\|(I_{\xi^+} - \frac{c\lambda}{\sqrt{u}}(V^+)^{-1}(x,\frac{y}{\sqrt{u}}))^{-1}\|^2.
\end{aligned}$$

But for any $\lambda \neq 0$, we have

$$\inf_{d\in\mathbf{R}}|1-d\lambda|^2 = \frac{|\mathrm{Im}\lambda|^2}{|\lambda|^2}.$$

Hence, for $\lambda \in \Gamma$, i.e., $|\mathrm{Im}\lambda| = 1$, we have

$$\|(I_{\xi^+} - \frac{\lambda}{\sqrt{u}}(V^+)^{-1}(x,\frac{y}{\sqrt{u}}))^{-1} - I_{\xi^+}\| \leq C\frac{|\lambda|^3}{\sqrt{u}}.$$

Also by finite increments, we get

$$\|(V^+(x,\frac{y}{\sqrt{u}}))^{-1} - (V^+(x,0))^{-1}\| \leq \frac{C|y|}{\sqrt{u}}.$$

Therefore, put the above all together, we know that if $x \in \mathcal{V}$, $|y| \le \varepsilon\sqrt{u}$, $|\mathrm{Im}\lambda| = 1$,

$$\|A^{+}(u,x,y,\lambda)\| \le \frac{C}{u}(|y| + |\lambda|^3).$$

Now we deal with the estimates for A^-. We define $B(u,x,y)$ by

$$\sqrt{u}V^-(x,\frac{y}{\sqrt{u}}) =: \nabla_Y^{\oplus}V^-(x) + \frac{1}{2\sqrt{u}}\nabla_Y^{\oplus}\nabla_Y^{\oplus}V^-(x) + B(u,x,y).$$

Obviously, if $x \in \mathcal{V}, |y| \le \varepsilon\sqrt{u}$, we have

$$\|B(u,x,y)\| \le \frac{C|y|^3}{u},$$

and hence

$$\|\sqrt{u}V^-(x,\frac{y}{\sqrt{u}}) - \nabla_Y^{\oplus}V^-(x)\| \le C\varepsilon|y|.$$

On the other hand, define $D(u,x,y,\lambda)$ by

$$\begin{aligned}(\lambda I_{\xi^-} - \sqrt{u}V^-(x,\frac{y}{\sqrt{u}}))^{-1} =:&(\lambda I_{\xi^-} - \nabla_Y^{\oplus}V^-(x))^{-1}\\ &+(\lambda I_{\xi^-} - \nabla_Y^{\oplus}V^-(x))^{-1}(\sqrt{u}V^-(x,\frac{y}{\sqrt{u}}) - \nabla_Y^{\oplus}V^-(x))\\ &(\lambda I_{\xi^-} - \nabla_Y^{\oplus}V^-(x))^{-1} + D(u,x,y,\lambda).\end{aligned}$$

By finite increments again, we find that

$$\begin{aligned}\|D(u,x,y,\lambda)\| \le \sup\nolimits_{c\in[0,1]}&\{\|(\lambda I_{\xi^-} - c\nabla_Y^{\oplus}V^-(x) - (1-c)\sqrt{u}V^-(x,\frac{y}{\sqrt{u}}))^{-1}\|^3\}\\ &\|\sqrt{u}V^-(x,\frac{y}{\sqrt{u}}) - \nabla_Y^{\oplus}V^-(x)\|^2.\end{aligned}$$

Therefore by the fact that in $\mathrm{End}\,\xi_x^-$, $\nabla_Y^{\oplus}V^-(x)$ and $V^-(x,\frac{y}{\sqrt{u}})$ are self-adjoint, together with the estimate for $B(u,x,y)$, we have

$$\|D(u,x,y,\lambda)\| \le \frac{C}{u}|y|^4.$$

In particular, if $x \in \mathcal{V}, |y| \le \varepsilon\sqrt{u}, \lambda \in \Delta$, we have

$$\|A^-(u,x,y,\lambda)\| \le \frac{C}{u}(|y|^4 + |y|^3).$$

So we have the estimate (b). This completes the proof of sublemma 3.

Proof of Sublemma 2. (a) If f is a smooth section of F, $Vf = 0$. Hence if $X \in T_{\mathbf{R}}M'$,

$$(\nabla^{\xi}_X V)f + V\nabla^{\xi}_X f = 0.$$

In particular, we get $(\nabla^{\xi}_X V)f \in F^{\perp}$. But on $\mathcal{V} \times D_\varepsilon$, if $U \in T_{\mathbf{R}}M$,

$$\nabla^{\xi}_U V = \nabla^{\oplus}_U V + S(U)V - VS(U).$$

Hence by the fact that on $\mathcal{V}$, $\nabla^{\oplus}_U V$ preserves the splitting $\xi|_{\mathcal{V}} = F \oplus F^{\perp}$, we have

$$P\nabla^{\xi}_U VQ = PS(U)VQ, \quad Q\nabla^{\xi}_U VP = -QVS(U)P.$$

So we have the structure formula for $S_x(U)$. Others may be obtained from a direct computation.

(b) By definition, we have

$$\nabla^{\oplus}_Y(\nabla^{\xi}_H V)(x) = \nabla^{\xi}_Y(\nabla^{\xi}_H V)(x) - [S(Y), \nabla^{\xi}_H V](x)$$

and

$$\nabla^{\xi}_Y(\nabla^{\xi}_H V)(x) = \nabla^{\xi}_H \nabla^{\xi}_Y V(x) + [(\nabla^{\xi})^2(Y, N), V](x),$$

since Y commutes with the vector fields $\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial \bar{x}^\alpha}$. But (a) tells us that $\nabla^{\xi}_H V(x)$ interchanges F_x and $F_x^{\perp}$. So, from above, we get

$$P\nabla^{\oplus}_Y(\nabla^{\xi}_H V)(x) = P\{\nabla^{\xi}_H \nabla^{\xi}_Y V(x) - S(Y)\nabla^{\xi}_H V(x) + \nabla^{\xi}_H V(x)S(Y)\}P.$$

Hence, by the fact $P\nabla^{\xi}_Y V(x)P = \nabla^{\oplus}_Y V(x)P$, we have

$$P\nabla^{\xi}_Y V(x)P = (\nabla^{\xi}_Y V(x) + V(x)S(Y))P.$$

Thus for any smooth section f of F on $\mathcal{V}$, since $\nabla^F f = P\nabla^{\xi} f$ as stated before the lemma, we have

$$(\nabla^F_H(P\nabla^{\xi}_Y VP))f = P\{\nabla^{\xi}_H(\nabla^{\xi}_Y V + VS(Y))f - (\nabla^{\xi}_Y V)(\nabla^{\xi}_H f - S(H)f)\}.$$

Therefore

$$\nabla^{\xi}_H(P\nabla^F_Y VP) = P(\nabla^{\xi}_H \nabla^{\xi}_Y V + (\nabla^{\xi}_H V)S(Y) + (\nabla^{\xi}_Y V)S(H))P.$$

Using again the fact that $\nabla^{\xi}_H V(x)$ maps F_x into $F_x^{\perp}$, we finally have

$$P(S(Y)\nabla^{\xi}_H V(x))P = P(\nabla^{\xi}_Y V)(x)(V^+)^{-1}(x)(\nabla^{\xi}_H V(x))P$$

$$P(\nabla^{\xi}_Y V(x)S(H))P = -P(\nabla^{\xi}_Y V(x))(V^+)^{-1}(x)(\nabla^{\xi}_H V(x))P$$

This completes the proof of Sublemma 2 and hence part (2) of the theorem for $k = 0$.

Now we show how one can get similar estimates for a general k. Using a partition of unity, we need only discuss the case in which μ has its support as in case (b). Let $x = (x_1, x_2)$ be a holomorphic system of coordinates on V such that $\pi(x) = x_1$. Then by our choice of the coordinate y, $\pi(x_1, x_2, y) = x_1$. Therefore we may lift any smooth real vector field X_1 on B to the vector field $(X_1, 0, 0)$ in the coordinate system (x_1, x_2, y). For short, we also denote it as X_1. Now, we need to study the behavior of

$$(L_{X_1})^k \int_{V \times D_{\epsilon\sqrt{u}}} (\sigma_u^* \mu) \mathrm{Tr}_s[\exp(-(\nabla_u^\xi + \sqrt{u} V(x, \frac{y}{\sqrt{u}}))^2)].$$

Obviously, if $|\mathrm{Im}\lambda| = 1$,

$$\begin{aligned}&\nabla^\oplus_{X_1} (\lambda I_\xi - \sqrt{u} V(x, \frac{y}{\sqrt{u}}))^{-1} \\ =&(\lambda I_\xi - \sqrt{u} V(x, \frac{y}{\sqrt{u}}))^{-1} \sqrt{u} (\nabla^\oplus_{X_1} V)(x, \frac{y}{\sqrt{u}}) (\lambda I_\xi - \sqrt{u} V(x, \frac{y}{\sqrt{u}}))^{-1}.\end{aligned}$$

Now V and $\nabla^\oplus$ preserve the splitting $\xi = \xi^+ \oplus \xi^-$; while on M', V vanishes on $\xi^- = F$ and maps $\xi^+ = F^\perp$ into itself. Therefore, $\nabla^\oplus_{X_1} V(x), \ldots, (\nabla^\oplus_{X_1})^k V(x), \ldots$ all vanish on F and map $F^\perp$ into itself. With the same method in the proof of Sublemma 3.(b), as $u \to \infty$, we get a similar asymptotic expansion for the left hand side of the above relation. Thus we see that the form $(L_{X_1})^k \int_Z \mu \mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]$ has a limit as $u \to \infty$, and that the norm of the difference in $C^0(B)$ with the limit can be dominated by $\frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}}(M)$. So we have the result for a general k.

The Proof of Part 3. From the proof above, this estimate is not difficult to obtain. The point is to replace the corresponding concept at the right place.

Now we briefly explain how this to be done. With the notation as above, recall that $\dim M_j' = l_j + l'$. Let $\hat{x} \in \mathbf{C}^{l_j + l'}$ and $\hat{y} \in \mathbf{C}^{e_j}$ be variables conjugate to x and y and set $z = (x, y)$, $\xi = (\hat{x}, \hat{y})$. Denote by $< z, \xi >$ the real scalar product of z and ξ. Let ϕ be a smooth current with support in U. Take $\alpha > 0$ and let Γ^α be the cone

$$\Gamma^\alpha := \{(\hat{x}, \hat{y}) \in \mathbf{C}^{l + l'} : |\hat{y}| \leq \alpha |\hat{x}|\}.$$

Then obviously, our assertion comes from the following

Lemma. For any natural number m, there exists $C > 0$ such that if $\xi \in \Gamma^\alpha$, $u \geq 1$, we have

$$|\xi|^{2m-1} \Big| \int_M e^{i<z,\xi>} \phi(z) \mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] - \int_{M'} e^{i<x,\hat{x}>} i^* \phi(z) \int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)] \Big| \leq \frac{C}{\sqrt{u}}.$$

Proof of the lemma. Let P_m be the differential operator with constant coefficients in the variable x such that

$$|\hat{x}|^{2m} e^{i<z,\xi>} = P_m(e^{i<z,\xi>}).$$

Integrating by parts, we get

$$
\begin{aligned}
&|\hat{x}|^{2m}\Big|\int_M e^{i<z,\xi>}\phi(z)\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] - \int_{M'} e^{i<x,\hat{x}>} i^*\phi(z)\int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)]\Big| \\
&=\Big|\int_M e^{i<z,\xi>} P_m(\phi\,\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]) - \int_{M'\times\mathbf{C}^{*j}} e^{i<x,\hat{x}>} P_m(i^*\phi\int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)])\Big|.
\end{aligned}
$$

On the other hand, if $\mathcal{A}$ is a partial differential operator with coefficients on $\mathbf{C}^{l+l'}$, $\mathcal{A}$ may be seen as acting on the variable $x \in \mathcal{V}$. Hence we can apply the results in the proof of 2 to the fibration $\mathcal{V}\times D_\varepsilon \to \mathcal{V}$ with fiber D_ε. That is, if $\mathcal{V}$ is small enough, there exist $C>0,\ C'>0$ such that if $x\in\mathcal{V},\ u\geq 1,\ y\in\mathbf{C}^e,\ |y|\leq\varepsilon\sqrt{u}$, then

$$
|\mathcal{A}\sigma_u^*\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] - \mathcal{A}\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]| \leq \frac{C}{\sqrt{u}}\exp(-C'|y|^2).
$$

Now by the fact that on compact subsets of $M-M'$, as $u\to\infty$, the form $\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]$ converges uniformly to 0 faster than $\exp(-Cu)$ for certain $C>0$ and similarly for the derivatives, we may assume that U is a small open neighborhood of $x_0\in M_j'$ of the form $\mathcal{V}\times D_\varepsilon$ chosen as before. Let μ be a differential form on U. Denote by $\|\mu\|$, $\|\frac{\partial\mu}{\partial y}\|$ the sup of the norms of μ and of the partial derivative on U. Hence if $|y|\leq\varepsilon\sqrt{u}$,

$$
|\sigma_u^*\mu) - (\sigma_\infty^*\mu)(x,y)| \leq \frac{\|\mu\| + \|\frac{\partial\mu}{\partial y}\|}{\sqrt{u}}(1+|y|).
$$

Therefore, by the fact that the form $\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]$ and its derivatives decay as $|y|\to\infty$ faster than $\exp(-C''|y|^2)$ with $C''>0$, we get

$$
\begin{aligned}
&\Big|\int_U \mu\,\mathcal{A}\sigma_u^*\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] - \int_{\mathcal{V}\times\mathbf{C}^{*j}} i^*\mu\,\mathcal{A}\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]\Big| \\
&\leq \frac{C}{\sqrt{u}}(\|\mu\| + \|\frac{\partial\mu}{\partial y}\|).
\end{aligned}
$$

In particular, put $\mathcal{A}=P_m$, then by the fact that the first derivative of the function $e^{i<z,\xi>}$ in the variable y is bounded by $|\hat{y}|$, we get

$$
\begin{aligned}
&|\hat{x}|^{2m}\Big|\int_M e^{i<z,\xi>}\phi(z)\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] - \int_{M'} e^{i<x,\hat{x}>} i^*\phi(z)\int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)]\Big| \\
&\leq \frac{C}{\sqrt{u}}(1+|\hat{y}|).
\end{aligned}
$$

But $|\hat{y}|\leq\alpha|\hat{x}|$, so we have the assertion, which completes the proof of the lemma and hence the theorem.

§I.8.2 Number Operators and the Double Transgression Formula

I.8.2.a General Double Transgression Formula

In this section, we will give a general double transgression formula. With the same notation as above, for $0 \le k \le m$, let $\mathcal{M}_k$ be the set of smooth hermitian metrics on the vector bundle ξ_k. Set $\mathcal{M} =: \prod_0^m \mathcal{M}_k$. If $g_k \in \mathcal{M}_k, x \in M$, let $\mathcal{B}_x^{g_k}$ be the set of endomorphisms of $\xi_{k,x}$ which are self-adjoint with respect to the metric g_k. We also identify the tangent space $T_{g_k}\mathcal{M}_k$ with the set of smooth sections of $\mathcal{B}^{g_k}$ on M: In fact, if $g'_k \in \mathrm{End}(\xi_k, \xi_k^*)$ is an infinitesimal deformation of h_k in $\mathcal{M}_k$, then $g_k^{-1}g'_k$ is the corresponding element in $\mathcal{B}^{g_k}$. Let $d_{\mathcal{M}}$ be the exterior differentiation operator on $\mathcal{M}$. For $g^\xi \in \mathcal{M}$, let ω be the connection-form associated with the corresponding canonical connection ∇^ξ. Then $d_{\mathcal{M}}\omega$ is a 2-form which is the equivalent representation of a 2-form γ on $M \times \mathcal{M}$. Obviously, we know that

$$\gamma = -[\nabla^{\xi'}, (g^\xi)^{-1} d_{\mathcal{M}} g^\xi]$$

with values in $\mathrm{End}\,\xi$. Recall that $\mathbf{A} = \mathbf{A}_1 = \nabla^\xi + V$. We let

$$\mathbf{A}' = \nabla^{\xi'} + \nu^*, \quad \mathbf{A}'' = \nabla^{\xi''} + \nu.$$

Then, we have the following

Proposition. 1. The following identities hold

$$\begin{aligned} d_{\mathcal{M}}\mathrm{Tr}_s[\exp(-\mathbf{A}^2)] =& \bar\partial \mathrm{Tr}_s[\gamma - [\nu^*, (g^\xi)^{-1} d_{\mathcal{M}} g^\xi]) \exp(-\mathbf{A}^2)]; \\ \mathrm{Tr}_s[(\gamma - [\nu^*, (g^\xi)^{-1} d_{\mathcal{M}} g^\xi]) \exp(-\mathbf{A}^2) =& -\partial\,\mathrm{Tr}_s[((g^\xi)^{-1} d_{\mathcal{M}} g^\xi) \exp(-\mathbf{A}^2)]. \end{aligned}$$

Therefore,

$$d_{\mathcal{M}}\mathrm{Tr}_s[\exp(-\mathbf{A}^2)] = -\bar\partial\partial\,\mathrm{Tr}_s[((g^\xi)^{-1} d_{\mathcal{M}} g^\xi)\exp(-\mathbf{A}^2)].$$

2. Similarly, for any odd Grassmannian variable z, we have

$$\begin{aligned} & d_{\mathcal{M}}\mathrm{Tr}_s[\exp(-\mathbf{A}^2 + z(g^\xi)^{-1} d_{\mathcal{M}} g^\xi)] \\ =& \frac{1}{2}\partial\,\mathrm{Tr}_s[[\mathbf{A}'', (g^\xi)^{-1} d_{\mathcal{M}} g^\xi] \exp(-\mathbf{A}^2 + z(g^\xi)^{-1} d_{\mathcal{M}} g^\xi)] \\ & - \frac{1}{2}\bar\partial\,\mathrm{Tr}_s[[\mathbf{A}', (g^\xi)^{-1} d_{\mathcal{M}} g^\xi \exp(-\mathbf{A}^2 + z(g^\xi)^{-1} d_{\mathcal{M}} g^\xi)]. \end{aligned}$$

Proof. Consider ξ as a vector bundle on $M \times \mathcal{M}$. There is a natural canonical connection $\tilde\nabla^\xi$ on ξ, which restricts to the canonical connection ∇^ξ with $g^\xi \in \mathcal{M}$ on $M \times \{g^\xi\}$, and is trivial on $\{0\} \times T\mathcal{M} \subset T(M \times \mathcal{M})$. As a superconnection on ξ over $M \times \mathcal{M}$, for $\tilde\nabla^\xi + V$, we have

$$(\tilde\nabla^\xi + V)^2 = (\nabla_V^\xi)^2 + \gamma - [\nu^*, (g^\xi)^{-1} d_{\mathcal{M}} g^\xi].$$

By Duhamel's formula,

$$\begin{aligned} & \mathrm{Tr}_s[\exp(-(\tilde\nabla^\xi + V)^2)] \\ =& \mathrm{Tr}_s[\exp(-\mathbf{A}^2)] - \mathrm{Tr}_s[(\gamma - [\nu^*, (g^\xi)^{-1} d_{\mathcal{M}} g^\xi])\exp(-\mathbf{A}^2)] + C \end{aligned}$$

with C being of degree ≥ 2 in the Grassmannian variables in $T^*\mathcal{M}$. So

$$d_{\mathcal{M}}\mathrm{Tr}_s[\exp(-\mathbf{A}^2)] = d\mathrm{Tr}_s[(\gamma - [\nu^*, (g^\xi)^{-1}d_{\mathcal{M}}g^\xi])\exp(-\mathbf{A}^2)].$$

Therefore, by counting the degree, we have

$$d_{\mathcal{M}}\mathrm{Tr}_s[\exp(-\mathbf{A}^2)] = \bar{\partial}\mathrm{Tr}_s[(\gamma - [\nu^*, (g^\xi)^{-1}d_{\mathcal{M}}g^\xi])\exp(-\mathbf{A}^2)]$$

$$\partial\mathrm{Tr}_s[(\gamma - [\nu^*, (g^\xi)^{-1}d_{\mathcal{M}}g^\xi])\exp(-\mathbf{A}^2)] = 0.$$

But by definition, we know that

$$\partial\mathrm{Tr}_s[(g^\xi)^{-1}d_{\mathcal{M}}g^\xi\exp(-\mathbf{A}^2)] = -\mathrm{Tr}_s[[\mathbf{A}', (g^\xi)^{-1}d_{\mathcal{M}}g^\xi]\exp(-\mathbf{A}^2)],$$

so we have the second equality.

The proof for 2 is very similar and is left to the reader.

I.8.2.b. Number Operators And The Double Transgression Formula

Let $N_H : \xi \to \xi$ be the operator in $\mathrm{End}(\xi)$ which maps $f \in \xi_k$ into $kf \in \xi_k$. We call N_H the **number operator** for ξ. As an application of the result above, we have the following

Theorem. For any $u \geq 0$, the smooth forms $\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]$ and $\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)]$ are in P_M. Moreover for any $u > 0$, we have

$$\begin{aligned}\frac{\partial}{\partial u}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] &= -\,\bar{\partial}\mathrm{Tr}_s[\frac{\nu^*}{\sqrt{u}}\exp(-\mathbf{A}_u^2)]\\ &= -\,\partial\mathrm{Tr}_s[\frac{\nu}{\sqrt{u}}\exp(-\mathbf{A}_u^2)];\\ \mathrm{Tr}_s[\frac{\nu^*}{\sqrt{u}}\exp(-\mathbf{A}_u^2)] &= -\,\partial\mathrm{Tr}_s[\frac{N_H}{u}\exp(-\mathbf{A}_u^2)]\\ \mathrm{Tr}_s[\frac{\nu}{\sqrt{u}}\exp(-\mathbf{A}_u^2)] &= \bar{\partial}\mathrm{Tr}_s[\frac{N_H}{u}\exp(-\mathbf{A}_u^2)].\end{aligned}$$

Therefore, for $u > 0$,

$$\frac{\partial}{\partial u}\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)] = \frac{1}{u}\bar{\partial}\partial\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)].$$

Proof. For $u > 0$, replace the metrics $h^\xi = (h^{\xi_0}, \ldots, h^{\xi_m})$ by

$$h_u^\xi = (h^{\xi_0}, u^{-1}h^{\xi_1}, \ldots, u^{-m}h^{\xi_m}).$$

Then the adjoint of ν with respect to h_u^ξ is $u\nu^*$, where ν^* is the one with respect to h^ξ. Also

$$(h_u^\xi)^{-1}\frac{\partial}{\partial u}h_u^\xi = -\frac{N_H}{u}, \quad [u\nu^*, (h_u^\xi)^{-1}\frac{\partial}{\partial u}h_u^\xi] = \nu^*.$$

If we let $T_u \in \operatorname{End}\xi$ be defined by $T_u f = u^{1/2} f$ for $f \in \xi_k$, we have

$$T_u^{-1}\nu T_u = \sqrt{u}\nu, \quad T_u^{-1}\nu^* T_u = \frac{\nu^*}{\sqrt{u}}$$

Hence, by result in the last subsection, we get the first and the third relations. Similarly, by interchanging holomorphic part and anti-holomorphic part, we get the others.

I.8.2.c **Convergence Properties**

Since, later, we will use the Mellin transform to define the relative Bott-Chern secondary characteristic currents, so we need to asymptotic expansions. In this subsection, we give convergence results for

$$\mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)], \quad \mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)].$$

There are the following

Theorem. (1) There exists a constant $C > 0$ such that for any $k \in \mathbf{N}$, any smooth differential form μ on M and $u \geq 1$,

$$\|\int_Z \mu\mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)]\|_{C^k(B)} \leq \frac{C}{\sqrt{u}}\|\mu\|_{C^{k+1}(M)}$$

and

$$\|\int_Z \mu\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)] - \int_Y i^*\mu\int_N \mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)]\|_{C^k(B)} \leq \frac{C_k}{\sqrt{u}}\|\mu\|_{C^{k+1}(M)}.$$

(2) As $u \to \infty$, we have the following convergence for currents on M:

$$\mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)] \to 0,$$

and

$$\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)] \to [\int_N \mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)]]\delta_{M'}$$

in $\mathcal{D}'_{N_{\mathbf{R}}^*}(M)$;

(3) If U, Γ, φ, m are taken as in 7.1.d, there exist constants $C, C' > 0$ such that for $u \geq 1$,

$$p_{U,\Gamma,\varphi,m}(\mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)]) \leq \frac{C}{\sqrt{u}},$$

and

$$p_{U,\Gamma,\varphi,m}(\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)]-[\int_N \mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)]\delta_{M'})\leq \frac{C'}{\sqrt{u}}.$$

Proof. By the fact that if K is any smooth section of $\mathrm{End}(\xi)$, then the analogue of Theorem 8.1 still holds for

$$\mathrm{Tr}_s[K\exp(-\mathbf{A}_u)^2)],$$

whose limit as $u\to infty$ will then be the current $\int_N \mathrm{Tr}_s[PKP\exp(-\mathbf{B}^2)]\delta_{M'}$. we see that the statements about the relations concerning N_H may be similarly proved by following the proof for Theorem 8.1, we will not do it here. Instead, we use Theorem 8.1 to prove the assertions for $\sqrt{u}V$.

For $(x,a)\in M\times \mathbf{C}^*$, set $\tilde{\pi}(x,a)=(\pi x,a)\in B\times\mathbf{C}^*$. Then $\tilde{\pi}$ has essentially the same properties as π. Let j be the embedding $M'\times\mathbf{C}^*\hookrightarrow M\times\mathbf{C}^*$. The vector bundles ξ_k (resp. η) extend naturally to $M\times\mathbf{C}$ (resp. $M'\times\mathbf{C}$). Then on $M\times\mathbf{C}^*$, we have the exact sequence of sheaves

$$0\to\mathcal{O}_{M\times\mathbf{C}^*}(\xi_m)\xrightarrow{a\nu}\mathcal{O}_{M\times\mathbf{C}^*}(\xi_{m-1})\xrightarrow{a\nu}\dots\xrightarrow{a\nu}\mathcal{O}_{M\times\mathbf{C}^*}(\xi_0)\xrightarrow{r}j_*\mathcal{O}_{M\times\mathbf{C}^*}(\eta)\to 0.$$

The natural canonical connection on ξ over $M\times\mathbf{C}^*$ is given by $\nabla^\xi+da\frac{\partial}{\partial a}+d\bar{a}\frac{\partial}{\partial\bar{a}}$. Thus if $a\in\mathbf{C}^*$ with $|a-1|<\frac{1}{2}$, by Theorem 8.1, we know that

$$\begin{aligned}\lim_{u\to+\infty}&\int_Z\mu\mathrm{Tr}_s[\exp(-(\nabla^\xi+da\frac{\partial}{\partial a}+d\bar{a}\frac{\partial}{\partial\bar{a}}+\sqrt{u}(a\nu+\bar{\nu}^*))^2)]\\ &=\int_Y i^*\mu\int_N\mathrm{Tr}_s[\exp(-(\nabla^F+da\frac{\partial}{\partial a}+d\bar{a}\frac{\partial}{\partial\bar{a}}+a\partial_y\nu+\bar{a}\partial_{\bar{y}}\nu^*)^2)].\end{aligned}$$

Also, with the correct parameters, we know that the difference between the expressions appearing in both sides above is dominated by $\frac{C_k}{\sqrt{u}}\|\mu\|_{C^{k+1}(M)}$. But by Duhamel's formula, we know that there exist forms γ_u and γ on B, such that over $B\times\{1\}$,

$$\begin{aligned}&\int_Z\mu\mathrm{Tr}_s[\exp(-(\nabla^\xi+da\frac{\partial}{\partial a}+d\bar{a}\frac{\partial}{\partial\bar{a}}+\sqrt{u}(a\nu+\bar{\nu}^*))^2)]\\ &\quad=\int_Z\mu\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]-\int_Z\mu\,\mathrm{Tr}_s[\sqrt{u}\nu\exp(-\mathbf{A}_u^2)]\,da\\ &\quad\quad-\int_Z\mu\,\mathrm{Tr}_s[\sqrt{u}\nu^*\exp(-\mathbf{A}_u^2)]\,d\bar{a}+\gamma_u\,dad\bar{a};\\ &\int_Y i^*\mu\int_N\mathrm{Tr}_s[\exp(-(\nabla^F+da\frac{\partial}{\partial a}+d\bar{a}\frac{\partial}{\partial\bar{a}}+a\partial_y\nu+\bar{a}\partial_{\bar{y}}\nu^*)^2)]\\ &\quad=\int_Y i^*\mu\int_N\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]-\int_Y i^*\mu da\int_N\mathrm{Tr}_s[\partial_y\nu\exp(-\mathbf{B}^2)]\\ &\quad\quad-\int_Y\mu d\bar{a}\int_N\mathrm{Tr}_s[\partial_{\bar{y}}\nu^*\exp(-\mathbf{B}^2)]+\gamma\,dad\bar{a}.\end{aligned}$$

Thus, assertion (1) is a direct consequence of the following

Lemma. With the same notation as above,

$$\int_N \mathrm{Tr}_s[\partial_y \nu \exp(-\mathbf{B}^2)] = 0, \quad \int_N \mathrm{Tr}_s[\partial_{\bar{y}} \nu^* \exp(-\mathbf{B}^2)] = 0.$$

Proof. Note that if $y \in N$ is considered as a vector field on the total space of N, then clearly $\iota_y(\nabla^F)^2 = 0$. Also

$$\mathbf{B}^2 = (\nabla^F)^2 + [\nabla^F, (P\nabla_Y^\xi VP)] + (P\nabla_Y^\xi VP)^2,$$

and then

$$\iota_y(\mathbf{B}^2) = \partial_y \nu.$$

But ι_y is a derivation of the $\mathbf{Z}_2$-graded algebra $\wedge T^* N_{\mathbf{R}} \hat{\otimes} \mathrm{End} F$; by Duhamel's formula, and the fact that Tr_s vanishes on supercommutators, we have

$$\iota_y \mathrm{Tr}_s[\exp(-\mathbf{B}^2)] = -\mathrm{Tr}_s[\partial_y \nu \exp(-\mathbf{B}^2)].$$

Therefore the form $\mathrm{Tr}_s[\partial_y \nu \exp(-\mathbf{B}^2)]$ has no component of maximal degree in the direction of the vector fibers N, and so

$$\int_N \mathrm{Tr}_s[\partial_y \nu \exp(-\mathbf{B}^2)] = 0.$$

Similarly, we could have

$$\int_N \mathrm{Tr}_s[\partial_{\bar{y}} \nu^* \exp(-\mathbf{B}^2)] = 0.$$

In this manner, by a similar process as in the proof of Theorem 8.1, we may aslo get the rest.

§I.8.3 The Construction Of Relative Bott-Chern Secondary Characteristic Currents With Respect To Closed Immersions

I.8.3.a. A Construction

By Theorem 7.1 and Theorem 7.2, we know that for $u \to \infty$, the difference

$$\mathrm{Tr}_s[N_H \exp(-\mathbf{A}_u^2)] - (\int_N \mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)])\delta_{M'}$$

is bounded by $u^{-1/2}$. Also, we know that when $u \to 0^+$, the asymptotic expansion exists from the general discussion for the heat kernels. Therefore its associated Mellin transform makes sense. With this, we easily have the following

Proposition-Definition. 1. For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < \frac{1}{2}$, let

$$\zeta_\xi(s) =: \frac{1}{\Gamma(s)} \int_0^{+\infty} u^s \{\mathrm{Tr}_s[N_H \exp(-\mathbf{A}_u^2)] - (\int_N \mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)])\delta_{M'}\} \frac{du}{u}.$$

Then $\zeta_\xi(s)$ is a well-defined current on M.

2. For any smooth form μ on M, there exists a unique meromorphic extension of $\int_0^{+\infty} \mu\zeta_\xi(s)$ to the whole complex plane such that this extension is holomorphic at $s = 0$.
3. Let $\zeta_\xi'(0)$ denote the current on M such that for any smooth differential form μ on M,

$$\int_M \mu\zeta_\xi'(0) = \frac{\partial}{\partial s}[\int_M \mu\zeta_\xi(s)]_{s=0}.$$

Then

$$\begin{aligned}\zeta_\xi'(0) = &\int_0^1 \{\mathrm{Tr}_s[N_H(\exp(-\mathbf{A}_u^2) - \exp(-\mathbf{A}_0^2))]\} \frac{du}{u} \\ &+ \int_1^{+\infty} \{\mathrm{Tr}_s[N_H \exp(-\mathbf{A}_u^2)] - (\int_N \mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)])\delta_{M'}\} \frac{du}{u}\cdot \\ &-\Gamma'(1)\{\mathrm{Tr}_s[N_H \exp(-\mathbf{A}_0^2)] - (\int_N \mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)])\delta_{M'}\}.\end{aligned}$$

If, as before, we let $[2\pi\mathrm{i}]$ be the natural operation on the graded algebra $D(M)$ such that

$$[2\pi\mathrm{i}]f_{[k]} = (2\pi i)^k f_{[k]},$$

then we define the **relative Bott-Chern secondary characteristic current**, denoted by $\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, \rho_i)$, to be $[2\pi\mathrm{i}]\zeta_\xi'(0)$, but with g^ξ defined by Bismut condition (A). Usually, if we do not have Bismut condition (A) on the metrics g^ξ, we denote the corresponding current by

$$\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, \rho_i; \xi, g^\xi).$$

From section 8.1, we know that for a given vector sheaf resolution of $i_*\eta$, there are metrics on ξ_k, such that the Bismut condition (A) is satisfied. Hence the above definition for relative Bott-Chern secondary characteristic current makes sense. Next, we have to check the axioms, from which it follows that $\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, \rho_i)$ does not depend on some of the special data used in the definition.

§I.8.4. Checking The Axioms

In this section, we check the axioms listed in section 7.2. As stated at the beginning of this chapter, we first work without Bismut assumption (A), but once this assumption is made, the translation can be easily done by using Theorem 7.1.c.

I.8.4.a. **Axiom 1.**

Let
$$\gamma(g^{\xi}) =: \frac{1}{(2\pi i)^{1/2}}\int_0^{\infty}[2\pi \mathbf{i}](\mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)])\frac{du}{2u}.$$
Here, we fix once for all one square root for $2\pi i$. Then the transgression axiom is verified by the following

Theorem. (1) The current $\mathrm{ch}_{\mathrm{BC}}(\eta, g^{\eta}; i, \rho_i; \xi, g^{\xi})$ lies in $P_{M'}^{M}$.
(2) The following identities of currents hold

$$d^c\mathrm{ch}_{\mathrm{BC}}(\eta, g^{\eta}; i, \rho_i; \xi, g^{\xi}) = -\,\gamma(g^{\xi});$$
$$d\,\gamma(g^{\xi}) = \mathrm{ch}(\xi, g^{\xi}) - \Big(\int_N [2\pi\mathbf{i}]\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]\Big)\delta_{M'}.$$

Therefore,

$$dd^c\mathrm{ch}_{\mathrm{BC}}(\eta, g^{\eta}; i, \rho_i; \xi, g^{\xi}) = \Big(\int_N [2\pi\mathbf{i}]\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]\Big)\delta_{M'} - \mathrm{ch}(\xi, g^{\xi}).$$

If g^{ξ} satisfies Bismut condition (A), we have

$$dd^c\mathrm{ch}_{\mathrm{BC}}(\eta, g^{\eta}; i, \rho_i) = \mathrm{td}^{-1}(g^N)\mathrm{ch}(g^{\eta})\delta_{M'} - \mathrm{ch}(\xi, g^{\xi}).$$

Proof. (1) By Theorem 2.b and Proposition 3.a.3, we see that $\mathrm{ch}_{\mathrm{BC}}(\eta, g^{\eta}; i, \rho_i; \xi, g^{\xi})$ is a combination of currents of type (p,p). So, for the first assertion, we only need to prove that $\mathrm{WF}(\zeta_{\xi}'(0))$ is included in $N_{\mathbf{R}}^*$. To do so, by Proposition 3.a.3, it is enough to prove that the term for $\int_1^{+\infty}$ has its kernel in $N_{\mathbf{R}}^*$. For short, we also denote this term by $\int_1^{+\infty}$, i.e.,

$$\int_1^{+\infty} := \int_1^{+\infty}(\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)] - \Big(\int_N [N_H\exp(-\mathbf{B}^2)]\Big)\delta_{M'})\frac{du}{u}.$$

By Duhamel's formula, we know that $\int_1^{+\infty}$ is smooth on $M - M'$. On the other hand, if we choose U, Γ, ϕ, m as in subsection 2.c, we know that

$$p_{U,\Gamma,\phi,m}\Big(\int_1^{+\infty}\Big) \le C\int_1^{+\infty}\frac{du}{u^{3/2}} < +\infty.$$

Hence, we have $WF(\mathrm{ch}_{\mathrm{BC}}(\eta, h^{\eta}; i, \rho_i)) \subset N_{\mathbf{R}}^*$.

(2) We use the result in section 2. We let

$$\gamma_{\xi} =: (2\pi i)^{1/2}[2\pi\mathbf{i}]^{-1}\gamma(g^{\xi}),$$

then for any smooth form μ,

$$\int_M \mu d\gamma_\xi = -\int_M d\mu\gamma_\xi.$$

But

$$\begin{aligned}\int_M d\mu\gamma_\xi =& \lim_{a\to+\infty}\int_0^a\{\int_M d\mu \mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)]\}\frac{du}{2u}\\ =& -\lim_{a\to+\infty}\int_0^a\{\int_M \mu d\mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)]\}\frac{du}{2u}.\end{aligned}$$

Thus by Theorem 2.b, and Theorem 2.c, we have

$$\int_M \mu d\gamma_\xi = \int_M \mu \mathrm{Tr}_s[\exp(-\mathbf{A}_0^2)] - \int_{M'} i^*\mu \int_N \mathrm{Tr}_s[\exp(-\mathbf{B}^2)].$$

This proves the second equality. Similarly, by definition, we know that

$$\int_M \mu d^c\zeta_\xi'(0) = \int_M d^c\mu\zeta_\xi'(0).$$

Now by the closed properties of associated currents, we know that

$$\begin{aligned}\int_M d^c\mu\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_0^2)] =&0\\ \int_{M'} i^*(d^c\mu)(\int_N \mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)]) =&0.\end{aligned}$$

Thus, similarly, as we did above for the second eqality, we may also have the first one.

I.8.4.b. **Axiom 2.**

By Proposition 2.a.3, we know that if we let

$$\begin{aligned}\zeta_\xi'^{\,a}(0) =& \int_0^1 \{\mathrm{Tr}_s[N_H(\exp(-\mathbf{A}_u^2)-\exp(-\mathbf{A}_0^2))]\}\frac{du}{u}\\ &+\int_1^a\{\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)] - (\int_N \mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)])\delta_{M'}\}\frac{du}{u}\\ &-\Gamma'(1)\{\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_0^2)] - (\int_N \mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)])\delta_{M'}\},\end{aligned}$$

then in $\mathcal{D}'_{N_{\mathbb{R}}^*}(M)$, by Theorem 2.c and Theorem 4.a,

$$\lim_{a\to+\infty}\zeta_\xi'^{\,a}(0) = \zeta_\xi'(0).$$

Therefore, what we need to check is the functorial property of each term in $\{\ \}$. But this is quite easy. Since f is transversal to M', we know that $f^*\xi$ provides a resolution for $f^*\eta$

and $N = f_* \tilde{N}$. By the functorial properties of characteristic forms, we only need to show that

$$f^* \delta_{M'} = \delta_{\tilde{M}'}.$$

But, such a result can be proved by using a local approximation of $\delta_{M'}$ with a sequence of smooth currents, which converge 'transversely' as in example 8.2.8 of [Hö 86]. Hence we have the assertion, again by the fact that f is transversal to M'.

I.8.4.c. **Axiom 3.**

In the same way, as for smooth morphisms, we use the $\mathbf{P}^1$-deformation technique to deduce the most general triangle relation from the degenerate triangle relation, i.e. we let $\eta_3 = 0$. As an illustration, we give the following

Theorem. (1) Let $\theta(g^F)$ be the smooth form on M' defined by

$$\theta(g^F) =: \int_N [2\pi i](\mathrm{Tr}_s[\exp(-\mathbf{B}^2)]).$$

Then $\theta(g^F) \in P^{M'}$ is closed.
(2) Suppose $t \in \mathbf{R} \mapsto g_t^F \in \mathcal{M}^F$ is a smooth map. Let $t \in \mathbf{R} \mapsto \mathbf{B}_t$ be the corresponding family of superconnections on the graded vector bundle F. Define

$$\chi(g_t^F) := \int_0^1 dt \int_N [2\pi i](\mathrm{Tr}_s[(g_t^F)^{-1} \frac{(\partial g_t^F)^{-1}}{\partial t} \exp(-\mathbf{B}_t^2)]).$$

Then $\chi(g_t^F) \in P^{M'}$ and its class in $P^{M'}/P^{M',0}$ only depends on g_0^F, g_1^F. So we may define this class as $\chi(h_0^F, h_1^F)$. Moreover

$$dd^c \chi(g_t^F) = \theta(g_1^F) - \theta(g_0^F).$$

(3) We have the following relation

$$\mathrm{ch}_{\mathrm{BC}}(\eta, \rho; g_1^\xi; i, \rho_i) - \mathrm{ch}_{\mathrm{BC}}(\eta, \rho; g_0^\xi; i, \rho_i) = \chi(g_0^F, g_1^F)\delta_{M'} - \mathrm{ch}_{\mathrm{BC}}(\xi; g_0^\xi, g_1^\xi).$$

(4) If Bismut assumption (A) holds for the corresponding metrics for all situations, then

$$\chi(g_0^F, g_1^F) = \mathrm{td}_{\mathrm{BC}}^{-1}(N; g_0^N, g_1^N)\mathrm{ch}(\eta, g_0) + \mathrm{td}^{-1}(N, g_1^N)\mathrm{ch}_{\mathrm{BC}}(\eta; g_0, g_1)$$

in $P^{M'}/P^{M',0}$.

Proof. The first two statements may be proved by the method similar to these for Theorem 4.5.a and Theorem 6.1.d. We leave the details to the reader. For (3), we have to use $\mathbf{P}^1$-deformation.

On $M \times \mathbf{P}^1$, we equip $(\xi.)$ with hermitian metrics g^ξ which restrict to the metrics g_0^ξ and g_1^ξ on $M \times \{0\}$ and $M \times \{\infty\}$ respectively. On $M \times \mathbf{P}^1$, the wave front set $\mathrm{WF}(\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, \rho_i; \xi, g^\xi))$ is included in $N_{\mathbf{R}}^*$, and

$$\mathrm{WF}(\log|z|^2) \cap \mathrm{WF}(\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, \rho_i; \xi, g^\xi)) = \emptyset.$$

Hence, by Theorem 8.2.10 of [Hö 86], we know that the product of currents

$$\operatorname{Log}|z|^2\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)$$

is well-defined. Also the usual rules of differential calculus can be used. Thus,

$$\begin{aligned}
&\frac{\bar\partial}{2\pi i}(\partial[\log|z|^2]\,\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)+\frac{\partial}{2\pi i}(\operatorname{Log}|z|^2\bar\partial\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi))\\
&=\frac{\bar\partial\partial}{2\pi i}(\operatorname{Log}|z|^2)\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)-\operatorname{Log}|z|^2\frac{\bar\partial\partial}{2\pi i}\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)\\
&=\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)\delta_{M\times\{0\}}-\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)\delta_{M\times\{\infty\}}\\
&\quad-\operatorname{Log}|z|^2(\theta(g^F)\delta_{M'\times\mathbf{P}^1}-\mathrm{ch}(g^\xi)).
\end{aligned}$$

If we integrate the above equality along the fiber of $M\times\mathbf{P}^1\to M$, we have

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)-\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)\\
&-[\int_{\mathbf{P}^1}\operatorname{Log}|z|^2\theta(g^F)]\delta_{M'}+\int_{\mathbf{P}^1}\operatorname{Log}|z|^2\mathrm{ch}(g^\xi)\in P_{M'}^{M,0}.
\end{aligned}$$

By the similar process for $\chi(g_0^F,g_1^F)$, instead of $\mathrm{ch}_{\mathrm{BC}}(\eta,g^\eta;i,\rho_i;\xi,g^\xi)$, we get

$$\chi(g^0,g^1)+\int_{\mathbf{P}^1}\operatorname{Log}|z|^2\theta(g^F)\in P^{M',0}.$$

Hence, we have (3), since

$$\int_{\mathbf{P}^1}[\log|z|^2\,\mathrm{ch}(g^\xi)=-\mathrm{ch}_{\mathrm{BC}}(g_0^\xi,g_1^\xi).$$

(4) If g_i^F satisfies Bismut assumption (A) with respect to (g_i^N,g_i) for $i=0,1$, we can find a smooth family of metrics $t\mapsto(g_t^N,g_t)$ on N and η which interpolate between the two initial points above. If g_t^F is the metric on F associated with the metric (g_t^N,g_t), the family $t\mapsto g_t^F$ interpolates between g_0^F and g_i^F. Also, the operator $(g_t^N)^{-1}\frac{\partial g_t^N}{\partial t}$ acts naturally on the exterior algebra $\wedge N^*$. One verifies easily that since $F=\wedge N^*\otimes\eta$,

$$(h_t^F)^{-1}\frac{\partial h_t^F}{\partial t}=(g_t^N)^{-1}\frac{\partial g_t^N}{\partial t}\otimes 1+1\otimes(g_t)^{-1}\frac{\partial g_t}{\partial t}.$$

Let $(\nabla_t^N)^2$, $(\nabla_t^\eta)^2$ be the curvatures of the corresponding canonical connections for $(N,g^N),(\eta,g^\eta)$. As in subsection 7.1.c, we find that

$$\begin{aligned}
&-\int_N\mathrm{Tr}_s[(g_t^F)^{-1}\frac{\partial g_t^F}{\partial t}\exp(-\mathbf{B}_t^2)]\\
&=(2\pi i)^{\dim N}\frac{\partial}{\partial b}[\mathrm{td}^{-1}(-(\nabla_t^N)^2-b(g_t^N)^{-1}\frac{\partial g_t^N}{\partial t})\,\mathrm{Tr}[\exp(-(\nabla_t^\eta)^2)]\\
&+\mathrm{td}^{-1}(-(\nabla_t^N)^2)\,\mathrm{Tr}[\exp(-(\nabla_t^\eta)^2-b(g_t)^{-1}\frac{\partial g_t}{\partial t})]]_{b=0}.
\end{aligned}$$

By the results in subsection 4.5.b for the classical Bott-Chern secondary characteristic forms with respect to Todd characteristic forms, we have the assertion. So we have the proof of this theorem. In particular, (3) and (4) verify axiom 3.

I.8.4.d. **Axiom 4.**

Let $\tilde{M}'$ be another complex submanifold of M with $\tilde{i} : \tilde{M}' \hookrightarrow M$ the closed immersion. We set $M'' =: M' \cap \tilde{M}'$. As for the immersion i, we could give the same results: We now only have to put $\tilde{\ }$ and $.''$ on the notation for $\tilde{M}'$ and M'' respectively. Since M' and $\tilde{M}'$ are transversal, we know that

$$N'' = N|_{M''} \oplus \tilde{N}|_{M''}.$$

Also $N|_{M''}$ and $\tilde{N}|_{M''}$ are exactly the normal bundles to $j : M'' \hookrightarrow M'$ and $\tilde{j} : M'' \hookrightarrow \tilde{M}'$ respectively. Thus if we let

$$(\xi'', \nu'') := (\xi \hat{\otimes} \tilde{\xi}, \nu \hat{\otimes} 1 + 1 \hat{\otimes} \tilde{\nu}),$$

and

$$\eta'' := \eta|_{M''} \hat{\otimes} \tilde{\eta}|_{M''},$$

with $r'' = r \hat{\otimes} \tilde{r}$, we have the following exact sequences of sheaves

$$0 \to \mathcal{O}_M(\xi'', \nu'') \xrightarrow{r''} i''_* \mathcal{O}_{M''}(\eta'') \to 0,$$

$$0 \to \mathcal{O}_{\tilde{M}'}((\xi, \nu)|_{\tilde{M}'}) \xrightarrow{r} \tilde{j}_* \mathcal{O}_{M''}(\eta|_{M''}) \to 0,$$

$$0 \to \mathcal{O}_{M'}((\tilde{\xi}, \tilde{\nu})|_{M'}) \xrightarrow{r'} j_* \mathcal{O}_{M''}(\eta'|_{M''}) \to 0.$$

Moreover, for the homology groups, we have the identification of holomorphic vector bundles

$$F'' = F|_{M''} \hat{\otimes} \tilde{F}|_{M''}.$$

Thus, similarly, we have the relative Bott-Chern secondary characteristic currents with respect to $i, \tilde{i}, i''$ respectively. Note that i and $\tilde{i}$ are transversal and we know that the pull-back currents $i^* \mathrm{ch}_{\mathrm{BC}}(\tilde{\eta}, g^{\tilde{\eta}}; \tilde{i}, \rho_{\tilde{i}})$ and $\tilde{i}^* \mathrm{ch}_{\mathrm{BC}}(\eta, g^{\eta}; i, \rho_i)$ on M' and $\tilde{M}'$ are well-defined, by Theorem 8.2.4 of [Hö 86], as, on wave fronts, statements are affected. Thus by axiom 2, we have

$$\mathrm{ch}_{\mathrm{BC}}(\tilde{i}^* \eta, \tilde{i}^* \rho^{\eta}; \tilde{i}, \rho^{\tilde{i}}) = \tilde{i}^* \mathrm{ch}_{\mathrm{BC}}(\eta, g^{\eta}; i, \rho_i),$$
$$\mathrm{ch}_{\mathrm{BC}}(i^* \tilde{\eta}, i^* g^{\tilde{\eta}}; i, \rho_i) = i^* \mathrm{ch}_{\mathrm{BC}}(\tilde{\eta}, g^{\tilde{\eta}}; \tilde{i}, \rho_{\tilde{i}}).$$

In particular, the corresponding wave front sets are included in $\tilde{N}^*_{\mathbb{R}}$ and $N^*_{\mathbb{R}}$ respectively. Now by theorem 8.2.13 of [Hö 86], the currents of both sides in the axiom 4 are actually elements in $P^M_{M' \cup \tilde{M}'}$, and axiom 4 makes sense. Next, we check the relation.

Let

$$\alpha_u =: \mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)], \quad \beta_u =: \mathrm{Tr}_s[N_H \exp(-\mathbf{A}_u^2)],$$

etc. Since

$$N''_H = N_H \hat{\otimes} 1 + 1 \hat{\otimes} \tilde{N}_H,$$

by definition, we have

$$\alpha''_u = \alpha_u \tilde{\alpha}_u, \quad \beta''_u = \alpha_u \tilde{\beta}_u + \tilde{\alpha}_u \beta_u.$$

Furthermore, by Theorem 1 and Theorem 2.c, we know that

(1) As $u \to \infty$, the currents $\alpha_u, \beta_u, \tilde{\alpha}_u, \tilde{\beta}_u, \alpha''_u, \beta''_u$ have limits $\alpha_\infty, \beta_\infty, \tilde{\alpha}_\infty, \tilde{\beta}_\infty, \alpha''_\infty, \beta''_\infty$, by Theorem 1, Theorem 2.c.

(2) $\beta_0, \beta_\infty, \tilde{\beta}_0, \tilde{\beta}_\infty, \beta''_0, \beta''_\infty$ are closed.

(3) $\frac{\partial}{\partial u}\tilde{\alpha}_u = \frac{1}{u}\bar{\partial}\partial\tilde{\beta}_u$, by Theorem 2.b.

(4) Set

$$\tilde{\eta}_u = \begin{cases} \int_0^u (\tilde{\beta}_t - \tilde{\beta}_0)\frac{dt}{t}, & \text{for } u \leq 1; \\ \int_0^1 (\tilde{\beta}_t - \tilde{\beta}_0)\frac{dt}{t} + \int_1^u (\tilde{\beta}_t - \tilde{\beta}_\infty)\frac{dt}{t}, & \text{for } 1 \leq u \leq +\infty; \end{cases}$$

etc. Then, by (2), (3),

$$\begin{aligned} \tilde{\alpha}_u &= \tilde{\alpha}_0 + \bar{\partial}\partial\tilde{\eta}_u, \\ \alpha_u &= \alpha_0 + \bar{\partial}\partial\eta_u. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \alpha''_\infty &= \alpha_\infty \tilde{\alpha}_\infty; \\ \beta''_\infty &= \alpha_\infty \tilde{\beta}_\infty + \tilde{\alpha}_\infty \beta_\infty; \\ \int_0^1 (\beta''_u - \beta''_0)\frac{du}{u} &= \tilde{\alpha}_0 \int_0^1 (\beta_u - \beta_0)\frac{du}{u} + \alpha_1\tilde{\eta}_1 + \tilde{\beta}_0 \int_0^1 (\alpha_u - \alpha_0)\frac{du}{u} \\ &\quad - \bar{\partial}\int_0^1 \partial(\beta_u)\tilde{\eta}_u \frac{du}{u} - \partial \int_0^1 \beta_u \bar{\partial}\tilde{\eta}_u \frac{du}{u} \\ &= \tilde{\alpha}_0 \int_0^1 (\beta_u - \beta_0)\frac{du}{u} + \alpha_1\tilde{\eta}_1 + \bar{\partial}\partial(\tilde{\beta}_0 \int_0^1 \eta_u \frac{du}{u}) \\ &\quad - \bar{\partial}\int_0^1 \partial(\beta_u)\tilde{\eta}_u \frac{du}{u} - \partial \int_0^1 \beta_u \bar{\partial}\tilde{\eta}_u \frac{du}{u}; \end{aligned}$$

And, similarly, for $A \in [1,+\infty[$,

$$\begin{aligned}\int_1^A (\beta''_u - \beta''_\infty)\frac{du}{u} =&\tilde{\alpha}_0 \int_1^A (\beta_u - \beta_\infty)\frac{du}{u} + \alpha_A \tilde{\eta}_A - \tilde{\alpha}_1\tilde{\eta}_1 \\ &+ \int_1^A (\alpha_u - \alpha_\infty)\tilde{\beta}_\infty \frac{du}{u} \\ &- \bar{\partial}\int_1^A \partial(\beta_u - \beta_\infty)\tilde{\eta}_u \frac{du}{u} - \partial \int_1^A (\beta_u \bar{\partial}\tilde{\eta}_u - \beta_\infty \bar{\partial}\tilde{\eta}_\infty)\frac{du}{u} \\ =&\tilde{\alpha}_0 \int_1^A (\beta_u - \beta_\infty)\frac{du}{u} + \alpha_A \tilde{\eta}_A - \tilde{\alpha}_1\tilde{\eta}_1 \\ &+ \bar{\partial}\partial \int_1^A (\eta_u - \eta_\infty)\tilde{\beta}_\infty \frac{du}{u} \\ &- \bar{\partial}\int_1^A \partial(\beta_u - \beta_\infty)\tilde{\eta}_u \frac{du}{u} - \partial \int_1^A (\beta_u \bar{\partial}\tilde{\eta}_u - \beta_\infty \bar{\partial}\tilde{\eta}_\infty)\frac{du}{u}.\end{aligned}$$

Now we take the limit of the above expressions for each case as $A \to +\infty$. Obviously, by Theorem 1, and Theorem 2.c, the only problem will come from the last two terms. We check them separately.

For any smooth differential form μ on M, we know that if $u \geq 1$, from definition and Theorem 2.c,

$$|\int_M \mu(\eta_u - \eta_\infty)| \leq \frac{C}{\sqrt{u}}||\mu||_{C^1(M)}.$$

Hence in $\mathcal{D}'(M)$,

$$\int_1^A (\eta_u - \eta_\infty)\frac{du}{u} \to \int_1^{+\infty} (\eta_u - \eta_\infty)\frac{du}{u}.$$

We also know that

$$p_{U,\Gamma,\varphi,m}(\eta_u - \eta_\infty) \leq \frac{C}{\sqrt{u}},$$

so in $\mathcal{D}'_{N^*_{\mathbb{R}}}(M)$, we have

$$\int_1^A (\eta_u - \eta_\infty)\frac{du}{u} \to \int_1^{+\infty} (\eta_u - \eta_\infty)\frac{du}{u}.$$

By the fact that i and $\tilde{i}$ are transversal, we know that, in $\mathcal{D}'_{N^*_{\mathbb{R}}+\tilde{N}^*_{\mathbb{R}}}(M)$,

$$\int_1^A (\eta_u - \eta_\infty)\tilde{\beta}_\infty \frac{du}{u} \to \int_1^{+\infty} (\eta_u - \eta_\infty)\tilde{\beta}_\infty \frac{du}{u}.$$

Thus, we get

$$\bar{\partial}\partial \int_1^A (\eta_u - \eta_u)\tilde{\beta}_\infty \frac{du}{u} \to \bar{\partial}\partial \int_1^{\infty} (\eta_u - \eta_u)\tilde{\beta}_\infty \frac{du}{u}$$

in $\mathcal{D}'_{N^\bullet_{\mathbb{R}}+\bar{N}^\bullet_{\mathbb{R}}}(M)$. On the other hand, since β_∞ is closed, we have

$$\beta_u\bar{\partial}\tilde{\eta}_u - \beta_\infty\bar{\partial}\tilde{\eta}_\infty = (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u + \bar{\partial}(\beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)).$$

Similarly, in $\mathcal{D}'_{N^\bullet_{\mathbb{R}}+\bar{N}^\bullet_{\mathbb{R}}}(M)$,

$$\int_1^A \beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\frac{du}{u} \to \int_1^{+\infty} \beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\frac{du}{u}.$$

In particular, we may also have

$$\bar{\partial}\partial\int_1^A \beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\frac{du}{u} \to \bar{\partial}\partial\int_1^{+\infty} \beta_\infty(\tilde{\eta}_u - \tilde{\eta}_\infty)\frac{du}{u},$$

in $\mathcal{D}'_{N^\bullet_{\mathbb{R}}+\bar{N}^\bullet_{\mathbb{R}}}(M)$. Now we need the following

Lemma. With the same notation as above,
(1) $\int_1^A(\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u\frac{du}{u} \to \int_1^\infty(\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u\frac{du}{u}$, in $\mathcal{D}'_{N^\bullet_{\mathbb{R}}+\bar{N}^\bullet_{\mathbb{R}}}(M)$. In particular,

$$\partial\int_1^A (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u\frac{du}{u} \to \partial\int_1^\infty (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u\frac{du}{u},$$

in $\mathcal{D}'_{N^\bullet_{\mathbb{R}}+\bar{N}^\bullet_{\mathbb{R}}}(M)$.
(2) $\int_1^A \partial(\beta_u - \beta_\infty)\tilde{\eta}_u\frac{du}{u} \to \int_1^\infty \partial(\beta_u - \beta_\infty)\tilde{\eta}_u\frac{du}{u}$, in $\mathcal{D}'_{N^\bullet_{\mathbb{R}}+\bar{N}^\bullet_{\mathbb{R}}}(M)$. In particular,

$$\bar{\partial}\int_1^A \partial(\beta_u - \beta_\infty)\tilde{\eta}_u\frac{du}{u} \to \bar{\partial}\int_1^\infty \partial(\beta_u - \beta_\infty)\tilde{\eta}_u\frac{du}{u},$$

in $\mathcal{D}'_{N^\bullet_{\mathbb{R}}+\bar{N}^\bullet_{\mathbb{R}}}(M)$.

Suppose we have this lemma, then we may take the limit for A at any place above. In particular, we get in $P^M_{M'\cup\tilde{M}'}/P^{M,0}_{M'\cup\tilde{M}'}$,

$$\begin{aligned}&\int_0^1(\beta''_u - \beta''_0)\frac{du}{u}+\int_1^{+\infty}(\beta''_u - \beta''_\infty)\frac{du}{u}\\ &= \tilde{\alpha}_0\{\int_0^1(\beta_u - \beta_0)\frac{du}{u}+\int_1^{+\infty}(\beta_u - \beta_\infty)\frac{du}{u}\}\\ &+\alpha_\infty\{\int_0^1(\tilde{\beta}_u - \tilde{\beta}_0)\frac{du}{u}+\int_1^{+\infty}(\tilde{\beta}_u - \tilde{\beta}_\infty)\frac{du}{u}\}.\end{aligned}$$

From here, we easily have the first assertion by certain trivial substitution.

Proof of the lemma. We only prove (1), as the rest is very similar. Obviously, the difficulties in this convergence only occur near M''. For any $x \in M''$, let U be a small open neighborhood of x, and let Γ, φ, m be as usual but with respect to M'. We know that, by Theorem 2.c,

$$|\varphi(\widehat{\beta_u - \beta_\infty})(\xi)| \leq \frac{C}{\sqrt{u}}(1 + |\xi|),$$

$$p_{U,\Gamma,\varphi,m}(\beta_u - \beta_\infty) \leq \frac{C}{\sqrt{u}},$$

$$|\widehat{\tilde{\varphi}\bar{\partial}\tilde{\eta}_u}(\xi)| \leq C(1 + |\xi|^2),$$

$$p_{U,\tilde{\Gamma},\tilde{\varphi},\tilde{m}}(\tilde{\eta}_u) \leq C.$$

Therefore, by definition and the condition $N^*_{\mathbf{R}}|_{M'} \cap \tilde{N}_{\mathbf{R}}|_{M'} = \{0\}$, from Theorem 8.7.4 of [Hö 83], we know that

(1) There exists a natural number k such that for $u \geq 1$,

$$|\varphi(\widehat{\beta_u - \beta_\infty})\tilde{\varphi}\bar{\partial}\tilde{\eta}_u(\xi)| \leq \frac{C}{\sqrt{u}}(1 + |\xi|)^k.$$

(2) For any smooth current θ with compact support in U, if Δ is a closed cone in $\mathbf{R}^{2l}$ such that

$$\Delta \cap (N^*_{\mathbf{R}} + \tilde{N}^*_{\mathbf{R}}) = \{0\}$$

on $(M' \cup \tilde{M}') \cap U$, then for $u \geq 1$,

$$p_{U,\Delta,\theta,m''}((\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u) \leq \frac{C}{\sqrt{u}}.$$

Therefore, in $\mathcal{D}'_{N^*_{\mathbf{R}} + \tilde{N}^*_{\mathbf{R}}}(M)$,

$$\int_1^A (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u \frac{du}{u} \to \int_1^\infty (\beta_u - \beta_\infty)\bar{\partial}\tilde{\eta}_u \frac{du}{u},$$

which completes the proof of the lemma and hence the theorem.

We have now finished our account of the theory of Bott-Chern secondary characteristic objects in various contexts, except for the uniquness, which will be proved after the proof of arithmetic Riemann-Roch theorem for l.c.i. morphisms. Next, we discuss the singularities of the relative Bott-Chern secondary characteristic currents with respect to closed immersions, which will be used in the ternary theory.

§I.8.5. The Singularities of Relative Bott-Chern Secondary Characteristic Currents

In this section, we prove that, in general, the current $\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, g_i; \xi, g^\xi)$ is smooth on $M - M'$ and not locally integrable on M. In the process, we determine the singularities of $\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, g_i; \xi, g^\xi)$ near M'. In particular, if Y is a normal coordinate to M', then near M', $\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, g_i; \xi, g^\xi) \sim |Y|^{-2\dim N}$, hence the integral of $\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, g_i; \xi, g^\xi)$ on the complement of a δ neighborhood of M' in M is equivalent to $c\,\mathrm{Log}\delta$ as $\delta \to 0$. Furthermore, we can calculate $\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, g_i; \xi, g^\xi)$ as a finite part by subtracting off the logarithmic divergence. All the results in this section will be used in next chapter to discuss the deformation theory of relative Bott-Chern secondary characteristic currents.

I.8.5.a. **The Singularities**

We use the same notation as above. Identifying N with the orthogonal to TM' in TM as C^∞ bundle, we assume that g^N is exactly the metric induced by g^{TM}. For $x_0 \in M'$, let U be an open neighborhood of x_0 in M, and $z = (z^1, \ldots, z^l)$ be a holomorphic system of coordinates on U such that $V = M' \cap U$ is represented by $z^{k+1} = \ldots = z^l = 0$. Set $x = (z^1, \ldots, z^k)$, $y = (z^{k+1}, \ldots, z^l)$. Then x is a coordinate system on V. For $\varepsilon > 0$, let D_ε be the open ball with the center 0 and radius ε in $\mathbf{C}^e$. Then for ε small enough, we have $V \times D_\varepsilon \subset U$. Usually, we use the convention that $x \in V, y \in D_\varepsilon$. If $(x, y) \in V \times D_\varepsilon$, we consider y as an element of N_x.

On $V \times D_\varepsilon$, we have

$$\wedge^p(T^*_{\mathbf{R}}M) = \oplus_{i+j=p} \wedge^i (T^*_{\mathbf{R}}M') \hat{\otimes} \wedge^j (N^*_{\mathbf{R}}).$$

Thus, for any α, we have the decomposition $\alpha = \sum_{p=0}^{2\dim N} \alpha^p$ according to the partial degree in the Grassmannian variables of $N^*_{\mathbf{R}}$. We denote the maximal degree element in this expression by $\alpha^{\max}$. In particular, on M', i.e. at $(x, 0)$, $\alpha^{\max}$ does not depend on the coordinate system (x, y).

For $u > 0$, we have the superconnection

$$\mathbf{B}_u = \nabla^F + \sqrt{u}\partial_Y V.$$

Then we have the differential form $\mathrm{Tr}_s[N_H \exp(-\mathbf{B}_u^2)]$ on N, which in the sequel, we consider as a form on $V \times \mathbf{C}^e$. Thus

$$\mathrm{Tr}_s[N_H \exp(-(\nabla^F)^2)]^{\max} = 0,$$

and as $u \to 0^+$,

$$\mathrm{Tr}_s[N_H \exp(-\mathbf{B}_u^2)]^{\max} = O(u).$$

Hence on $N - \{0\}$, we have a smooth form

$$\beta_F := \int_0^{+\infty} [2\pi \mathrm{i}] \mathrm{Tr}_s[N_H \exp(-\mathbf{B}_u^2)]^{\max} \frac{du}{u}.$$

In fact, as $u \to +\infty$, the form $\mathrm{Tr}_s[N_H \exp(-\mathbf{B}_u^2)]$ decays exponentially, so β_F is well-defined and depends on the coordinate system (x, y).

Theorem. (1) For any $a > 0$, $y \in N - \{0\}$,

$$\beta_F(ay) = \frac{1}{a^{2\dim N}}\beta_F(y).$$

(2) Let $\omega(\xi, h^\xi)$ be the smooth form on $M - M'$ defined by the restriction of the current

$$\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, g_i; \xi, g^\xi)$$

to $M - M'$. Then there exists $C > 0$ such that, if $(x, y) \in V \times D_\epsilon$,

$$|y|^{2\dim N}|\omega(\xi, g^\xi)(x, y) - \beta_F(x, y)| \leq C|y|.$$

Hence the current $\omega(\xi, g^\xi) - \beta_F$ is integrable on $V \times D_\epsilon$.
(3) Let $\theta(\xi, g^\xi)$ be the smooth form on $M - M'$ defined by the restriction of the current

$$\gamma(\eta, g^\eta; i, g_i; \xi, g^\xi)$$

to $M - M'$. Here γ is defined in subsection 4.a. Then $\theta(\xi, g^\xi)$ is locally integrable on M, and coincides as a current with

$$\gamma(\eta, g^\eta; i, g_i; \xi, g^\xi).$$

(4) If the metric g^ξ satisfies Bismut condition (A) with respect to g^N and g^η, then

$$\beta_F = -(\dim N - 1)!\frac{1}{|Y|^{2\dim N}}(\mathrm{td}^{-1})'(N, g^N)\mathrm{ch}(\eta, g^\eta)\frac{\lambda}{\pi^{\dim N}},$$

where λ is the volume form on N with respect to the metric g^N.

Proof. (1) Let ϕ_t be a group of diffeomorphisms of N: $y \mapsto e^t y$. Then the group ϕ_t is generated by the vector field $Y = y + \bar{y}$. Also

$$\mathrm{Tr}_s[N_H \exp(-\mathbf{B}_u^2)] = \phi^*_{\mathrm{Log}\,u/2}\mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)].$$

So if τ_a is the map $(x, y) \mapsto (x, ay)$, then $\tau_a^*\beta_F = \beta_F$. But naturally we have

$$\tau_a^*\beta_F(x, y) = a^{2\dim N}\beta_F(x, ay).$$

Hence we have (1).

(2) In the sequel, the constants C may vary from line to line. In the definition of $\mathrm{ch}_{\mathrm{BC}}$ in section 3, the first integral $\int_0^1\{\ \}\frac{du}{u}$ defines a smooth form on M, and so does not contribute to the singular part of the current $\zeta_\xi'(0)$ near M'. So we only need to consider the other terms.

Let α_u and δ_u be the forms on M and N respectively given by

$$\alpha_u := \mathrm{Tr}_s[N_H \exp(-\mathbf{A}_u^2)], \quad \delta_u := \mathrm{Tr}_s[N_H \exp(-\mathbf{B}_u^2)].$$

Then

$$\int_1^{+\infty} \alpha_u(x,y)\frac{du}{u} = \int_{|y|^2/\varepsilon^2}^{+\infty} \alpha_{\varepsilon^2 u/|y|^2}(x,y)\frac{du}{u}.$$

On the other hand, if we denote $\sigma_u = \tau_{1/\sqrt{u}}$, then by the proof of Lemma 8.1, we know that if V and ε are small enough, there exist C, $C' > 0$ such that for $u \geq 1, x \in V, |y| \leq \varepsilon\sqrt{u}$, we have

$$|\sigma_u^*\alpha_u(x,y) - \delta_1(x,y)| \leq \frac{C}{\sqrt{u}}\exp(-C'|y|^2).$$

Thus by the fact that

$$(\sigma_u^*\alpha)(x,y) = \sum_0^{2\dim N} u^{-p/2}\alpha_u^p(x, \frac{y}{\sqrt{u}}),$$

we know that for $0 < \eta \leq 1, |y| \leq \varepsilon$,

$$|\sum_0^{2\dim N} \eta^p \alpha_{1/\eta^2}^p(x,y) - \delta_1(x,\frac{y}{\eta})| \leq C\eta\exp(-C'\frac{|y|^2}{\eta^2}).$$

So if $|y| \leq \varepsilon\sqrt{u}$, then $\eta \leq 1$, and

$$|\sum_0^{2\dim N} (\frac{y}{\varepsilon\sqrt{u}})^p \alpha_{\varepsilon^2 u/|y|^2}^p(x,y) - \delta_1(x,\varepsilon\sqrt{u}\frac{y}{|y|})| \leq C\frac{|y|}{\varepsilon\sqrt{u}}\exp(-C'\varepsilon^2 u).$$

So we have proved

Lemma 1. With the same notation as above, if $|y| \leq \varepsilon$, $0 \leq p \leq 2\dim N$,

$$\begin{aligned} ||y|^p \int_1^{+\infty} \alpha_u^p(x,y)\frac{du}{u} - \int_{|y|^2/\varepsilon^2}^{+\infty} (\varepsilon\sqrt{u})^p \delta_1^p(x,\varepsilon\sqrt{u}\frac{y}{|y|})\frac{du}{u}| \\ \leq C|y| \int_{|y|^2/\varepsilon^2}^{+\infty} \exp(-C'\varepsilon^2 u)(\varepsilon\sqrt{u})^{p-1}\frac{du}{u}. \end{aligned}$$

Now we can deduce assertion 2 from the following

Lemma 2. As $|y| \to 0$,
(a) $|y|^{2\dim N}|\int_1^{+\infty} \alpha_u^0(x,y)\frac{du}{u}| \leq C|y|$.
(b) For $1 \leq p \leq 2\dim N - 1$,

$$|y|^{2\dim N}|\int_1^{+\infty} \alpha_u^p(x,y)\frac{du}{u}| \leq C|y|.$$

(c) $|y|^{2\dim N}|\int_1^{+\infty} [2\pi \mathbf{i}]\alpha_u^{2\dim N}(x,y)\frac{du}{u} - \beta_F(x,\frac{y}{|y|})| \leq C|y|$.

Suppose we have this lemma, then note that since the function $1/|y|^{2\dim N-1}$ is integrable near 0, by (1), we have (2).

(3) First, we have

$$\sqrt{u}\mathrm{Tr}_s[\partial_Y V\exp(-\mathbf{B}_u^2)] = -i_Y \mathrm{Tr}_s[\exp(-\mathbf{B}_u^2)].$$

Therefore

$$\sqrt{u}\mathrm{Tr}_s[\partial_Y V\exp(-\mathbf{B}_u^2)]^{\max} = 0.$$

Similarly as in Lemma 2, we see that

$$|y|^{2\dim N}|\theta(\xi, g^\xi)(x,y)| \le C|y|.$$

As a consequence, $\theta(\xi, g^\xi)$ is integrable near M'.

Replacing in the analogue the integrals $\int_1^{+\infty}\{\ \}\frac{du}{u}$ by integrals $\int_1^a\{\ \}\frac{du}{u}$, we find that if $\theta^a(\xi, g^\xi)$ is the density of the smooth approximating current $\gamma^a(\xi, g^\xi)$, then we have the uniform estimate for $a \ge 1$, $x \in V$, $|y| \le \varepsilon$,

$$|y|^{2\dim N}|\theta^a(\xi, g^\xi)(x,y)| \le C|y|.$$

Thus as $u \to \infty$, $\theta^a(\xi, g^\xi) \to \theta(\xi, g^\xi)$. But $\gamma^a(\xi, g^\xi) = \theta^a(\xi, g^\xi)$ and $\gamma^a(\xi, g^\xi) \to \gamma^a(\xi, g^\xi)$, so $\theta(\xi, g^\xi) = \gamma(\xi, g^\xi)$.

(4) This follows from the fact that

$$\begin{aligned}&\mathrm{Tr}_s[N_H\exp(-\mathbf{B}_u^2)]^{\max}\\ &= -(iu)^{\dim N}(\mathrm{td}^{-1})'(-(\nabla^F)^2)\,\mathrm{Tr}[\exp(-(\nabla^\eta)^2)]\exp\left(\frac{-u|Y|^2}{2}\right)\lambda,\end{aligned}$$

which may be proved as what we did for Theorem 7.1.c.

Finally we end this subsection by the following

Proof of Lemma 2. Since

$$\delta_a = \tau_{\sqrt{a}}^*\delta_1,$$

if $y \ne 0$, we have

$$(\varepsilon\sqrt{u})^p\,\delta_1^p(x, \varepsilon\sqrt{u}\frac{y}{|y|}) = \delta_{\varepsilon^2 u}^p(x, \frac{y}{|y|}).$$

(a) By Lemma 1, the assertion is a consequence of the facts that

$$\int_{|y|^2/\varepsilon^2}^{+\infty} \delta_{\varepsilon^2 u}^0(x, \frac{y}{|y|}) \le C(\mathrm{Log}|y| + C'),$$

and $2\dim N \ge 2$.

(b) By Lemma 1, note that for $p \geq 1$, the integral

$$\int_{|y|^2/\varepsilon^2}^{+\infty} (\varepsilon\sqrt{u})^p \delta_1^p(x, \varepsilon\sqrt{u}\frac{y}{|y|})\frac{du}{u}$$

is bounded as $|y| \to 0$. Also, the expression on the right hand side of lemma 1 is bounded, so we have (b).

c. Since $2\,\mathrm{dim}N \geq 2$, we have

$$|\int_0^{|y|^2/\varepsilon^2} (\varepsilon\sqrt{u})^{2\mathrm{dim}N}\delta_1^{2\mathrm{dim}N}(x, \varepsilon\sqrt{u}\frac{y}{|y|})\frac{du}{u}| \leq C|y|^2.$$

Thus by Lemma 1, we have

$$||y|^{2\mathrm{dim}N}\int_1^{+\infty}[2\pi\mathrm{i}]\alpha_u^{2\mathrm{dim}N}(x,y)\frac{du}{u} - \int_0^{+\infty}\delta_{\varepsilon^2 u}^{2\mathrm{dim}N}(x, \frac{y}{|y|})\frac{du}{u} \leq C|y|.$$

So we have the assertion (c).

I.8.5.b. **The Principle Part.**

The form $\omega(\xi, g^\xi)$ is not integrable on M in general. However, by Theorem a.(2), we know that it has a well-defined principal part, which defines a current. We now compare the current $\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, g^i; \xi, g^\xi)$ with this principal part.

Theorem. (1) For $\eta > 0$, let M^η denote the set of points of M whose Riemannian distances to M' are greater than η. Let μ be a smooth even form on M, then as $\eta > 0$ converges to 0,

$$\int_{M^\eta}\mu\,\omega(\xi, g^\xi) + 2\,\mathrm{Log}\eta\int_{M'} i^*\mu(\int_N [2\pi\mathrm{i}]\mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)])$$

has a limit, which we denote by $\int_M \mu\omega^c(\xi, g^\xi)$

(2) We have

$$\int_M \mu\mathrm{ch}_{\mathrm{BC}}(\eta, g^\eta; i, g^i; \xi, g^\xi)$$
$$= \int_M \mu\omega^c(\xi, g^\xi) - \int_{M'} i^*\mu\int_N (2\mathrm{Log}|Y| - \Gamma'(1))[2\pi\mathrm{i}](\mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)]).$$

(3) If the metrics g^ξ satisfy Bismut condition (A) with respect to g^N and g^η, then

$$\int_N (2\mathrm{Log}|Y| - \Gamma'(1))[2\pi\mathrm{i}](\mathrm{Tr}_s[N_H\exp(-\mathbf{B}^2)])$$
$$= -(\mathrm{td}^{-1})'(N, g^N)\mathrm{ch}(\eta, g^\eta)(\sum_{k=1}^{\mathrm{dim}N-1}\frac{1}{k} + \mathrm{Log}2).$$

Proof. With the same notation as above, we choose geodesic coordinates in the directions of $T_{\mathbf{R}}M$, which are normal to $T_{\mathbf{R}}M'$ with respect to the given Euclidean scalar product of $T_{\mathbf{R}}M$. For $\varepsilon > 0$, set $B_\varepsilon^{\mathbf{R}} := \{Y \in \mathbf{R}^{2e}; |Y| \le \varepsilon\}$; $U = V \times B_\varepsilon^{\mathbf{R}}$ is then a small neighborhood of x in M. We identify $\mathbf{R}^{2e}$ with the real normal bundle $N_{\mathbf{R}}$ to M' in M. Set

$$\omega := [2\pi \mathrm{i}]^{-1}\omega(\xi, g^\xi), \quad \beta := [2\pi \mathrm{i}]^{-1}\beta_F(\xi, g^\xi).$$

(1) We may assume that the support of μ is contained in U. The form $i^*\mu$ on V lifts naturally to a form on $V \times \mathbf{R}^{2e}$. Moreover, $i^*\mu$ has partial vertical degree 0 and coincides with $\mu^0(x,0)$. Hence it makes sense to consider the following identity

$$\int_{M^\eta} \mu\omega = \int_{M^\eta} (\mu - 1_{|Y|\le\epsilon} i^*\mu)\omega + \int_{M^\eta} 1_{|Y|\le\epsilon}(i^*\mu)\omega.$$

By the condition that μ is smooth, we have

$$|\mu^0(x,y) - \mu^0(x,0)| \le C|y|.$$

Thus, by Theorem 5.a, we have

$$\lim_{\eta\to 0} \int_{M^\eta} (\mu - 1_{|Y|\le\epsilon} i^*\mu)\omega = \int_M (\mu - 1_{|Y|\le\epsilon} i^*\mu)\omega.$$

Moreover

$$\int_{M^\eta} 1_{|Y|\le\epsilon}(i^*\mu)\omega = \int_{M^\eta} 1_{|Y|\le\epsilon}(i^*\mu)(\omega - \beta) + \int_{M^\eta} 1_{|Y|\le\epsilon}(i^*\mu)\beta.$$

So by the same theorem, we have

$$\lim_{\eta\to 0} \int_{M^\eta} 1_{|Y|\le\epsilon}(i^*\mu)(\omega - \beta) = \int_M 1_{|Y|\le\epsilon}(i^*\mu)(\omega - \beta).$$

But by Theorem 5.a.(1), we have that if $\eta \le \varepsilon$, then

$$\int_{M^\eta} 1_{|Y|\le\epsilon}(i^*\mu)\beta(y) = \int_{M^\eta} 1_{|Y|\le\epsilon}(i^*\mu)|Y|^{-2\dim N}\beta(\frac{y}{|Y|}).$$

Also,

$$\int_{M^\eta} 1_{|Y|\le\epsilon}(i^*\mu)\beta(y) = (\mathrm{Log}\,\varepsilon - \mathrm{Log}\,\eta)\int_{M'} i^*\mu \int_{S_N} i_n\beta.$$

Now the assertion is a direct consequence of the following local result:

Lemma. Let $S_N := \{Y \in N_{\mathbf{R}}; |Y| = 1\}$ be the unit sphere in $N_{\mathbf{R}}$. S_N is naturally oriented with n the unit vector in $N_{\mathbf{R}}$ normal to S_N and pointing outwards. Suppose

α is a smooth form on N such that there exist $c, C > 0$ for which $|\alpha| \leq c\exp(-C|y|^2)$. Then, on M', we have

$$\int_N \alpha = \int_{S_N} \int_{-\infty}^{+\infty} i_n \phi_s^* \alpha ds.$$

In fact, by this lemma, we know that

$$\int_{S_N} i_n \beta_F = 2 \int_N [2\pi \mathbf{i}] \mathrm{Tr}_s [N_H \exp(-\mathbf{B}^2)].$$

Therefore, as $u \to 0^+$,

$$\int_{M^\eta} \mu\omega(\xi, g^\xi) + 2 \operatorname{Log} \eta \int_{M'} i^* \mu \int_N [2\pi \mathbf{i}] \mathrm{Tr}_s [N_H \exp(-\mathbf{B}^2)])$$

has a limit

$$\int_M \mu\omega^c(\xi, g^\xi) = \int_M (\mu - 1_{|Y| \leq \varepsilon} i^* \mu)\omega + \int_M 1_{|Y| \leq \varepsilon} (i^* \mu)(\omega - \beta)$$
$$+ 2 \operatorname{Log} \varepsilon \int_{M'} i^* \mu (\int_N \mathrm{Tr}_s [N_H \exp(-\mathbf{B}^2)]).$$

Proof of the lemma. First, by the hypothesis, we know that the integrate

$$\int_{-\infty}^{+\infty} i_n \phi_s^* \alpha ds$$

exists. Let $F : \mathbf{R} \times S_N \to N_\mathbf{R}$ be defined by $(s, y) \mapsto \phi_s(y)$, then

$$\int_N \alpha = \int_{\mathbf{R} \times S_N} F^* \alpha.$$

Thus if j is the embedding $S_N \hookrightarrow N_\mathbf{R}$,

$$(F^* \alpha)(s, y) = j^* \phi_s^* \alpha + ds\, j^* (i_Y \phi_s^* \alpha).$$

This proves the lemma.

(2) By definition, we know that

$$\int_M \mu \int_0^T \alpha_u \frac{du}{u} = \int_M \int_{|Y|^2/\varepsilon^2}^{T|Y|^2/\varepsilon^2} \mu \alpha_{\varepsilon^2 u/|Y|^2} \frac{du}{u}.$$

We discuss the various cases.

(a) If $0 \le p \le 2N-1$, as $T \to +\infty$, by the latest relation before Lemma 5.a.1, we get

$$\int_M \mu \int_1^T \alpha_u^p \frac{du}{u} \to \int_M \mu \int_1^{+\infty} \alpha_u^p \frac{du}{u},$$

which is a locally integrable current.

(b) Set

$$\bar\alpha := \alpha^{\max}, \quad \bar\delta = \delta^{\max}.$$

Then

$$\begin{aligned}\int_M \int_0^T \mu\bar\alpha_u \frac{du}{u} = &\int_M \int_1^T (\mu^0 - 1_{|Y|\le\epsilon} i^*\mu)\bar\alpha_u \frac{du}{u}\\ &+ \int_M \int_1^T 1_{|Y|\le\epsilon}\mu^0(x,0)(\bar\alpha_u - \bar\delta_u)\frac{du}{u}\\ &+ \int_M \int_1^T 1_{|Y|\le\epsilon}\mu^0(x,0)\bar\delta_u \frac{du}{u}.\end{aligned}$$

We consider this term by term.

(b.1) By the same relation as used in (a), as $T \to +\infty$, we have

$$\int_M \int_1^T (\mu^0 - 1_{|Y|\le\epsilon} i^*\mu)\bar\alpha_u \frac{du}{u} \to \int_M \int_1^{+\infty} (\mu^0 - 1_{|Y|\le\epsilon} i^*\mu)\bar\alpha_u \frac{du}{u}$$

and

$$\int_M \int_1^T 1_{|Y|\le\epsilon}\mu^0(x,0)(\bar\alpha_u - \bar\delta_u)\frac{du}{u} \to \int_M \int_1^{+\infty} 1_{|Y|\le\epsilon}\mu^0(x,0)(\bar\alpha_u - \bar\delta_u)\frac{du}{u}.$$

(b.2) Since

$$\bar\delta_u(x,y) = \frac{1}{|Y|^{2\dim N}}\bar\delta_{u|Y|^2}(x, \frac{y}{|Y|}),$$

we have

$$\int_M \int_1^T 1_{|Y|\le\epsilon}\mu^0(x,0)\bar\delta_u \frac{du}{u} = \int_M \int_{|Y|^2}^{T|Y|^2} 1_{|Y|\le\epsilon}\mu^0(x,0)\frac{1}{|Y|^{2\dim N}}\bar\delta_u(x, \frac{y}{|Y|})\frac{du}{u}.$$

But for $0 \le T \le T\varepsilon^2$,

$$\begin{aligned}&\int_M 1_{\sqrt{u/T}\le|Y|\le(\sqrt{u}/\epsilon)}\mu^0(x,0)\frac{1}{|Y|^{2\dim N}}\bar\delta_u(x, \frac{y}{|Y|})\\ =&(\operatorname{Log}(\sqrt{u}/\varepsilon) - \operatorname{Log}(\sqrt{\frac{u}{T}}))\int_{M'} i^*\mu \int_{S_N} i_n\bar\delta_u(x,y).\end{aligned}$$

So

$$\int_M \int_1^T 1_{|Y|\leq\epsilon}\mu^0(x,0)\bar{\delta}_u \frac{du}{u} = \frac{1}{2}\operatorname{Log} T \int_Y i^*\mu \int_{S_N}\int_0^{T\epsilon^2} i_n\bar{\delta}_u \frac{du}{u} + \int_Y i^* u \int_{S_N}\int_{\epsilon^2}^{T\epsilon^2} \frac{1}{2}\operatorname{Log}(\frac{\varepsilon^2}{u}) i_n \bar{\delta}_u \frac{du}{u}.$$

Furthermore, by the fact that there exists a $C > 0$ so that for $u \geq 1$

$$|\int_{S_N} i_n\bar{\delta}_u| \leq \exp(-Cu),$$

we have

$$\int_{S_N}\int_{\epsilon^2}^{+\infty} \frac{1}{2}\operatorname{Log}(\frac{\varepsilon^2}{u}) i_n\bar{\delta}_u \frac{du}{u} = -\int_{S_N}\int_{\epsilon^2}^{+\infty} 1_{\epsilon<|Y|\leq\sqrt{u}} \frac{1}{|Y|^{2\dim N}}\bar{\delta}_u(x, \frac{y}{|Y|})\frac{du}{u} = -\int_1^{+\infty}\frac{du}{u}\int_N 1_{|Y|\geq\epsilon}\bar{\delta}_u.$$

Therefore, put (b.1) and (b.2) together, we finally have

$$\lim_{T\to+\infty}\{\int_M\int_1^T 1_{|Y|\leq\epsilon}\mu^0(x,0)\bar{\delta}_u\frac{du}{u} - \operatorname{Log} T\int_Y i^*\mu\int_N \delta_1\} = -\int_Y i^*\mu\int_N 1_{|Y|\geq\epsilon}\int_1^{+\infty}\bar{\delta}_u\frac{du}{u}.$$

Hnece, by definition, we have

$$\int_M \mu\,\zeta_\xi'(0) = \lim_{T\to+\infty}\{\int_0^1\int_M \mu(\alpha_u - \alpha_0)\frac{du}{u} + \int_1^T\int_M \mu\alpha_u\frac{du}{u} - \Gamma'(1)(\int_M \mu\alpha_0 - \int_Y i^*\mu\int_N \delta_1) - \operatorname{Log} T\int_Y i^*\mu\int_N \bar{\delta}_1\}.$$

Thus by (a) and (b) above, we have

$$\begin{aligned}\int_M \mu\,\zeta_\xi'(0) = &\int_M\int_0^1 \mu(\alpha_u-\alpha_0)\frac{du}{u} + \sum_0^{2\dim N-1}\int_M \mu\int_1^{+\infty}\alpha_u^p\frac{du}{u}\\ &+\int_M(\mu^0 - 1_{|Y|\leq\epsilon}i^*\mu)\int_1^{+\infty}\tilde{\alpha}_u\frac{du}{u}\\ &+\int_M 1_{|Y|\leq\epsilon}i^*\mu\int_1^{+\infty}(\tilde{\alpha}_u-\bar{\delta}_u)\frac{du}{u} - \int_M 1_{|Y|>\epsilon}i^*\mu\int_1^{+\infty}\bar{\delta}_u\frac{du}{u}\\ &-\Gamma'(1)(\int_M\mu\alpha_0 - \int_{M'} i^*\mu\int_N \delta_1).\end{aligned}$$

So, by (1), we have

$$\int_M \mu\,\zeta_\xi'(0) = \int_M \mu\,\omega^c + \int_{M'} i^*\mu \int_N [1_{|Y|\le\epsilon}(\beta - \int_1^{+\infty} \bar\delta_u \frac{du}{u}) - 1_{|Y|>\epsilon}\int_1^{+\infty} \bar\delta_u \frac{du}{u} + (\Gamma'(1) - 2\,\mathrm{Log}\,\epsilon)\delta_1].$$

Now

$$\int_N [1_{|Y|\le\epsilon}(\beta - \int_1^{+\infty} \bar\delta_u \frac{du}{u}) - 1_{|Y|>\epsilon}\int_1^{+\infty} \bar\delta_u \frac{du}{u}] = \int_N [1_{|Y|\le\epsilon}\int_0^1 \bar\delta_u \frac{du}{u} - 1_{|Y|>\epsilon}\int_1^{+\infty} \bar\delta_u \frac{du}{u}].$$

So by the fact that $\bar\delta_u = \phi^*_{(\mathrm{Log}\,u)/2}\bar\delta_1$, we have

$$\begin{aligned}\int_N 1_{|Y|\le\epsilon}\int_0^1 \bar\delta_u \frac{du}{u} &= \int_0^1 \frac{du}{u}\int_N 1_{|Y|\le\epsilon\sqrt{u}}\bar\delta_1 \\ &= 2\int_N 1_{|Y|\le\epsilon}(\mathrm{Log}\,\epsilon - \mathrm{Log}\,|Y|)\bar\delta_1; \text{ and,} \\ \int_N 1_{|Y|>\epsilon}\int_1^{+\infty} \bar\delta_u \frac{du}{u} &= \int_1^{+\infty} \frac{du}{u}\int_N 1_{|Y|>\epsilon\sqrt{u}}\bar\delta_1 \\ &= 2\int_N 1_{|Y|>\epsilon}(\mathrm{Log}\,\epsilon - \mathrm{Log}\,|Y|)\bar\delta_1.\end{aligned}$$

Hence

$$\int_N [1_{|Y|\le\epsilon}(\beta - \int_1^{+\infty} \bar\delta_u \frac{du}{u}) - 1_{|Y|>\epsilon}\int_1^{+\infty} \bar\delta_u \frac{du}{u}] = 2\int_N (\mathrm{Log}\,\epsilon - \mathrm{Log}\,|Y|)\bar\delta_1.$$

Therefore, we have (2).

(3) By an argument similar to the proof of Theorem 7.1.c, we have

$$\begin{aligned}&\int_N \exp(\frac{-t|Y|^2}{2})\,\mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)] \\ &= -\frac{1}{(1+t)^{\dim N}}(2\pi i)^{\dim N}(\mathrm{td}^{-1})'(-(\nabla^N)^2)\mathrm{Tr}[\exp(-(\nabla^\eta)^2)].\end{aligned}$$

But for $0 < \mathrm{Re}(s) < \dim N$, we have

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^{+\infty} t^s \exp(\frac{-t|Y|^2}{2})\frac{dt}{t} &= (\frac{2}{|Y|^2})^s; \\ \frac{1}{\Gamma(s)}\int_0^{+\infty} \frac{t^s}{(1+t)^{\dim N}}\frac{dt}{t} &= \frac{\Gamma(\dim N - s)}{\Gamma(\dim N)}.\end{aligned}$$

So, for $0 < \mathrm{Re}(s) < \dim N$, we have

$$\int_N \left(\frac{2}{|Y|^2}\right)^s \mathrm{Tr}_s[N_H \exp(-\mathbf{B}^2)]$$
$$= -(2\pi i)^{\dim N} (\mathrm{td}^{-1})'(-(\nabla^N)^2) \mathrm{Tr}[\exp(-(\nabla^\eta)^2)] \frac{\Gamma(\dim N - s)}{\Gamma(\dim N)}.$$

Each side of this equation extends to a meromorphic function of s which is holomorphic at $s = 0$. Thus

$$\int_N (2\mathrm{Log}\,|Y| - \mathrm{Log}\,2) \mathrm{Tr}_s[N_H \exp(-B^2)]$$
$$= -(2\pi i)^{\dim N} (\mathrm{td}^{-1})'(-(\nabla^N)^2)\,\mathrm{Tr}[\exp(-(\nabla^\eta)^2)] \frac{\Gamma'(\dim N)}{\Gamma(\dim N)}.$$

Now by the fact that $\Gamma(s+1) = \Gamma(s)$, we get

$$\frac{\Gamma'(\dim N)}{\Gamma(\dim N)} = \Gamma'(1) + \sum_{k=0}^{\dim N - 1} \frac{1}{k},$$

which completes the proof.

Chapter I.9.
Ternary Objects And Deformation To The Normal Cone

In the previous chapters, we gave the theory about the Bott-Chern secondary characteristic objects for various situations. Among others, the very important property for the secondary characteristic objects is that they measure the change of characteristic objects. Thus if we want to measure the change of secondary characteristic objects, naturally, we need a kind of the so-called ternary objects.

Since we do not find any further application at the present time for the ternary objects in general, in this chapter, we only deal with a special situation, which will be used in the proof of the arithmetic Riemann-Roch theorem in part II: We discuss the deformation theory of relative Bott-Chern secondary characteristic objects.

Before reading the first section, it may be helpful to look at the beginning of I.9.2. Also, if the reader is only interested in the deformation theory for the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms, he or she may just read I.9.2. References here are [BGS 91], [Fa 92], [We 91].

§I.9.1. Basic Constructions And Facts

We recall the construction of the deformation to the normal cone for certain closed immersions $i : X \hookrightarrow Y$ and some associated facts. The advantage of this construction is that, to study a property for an arbitrary closed immersion, we only need to study the property for a section of a projective bundle. For topologists, this means that in a homotopy class, one can find a good representative so that certain properties are very easy to check. The references here are [SGA 6], [Ha 77] and [BFM 75].

I.9.1.a. Projective Bundles And Koszul Complexes

Let $i : Y \hookrightarrow X$ be a closed embedding of complex manifolds. Denote by $\mathcal{I}_i$ the **ideal sheaf** of Y in X. It is a standard fact that if X is of codimension one, then $\mathcal{I}_i \simeq \mathcal{O}_X(-Y)$ and there is a canonical inclusion $\mathcal{O}_X \subset \mathcal{O}_X(-Y)$. Also, there is a canonical normal bundle

N_i associated with i. We then have the following canonical isomorphisms:

$$i^*\mathcal{I}_i \simeq \mathcal{N}_i^* \simeq \mathcal{I}_i/\mathcal{I}_i^2,$$

where $\mathcal{N}^*$, the **conormal sheaf**, is the dual of the sheaf of the sections of N_i. Moreover, we have

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \wedge^i \mathcal{N}_i.$$

(The definition of Tor is given in II.1.)

Let $\varphi : \mathcal{V} \to \mathcal{O}_X$ be a homomorphism of vector sheaves: Then the **differential** in the associated **Koszul complex** $K(\varphi) := (\wedge^{\cdot}\mathcal{V})$ is the complex defined by

$$d_i^{\varphi} : \wedge^i \mathcal{V} \to \wedge^{i-1}\mathcal{V}$$
$$e_1 \wedge \ldots \wedge e_i \mapsto \sum_{k=1}^{i} (-1)^{k-1} \varphi(e_k) e_1 \wedge \ldots \wedge \hat{e}_k \wedge \ldots \wedge e_i$$

Now let $p : E \to X$ be a vector bundle with $\mathcal{E}$ the sheaf of holomorphic sections. There is a canonical homomorphism $\iota : p^*\mathcal{E} \to \mathcal{O}$ dual to the tautological section of p^*E. This homomorphism vanishes along the zero section $s : X \to E$, and the associated **Koszul complex** $K(\iota) := (\wedge^{\cdot} p^*\mathcal{E}^*)$ is a resolution of $s_*\mathcal{O}_X$. This fact has the following explaination:

Let $\hat{p} : P := \mathbf{P}_X(E \oplus \mathbf{C}) \to X$ be the projective bundle of $E \oplus \mathbf{C}$ over X. On P, there is a **universal exact sequence**:

$$0 \to \mathcal{H} \to \hat{p}^*\mathcal{E}^* \oplus \mathcal{O}_P \to \mathcal{O}_P(1) \to 0.$$

As a divisor on P, $\mathbf{P}_X(E)$ is given by the vanishing set of the map $\mathcal{O}_P \to \mathcal{O}_P(1)$ induced from the inclusion $\mathcal{O}_P \subset \hat{p}^*\mathcal{E}^* \oplus \mathcal{O}_P$. That is, $\mathbf{P}_X(E)$ is the locus on which $\mathcal{O}_P \subset \mathcal{H}$. Since a line $L \subset E \oplus \mathbf{C}$, which maps surjectively to $\mathbf{C}$, is equivalent to a homomorphism $\mathbf{C} \to E$, the complement of $\mathbf{P}_X(E)$ is canonically isomorphic to E.

On the other hand, the map $\theta : \mathcal{H} \to \hat{p}^*\mathcal{E}^*$ induced by the projection from $\hat{p}^*\mathcal{E}^* \oplus \mathcal{O}_P$ is an isomorphism on E, and on E, if we compose θ^{-1} with the homomorphism induced by the negative of the second projection, we obtain ι. Let $\varphi : \mathcal{H} \to \mathcal{O}_P$ be the homomorphism induced by the negative of the projection $\hat{p}^*\mathcal{E}^* \oplus \mathcal{O}_P$. By the fact that this map is surjective on $\mathbf{P}(E)$ and is equal to ι on E, we know that the associated Koszul complex $K(\varphi)$ is a resolution of $s_*\mathcal{O}_X$, where $s : X \hookrightarrow E \subset \mathbf{P}(E \oplus \mathbf{C})$ is the zero section.

I.9.1.b. The Construction Of The Deformation To The Normal Cone

Let $i : Y \hookrightarrow X$ be a closed embedding of complex manifolds of pure codimension n. Denote by

$$\pi : W := B_{Y \times \{\infty\}} X \times \mathbf{P}^1 \to X \times \mathbf{P}^1,$$

where $B_{Y\times\{\infty\}}X \times \mathbf{P}^1$ denotes the blowing-up of $X \times \mathbf{P}^1$ along $Y \times \{\infty\}$. Let P be the exceptional divisor. Then the map $q_W : W \to \mathbf{P}^1$, obtained by composing π with the projection $q : X \times \mathbf{P}^1 \to \mathbf{P}^1$, is flat. For $z \in \mathbf{P}^1$:

$$q^{-1}(z) = \begin{cases} X, & \text{for } z \neq \infty, \\ P \cup B_Y X, & \text{for } z = \infty. \end{cases}$$

Also $P \cap B_Y X$ is the divisor at ∞ on P, which may be identified with the exceptional divisor on $B_Y X$.

We know that $P \simeq \mathbf{P}(N_{Y\times\{\infty\}/X\times\mathbf{P}^1})$ and

$$N_{Y\times\{\infty\}/X\times\mathbf{P}^1} \simeq p_Y^* N_i \oplus p_\infty^* N_{\infty/\mathbf{P}^1},$$

where $p_Y : Y \times \{\infty\} \to Y$ and $p_\infty : Y \times \{\infty\} \to \{\infty\}$ are the projections. Thus we have

$$P \simeq \mathbf{P}(N_i \otimes N_{\infty/\mathbf{P}^1}^{-1} \oplus \mathbf{C}).$$

The bundle $N_{\infty/\mathbf{P}^1}$, while trivial, is not canonically trivial. Hence P is the projective completion of $N_i \otimes N_{\infty/\mathbf{P}^1}^{-1}$ with the divisor $\mathbf{P}(N_i \otimes N_{\infty/\mathbf{P}^1}^{-1}) \simeq \mathbf{P}(N_i)$.

In particular, we have the following diagram:

$$\begin{array}{ccc} P & \overset{i_\infty}{\hookrightarrow} & W = P \cup B_Y X \\ \pi_P \downarrow & & \downarrow \pi \\ Y \times \{\infty\} & \overset{i_\infty}{\hookrightarrow} & X \times \mathbf{P}^1. \end{array}$$

I.9.1.c. Deformation of The Resolution

At the beginning we should say that this subsection will only be used in section 4.

With the same notation as in subsection c, let η be a vector sheaf on Y and let $\xi \to i_*\eta \to 0$ be a resolution of $i_*\eta$ by a bounded complex of vector sheaves on X. We deform ξ through a complex $\tilde{\xi}$ on W to a Koszul type resolution of $s_*\eta$ on P, where $s : Y \to N_i \times N_{\infty/\mathbf{P}^1}^{-1} \subset P$ is the zero section.

First we construct the complex $(\tilde{\xi}, d^{\tilde{\xi}})$. By the canonical exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1}(-\infty) \to \mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_\infty \to 0,$$

if we let K^∞ be the complex $\mathcal{O}_{\mathbf{P}^1}(-\infty) \to \mathcal{O}_{\mathbf{P}^1}$ with $\mathcal{O}_{\mathbf{P}^1}$ of degree zero, we know that $p^*\xi \otimes q^*K^\infty$ is a resolution of $i_{\infty *}\eta$. Thus, by exp. VII, lemma 3.2 [SGA 6], if we let

$$\mathcal{G} := \pi^*(p^*\xi \otimes q^*K^\infty)$$

be a bounded complex of vector sheaves on W and

$$\mathcal{H} := \mathrm{Ker}(\pi_P^*(\mathcal{N}_i^* \oplus \mathcal{N}_{\infty/\mathbf{P}^1}^*) \to \mathcal{O}_P(1))$$

be the vector sheaf on P, we see that

$$\mathcal{H}_p(\mathcal{G}) \simeq (j_\infty)_*(\wedge^p \mathcal{H} \otimes \mathcal{O}_P(\pi_P^* \eta)).$$

Here, $\mathcal{H}_p(\mathcal{G})$ denote the cohomological sheaves of $\mathcal{G}$. In particular, it is locally of projective dimension one, by the fact that P is a divisor of W. Hence if $\mathcal{E}$ is a vector sheaf and

$$\alpha : \mathcal{E} \to (j_\infty)_*(\wedge^p \mathcal{H} \otimes \mathcal{O}_P(\pi_P^* \eta))$$

is an epimorphism, then $\mathrm{Ker}\,\alpha$ is also a vector sheaf. As a consequence, and by an induction on $i \geq 0$, we know that $\mathrm{Ker}(d_i^{\mathcal{G}})$ is a vector sheaf on W. With this, we may define $\tilde{\xi}$ as follows:

For $k \geq 0$, $\tilde{\xi}_k := \mathrm{Ker}\,(d_k^{\mathcal{G}}) \otimes_{\mathcal{O}_W} \mathcal{O}_W(k\infty)$. Here

$$\mathcal{O}_W(\infty) := q_W^* \mathcal{O}_{\mathbf{P}^1}(\infty), \quad \mathcal{O}_W(k\infty) := \mathcal{O}_W(\infty)^{\otimes k}.$$

Here $q : X \times \mathbf{P}^1 \to \mathbf{P}^1$ is a projection to $\mathbf{P}^1$, and $q_W := q \circ \pi$. Similarly, we denote the projection $X \times \mathbf{P}^1 \to X$ by p and $p_W := p \circ \pi$.

Next we must give the definition of the boundary maps. By definition, we know that

$$\mathcal{G}_i \simeq p_W^* \xi_i \oplus p_W^* \xi_{i-1}(-\infty), \quad d_i^{\mathcal{G}}(x, y) = (d_i^{\xi}(x) + (-1)^i y, d_{i-1}^{\xi}(y)),$$

where we identify $y \in p_W^* \xi_{i-1}(-\infty)$ with its image in $p_W^* \xi_i$ under the natural inclusion. Hence, $\tilde{\xi}_i$ is isomorphic to the fiber product of the diagram:

$$\begin{array}{ccc} \tilde{\xi}_i & \to & p_W^* \xi_{i-1}((i-1)\infty) \\ \downarrow & & \downarrow \\ p_W^* \xi_i(i\infty) & \to & p_W^* \xi_{i-1}(i\infty). \end{array}$$

The differential $p_W^* d_i$ extends to a homomorphism $d_i^{\tilde{\xi}} : \tilde{\xi}_i \to \tilde{\xi}_{i-1}$. Then, the restriction of $p_W^* d_i^{\xi} \otimes \mathrm{Id}_{\mathcal{O}_W(i\infty)}$ to $\tilde{\xi}_i$ has its image in $p_W^* \xi_{i-1}((i-1)\infty)$. But $(d^{\xi})^2 = 0$, so the image is contained in $\tilde{\xi}_{i-1} \subset p_W^* \xi_{i-1}((i-1)\infty)$. On the other hand, $d_i^{\tilde{\xi}}|_{W - W_\infty} = p_W^*(d_i^{\xi})$. So $d_i^{\tilde{\xi}2}$ vanishes on $W - W_\infty$, and hence on W. In this way, we get a complex $(\tilde{\xi}, d^{\tilde{\xi}})$ on W.

In addition, we know the following

Corollary. Let $U = X - Y$. The restriction of $\tilde{\xi}$ to $U \times \mathbf{P}^1 \subset W$ restricts to a split acyclic complex on $U \times \{\infty\} \subset B_Y X \subset W_\infty$.

Indeed, this comes from the fact that if ξ is acyclic, then $\tilde{\xi}$ is the pull-back, via π, a complex on $X \times \mathbf{P}^1$, which is acyclic when restricted to $X \times \{0\}$.

Now we discuss $(\tilde{\xi}, d^{\tilde{\xi}})$ in more detail. Let $I : Y \times \mathbf{P}^1 \hookrightarrow W$ be the natural inclusion. Since $I_*\mathcal{O}_{Y\times\mathbf{P}^1}$ and $\mathcal{O}_{W_\infty}$ are Tor independent, the above construction of $\tilde{\xi}$ commutes with the restriction to $Y \times \mathbf{P}^1 \subset W$. The restriction of ξ to Y has locally free homology:

$$\begin{aligned}\mathcal{H}_p(i^*\xi) &= \mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_Y, \eta)\\ &\simeq \mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \eta \simeq \wedge^p \mathcal{N}_i^* \otimes_{\mathcal{O}_Y} \eta.\end{aligned}$$

Hence, associated with $i^*\xi$, we have two natural short exact sequences of vector sheaves associated with its kernel and its image:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{Z}_i & \to & i^*\xi_i & \to & \mathcal{B}_i & \to & 0;\\ 0 & \to & \mathcal{B}_{i+1} & \to & \mathcal{Z}_i & \to & \mathcal{H}_i(i^*\xi) & \to & 0.\end{array}$$

Therefore, $\tilde{\xi}_i|_{Y\times\mathbf{P}^1}$ is obtained by pulling back the extension from the inclusion $\mathcal{B}_i(-\infty) \hookrightarrow \mathcal{B}_i$ and twisting with $\mathcal{O}(i\infty)$. Thus, restricted to $Y \times \{\infty\} \subset Y \times \mathbf{P}^1$, we find that since $\mathcal{O}_{\mathbf{P}^1}(\infty)|_\infty \simeq \mathcal{N}_{\infty/\mathbf{P}^1}$,

$$\tilde{\xi}_i|_{Y\times\mathbf{P}^1} \simeq (\mathcal{Z}_i \otimes \mathcal{N}^i_{\infty/\mathbf{P}^1}) \oplus (\mathcal{B}_i \otimes \mathcal{N}^{i-1}_{\infty/\mathbf{P}^1}).$$

Also the differential $d_i^{\tilde{\xi}}$ restricts to the map $d_i : (x, y) \mapsto (\gamma_i(y), 0)$, where

$$\gamma_i : \mathcal{B}_i \otimes \mathcal{N}^{i-1}_{\infty/\mathbf{P}^1} \hookrightarrow \mathcal{Z}_{i-1} \otimes \mathcal{N}^{i-1}_{\infty/\mathbf{P}^1}$$

is the natural inclusion. So if we let $\mathcal{L}$ be the split acyclic complex with

$$\mathcal{L}_i := (\mathcal{B}_{i+1} \otimes \mathcal{N}^i_{\infty/\mathbf{P}^1}) \oplus (\mathcal{B}_i \otimes \mathcal{N}^{i-1}_{\infty/\mathbf{P}^1})$$

and the differential $d_i(x, y) = (y, 0)$, then we have an exact sequence:

$$0 \to \mathcal{L} \to \tilde{\xi}|_{Y\times\{\infty\}} \to \oplus_{i\geq 0} \wedge^i (\mathcal{N}_i^* \otimes \mathcal{N}_{\infty/\mathbf{P}^1}) \otimes_{\mathcal{O}_Y} \eta \to 0.$$

Before we go further, let us look at an example with $\eta = \mathcal{O}_Y$. Suppose that $i : Y \hookrightarrow X$ is defined by equations $x_1 = \ldots = x_n = 0$ with the x_j part of a system of coordinates on X. Let $\xi_. = K_.(\mathbf{x})$ be the Koszul complex associated with the map $\mathcal{O}_X^n \to \mathcal{O}_X$ which sends $\mathbf{a}$ to $\mathbf{ax}$. Here $\mathbf{x} := (x_1, \ldots, x_n)$. Then $K_.(\mathbf{x})$ is a resolution of $i_*\mathcal{O}_Y$. On W, we have an epimorphism

$$\begin{array}{ccc} \mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) & \to & \mathcal{O}_W(-P)\\ (\mathbf{a}, b) & \mapsto & \mathbf{ax} + b.\end{array}$$

By considering the projective dimension, we know that the kernel of this morphism is a vector sheaf too.

Proposition. There is a canonical isomorphism of complexes $\tilde{\xi} \to K_.(\varphi)$ with

$$\varphi : p_W^*\mathcal{I}(\infty) \to \mathcal{O}_W$$

induced by the negative of the projection map $\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) \to \mathcal{O}_W(-\infty)$. Furthermore, we have

(1) $\tilde{\xi}_\cdot|_{B_Y X}$ is split acyclic.
(2) $\tilde{\xi}_\cdot|_{\mathcal{N}_i \otimes \mathcal{N}^{-1}_{\infty/\mathbf{P}^1}}$ is the tautological Koszul complex.
(3) $\tilde{\xi}_\cdot$ is a resolution of $I_*\mathcal{O}_{Y\times\mathbf{P}^1}$.

Proof. The complex $p_W^*\xi_\cdot \otimes K_\cdot^\infty$ is the Koszul complex associated with the maps of sheaves

$$\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) \to \mathcal{O}_W$$

induced by the inclusion $\mathcal{O}_W(-P) \subset \mathcal{O}_W$. Hence the Koszul differential

$$\wedge^i \mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) \to \wedge^{i-1}(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty))$$

is the composition of the following maps: the canonical map

$$\wedge^i(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty)) \to \wedge^{i-1}(\mathcal{O}_W^n \oplus \mathcal{O}_W(-P)),$$

which has the kernel $\wedge^i\mathcal{I}$; the injective multiplication

$$\wedge^{i-1}(\mathcal{I}) \otimes \mathcal{O}_W(-P) \to \wedge^{i-1}(\mathcal{I});$$

and the natural inclusion

$$\wedge^{i-1}(\mathcal{I}) \to \wedge^{i-1}(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty)).$$

Hence the inclusion

$$\wedge^i(\mathcal{I}) \to \wedge^i(\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty))$$

identifies $\wedge^i(\mathcal{I})$ with $\operatorname{Ker}(d_i^g)$. In particular, we may identify $\wedge^i(\mathcal{I}(\infty))$ with $\tilde{\xi}_i$. Next, we check (1), (2) and (3).

The negative of the projection $\mathcal{O}_W^n \oplus \mathcal{O}_W(-\infty) \to \mathcal{O}_W(-\infty)$ induces a surjective map

$$\mathcal{O}_W(-\infty) \oplus \mathcal{O}_W \to \mathcal{O}_W,$$

and hence, a map $\mathcal{I}(\infty) \to \mathcal{O}_W$. On the other hand, there is an induced morphism between Koszul complexes $\wedge^\cdot\mathcal{I}(\infty) \to \wedge^\cdot(\mathcal{O}_W^n(\infty) \oplus \mathcal{O}_W)$ which gives the commutative diagram

$$\begin{array}{ccccccc}
\wedge^i(\mathcal{O}_W^n) = \xi_i & \to & \wedge^i(\mathcal{I}(\infty)) & \to \wedge^i(\mathcal{O}_W^n(\infty) \oplus \mathcal{O}_W) = & \wedge^i(\mathcal{O}_W(\infty)) \oplus \wedge^{i-1}(\mathcal{O}_W(\infty)) \\
\downarrow & & \downarrow & & \downarrow \\
\wedge^{i-1}(\mathcal{O}_W^n) = \xi_{i-1} & \to & \wedge^{i-1}(\mathcal{I}(\infty)) & \to \wedge^{i-1}(\mathcal{O}_W^n(\infty) \oplus \mathcal{O}_W) = & \wedge^{i-1}(\mathcal{O}_W(\infty)) \oplus \wedge^{i-1}(\mathcal{O}_W(\infty))
\end{array}$$

by

$$\begin{array}{ccc}
a & \to & (a, (-1)^i d_i(a)) \\
\downarrow & & \downarrow \\
da & \to & (d_i(a), 0).
\end{array}$$

(1) Since $\mathcal{O}_W(-P)\otimes\mathcal{O}_W(-\infty)=\mathcal{O}_W(B_YX)$, we have an exact sequence

$$0\to\mathcal{I}(\infty)\to\mathcal{O}_W^n(\infty)\oplus\mathcal{O}_W\to\mathcal{O}_W(B_YX)\to 0.$$

The restriction to B_YX of the map $\mathcal{O}_W|_{B_YX}\subset(\mathcal{O}_W^n(\infty)\oplus\mathcal{O}_W)|_{B_YX}$ to $\mathcal{O}_W(B_YX)|_{B_YX}$ vanishes. Therefore, on restricting to B_YX, $\mathcal{I}(\infty)$ splits as a direct sum

$$\mathcal{I}(\infty)|_{B_YX}=\mathcal{I}_1\oplus\mathcal{O}_{B_YX}\subset\mathcal{O}_W(\infty)^n|_{B_YX}\oplus\mathcal{O}_{B_YX}.$$

The Koszul complex associated with $\mathcal{I}(\infty)\to\mathcal{O}_W$ therefore restricts to the Koszul complex for the negative of the projection $\mathcal{I}_1\oplus\mathcal{O}_{B_YX}\to\mathcal{O}_{B_YX}$, which is split acyclic.

(2) This comes from the fact that on P, $\mathcal{I}(\infty)$ restricts to the kernel of the map

$$\pi_P^*(\mathcal{N}_i\otimes\mathcal{N}_{\infty/\mathbf{P}^1})\oplus\mathcal{O}_P\to\mathcal{O}_P(B_YX\cap P),$$

i.e., to the analogue of the vector sheaf $\mathcal{H}$ above. Hence, with a similar discussion, we have (2)

(3) It is sufficient to show that the map $\tilde{\xi}.\to I_*\mathcal{O}_{Y\times\mathbf{P}^1}$, induced by the map

$$\tilde{\xi}_0=\mathcal{O}_W\to i_*\mathcal{O}_{Y\times\mathbf{P}^1},$$

is a quasi-isomorphism of complexes. By the corollary above, we may deduces it in a neighborhood U of $Y\times\{\infty\}\subset W$. Since $\tilde{\xi}.|_{W-W_\infty}=p_W^*\xi.$ is a resolution of $I_*\mathcal{O}_{Y\times\mathbf{C}}$ and $\xi.|_{B_YX}$ is acyclic, if we choose a local system of equations for ∞ in $\mathbf{P}^1$, then $x_1=\ldots=x_n=t=0$ is a local system of equations for $Y\times\{\infty\}\subset X\times\mathbf{P}^1$. Hence $\mathcal{I}$ is isomorphic to the kernel of the map $\mathcal{O}_W^{n+1}\to\mathcal{O}_W$ defined by

$$(a_1,\ldots,a_n,b)\mapsto\sum_{i=1}^n a_ix_1+bt.$$

Hence we may choose U so that $\frac{x_1}{t},\ldots,\frac{x_n}{t},t$ is part of a system of coordinates on U, with $Y\times\mathbf{P}^1\subset U$ given by $\frac{x_1}{t}=\ldots=\frac{x_n}{t}=t=0$. Therefore the map $\mathcal{O}_W^n\to\mathcal{O}_W^{n+1}$ sending $(a_1,\ldots,a_n)$ to $(a_1,\ldots,a_n,-\sum a_ix_i/t)$ is an isomorphism onto $\mathcal{I}(\infty)$. Composing with the negative of the projection $\mathcal{O}_W^{n+1}=\mathcal{O}_W^n\oplus\mathcal{O}_W\to\mathcal{O}_W$, we can identify the Koszul complex $\wedge\mathcal{I}(\infty)$ with the Koszul complex $K.(\frac{x_1}{t},\ldots,\frac{x_n}{t})$, which is a resolution of $j_*\mathcal{O}_{Y\times\mathbf{P}^1}$. This completes the proof.

Now we deal with the general situation.

Theorem. With the same notation as above, we have
(1) $\tilde{\xi}$ is a resolution of $i_*(p_Y^*\eta)$.
(2) $\tilde{\xi}|_{B_YX}$ is split acyclic.
(3) There is a natural exact sequence of complexes of vector sheaves on P:

$$0\to\pi_P^*\mathcal{L}.\to f^*\tilde{\xi}\to K.(\varphi)\otimes\pi_P^*\eta\to 0.$$

Proof. (1) This is a local problem on X. So we may suppose that $\xi = \zeta \oplus (\underline{K(\mathbf{x})} \otimes \mathcal{V})$, where ζ is acyclic and $\mathcal{V}$ is a vector sheaf on X such that $j^*\mathcal{V} \simeq \eta$. Thus $\tilde{\xi} \simeq \tilde{\zeta} \oplus \widetilde{K(\mathbf{x})} \otimes p_W^* \mathcal{V}$ with $\tilde{\zeta}$ acyclic and $\widetilde{K(\mathbf{x})}$ is a resolution of $I_* \mathcal{O}_{Y \times \mathbf{P}^1}$. Hence $\tilde{\xi}$ is a resolution of $\mathcal{O}_{Y \times \mathbf{P}^1} \otimes p_W^* \mathcal{V} \simeq I_*(p_Y^* \eta)$, which gives (1).

Note that $\tilde{\xi}.|_{W_\infty}$ is split acyclic, and hence $\tilde{\xi}.|_{B_Y X}$ is split acyclic too. So by (1) of the previous proposition, we see that at least locally, $\tilde{\xi}.|_{B_Y X}$ is split acyclic. However, by the construction, we get $\tilde{\xi}|_{B_Y X - B_Y X \cap P}$ is split acyclic. So (2) comes from the fact that all the splitting described above are compatible, and uniquely determined.

(3) By the construction of $\tilde{\xi}$, we know that there is an epimorphism of vector sheaves on P:

$$\varepsilon : f^* \tilde{\xi}_i \to \mathcal{H}_i(p_W^* \xi \otimes (\mathcal{O}_W(-\infty) \to \mathcal{O}_W))(i\infty) \simeq \wedge^i(\mathcal{H}) \otimes \pi_P^* \eta,$$

where $\mathcal{H}$ is the kernel of $\mathcal{N}_i^* \otimes \mathcal{N}_{\infty/\mathbf{P}^1} \oplus \mathcal{O}_P \to \mathcal{O}_P(B_Y X)$. Also by the expression in the proof for (1), (2), we know that this process is compatible with differentials. Hence ε is a morphism of complexes and, on X, its kernel is $\tilde{\zeta}|_P$. On the other hand, by the inclusion

$$\pi_P^*(B_{i+1} \otimes \mathcal{N}^i_{\infty/\mathbf{P}^1}) \subset \pi_P^*(i^* \xi_i \otimes \mathcal{O}_{\mathbf{P}^1}(\infty)) = (j_\infty)^* \pi^* (\xi_i(i\infty)),$$

we have a morphism of complexes $(j_\infty)^* \pi_P^* \eta \to (j_\infty)^* (\tilde{\xi} \otimes (\mathcal{O}_W \to \mathcal{O}_W(\infty)))$. Hence it is enough to prove that the image of this morphism is contained in $\tilde{\xi}$ and is equal to the kernel of ε. Again this is a local problem. Hence we may use the expression $\tilde{\xi} \simeq \tilde{\zeta} \oplus \widetilde{K(\mathbf{x})} \otimes p_W^* \mathcal{V}$ and we see that $\mathcal{L} \simeq \tilde{\zeta}|_{Y \times \mathbf{P}^1} = \operatorname{Ker} \varepsilon$, which completes the proof.

§I.9.2. Deformation To the Normal Cone: Smooth Situations

In this section, we discuss the deformation theory for the relative Bott-Chern secondary characteristic forms with respect to smooth morphisms.

I.9.2.a An Axiom

Let $i : X \hookrightarrow Z$ be a closed immersion over Y with smooth structure morphisms of regular arithmetic varieties $f : X \to Y$ and $g : Z \to Y$. Then we have the following diagram for the deformation to the normal cone as stated in subsection 1.b:

$$\begin{array}{ccccccc}
X & \leftarrow & X \times \{\infty\} & \overset{i_\infty}{\hookrightarrow} & \mathbf{P}(\mathcal{N} \oplus \mathcal{O}_X) + B_X Z = W_\infty & \to & \{\infty\} \\
\downarrow & & \downarrow & & j_\infty \downarrow & & \downarrow \\
X & \overset{p}{\leftarrow} & X \times \mathbf{P}^1 & \overset{I}{\hookrightarrow} & B_{X \times \{\infty\}} Z \times \mathbf{P}^1 = W & \overset{p}{\to} & \mathbf{P}^1 \\
\uparrow & & \uparrow & & j_0 \uparrow & & \uparrow \\
X & \leftarrow & X \times \{0\} & \overset{i_0}{\hookrightarrow} & Z \times \{0\} = W_0 & \to & \{0\}.
\end{array} \qquad (*)$$

Moreover, for $f = g \circ i$, we may have the associated picture

$$\begin{tikzcd}
X \times \{\infty\} \arrow[rr, hook, "i_\infty"] \arrow[dr, "f_\infty"] \arrow[dd] & & \mathbf{P}(\mathcal{N} \oplus \mathcal{O}_X) \arrow[dl, "g_\infty"'] \arrow[dd] \\
& Y \times \{\infty\} & \\
X \times \mathbf{P}^1 \arrow[rr, hook, "I"] \arrow[dr, "F"] & & W \arrow[dl, "G"'] \\
& Y \times \mathbf{P}^1 & \\
X \times \{0\} \arrow[rr, hook, "i_0"] \arrow[dr, "f_0"] \arrow[uu] & & W_0 \arrow[dl, "g_0"'] \arrow[uu] \\
& Y \times \{0\} &
\end{tikzcd}$$

,

where $f_0 = F_0 = f$, $g_0 = G_0 = g$, $i_0 = I_0 = i$ and f_∞, g_∞, i_∞ are the restrictions of F_∞, G_∞, I_∞ to $\mathbf{P}(\mathcal{N} \oplus \mathcal{O}_X)$, respectively. Let $\partial W_\infty := W_\infty^1 \cap W_\infty^2$ with $W_\infty^1 := \mathbf{P}(\mathcal{N} \oplus \mathcal{O}_X)$ and $W_\infty^2 := B_X Z$. Then W_∞^1 and W_∞^2 intersect transversally along ∂W_∞. Choose hermitian metrics on the normal bundle of i and relative tangent bundles of f and g. Let $(\mathcal{E}, \rho)$ be a hermitian vector sheaf on X. Assume that

$$0 \to \mathcal{F}. \to i_*\mathcal{E} \to 0$$

is a vector sheaf resolution of $i_*\mathcal{E}$. Put hermitian metrics on $\mathcal{F}.$ so that Bismut condition (A) is satisfied. Further, choose hermitian metrics for each pair so that Bismut condition (A) holds. By the results in Chapter 6 and Chapter 8, with certain acyclic conditions, we have the following correspondences: (Here, for simplicity, we will omit the pull-back symbol by the projection p.)

(1) With respect to the smooth morphism F, we have

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; f_\infty, \rho_\infty) \longleftrightarrow \\
&\qquad f_{\infty *}(\mathrm{ch}(\mathcal{E}, \rho)\,\mathrm{td}(f_\infty, \rho_\infty)) - \mathrm{ch}(f_{\infty *}\mathcal{E}, f_{\infty *}\rho); \\
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; F, \rho_F) \longleftrightarrow \\
&\qquad F_*(\mathrm{ch}(\mathcal{E}, \rho)\,\mathrm{td}(F, \rho_F)) - \mathrm{ch}(F_*\mathcal{E}, F_*\rho); \\
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; f_0, \rho_0) \longleftrightarrow \\
&\qquad f_{0*}(\mathrm{ch}(\mathcal{E}, \rho)\,\mathrm{td}(f_0, \rho_0)) - \mathrm{ch}(f_{0*}\mathcal{E}, f_{0*}\rho).
\end{aligned}$$

(2) With respect to the closed immersion I, we have

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; i_\infty, \rho_\infty) \longleftrightarrow \\
&\qquad \mathrm{td}^{-1}(\mathcal{N}_\infty, \rho_\infty)\,\mathrm{ch}(\mathcal{E}, \rho)\delta_\infty - \mathrm{ch}(\mathcal{F}_\infty., \rho_\infty.) \\
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; I, \rho_I) \longleftrightarrow \\
&\qquad \mathrm{td}^{-1}(\mathcal{N}_I, \rho_I)\,\mathrm{ch}(\mathcal{E}, \rho)\delta_I - \mathrm{ch}(\mathcal{F}_I., \rho_I.) \\
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{E}, \rho; i_0, \rho_0) \longleftrightarrow \\
&\qquad \mathrm{td}^{-1}(\mathcal{N}_0, \rho_0)\,\mathrm{ch}(\mathcal{E}, \rho)\delta_0 - \mathrm{ch}(\mathcal{F}_0., \rho_0.).
\end{aligned}$$

(3) With respect to the smooth morphism G, we have

$$\begin{aligned}
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}_{\infty}.,\rho_{\infty}.;g_{\infty},\rho_{\infty})\longleftrightarrow\\
&\qquad g_{\infty*}(\mathrm{ch}(\mathcal{F}_{\infty}.,\rho_{\infty}.)\,\mathrm{td}(g_{\infty},\rho_{\infty}))-\mathrm{ch}(g_{\infty*}\mathcal{F}_{\infty}.,g_{\infty*}\rho_{\infty}.);\\
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{F},\rho.;G,\rho_G)\longleftrightarrow\\
&\qquad G_*(\mathrm{ch}(\mathcal{F}.,\rho.)\,\mathrm{td}(G,\rho_G))-\mathrm{ch}(G_*\mathcal{F}.,G_*\rho.);\\
&\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}_0.,\rho_0.;g_0,\rho_0)\longleftrightarrow\\
&\qquad g_{0*}(\mathrm{ch}(\mathcal{F}_0.,\rho_0.)\,\mathrm{td}(g_0,\rho_0))-\mathrm{ch}(g_{0*}\mathcal{F}_0.,g_{0*}\rho_0.).
\end{aligned}$$

Obviously, by the construction of relative Bott-Chern secondary characteristic forms with respect to smooth morphisms, we know that, for (1), there is the following relation:

$$\begin{aligned}
dd^c\int_{\mathbf{P}^1}&[\log|z|^2]\,\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;F,\rho_F)\\
&=\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f_0,\rho_0)-\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;f_{\infty},\rho_{\infty})\\
&\quad+\int_{\mathbf{P}^1}[\log|z|^2]\,dd^c\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;F,\rho_F).
\end{aligned}$$

This may be thought of as a trivial deformation theory of the secondary characteristic forms, since both sides now are just zero. Nevertheless, this sheds a light about the theory of ternary characteristic objects. So basically, we may hope to have the following relations:

$$\begin{aligned}
dd^c\int_{\mathbf{P}^1}&[\log|z|^2]\,\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;I,\rho_I)\\
&=\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i_0,\rho_0)-\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;i_{\infty},\rho_{\infty})\\
&\quad+\int_{\mathbf{P}^1}[\log|z|^2]\,dd^c\mathrm{ch}_{\mathrm{BC}}(\mathcal{E},\rho;I,\rho_I);\\
dd^c\int_{\mathbf{P}^1}&[\log|z|^2]\,\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}.,\rho.;G,\rho_G)\\
&=\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}_0.,\rho_0.;g_0,\rho_0)-\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}_{\infty}.,\rho_{\infty}.;g_{\infty},\rho_{\infty})\\
&\quad+\int_{\mathbf{P}^1}[\log|z|^2]\,dd^c\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}.,\rho.;G,\rho_G).
\end{aligned}$$

In this section, we only study (3). We delay the detailed discussion about (2) till the end of this chapter.

For (3), in order to make the situation simple, we will assume that the closed immersion is a *codimensional one* closed immersion: In this case, we have a natural vector sheaf resolution for $i_*\mathcal{E}$, that is,

$$0\to\mathcal{I}_i\mathcal{E}\to\mathcal{E}\to i_*\mathcal{E}\to 0,$$

which comes from the structure exact sequence

$$0\to\mathcal{I}\to\mathcal{O}_Z\to\mathcal{O}_X\to 0.$$

(On the other hand, for (2), since i_∞ is just the zero section of the projective bundle, we may use the Koszul complex to make a precise calculation.)

In the following, let $E^\cdot$ be the complex $\mathcal{I}_I\mathcal{E}(B_XZ) \hookrightarrow \mathcal{E}(B_XZ)$ of vector sheaves on W. We endow $\mathcal{E}$ and $\mathcal{I} := \mathcal{I}_I$ with hermitian metrics in such a way that on an open neighborhood U of $W_\infty^2 = B_XZ$, the metric of $\mathcal{I}_I$ coincides with that on $\mathcal{O}_W$. (This is possible since $X \times \mathbf{P}^1$ does not meet B_XZ in W.) Hence we also have a metric on $\mathcal{I}\,\mathcal{E}$.

With this, from the above discusssion for (3), we introduce the following

Axiom. (Upstairs Rule) Let $i : X \hookrightarrow Z$ be a *codimension one* closed immersion. Let $(\mathcal{E}, \rho)$ be an F-acyclic hermitian vector sheaf. Suppose that $i_{0*}\mathcal{E}$ (resp. $i_{\infty *}\mathcal{E}$) is a g_0 (resp. g_∞)-acyclic vector sheaf. Then, with the same notation as above, we have

$$
\begin{aligned}
d_Y d_Y^c \int_{\mathbf{P}^1} & \mathrm{ch}_{\mathrm{BC}}(E^\cdot, \rho_{E^\cdot}; G, \rho_G)\,[\log|z|^2] \\
=&\mathrm{ch}_{\mathrm{BC}}(E^\cdot|_{W_0}, \rho_{E^\cdot|_{W_0}}; g_0, \rho_0) - \mathrm{ch}_{\mathrm{BC}}(E^\cdot|_{W_\infty^1}, \rho_{E^\cdot|_{W_\infty^1}}; g_\infty, \rho_\infty) \\
&+ \int_{\mathbf{P}^1} G_*(\mathrm{ch}(E^\cdot, \rho_{E^\cdot})\,\mathrm{td}(\mathcal{T}_G(-\log\infty), \rho_G))\,[\log|z|^2] - \int_{\mathbf{P}^1} \mathrm{ch}(G_*E^\cdot, G_*\rho_{E^\cdot})\,[\log|z|^2].
\end{aligned}
$$

Essentially, this axiom is the downstairs rule for ternary objects (in this special content). In general, by the fact that the secondary object may be thought of as the one which measures the change of the first level objects, e.g. $\mathrm{ch}_{\mathrm{BC}}$ measures changes in ch via dd^c, so for the secondary objects, we may also want to construct a ternary object which measures the change of them. But we do not want to go further here as there is no further application now. Instead, let us say a few words for the structure of the supposed axioms: By comparing Proposition I.1.3, or better II.1.3.a, with axioms for the secondary objects, we may say that the axioms for ternary characteristic objects should have the same structure as these for secondary objects, i.e. the dd^c equation to measure the change of secondary objects, the functorial rule, and the uniqueness rule. Furthermore, one may use the so-called b-calculus developed by Melrose and others to check the above axiom, with the special attention on ∂W_∞. But from my point of view, all of this needs another book. So we will not really do in this way. Alternatively, we go directly following Faltings.

To do so, the first problem we meet is that the projection from W to $Y \times \mathbf{P}^1$ is not smooth: How we can define the relative Bott-Chern secondary characteristic forms with respect to this projection? Note that now we are working with a very special situation, i.e. the deformation to the normal cone with a codimension one closed immersion, we see that once we use the logarithmic relative tangent vactor sheaf at infinity, we may do the same thing as what we did in Chapter I.6. For more details, see the later part of this section.

Next we denote by $F^\cdot$ the complex $f_*(\mathcal{I}_i\mathcal{E}) \hookrightarrow f_*(\mathcal{E})$ on Y. Then, on $Y \times \mathbf{P}^1$, we have an augmentation $\nu : F^\cdot(\to G_*(E^\cdot)) \to G_*(E^\cdot \otimes \wedge\Omega_G^{0,1})$, which vanishes on B_XZ. (Note that here $F^\cdot \to G_*(E^\cdot)$ is in fact a quasi-isomorphism.) For each $t \in \mathbf{P}^1 - \{\infty\}$, let ν_t be the restriction of ν at t, i.e. $\nu_t : F^\cdot \to F_{t*}(E^\cdot|_{W_t})$. Also let $\nu_\infty : F_{\infty}{}^\cdot \to G_*(E^\cdot|_{W_\infty^1})$ denote the augmentation at infinity.

Note that

$$dd^c[\log|z|^2] = \delta_0 - \delta_\infty,$$

so to check the axiom above, as in 3.7.b for the construction of the classical Bott-Chern secondary characteristic forms, we need to show that for the section of the augmentation ν_t of ν at $t \in \mathbf{P}^1 - \{\infty\}$, we have the following

Theorem. With the same notation as above, then

$$\lim_{t\to\infty}\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\nu_t)) = \mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\nu_\infty)),$$

where $\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(\nu_.))$ is defined by using the same process as in section 6.2 for the morphism $\nu_.$.

I.9.2.b **The Proof Of The Theorem**

We prove the latest theorem in the following steps.

Step I. Find A Uniform Coordinate System:

From the previous discussion towards the relative Bott-Chern object, we saw that a suitable local discussion is necessary: For doing so, we need use the normal coordinate system. So to prove the theorem, we should have a uniform way to choose such coordinates.

But this is not an easy task. In fact, when t goes to the infinity, the hermitian metric on the relative tangent bundle of G_t becomes to singular near $\partial W_\infty := W_\infty^1 \cap W_\infty^2$. So, the natural choice for the metrics does not work directly. To solve this problem, as we said before, instead of using the relative tangent bundle of G_t, we consider these associated with the logarithmic tangent bundle at infinity, i.e., we take the dual of $\Omega_{W/Y\times\mathbf{P}^1}(\log\infty)$. In this way, by the fact that the closed immersion is of codimension one, we know the resulting new metric on W_t looks like a small perturbation of a translation-invariant metric on $X \times \mathbf{C}$. Thus we see that now we can take the limit at infinity with respect to this new metric. Hence, one may hope that we can start from the very beginning to establish estimates with respect to this new metric.

But by doing things like this, we meet other two problems: The first is that, at infinity, the metric is no longer Kähler; while the second is that as $t \to \infty$, the volume of W_t with respect to the new metric become infinite; therefore, we cannot make the L^2-estimate by just considering the usual sup-norm estimate as in section 3.4. Fortunately, these two problems only happen near infinity around ∂W_∞, where the associated complex is in fact split. Hence, the problems are not that serious.

More precisely, we go as follows: With respect to the new metric, we find a good finite system of coordinate charts, uniform for each W_t in the sense of estimations, by the fact that locally W is isomorphic to the product of $X - \partial W_\infty$ with the product of two unit disks $\{(z,w) : |z| < 1, |w| < 1\}$, and the projection is given by $\frac{1}{t} = zw$, and hence if we use $\log(z)$ and $\log(w)$ instead, and identify the punctured disc via $\exp(2\pi i\tau)$ with $\mathcal{H}/\mathbf{Z}$, where $\mathcal{H}$ is the upper half-plane, then the metric on W_t looks like a small perturbation of a translation-invariant metric on $X \times \mathbf{C}$ and similar for $W^1_\infty - \partial W_\infty$ and $W^2_\infty - \partial W_\infty$. In particular, we can define Sobolev-norms with them (see Chapter 3): For any positive integer $s \geq 0$, H_s denotes the completion of C_0^∞ under the square-integration of all derivatives up to order s, and H_{-s} is its dual. Also we see that the new metric is complete. Since C_0^∞ is dense in H_{-s}, there is a uniform Garding inequality for the $\bar{\partial}$-Laplacian Δ, and Δ extends to a self-adjoint operator. Therefore, all process are quite regular as $t \to \infty$. However, as we said above, the price we pay for this is that the volume of W_t approaches infinity as $t \to \infty$, so that the estimates in sup-norm do not imply L^2-estimates as easily as before. Also an integral operator need not to be a trace-class anymore.

Step II. Construct The Relatice Bott-Chern Secondary Characteristic Forms $\mathrm{ch}_{\mathrm{BC}}$.

As in section 6.2, we know that there is a family of super-Laplacians $\mathbf{A}_u^2$, defined as the limit of the ordinary Laplacians coming from blowing-up the metric on Y together with a rescaled factor u for the total metric (or better, for the fibre metric). So finally we can define the $\mathrm{ch}_{\mathrm{BC}}$-class via the Mellin transform, i.e. the regularized integral

$$\int_0^{+\infty} \mathrm{Tr}_s[\mathrm{N}\exp(-\mathbf{A}_u^2)]\frac{du}{u}.$$

Then via the cone construction, with this process for the augmentation v_t with $t \in \mathbf{P}^1$, we could get $\mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(v_t))$. With this, what we intend to show becomes

$$\lim_{t\to\infty} \mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(v_t)) = \mathrm{ch}_{\mathrm{BC}}(\mathrm{cone}(v_\infty)).$$

Step III. Relate the Objects with respect to the New Metric And the Old Metric.

We then have the associated relative Bott-Chern secondary characteristic forms with respect to two kinds of metrics mentioned above. Now we discuss their relations.

Under an infinitesimal change of metrics, the derivative of the regularized integral

$$\int_0^{+\infty} \mathrm{Tr}_s[\mathrm{N}\exp(-\mathbf{A}_u^2)]\frac{du}{u}$$

is the Schwartz-limit

$$\lim^{\mathrm{Sch}}_{u\to 0+}(Q\exp(-\mathbf{A}_u^2)),$$

where Q denotes the hermitian operator describing the change of metrics. If Q has support in U, this vanishes because of the splitting, except for terms related to the augmentation ν. If we replace ν by $s^{\frac{1}{2}}\nu$, with a parameter s between 0 and 1, the derivative $s\frac{d}{ds}$ of this class is equal to the Schwartz-limit $\lim^{\mathrm{Sch}}_{u\to 0+}$ of the derivative (described by Q) of $\mathrm{Tr}_s(N_F\exp(-\mathbf{A}_u^2))$, where N_F denotes the number-operator which is identically 1 on $F\cdot$

and vanishes on $E^\cdot$. Integrating we obtain that the $s\frac{d}{ds}$-derivative of the effect of the change of the metrics is given by the difference in Schwartz-limits

$$\lim^{\rm Sch}_{u\to 0+}\mathrm{Tr}_s(N_F\exp(-\mathbf{A}_u^2)),$$

taken once for the origianl Kähler metric and once for the logarithmic non-Kähler metric. But for the original metric this term does not depend on s and is equal to the Chern form of $F^\cdot$ with the associated L^2-metric, while for the new metric the term converges nicely as we move our family to the infinity along $\mathbf{P}^1$. (See the last step for more details.) The similar result also holds for the integration against $\frac{ds}{s}$. So, with respect to the change of the metrics, the terms contributes a correction-term which converges for the family move to the infinity along $\mathbf{P}^1$ to the corresponding term at infinity, and thus does not affect the conclusion of the theorem. In particular, we only then need study the situation for the new metric. Thus, by definition, there is no problem at finite places: Only the singularities around ∂W_∞ cause difficulties.

Step IV. First Attack Around ∂W_∞.

By definition, at infinity, there are two irreducible (smooth) components. i.e. W^1_∞, which is a projective bundle over Y, and W^2_∞, which is the blowing-up of Z along X. On the other hand, passing to the limit above, only W^1_∞ is concerned. Hence, we need take care of the intersection of W^1_∞ and W^2_∞.

We now introduce some twisted objects: Define a second family of vector sheaves

$$E^{\cdot *} = (\mathcal{E}(B_XZ)\simeq\mathcal{E}(B_XZ)),\quad F^{\cdot *} = ((0)\to(0)).$$

The advantage for using these twisted objects is that, on one hand, the original theorem holds directly for $E^{\cdot *}$ and $F^{\cdot *}$ since everything is split and hence all $\mathrm{ch_{BC}}$-classes vanish, while, on the other hand, on U, $E^\cdot$ and $E^{\cdot *}$ are isomorphic, and the augmentation ν has a small norm. Thus if Δ defines a heat kernel $\exp(-u\Delta)$ on $W^1_\infty - \partial W_\infty$, by the fact that, for the new metric, the volume is not finite, we now have the problem that the kernel is not of trace-class. Nevertheless, if we identify $E^\cdot$ and $E^{\cdot *}$ on $U\cap W^1_\infty$, and consider the difference of the heat kernels $\exp(-u\Delta)$ and $\exp(-u\Delta^*)$, we expect that the singularities around ∂W_∞ will cancel out. The same thing should also hold for the super-analogue, if we consider the difference $\mathrm{ch_{BC}}_t - \mathrm{ch_{BC}}^*_t$: We shall see that the difference is integrable. In particular, we could introduce the combination $\mathrm{ch_{BC}}_\infty - \mathrm{ch_{BC}}^*_\infty$, although the individual terms are not defined.

Claim. With the above notation, if

$$\lim_{t\to\infty}(\mathrm{ch_{BC}}_t - \mathrm{ch_{BC}}^*_t) = \mathrm{ch_{BC}}_\infty - \mathrm{ch_{BC}}^*_\infty,$$

then the original theorem holds.

Proof. Suppose we have the equality, then we can repeat the same procedure for the embedding $X\hookrightarrow W^1_\infty$, which is the deformation of $i: X\hookrightarrow Z$, with respect to the original

Kähler metric. Nevertheless, for $X \hookrightarrow W_\infty^1$, the deformation to the normal cone gives that all fibers W_t are isomorphic to W_∞^1. Hence, the original theorem holds for it, since there is no essential change. Moreover, our construction for two closed immersions $X \hookrightarrow Z$ and $X \hookrightarrow W_\infty^1$ gives the same $\mathrm{ch}_{\mathrm{BC}_\infty} - \mathrm{ch}_{\mathrm{BC}_\infty^*}$ at infinity. Hence if we add up all the pieces, we have the claim and hence the theorem.

So for the proof of the theorem, it is sufficient to verify the following

Lemma. With the same notation as above, we have

$$\lim_{t\to\infty}(\mathrm{ch}_{\mathrm{BC}_t} - \mathrm{ch}_{\mathrm{BC}_t^*}) = \mathrm{ch}_{\mathrm{BC}_\infty} - \mathrm{ch}_{\mathrm{BC}_\infty^*}.$$

Step V. Prove Of The Lemma.

To prove the lemma, we need some technical results from the very beginning. Since the situation is quite similar for $E^{\cdot *}$, we only formulate them for $E^{\cdot}$.

Note that the cone construction of ν_t is associated with $F^{\cdot} \oplus F_{t*}(E^{\cdot} \otimes \wedge\Omega_{F_t}^{0,1})$, when we study the $\mathrm{ch}_{\mathrm{BC}}$-classes in the sequel, we usually ignore the $F^{\cdot}$-part, since it is on the base and hence it does not affect our discussions seriously.

Now what we need to study is to get the limit near W_∞ when $t \to \infty$. So basically, we must discuss the following two cases:

(1) Away from ∂W_∞.
(2) Near ∂W_∞.

We discuss (1) first. In order to explain the idea, we start with a primitive case: Let $D := \bar\partial + \bar\partial^* + \nu_t + \nu_t^*$ denote the Dirac operator, and $\Delta := D^2$. So note the fact that now we are working away from the singular part, so everything goes well. In particular, we may hope to have the following

Fact. Let $K_t(x;u;y)$ be the heat kernel $e^{-u\Delta}$ on W_t. Then, uniformly on any compact subset of $(W_\infty - \partial W_\infty)\times]0,\infty[\times (W_\infty - \partial W_\infty)$, the family $K_t(x,y)$ converges to the heat kernel $K_\infty(x,y)$ on $W_\infty^1 \coprod W_\infty^2$, where $K_\cdot(x,y)$ denotes the asymptotic expansion of $K_\cdot(x;u;y)$ when $u \to 0^+$.

Proof. First, by that fact that for any sequence of $\{t_n\}$ where $t_n \to \infty$, we can find a subsequence such that $K_{t_n}(x;u;y)$ converges in the C^∞-topology, uniformly on compact subsets of $(W_\infty - \partial W_\infty)\times]0,\infty[\times (W_\infty - \partial W_\infty)$, we only need show that this limit $K(x;u;y)$ is just $K_\infty(x;u;y)$.

For doing this, by taking $\lim_{t\to\infty}$, as C^∞-functions on $W_\infty - \partial W_\infty$, we have

$$(\frac{\partial}{\partial u} + \Delta)K(x;u;y) = 0.$$

Then, from the heat kernel analogue of the fact that for all λ,

$$1 - e^{-\lambda u} \leq \lambda u,$$

we have, for $\phi \in C_0^\infty(W_{t_n})$,

$$\|K_{t_n}(u)(\phi) - \phi\|_{L^2} \leq u\|\Delta\phi\|_{L^2}^2.$$

Therefore, for $\phi \in C_0^\infty(W_\infty - \partial W_\infty)$,

$$\lim_{u \to 0} K(u)\phi = \phi,$$

which gives the initial condition. Thus, from the uniquness of the heat kernel, by the fact that, for such a ϕ,

$$\|DK(u)\phi\|_{L^2}, \ \ \|\Delta K(u)\phi\|_{L^2}$$

are finite, so by a cut-off process, we see that as distributions, at infinity,

$$(\frac{\partial}{\partial u} + \Delta)K(x,y) = 0.$$

This completes the proof.

With this, we may state the results over $W_\infty - \partial W_\infty$ as the following

Theorem. On compact subsets in $W_\infty - \partial W_\infty$, we have
(a) For $u \to 0^+$, the asymptotic expansion of $K_t(u,x,y)$ converges to $K_\infty(u,x,y)$ uniformly.
(b) For $u \to +\infty$, $K_t(u,x,y)$ decays uniformly exponentially.
In particular, we have a complete control on $W - \partial W_\infty$, uniform on any compactum.

Proof. Obviously, (a) is a direct consequence of the fact above by the local nature of the expansion. For (b), we need to control the eigenvalues of the Laplacians on cone(ν_t). In fact, by a standard process, we see that (b) is a consequence of the following

Proposition. There exists a positive $\lambda > 0$, such that, uniformly on each fiber $f^{-1}(y)$, $y \in Y$, and for each $t \in \mathbf{P}^1 - \{\infty\}$, the Laplacian on the cone cone(v_t) has all eigenvalues at least λ. The same is true for cone(v_∞).

Proof. First, let us discuss the situation for ν_t. Obviously, the problem may only happen near ∞. Suppose near ∞, the assertion is wrong. Then there are a sequence $t_n \to \infty$ and eigenfunctions f_n on $W_n := W_{t_n}$ of eigenvalues λ_n so that $\lambda_n \to 0$ and $\|f_n\| = 1$. By Sobolev estimates, as in section 3.1, we see that f_n are uniformly bounded in the C^∞-topology. Hence, by the Rellich lemma, we may assume that f_n converges on each compactum in $W - \partial W_\infty$, and on Y. The limit must be annihilated by $\Delta_{\tilde{g}}$. Now we claim that the limit of f_n is identically zero.

Indeed, near W_∞^1, we may suppose again that we have a sequence of eigenfunctions f_n of eigenvelues λ_n such that $\lambda_n \to 0$ and $\|f_n\| = 1$. As above, we may also assume that f_n converges to f and that over $U \cap W_n$ f_n converges to zero. In particular, f should vanish ideitically on U, $\Delta_{\tilde{g}} f = 0$ and $\|f\| = 1$. Thus, with respect to the original Kähler metric, f is a non-trivial harmonic form. Thus, cone(ν_∞) should have non-trivial cohomologh, which contradicts the choice of $F^{\cdot}$. Therefore, we may assume that f_n concentrate their mass around W^2_∞, where $E^{\cdot}$ is metrically split. Thus, by a cut-off process, we may also assume that f_n is supported in $U \cap W_n$. Now, by the condition that $\lambda_n \to 0$, (that this limit must be annihilated by $\Delta_{\bar\delta}$,) we have $\|\Delta_{\bar\delta} f_n\| < 1$. On the other hand, since $E^{\cdot}$ is split on U, so there $\Delta_{\bar\delta}$ is the sum of Id and a positive operator. As a direct consequence,

$$< \Delta_{\bar\delta} f_n, \Delta_{\bar\delta} f_n > \geq < f_n, f_n > = 1,$$

which offers a contradiction. This completes the proof of the proposition, and hence the proof of the last theorem.

With the above primitive situation, the super-analogue is not vary hard to obtained: As in Chapter 6, using the perturbation expansion, the above assertions also hold for the super-case $\mathbf{A}^2_u$ as well as for the operators $\mathbf{A}^2_u + \alpha\, \mathbf{N}_u$, with α a small parameter. Moreover, the result is C^∞ in α. Some care is needed since for small u, $\mathbf{A}^2_u$ and $\mathbf{N}_u$ may contain Grassmannian terms which scale with negative powers of u as $u \to 0^+$. However, note that we now also have the right cancellation, we then could derive estimates for the asymptotic expansions. This completes the discussion for (1).

Next, we discuss case (2), i.e., consider what happens in an open neighborhood U of W^2_∞.

On U, we can first identify the two complexes $E^{\cdot}$ and $E^{\cdot *}$. Then, the Laplacian Δ on $E^{\cdot}$ is equal to the sum of a local operator Δ^*, which coincides with the Laplacian for $E^{\cdot *}$; and an integral operator Δ', which comes from the augmentation ν and is determined by global sections of $F^{\cdot}$. So, it offers a trace-class norm $O(\varepsilon)$ over a ε-neighborhood of W^2_∞ in the original metric. Furthermore, this primitive picture also holds for the super-analogue, since, for the super-analogue, the difference $\mathbf{A}^2_u - \mathbf{A}^{*2}_u$ is a sum of linear terms in u with coefficient an integral operator of Grasmannian degree zero and a constant term of u with an integral operator of Grassmannian degree ≥ 1: The linear term comes just as before; while the constant term is made up from covariant derivatives (in the Y-direction) of elements of $F^{\cdot}$. Thus note that now we use the logarithmic metrics, so the above difference vanishes on ∂W_∞. Hence again the difference operator has trace-class norm $O(\varepsilon)$. Next we use this conclusion to deduce the assertion in the lemma.

First we give its primitive form. Note that now we need to consider the kernel for the difference of the associated generalized Laplacians, we begin with an expression of this difference.

Let φ_ε be a cut-off function with its support in a 2ε-neighborhood of W^2_∞, which is 1 at the points with the distance at most ε from ∂W_∞. Also assume that φ_ε has support in U, that the C^∞-norm of φ_ε with respect to the logarithmic norm is uniformly bounded, and that φ_ε acts as zero on $F^{\cdot}$. With this, since, on U, we may identify $E^{\cdot}$ and $E^{\cdot *}$, it makes sense to consider

$$L_t(u) := \varphi_\varepsilon\, (K_t(u) - K^*_t(u))\, \varphi_\varepsilon$$

as an operator on $F_{t*}(E^{\cdot *} \otimes \wedge\Omega^{0,1}_{W_t/Y})$, where, as usual, K denotes the heat kernel. Thus, we have

$$
\begin{aligned}
(\Delta^* + \frac{\partial}{\partial u})L_t(u) =& \Delta^*\varphi_\epsilon(K_t(u) - K_t^*(u))\varphi_\epsilon + \frac{\partial}{\partial u}L_t(u) \\
=& \Delta^*\varphi_\epsilon(K_t(u) - K_t^*(u))\varphi_\epsilon \\
&+ \varphi_\epsilon(\frac{\partial}{\partial u}K_t(u) - \frac{\partial}{\partial u}K_t^*(u))\varphi_\epsilon \\
=& \Delta^*\varphi_\epsilon(K_t(u) - K_t^*(u))\varphi_\epsilon \\
&- \varphi_\epsilon(\Delta K_t(u) - \Delta^* K_t^*(u))\varphi_\epsilon \\
=& [\Delta^*, \varphi_\epsilon]\varphi_{2\epsilon}(K_t(u) - K_t^*(u))\varphi_\epsilon \\
&- \varphi_\epsilon \Delta' K_t(u)\varphi_\epsilon,
\end{aligned}
$$

where $\varphi_\epsilon \Delta'$ has trace-class norm $O(\epsilon)$.

From this relation, by the initial condition that $\lim_{u\to 0^+} L_t(u) = 0$, we have

$$
\begin{aligned}
L_t(u) =& u\int_{a+b=1} K_t^*(au)[\Delta^*, \varphi_\epsilon]\varphi_{2\epsilon}(K_t(bu) - K_t^*(bu))\varphi_\epsilon da \\
&- u\int_{a+b=1} K_t^*(au)\varphi_\epsilon \Delta' K_t(bu)\varphi_\epsilon da.
\end{aligned}
$$

With this expression for $L_t(u)$, we may deduce the sturcture of $L_t(u)$ as follows: The second term is of trace class norm $O(\epsilon u e^{-\lambda u})$ uniformly in t for u large, for small u, hence is $O(\epsilon)$ in the asymptotic sense, i.e. all terms in the asymptotic expansion will be of trace-class norm $O(\epsilon)$. To determine the first term, we apply the same procedure from the right, i.e. apply $(\frac{\partial}{\partial u} + \Delta^*)$ from the other side, then we have the following expression for $L_t(u)$

$$
\begin{aligned}
L_t(u) =& u^2\int_{a+b+c=1} \{K_t^*(au)[\Delta^*, \varphi_\epsilon]\varphi_{2\epsilon}(K_t(bu) - K_t^*(bu))\varphi_{2\epsilon}[\varphi_\epsilon, \Delta^*]K_t^*(cu)\}dadb \\
&+ O(\epsilon u^2 e^{-\lambda u}).
\end{aligned}
$$

Similarly, when we introduce the number operator, for $\mathrm{Tr}_s[N_u\, L_t(u)]$, we could have the follows:

$$
\begin{aligned}
&\mathrm{Tr}_s(N_u\, L_t(u)) \\
&= u^2\int_{a+b+c=1} \mathrm{Tr}_s[N_u K_t^*(au)[\Delta^*, \varphi_\epsilon]\varphi_{2\epsilon}(K_t(bu) - K_t^*(bu))\varphi_{2\epsilon}[\varphi_\epsilon, \Delta^*]K_t^*(cu)]dadb \\
&\quad + O(\epsilon u^2 e^{-\lambda u}) \\
&= u^2\int_{a+b+c=1} \mathrm{Tr}_s[K_t^*(cu)N_u K_t^*(au)[\Delta^*, \varphi_\epsilon]\varphi_{2\epsilon}(K_t(bu) - K_t^*(bu))\varphi_{2\epsilon}[\varphi_\epsilon, \Delta^*]]dadb \\
&\quad + O(\epsilon u^2 e^{-\lambda u}) \\
&= -u\int_{a+b=1} \mathrm{Tr}_s[\frac{\partial}{\partial \alpha}[e^{-au(\Delta^* + \alpha \hat{N})}]_{\alpha=0}[\Delta^*, \varphi_\epsilon]\varphi_{2\epsilon}(K_t(bu) - K_t^*(bu))\varphi_{2\epsilon}[\varphi_\epsilon, \Delta^*]]da \\
&\quad + O(\epsilon u^2 e^{-\lambda u}).
\end{aligned}
$$

So the above trace is an integral over $\mathrm{Supp}(d\varphi_\varepsilon)^2$. Thus, for a fixed ε, we are in a compact set of $W - \partial W_\infty$, so everything has good asymptotic expansions and decays nicely as $u \to +\infty$, etc. In particular, it makes sense to use the Mellin transform, i.e. to form the regularized integral

$$\int_0^\infty \mathrm{Tr}_s(N_t\, L_t(u))\frac{du}{u}.$$

Furthermore, by the above discussion, it follows that up to a term $O(\varepsilon)$, as $t \to \infty$, this converges to the corresponding integral at ∞.

By taking a limit for the blowing-up of the base metric on Y, we may also get similar assertions for the super-analogue as we did before: We need to replace $au\Delta^*$ by $a\mathbf{A}_u^{*2}$, etc. Thus, up to a term $O(\varepsilon)$, as $t \to \infty$, the regularized integral converges to the corresponding regularized integral at ∞. But these integrals have their contributions to $\mathrm{ch}_{\mathrm{BC}t} - \mathrm{ch}_{\mathrm{BC}t}^*$, which comes from integrating the appropriate kernel over $\mathrm{Supp}(\varphi_\varepsilon)$. So, finally, we have

$$\lim_{t\to\infty}(\mathrm{ch}_{\mathrm{BC}t} - \mathrm{ch}_{\mathrm{BC}t}^*) = \mathrm{ch}_{\mathrm{BC}\infty} - \mathrm{ch}_{\mathrm{BC}\infty}^*,$$

which completes the proof of Theorem a.

§I.9.3. Euler-Green Currents

From now on, we study the deformation theory of the relative Bott-Chern secondary characteristic currents with respect to closed immersions. Before doing so we give one example for such a current, one which will also be used in the deformation theory.

Let M be a complex manifold and (E,ρ) a hermitian vector bundle on M. Denote the total space of E by M^E, and let $i: M \hookrightarrow M^E$ be the natural embedding. Then, the Koszul complex $(\wedge E^*, i_y)$ gives a resolution of $i_*\mathcal{O}_M$ on M^E and the normal bundle to M in M^E is exactly E. Let g^E be the metric on $\wedge E^*$ induced by ρ. We are going to calculate the relative Bott-Chern secondary characteristic current $\mathrm{ch}_{\mathrm{BC}}(E,\rho; i, \rho_i)$ on M^E.

The basic idea is as follows: Let e be the Euler characteristic form. Then we know that $\mathrm{ch} = e\,\mathrm{td}^{-1}$. Therefore,

$$dd^c\mathrm{ch}_{\mathrm{BC}}(E,\rho; i,\rho_i) = \mathrm{td}^{-1}(E,\rho)(\delta_M - e(E,\rho)).$$

Here we look everything on the total space M^E. Thus it is natural for us to consider how to measure the difference $\delta_M - e(E,\rho)$. Therefore, we introduce the Bott-Chern secondary characteristic current with respect to this difference, say $e_{\mathrm{BC}}(E,\rho)$. Then we try to construct it. Thus finally, we may use it to deal with the original relative Bott-Chern secondary characteristic currents. The main reference here is [BGS 91].

I.9.3.a. The Bott-Chern Secondary Characteristic Current of A Koszul Complex.

Let M be a complex manifold of dimension l and (E,ρ) be a hermitian vector bundle of rank k on M. Denote the total space of E by M^E, $p : M^E \to M$ the natural projection, and $i : M \hookrightarrow M^E$ the natural embedding. Let E^* be the dual of E, and let $\wedge E^* = \oplus_{j=0}^k \wedge^j E^*$ be the exterior algebra of E^*. For any $y \in E$, the interior multiplication operator i_y acts naturally on $(\wedge E^*)_{p(y)}$. Then the Koszul complex $(\wedge E^*, i_y)$ gives a resolution of $i_*\mathcal{O}_M$ on M^E, i.e. we have the exact sequence of sheaves

$$0 \to \mathcal{O}_{M^E}(\wedge^k E^*) \xrightarrow{i_y} \dots \xrightarrow{i_y} \mathcal{O}_{M^E} \xrightarrow{r} i_*\mathcal{O}_M \to 0.$$

Let g^E be the metric on $\wedge E^*$ induced by ρ. Denote by ∇^E the canonical connection on both of them. Thus the adjoint i_y^* of i_y is $\bar{y}\wedge$. Since the normal bundle N to M in M^E is exactly E, we know that Bismut condition (A) is automatically satisfied in this special situation.

Even though M^E is noncompact, we can talk of a current on M^E which comes from a smooth form with a compact support on M^E. Also we see that the results in the last chapter are valid on M^E. In particular, we have $\mathrm{ch}_{\mathrm{BC}}(E,\rho;i,\rho)$ the Bott-Chern secondary characteristic current on M^E associated with $(\wedge E^*, i_y)$. So by the fact that $\mathrm{ch}_{\mathrm{BC}}(E,\rho;i,\rho_i) \in P_M^{M^E}$ depends only on ρ. For short, we denote it as $\mathrm{ch}_{\mathrm{BC}}(E,\rho)$.

Let e be the ad-invariant polynomial on (k,k) matrices $e : A \mapsto \mathrm{Det}A$. Then $e(E,\rho)$ and $\mathrm{td}(E,\rho)$ are smooth forms on M. We call $e(E,\rho)$ the **Euler characteristic form** of (E,ρ). By the classical theory for characteristic forms, we know that

$$\mathrm{ch} = e\,\mathrm{td}^{-1},$$

so when lifting such forms to M^E, we have

$$dd^c\mathrm{ch}_{\mathrm{BC}}(E,\rho) = \mathrm{td}^{-1}(E,\rho)(\delta_M - e(E,\rho)).$$

Next, we give a precise description for $\mathrm{ch}_{\mathrm{BC}}$. To do so, the basic idea is to use a locally integrable current, which comes from $\mathrm{td}^{-1}(E,\rho)$ and the difference $\delta_M - e(E,\rho)$ in the sense of dd^c.

If we let $\omega(E,\rho)$ be the restriction of $\mathrm{ch}_{\mathrm{BC}}(E,\rho)$ to $M^E - M$, we know that $\omega(E,g^E)$ is smooth on $M^E - M$ and by the finite result in section 8.5, we know that $\omega(E,\rho)$ entirely determines $\mathrm{ch}_{\mathrm{BC}}(E,\rho)$. We take this as a start point.

The canonical connection ∇^E defines a horizontal subspace $T^H M^E$ in TM^E so that $TM^E = T^H M^E \oplus E$. As usual, we let $\Omega^E = (\nabla^E)^2$ be its curvature.

Next we introduce Mathai-Quillen's convention. If $Y \in T_{\mathbf{R}}M^E$, let Y^V be the component in $E_{\mathbf{R}}$ with respect to above splitting. If A is an antisymmetric tensor in End $E_{\mathbf{R}}$, we identify A with the 2-form on $T_{\mathbf{R}}M^E$:

$$Y, Z \in T_{\mathbf{R}}M^E \mapsto < Y^V, AZ^V > .$$

We also denote this 2-form by A and its exterior powers by $A^2, \dots, A^k$. If A is invertible, we may also define the forms $A^{-1}, \dots, A^{-k}$. If $\mathrm{Pf}(A)$ is the **Pfaffian** of A, the forms $\mathrm{Pf}(A)A^{-1}, \dots, \mathrm{Pf}(A)A^{-k}$ are rational functions of A, which may be extended by continuity to any arbitrary A, which may not be invertible. In the following, we still denote them as above, even if A is not invertible. Similarly, we have the complex analogue: Let $J_E = \sqrt{-1}I_E$ be the complex structure of $E_{\mathbf{R}}$ with I_E the identity map of E, then for any $j \geq 0$,

$$\det\Big(\frac{\Omega^E}{2\pi i} + bI_E\Big)\Big(\frac{\Omega^E}{2\pi i} + bJ_E\Big)^{-j}$$

are defined, as forms on M^E.

Theorem. (1) On $M^E - M$, we have

$$\omega(E, g^E) = -\frac{\partial}{\partial b}\Big[\det\Big(I_E - \exp\Big(\frac{\Omega^E}{2\pi i} + bI_E\Big)\Big) \mathrm{Log}\Big(\frac{|Y|^2}{2} + \Big(2\pi\Big(\frac{\Omega^E}{2\pi i} + bJ_E\Big)\Big)^{-1}\Big)\Big]_{b=0}.$$

(2) Let $\gamma(E, g^E)$ on M^E be defined by $\gamma(E, g^E) := \frac{\bar\partial - \partial}{2}\mathrm{ch}_{\mathrm{BC}}(E, g^E)$. Then $\gamma(E, g^E)$ is a locally integrable current on M^E and

$$\gamma(E, g^E) = \frac{1}{2}\det\Big(I_E - \exp\Big(\frac{\Omega^E}{2\pi i} + bI_E\Big)\Big)\Big(-\frac{\Omega^E}{2\pi i}\Big)^{-1} Y\,\Big(\frac{|Y|^2}{2} + (\Omega^E)^{-1}\Big)^{-1}.$$

Proof. 1. This is a standard argument. Let N_H be the number operator, $V := i_y + i_{\bar y}^* = i_y + \bar y \wedge$. If $\mathbf{A}_u := \nabla^E + \sqrt{u}V$, then by the proof of the second relation in Theorem 7.1.c, we have

$$\mathrm{Tr}_s[N_H \exp(-\mathbf{A}_u^2)] = \frac{\partial}{\partial b}\Big[\det(I_E - \exp(\Omega^E + bI_E)) \exp\Big(-u\Big(\frac{|Y|^2}{2} + (\Omega^E + bI_E)^{-1}\Big)\Big)\Big]_{b=0}.$$

Hence on $M^E - M$, if $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 0$,

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^\infty u^s \mathrm{Tr}_s[N_H \exp(-\mathbf{A}_u^2)]\frac{du}{u}\\ &=\frac{\partial}{\partial b}\Big[\det(I_E - \exp(\Omega^E + bI_E))\Big(\frac{|Y|^2}{2} + (\Omega^E + bI_E)^{-1}\Big)^{-s}\Big]_{b=0}.\end{aligned}$$

As a consequence,

$$\begin{aligned}&\Big[\frac{1}{\Gamma(s)}\int_0^\infty u^s \mathrm{Tr}_s[N_H \exp(-\mathbf{A}_u^2)]\frac{du}{u}\Big]'(0)\\ &= -\frac{\partial}{\partial b}\Big[\det(I_E - \exp(\Omega^E + bI_E))\mathrm{Log}\Big(\frac{|Y|^2}{2} + (\Omega^E + bI_E)^{-1}\Big)\Big]_{b=0},\end{aligned}$$

which gives (1).

(2) By Theorem 8.5.b, the finite part theorem, we know that the current $\gamma(E, g^E)$ is locally integrable. Let

$$\eta := \int_0^\infty \mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)]\frac{du}{2u}.$$

Then η defines a locally integrable current on M^E, and

$$\frac{\bar\partial - \partial}{2}\mathrm{ch}_{\mathrm{BC}}(E,\rho) = (2\pi i)^{1/2}[2\pi\mathrm{i}]\eta.$$

On the other hand, by the fact that $i_Y(\Omega^E)^{-1} = -(\Omega^E)^{-1}Y$, we know that

$$\begin{aligned}\mathrm{Tr}_s[\sqrt{u}V\exp(-\mathbf{A}_u^2)] &= -i_Y\mathrm{Tr}_s[\exp(-\mathbf{A}_u^2)]\\ &= -\, i_Y\det(I-\exp(\Omega^E))\exp(-u(\frac{|Y|^2}{2}+(\Omega^E)^{-1}))\\ &= -\, u\det(I-\exp(\Omega^E))\,(\Omega^E)^{-1}Y\exp(-u(\frac{|Y|^2}{2}+(\Omega^E)^{-1})).\end{aligned}$$

So we get

$$\eta = -\frac{1}{2}\det(I-\exp(\Omega^E))\,(\Omega^E)^{-1}Y\,(\frac{|Y|^2}{2}+(\Omega^E)^{-1})^{-1}.$$

But the form $(\Omega^E)^{-1}Y$ is of formal degree -1. Thus

$$[2\pi\mathrm{i}]\eta = -\frac{1}{2}\det(I-\exp(\Omega^E))\,(2\pi i)^{1/2}(\Omega^E)^{-1}Y\,(\frac{|Y|^2}{2}+(\Omega^E)^{-1})^{-1},$$

which completes the proof.

I.9.3.b. **Several Intermediate Results**

We start with a description of exterior differentiation acting on smooth sections of $\wedge(T^*_{\mathbf{R}}M^E)$ on M^E. From the decomposition of $T_{\mathbf{R}}M^E$ induced by the canonical connection ∇^E, we know that

$$\wedge(T^*_{\mathbf{R}}M^E) \simeq \wedge(T^*_{\mathbf{R}}M)\hat\otimes(\wedge E^*_{\mathbf{R}}).$$

Hence, by antisymmetrization, we can define an operator ${}^a\nabla^E$ acting on the smooth sections of $\wedge(T^*_{\mathbf{R}}M^E)$ by letting ${}^a\nabla^E\beta$ be the corresponding $j+1$ form on M^E, and

$${}^a\nabla^E(\alpha\beta) := (d\alpha)\beta + (-1)^{|\alpha|}\alpha({}^a\nabla^E\beta),$$

where α, β are smooth sections on M^E of $\wedge(T^*_{\mathbf{R}}M)$ and $\wedge^j E^*_{\mathbf{R}}$, respectively. On the other hand, $\Omega^E Y$ is a 2-form on M with the values in $E_{\mathbf{R}} \subset T_{\mathbf{R}}M^E$. So the operator $i_{\Omega^E Y}$ acts on $\wedge T^*_{\mathbf{R}}M^E$.

Lemma. We have the following relation

$$d = {}^a\nabla^E + i_{\Omega^E Y}.$$

Proof. We denote the lifting of ∇^M on $T^H M^E$ also by ∇^M for any torsion free connection on $T_{\mathbf{R}}M$. Thus the connection $\nabla^\oplus := \nabla^M \oplus \nabla^E$ defines a connection on $T_{\mathbf{R}}M^E$ with torsion T. Hence if $Y \in M^E, U, U' \in (T^*_{\mathbf{R}}M^E)_Y$, then

$$T_Y(U, U') = \Omega^E(p_* U, p_* U')Y.$$

The connection $\nabla^\oplus$ induces an operator ${}^a\nabla^\oplus$ on $\wedge T^*_{\mathbf{R}}M^E$ similarly as above, we have

$$d = {}^a\nabla^\oplus + i_T.$$

Now the result comes by considering the actions on α, β as above. In fact, since $\Omega^E Y$ takes its values in $E_{|b/R}$, $i_{\Omega^E Y}\alpha = 0$ and so ${}^a\nabla\alpha = d\alpha$. Similarly, we may get $({}^a\nabla + i_{\Omega^E Y})\beta = d\beta$, and hence the lemma.

We can now introduce several families of differential forms on M^E. For any $u > 0$, let

$$\begin{aligned} a_u :=& \det(\frac{-\Omega^E}{2\pi i})\exp(-u(\frac{|Y|^2}{2} + (\Omega^E)^{-1})); \\ b_u :=& \frac{1}{2}\det(\frac{-\Omega^E}{2\pi i})(-\Omega^E)^{-1}Y\exp(-u(\frac{|Y|^2}{2} + (\Omega^E)^{-1})); \\ c_u :=& \frac{\partial}{\partial b}[\det(\frac{-\Omega^E}{2\pi i} + bI_E)\exp(-u(\frac{|Y|^2}{2} + (2\pi(\frac{\Omega^E}{2\pi i} + bI_E))^{-1}))]_{b=0}. \end{aligned}$$

Then we have

Theorem. (1) For any $u > 0$, the form a_u is closed and lies in P^{M^E}. Furthermore, a_u is integrable.
(2) (**Double Transgression Formula**) For any $u > 0$,

$$\begin{aligned} \frac{\partial}{\partial u}a_u &= -\frac{1}{2\pi i}db_u; \\ b_u &= \frac{\bar\partial - \partial}{2u}c_u. \end{aligned}$$

In particular,

$$\frac{\partial}{\partial u}a_u = -\frac{1}{u}dd^c c_u.$$

Proof. (1) By the definition, we know that a_u is a form of type (k, k).

By Bianchi's identity, we know that ${}^a\nabla^E\Omega^E = 0$. So ${}^a\nabla^\oplus(\Omega^E)^{-1} = 0$, and $(\Omega^E)^{-1}Y$ is a form of degree -1 taking values in E. Thus by the lemma above,

$$d(\Omega^E)^{-1}Y = -2(\Omega^E)^{-1} - |Y|^2.$$

So

$$d(\frac{|Y|^2}{2} + (\Omega^E)^{-1}) = 0.$$

As a corollary, $da_u = 0$.

(2) From the proof of (1), we know that

$$a_u = \det(-\frac{\Omega^E}{2\pi i})\exp(\frac{u}{2}d(\Omega^E)^{-1}Y).$$

Hence

$$\begin{aligned}\frac{\partial}{\partial u}a_u =&\frac{1}{2}\det(-\frac{\Omega^E}{2\pi i})\, d(\Omega^E)^{-1}Y\exp(-u(\frac{|Y|^2}{2} + (\Omega^E)^{-1})\\ =&d(\frac{1}{2}(\Omega^E)^{-1}Y\, a_u).\end{aligned}$$

Let θ be the Kähler form of $E_{\mathbf{R}}$. If $X, Y \in E_{\mathbf{R}}$, $\theta(X,Y) =< X, J_E Y >$ and the element in E^* corresponding to X is given by

$$\sqrt{-1}(-i_{X^{(0,1)}} + i_{X^{(1,0)}})\theta.$$

If for any invertible skew-adjoint matrix A in End(E), let θ^A be the (1,1)-form on E defined by

$$\theta^A(U,V) := \theta(A^{-1}U, A^{-1}V),$$

we have as an element in $E^*_{\mathbf{R}}$

$$A^{-1}Y = \sqrt{-1}(-i_{AY^{(0,1)}} + i_{AY^{(1,0)}})\theta^A.$$

On the other hand, if $d^E = \partial^E + \bar{\partial}^E$ is the exterior differential on E, set

$$\partial^E_A = \partial^E + i_{AY^{(0,1)}}, \quad \bar{\partial}^E_A = \bar{\partial}^E + i_{AY^{(1,0)}}.$$

Clearly $(\partial^E_A)^2 = 0$, $(\bar{\partial}^E_A)^2 = 0$. But the Lie derivative with respect to AY is given by

$$L_{AY} = (d^E + i_{AY})^2 = \bar{\partial}^E_A\partial^E_A + \partial^E_A\bar{\partial}^E_A.$$

Hence by the fact that $L_{AY}\theta^A = 0$, we have

$$\bar{\partial}^E_A\partial^E_A\,\theta^A = -\partial^E_A\bar{\partial}^E_A\,\theta^A.$$

Note that since θ^A is both ∂^E- and $\bar{\partial}^E$-closed, we have

$$\begin{aligned}\sqrt{-1}(\partial^E_A - \bar{\partial}^E_A)\theta^A =&A^{-1}Y,\\ \frac{|Y|^2}{2} + A^{-1} =& -(\partial^E_A + \bar{\partial}^E_A)\frac{A^{-1}Y}{2}\\ =& -\sqrt{-1}\bar{\partial}^E_A\partial^E_A\,\theta^A.\end{aligned}$$

In particular,

$$\partial_A^E(\frac{|Y|^2}{2}+A^{-1})=0,\quad \bar\partial_A^E(\frac{|Y|^2}{2}+A^{-1})=0.$$

Therefore

$$\begin{aligned}&\det(-\frac{A}{2\pi i})(-A)^{-1}Y\exp(-u(\frac{|Y|^2}{2}+A^{-1}))\\ =&2\pi(\partial_A^E-\bar\partial_A^E)[\det(-\frac{A}{2\pi i})\,\theta^A\exp(-u(\frac{|Y|^2}{2}+A^{-1}))].\end{aligned}$$

However, if A is not invertible, the form

$$\det(\frac{-A}{2\pi i})\,\theta^A\exp(-u(\frac{|Y|^2}{2}+A^{-1})$$

cannot be extended by continuity to a well-defined form. Nevertheless, the form

$$\begin{aligned}&\frac{\partial}{\partial b}[\det(-(\frac{A}{2\pi i}+bI_E))\exp(-u(\frac{|Y|^2}{2}+(2\pi(\frac{A}{2\pi}+bI_E))^{-1}))]_{b=0}\\ =&\frac{\partial}{\partial b}[\det(-(\frac{A}{2\pi i}+bI_E))]_{b=0}\exp(-u(\frac{|Y|^2}{2}+A^{-1}))\\ &-u\det(-\frac{A}{2\pi i})2\pi\theta^A\exp(-u(\frac{|Y|^2}{2}+A^{-1}))\end{aligned}$$

may be extended by continuity for arbitrary A. Thus by the fact that

$$\frac{\partial}{\partial b}[\det(-(\frac{A}{2\pi i}+bI_E))]_{b=0}\exp(-u(\frac{|Y|^2}{2}+A^{-1}))$$

is ∂_A^E, $\bar\partial_A^E$-closed, we have

$$\begin{aligned}&\det(-\frac{A}{2\pi i})\,(-A)^{-1}Y\exp(-u(\frac{|Y|^2}{2}+A^{-1}))\\ =&\frac{1}{u}\frac{\bar\partial_A^E-\partial_A^E}{2\pi i}\frac{\partial}{\partial b}[\det(-(\frac{A}{2\pi i}+bI_E))\exp(-u(\frac{|Y|^2}{2}+(2\pi i(\frac{A}{2\pi i}+bI_E))^{-1}))]_{b=0}.\end{aligned}$$

In particular, now both sides can be extended to arbitary A. So by the lemma above, we have the first two relations, while the last one is a direct consequence of these two.

Next, we establish the convergence of the above currents. For this, we have

Theorem. For any $n\in\mathbf{N}$, there exists a constant $C>0$ such that if μ is a smooth differential form on M^E with a compact support in $B_n:=\{Y\in M^E:|Y|\le n\}$, then, for $u\ge 1$,

$$\begin{aligned}|\int_{M^E}\mu(a_u-\delta_M)|\le&\frac{C}{\sqrt{u}}\|\mu\|_{C^1(M^E)};\\ |u\int_{M^E}\mu b_u|\le&\frac{C}{\sqrt{u}}\|\mu\|_{C^1(M^E)};\\ |\int_{M^E}\mu c_u|\le&\frac{C}{\sqrt{u}}\|\mu\|_{C^1(M^E)}.\end{aligned}$$

Furthermore, if U, Γ, φ, m are taken with respect to the embedding $i : M \hookrightarrow M^E$ as in 7.1.d, there exists $C' > 0$ such that for $u \geq 1$,

$$p_{U,\Gamma,\varphi,m}(a_u - \delta_M) \leq \frac{C}{\sqrt{u}};$$
$$p_{U,\Gamma,\varphi,m}(ub_u) \leq \frac{C}{\sqrt{u}};$$
$$p_{U,\Gamma,\varphi,m}(c_u) \leq \frac{C}{\sqrt{u}}.$$

Proof. We only prove the first part, as the proof of the second part is similar to the proof of the correspondine part of Theorem 8.1.

Let τ_u be the map $Y \mapsto \sqrt{u}Y$. Then

$$a_u = \tau_u^* a_1, \quad ub_u = \tau_u^* b_1, \quad c_u = \tau_u^* c_1.$$

Let $\sigma_u := \tau_u^{-1}$. We have

$$\int_{M^E} \mu a_u = \int_{M^E} (\sigma_u^* \mu) a_1;$$
$$\int_{M^E} \mu u b_u = \int_{M^E} (\sigma_u^* \mu) b_1;$$
$$\int_{M^E} \mu c_u = \int_{M^E} (\sigma_u^* \mu) c_1.$$

Hence, as $u \to +\infty$, we have

$$\int_{M^E} \mu a_u \to \int_{M^E} (i^* \mu) \int_E a_1;$$
$$\int_{M^E} \mu u b_u \to \int_{M^E} (i^* \mu) \int_E b_1;$$
$$\int_{M^E} \mu c_u \to \int_{M^E} (i^* \mu) \int_E c_1.$$

Now the result follows from the facts that

$$\int_E a_1 = 1, \quad \int_E b_1 = 0, \quad \int_E c_1 = 0.$$

I.9.3.c **A Special Bott-Chern Secondary Characteristic Current: The Euler-Green Current**

By the convergence of section 3.b, we can use the Mellin transform to go further: For $s \in \mathbf{C}, 0 < \mathrm{Re}(s) < \frac{1}{2}$, let $\varphi_{E,g^E}(s)$ be the current on M^E, defined by

$$\begin{aligned}\varphi_{E,\rho}(s) :=& \frac{1}{\Gamma(s)} \int_0^{+\infty} u^s c_u \frac{du}{u},\\ \phi(E,\rho) :=& \int_0^{+\infty} b_u du.\end{aligned}$$

From Theorem 3.b, we know that $\varphi_{E,\rho}(s)$ and $\phi(E,\rho)$ are well-defined. Further, $\varphi_{E,\rho}$ may be extended to a meromorphic function of s on the whole complex plane, which is holomorphic at $s = 0$. Hence $\varphi'_{E,\rho}(0)$ exists and is equal to

$$\int_0^1 (c_u - c_0) \frac{du}{u} + \int_1^{+\infty} c_u \frac{du}{u} - \Gamma'(1) c_0.$$

Now the Euler-Green current, a special Bott-Chern secondary characteristic form, denoted by e_{BC}, is defined by

$$e_{\mathrm{BC}}(E,\rho) := [2\pi \mathrm{i}] \varphi'_{E,\rho}(0).$$

Since E is the normal bundle to M in M^E and c_0 is a closed form, we have the following

Theorem. (1) The total degree of $\phi(E,\rho)$ is $2\dim E - 1$. The current $e_{\mathrm{BC}}(E,\rho)$ is of complex type $(\dim E - 1, \dim E - 1)$.
(2) The wave front sets of the currents $e_{\mathrm{BC}}(E,\rho)$ and $\phi(E,\rho)$ are contained in $E^*_{\mathbf{R}}$. Moreover, we have $e_{\mathrm{BC}}(E,\rho) \in P_M^{M^E}$.
(3) The following equations of currents hold on M^E:

$$\begin{aligned}\phi(E,\rho) =& \frac{1}{2} \frac{\bar\partial - \partial}{2\pi i} \varphi'_{E,\rho}(0);\\ d\phi(E,\rho) =& e(E,\rho) - \delta_M.\end{aligned}$$

In particular,

$$dd^c e_{\mathrm{BC}}(E,\rho) + \delta_M = e(E,\rho).$$

Moreover, we know that the singularities of $e_{\mathrm{BC}}(E,\rho)$ and $\phi(E,\rho)$ are given by the following

Theorem. (1) The currents $e_{\mathrm{BC}}(E,\rho)$ and $\phi(E,\rho)$ are locally integrable.
(2) The following equations hold

$$\begin{aligned}
e_{\mathrm{BC}}(E,\rho) =& -\frac{\partial}{\partial b}[\det(-(\frac{\Omega^E}{2\pi i}+bI_E)) \\
&\cdot \operatorname{Log}(\frac{|Y|^2}{2}+(2\pi i(\frac{\Omega^E}{2\pi i}+bI_E))^{-1})]_{b=0} \\
=& -\frac{\partial}{\partial b}[\det(-(\frac{\Omega^E}{2\pi i}+bI_E)) \\
&\{\operatorname{Log}(\frac{|Y|^2}{2})+\sum_{j=1}^{\dim E-1}\frac{2^j}{j|Y|^{2j}}((-2\pi i(\frac{\Omega^E}{2\pi i}+bI_E))^{-1})^j\}]_{b=0}; \\
\phi(E,\rho) =& \frac{1}{2}\det(-\frac{\Omega^E}{2\pi i})(-\frac{\Omega^E}{2\pi i})^{-1}Y(\frac{|Y|^2}{2}+(\Omega^E)^{-1})^{-1} \\
=& \frac{1}{2}\det(-\frac{\Omega^E}{2\pi i})(-\frac{\Omega^E}{2\pi i})^{-1}Y\sum_{j=1}^{\dim E}\frac{2^j}{|Y|^{2j}}((-\Omega^E)^{-1})^{j-1}.
\end{aligned}$$

In particular, if $\dim E = 1$,

$$\tilde{e}(E,\rho) = \operatorname{Log}(|y|^2).$$

Proof. First of all, we have

$$\begin{aligned}
c_u = \exp(-\frac{u|Y|^2}{2})\frac{\partial}{\partial b}[\det(-(\frac{\Omega^E}{2\pi i}+bI_E)) \\
\sum_{j=0}^{\dim E}((-2\pi i(\frac{\Omega^E}{2\pi i}+bI_E))^{-1})^j\frac{u^j}{j!}]_{b=0}.
\end{aligned}$$

Then, we know that in the sum, the last index is $\dim E-1$, not $\dim E$. Moreover, we see that

$$\int_1^{+\infty}\exp(-\frac{u|Y|^2}{2})u^k\frac{du}{u} \le \begin{cases} C(1+\operatorname{Log}\frac{1}{|Y|}), & \text{if } k=0; \\ |Y|^{-2k}, & \text{if } k>0. \end{cases}$$

But, $\operatorname{Log}|Y|$ is locally integrable on M^E, and for $1 \le k \le \dim E-1$, the function $|Y|^{-2k}$ is locally integrable. Hence $e_{\mathrm{BC}}(E,\rho)$ is locally integrable. Similarly, we can show that $\phi(E,\rho)$ is locally integrable. This completes the proof for (1). For (2), we only need to do the same thing as what we did for the proof of Theorem 3.a.

2. It also follows from the above discussion that 2 holds. Furthermore, if $\dim E = 1$, then

$$e_{\mathrm{BC}}(E,\rho) = [\frac{\partial}{\partial b}(\frac{\Omega^E}{2\pi i}+b)]_{b=0}\operatorname{Log}|y|^2 = \operatorname{Log}|y|^2.$$

I.9.3.d **An Explicit Formula For** $\mathrm{ch}_{\mathrm{BC}}(E,\rho)$

We now compare the current $\mathrm{ch}_{\mathrm{BC}}(E,\rho)$ with $e_{\mathrm{BC}}(E,\rho)$.

Theorem. We have the following relation

$$\mathrm{ch}_{\mathrm{BC}}(E,\rho)-\mathrm{td}^{-1}(E,\rho)\,e_{\mathrm{BC}}(E,\rho)\in P_M^{M^E,0}.$$

Proof. We can use the Mellin transform to obtain the following: For $s\in\mathbf{C}, 0<\mathrm{Re}(s)<\frac{1}{2}$, $\delta(s)$ is defined by

$$\delta(s)=\frac{1}{\Gamma(s)}\int_0^{+\infty}u^s([2\pi\mathbf{i}](\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)]) \\ -\mathrm{td}^{-1}(E,g^E)c_u+(\mathrm{td}^{-1})'(E,g^E)\delta_M)\frac{du}{u}.$$

Then, by the asymototic expansion of $\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)]$, we know that $\delta(s)$ is a well-defined current on M^E, which extends to a current so that it is a meromorphic function of $s\in\mathbf{C}$. Furthermore, the function is holomorphic at $s=0$ and

$$\delta'(0)=\mathrm{ch}_{\mathrm{BC}}(E,\rho)-\mathrm{td}^{-1}(E,\rho)\,e_{\mathrm{BC}}(E,\rho).$$

On the other hand, by the first equation in the proof of Theorem 3.a, we know that

$$\begin{aligned}&[2\pi\mathbf{i}]\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)]\\ &=\frac{\partial}{\partial b}[\mathrm{td}^{-1}(-(\frac{\Omega^E}{2\pi i}+bI_E))\det(-(\frac{\Omega^E}{2\pi i}+bI_E))\\ &\quad\exp(-u(\frac{|Y|^2}{2}+(2\pi i(\frac{\Omega^E}{2\pi i}+bI_E)^{-1})))]_{b=0}.\end{aligned}$$

Therefore

$$[2\pi\mathbf{i}](\mathrm{Tr}_s[N_H\exp(-\mathbf{A}_u^2)])-\mathrm{td}^{-1}(E,\rho)c_u=-(\mathrm{td}^{-1})'(E,\rho)\,a_u.$$

In particular, for $0<\mathrm{Re}(s)<\frac{1}{2}$, we have

$$\delta(s)=-(\mathrm{td}^{-1})'(E,\rho)\frac{1}{\Gamma(s)}\int_0^{+\infty}u^s(a_u-\delta_M)\frac{du}{u}.$$

But, by Theorem 3.b.1,

$$a_u-\delta_M=dd^c\int_u^{+\infty}c_\nu\frac{d\nu}{\nu}.$$

So, for $0<\mathrm{Re}(s)<\frac{1}{2}$,

$$\delta(s)=(\mathrm{td}^{-1})'(E,\rho)\,dd^c[\frac{1}{\Gamma(s+1)}\int_0^{+\infty}u^s c_u\frac{du}{u}].$$

Since

$$\frac{1}{\Gamma(s+1)}\int_0^{+\infty} u^s c_u \frac{du}{u} = \frac{\varphi_{E,\rho}(s)}{s},$$

and $\varphi_{E,\rho}(0) = c_0$ is a closed form, we have

$$\delta(s) = (\mathrm{td}^{-1})'(E,\rho)\, dd^c\Big(\frac{\varphi_{E,\rho}(s) - \varphi_{E,\rho}(0)}{s}\Big).$$

Hence

$$\delta'(0) = dd^c\Big((\mathrm{td}^{-1})'(E,\rho)\frac{\varphi''_{E,\rho}(0)}{2}\Big).$$

In particular,

$$\mathrm{ch}_{\mathrm{BC}}(E,\rho) - \mathrm{td}^{-1}(E,\rho)\, e_{\mathrm{BC}}(E,\rho) = dd^c\Big(((\mathrm{td}^{-1})'(E,\rho)\frac{\varphi''_{E,\rho}(0)}{2}\Big).$$

Thus, the theorem comes from the fact that the wave front of the current $\varphi''_{E,\rho}(0)$ is contained in E.

The advantage of this result is that we may choose a locally integrable representative current

$$\mathrm{td}^{-1}(E,\rho)\, e_{\mathrm{BC}}(E,\rho)$$

in the class of the non-locally integrable current $\mathrm{ch}_{\mathrm{BC}}(E,\rho)$. For certain purposes, this result is very useful.

Next we consider $e_{\mathrm{BC}}(E,\rho)$ as a function of ρ. By the fact that

$$\int_{\mathbf{P}^1} [\mathrm{Log}|z|^2]d\mu = 0,$$

we make the $\mathbf{P}^1$-deformation and have the following

Theorem. In $P_M^{M^E}/P_M^{M^E,0}$,

$$e_{\mathrm{BC}}(E,\rho) - e_{\mathrm{BC}}(E,\rho') = e_{\mathrm{BC}}(E,\rho,\rho').$$

I.9.3.e. Compatibility With Sections

We consider now the compatibility of the above process with a certain kind of sections, which will be used in Part II. Let s be a holomorphic section of E on M which is transversal to M in M^E. Namely, we assume that if $x \in M$ is such that $s(x) = 0$, and $d(s)$ is the differential of s at x, then $\mathrm{Im}[ds(x)] = E$. Let $M' := \{x \in M : s(x) = 0\}$, then on M', ds identifies $E|_{M'}$ with the normal bundle N to M'. Let i be the embedding $M' \hookrightarrow M$. Then the chain complex $(\wedge E^*, i_s) = s^*(\wedge E^*, i_y)$ provides a resolution of $i_*\mathcal{O}_{M'}$. Hence we can construct $\mathrm{ch}_{\mathrm{BC}}(E, g^E; s)$ on M associated with $(\wedge E^*, s)$. By axiom 2, we know that

$$\mathrm{ch}_{\mathrm{BC}}(E, g^E; s) = s^*\mathrm{ch}_{\mathrm{BC}}(E, g^E).$$

Also, the current $e_{\mathrm{BC}}(E, g^E)$ can be pulled back by the section s. Then, $s^*e_{\mathrm{BC}}(E, g^E)$ is a current on M which lies in $P^M_{M'}$ and

$$dd^c s^* e_{\mathrm{BC}}(E, g^E) = \delta_{M'} - e(E, g^E).$$

Furthermore, if E, E' are two holomorphic vector bundles on M, we can consider E and E' as sub-vector bundles of $E \oplus E'$. In the same way, the manifolds M^E and $M^{E'}$ are seen as submanifolds of $M^{E\oplus E'}$ which intersect transversely, and $M^E \cap M^{E'} = M$. The vector bundles E and E' lift naturally to $M^{E\oplus E'}$. If $z = (y, y') \in E \oplus E'$, set $\sigma(z) = y$, $\sigma'(z) = y'$. Then σ, σ' are holomorphic sections of E and E', which vanish exactly on $M^{E'}$ and M^E, respectively.

Let ρ and ρ' be hermitian metrics on E and E'. We equip $E \oplus E'$ with the metric $\rho^{\oplus} := \rho \oplus \rho'$. By above discussion, we know that $\sigma^* e_{\mathrm{BC}}(E, \rho)$ and $\sigma'^* e_{\mathrm{BC}}(E', \rho')$ are well-defined currents on $M^{E\oplus E'}$. If we imitate the proof of the axiom 4 for the relative Bott-Chern secondary characteristic current, we have the following

Theorem. In $P^{M^{E\oplus E'}}_{M^E \cup M^{E'}} / P^{M^{E\oplus E'},0}_{M^E \cup M^{E'}}$,

$$\begin{aligned} e_{\mathrm{BC}}(E \oplus E', \rho^{\oplus}) &= e(E', \rho')\sigma^* e_{\mathrm{BC}}(E, \rho) + e_{\mathrm{BC}}(E', \rho')\delta_{M^{E'}} \\ &= e(E, \rho)\sigma'^* e_{\mathrm{BC}}(E', \rho') + e_{\mathrm{BC}}(E, \rho)\delta_{M^E}. \end{aligned}$$

Accordingly, we let s, s' be the holomorphic sections on M of E, E', respectively, chosen as above, and let

$$M' := \{x \in M : s(x) = 0\}, \quad \tilde{M}' := \{x \in M : s'(x) = 0\}.$$

Then the section $s'' := (s, s')$ of $E \oplus E'$ is chosen for $E \oplus E'$. Let $M'' := M' \cap \tilde{M}'$. Then, in $P^M_{M' \cup \tilde{M}'} / P^{M,0}_{M' \cup \tilde{M}'}$ from the above theorem,

$$s''^* e_{\mathrm{BC}}(E \oplus E', \rho^{\oplus}) = e(E', \rho')s^* e_{\mathrm{BC}}(E, \rho) + s'^* e_{\mathrm{BC}}(E, \rho)\delta_{M'}.$$

§I.9.4 Deformation of Relative Bott-Chern Secondary Characteristic Currents

In this section, we give the deformation theory for the relative Bott-Chern secondary characteristic currents with respect to closed immersions. We will use the same notation as above. We know that there may be have two different relative Bott-Chern secondary characteristic currents with respect to i_0 and i_∞ respectively. As before, a quite natural question is how we can measure the difference and this leads us to introduce certain ternary objects. But as in the case of smooth morphisms, we only prove a weak result in this direction. We consider this more carefully. For short, we let $N = N_i, N' = N_{Y\times\mathbf{P}^1/W}$, $\varphi = p_Z : W \to Z$ and let $\phi = \pi_P : P \to X$ be the restriction to P. We fix hermitian metrics g^N, g_η on N, η and g^ξ on ξ, which satisfy Bismut assumption (A). If we put the standard Fubini-Study metric on $\mathcal{O}^*_{\mathbf{P}^1}(-\infty)$, then by the fact that

$$\mathcal{N}_{Y\times\mathbf{P}^1/W} \simeq p_X^*\mathcal{N}_{X/Z} \otimes q_X^*\mathcal{O}_{\mathbf{P}^1}(-\infty),$$

we have a hermitian metric $g^{N'}$ on N'. We also choose hermitian metrics $g^{\tilde{\xi}}$ on $\tilde{\xi}$, which satisfy Bismut assumption (A) with respect to g^η and $g^{N'}$. Since B_XZ does not meet $X\times\{\infty\}$ at W_∞, we may assume that the restriction of $g^{\tilde{\xi}}$ to $W_0 = Z\times\{0\}$ coincides with g^ξ, and that the restriction of $g^{\tilde{\xi}}$ to $B_XZ \subset W_\infty$ is split acyclic. Hence as a holomorphic hermitian vector bundle, the normal bundle of X in W_∞ coincides with N with the metric g^N.

Note that I, defined in 2.a, is also a closed immersion, and hence, there are associated relative Bott-Chern secondary characteristic currents. (As before, we omit the notation of the pull-back from the projection.) That is, we have a current

$$\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}}).$$

On the other hand, we can also consider the natural current $\mathrm{Log}\,|z|^2$ on W, which may be defined as the pull-back of $\mathrm{Log}\,|z|^2$ from $\mathbf{P}^1$ with z the standard coordinate of $\mathbf{P}^1$. Near $\mathbf{P}(N)$, we have the equation $z^{-1} = \frac{x_0}{x_i}y_i$ and we know that $\mathrm{Log}\,|z|^2$ is integrable on W. Thus by Theorem 8.2.4 of [Hö 86], since $\mathrm{WF}(\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}})) \subset N'^*_{\mathbf{R}}$, and $q_W : W \to \mathbf{P}^1$ is a submersion near $Y\times\mathbf{P}^1 \subset W$, we have that

$$\mathrm{WF}(\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}})) \cap \mathrm{WF}(\mathrm{Log}\,|z|^2) = \emptyset.$$

Hence by Theorem 8.2.10 of [Hö 86], the product of currents $\mathrm{Log}\,|z|^2\,(\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}}))$ is well-defined. The usual rules of differential calculus apply to this product. In particular,

$$\begin{aligned}&\frac{\bar\partial\partial}{2\pi i}(\mathrm{Log}\,|z|^2)(\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}})) - \mathrm{Log}\,|z|^2\,\frac{\bar\partial\partial}{2\pi i}(\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}}))\\ =&\frac{\bar\partial}{2\pi i}((\partial\mathrm{Log}\,|z|^2)\,\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}})) + \frac{\partial}{2\pi i}(\mathrm{Log}\,|z|^2\,(\bar\partial\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}}))) \in P^{W,0}.\end{aligned}$$

On the other hand, we have

$$\frac{\bar\partial\partial}{2\pi i}\,\mathrm{Log}\,|z|^2 = \delta_{W_0} - \delta_{W_\infty},$$

so the restrictions of the current $\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}})$ to W_0 and W_∞, respectively, are well-defined with

$$\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}})|_{W_0} = \mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; i, g^i; \xi, g^\xi).$$

Therefore

$$\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;I,g^I;\tilde\xi,g^{\tilde\xi})\delta_{W_0}-\mathrm{ch}_{\mathrm{BC}}(\eta,\rho;g_\eta;I,g^I;\tilde\xi,g^{\tilde\xi})\delta_{W_\infty}$$
$$-\mathrm{Log}|z|^2(\mathrm{td}^{-1}(N',g^{N'})\mathrm{ch}(\eta,g_\eta))\delta_{X\times\mathbf{P}^1}-\mathrm{ch}(\tilde\xi,g^{\tilde\xi}))\in P^{W,0}.$$

But, if R is the curvature of the canonical connection on $\mathcal{O}(-1)$, from the definition,

$$\mathrm{td}^{-1}(N',g^{N'})=\mathrm{td}^{-1}(N,g^N)+(\mathrm{td}^{-1})'(N,g^N)(\frac{-R}{2\pi i}).$$

Thus, by integrating along the fibers of $\varphi:W\to Z$ and noting that

$$\int_{\mathbf{P}^1}[\mathrm{Log}\,|z|^2]d\mu=0,$$

we have

$$\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;i,g^i;\xi,g^\xi)-\varphi_*[\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;I,g^I;\tilde\xi,g^{\tilde\xi})\delta_{W_\infty}]$$
$$+\varphi_*[\mathrm{Log}\,|z|^2\mathrm{ch}(\tilde\xi,g^{\tilde\xi})]-[\int_{\mathbf{P}^1}\mathrm{Log}|z|^2(\frac{-R}{2\pi i})]\,(\mathrm{td}^{-1})'(N,g^N)\,\mathrm{ch}(\eta,g^\eta)\delta_Y$$
$$=\frac{\bar\partial}{2\pi i}\varphi_*((\partial\,\mathrm{Log}\,|z|^2)\,\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;I,g^I;\tilde\xi,g^{\tilde\xi}))+\frac{\partial}{2\pi i}\varphi_*(\mathrm{Log}\,|z|^2\,(\bar\partial\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;I,g^I;\tilde\xi,g^{\tilde\xi}))).$$

So in order to investigate the weak deformation theory at the level P/P^0, we have to evaluate

$$\varphi_*(\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;I,g^I;\tilde\xi,g^{\tilde\xi})\,\delta_{W_\infty}).$$

But this is about the deformation at infinity and it is natural for us to use the Koszul complex to study it. So we need to recall a few facts from Section 1:

Let H be the associated vector bundle of $\mathcal{H}$ on P. Then, by the inclusion $\mathcal{H}\hookrightarrow\phi^*(\mathcal{N}^*_{X/Z}\oplus\mathcal{N}^*_{\infty/\mathbf{P}^1})$, we have an induced metric g^H on H. On $K_j(\varphi)=\wedge^j H$, we take the metric induced from g^H. Let σ be the canonical section of H^* and $\sigma^*(e_{\mathrm{BC}}(H^*,g^{H^*}))$ the corresponding Euler-Green current on P.

For each $j\geq 0$, we have the exact sequence of vector sheaves:

$$A_j:0\to\pi_P^*\mathcal{L}.\to f=j_\infty^*\tilde\xi\to K.(\varphi)\otimes\pi_P^*\eta\to 0.$$

Let L be the associated vector bundle of

$$\mathcal{L}_j=(\mathcal{B}_{j+1}\otimes\mathcal{N}^j_{\infty/\mathbf{P}^1})\oplus(\mathcal{B}_j\otimes\mathcal{N}^{j-1}_{\infty/\mathbf{P}^1}).$$

We can use the orthogonal direct sum of the induced metrics, so that the complex L attached to $\mathcal{L}$ becomes split acyclic, as a complex of hermitian holomorphic vector bundles. So we metrize the complex A_j and we get a smooth current $\mathrm{ch}_{\mathrm{BC}}(A_j,\rho_{A_j})$ on P.

Now we have

Claim. In $P_X^P/P_X^{P,0}$,

$$\begin{aligned}&\varphi_*(\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;I,g^I;\tilde{\xi},g^{\tilde{\xi}})\delta_{W_\infty})\\&=[\phi_*[\mathrm{td}^{-1}(H^*,g^{H^*})\sigma^*(e_{\mathrm{BC}}(H^*,g^{H^*}))]\mathrm{ch}(\eta,g_\eta)-\phi_*[\sum_{j=0}^m(-1)^j\mathrm{ch}_{\mathrm{BC}}(A_j,\rho_{A_j})]]\delta_X.\end{aligned}$$

When the claim is proved, or equivalently, we have the following

Theorem. (1) As currents, modulo the ∂- and $\bar{\partial}$-exact currents, we have the following relation

$$\begin{aligned}&\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;i,g^i;\xi,g^\xi)\\&\quad=[\phi_*[\mathrm{td}^{-1}(H^*,g^{H^*})\sigma^*(e_{\mathrm{BC}}(H^*,g^{H^*}))]\mathrm{ch}(\eta,g_\eta)\\&\quad\quad-\varphi_*[\mathrm{Log}|z|^2\mathrm{ch}(\tilde{\xi},g^{\tilde{\xi}})]-\phi_*[\sum_{j=0}^m(-1)^j\mathrm{ch}_{\mathrm{BC}}(A_j,\rho_{A_j})]]\delta_Y.\end{aligned}$$

(2) The integral $\phi_*[\mathrm{td}^{-1}(H^*,g^{H^*})\sigma^*(e_{\mathrm{BC}}(H^*,g^{H^*}))]$ along the fibers of ϕ is a smooth closed differential form on X, whose cohomology class does not depend on the choice of the metric g^N.

Proof. (1) By the earlier discussion, the first statement is a consequence of the certain statements for wave front sets: By Theorem 8.2.13 [Hö] and the fact that φ is a composition of an immersion and a submersion, if ω is a current on W, then

$$\mathrm{WF}(\varphi_*\omega)\subset\{p\in T^*_{\mathbf{R}}-\{0\}:\varphi^*p\in\{0\}\cup\mathrm{WF}(\omega)\}.$$

Thus it is sufficient to show that the wave front sets of

$$(\partial\mathrm{Log}|z|^2)\,\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;I,g^I;\tilde{\xi},g^{\tilde{\xi}})$$

and

$$\mathrm{Log}\,|z|^2\,(\bar{\partial}\mathrm{ch}_{\mathrm{BC}}(\eta,g_\eta;I,g^I;\tilde{\xi},g^{\tilde{\xi}}))$$

are in the sum of conormal bundles to W_∞ and to $X\times\mathbf{P}^1$ in W. This, by definition, is the conormal bundle to W_∞ on $\mathbf{P}(N)$, which is a direct consequence of Theorem 8.2.13 [Hö 86].

Before going further, we now we prove the claim. From the very beginning, we use the superconnection formalism for $\tilde{\ }$-system. We have $\tilde{A}_u$, etc. Let $k:W_\infty\hookrightarrow W$. Near $\mathbf{P}(N)$, W_∞ is the union of two smooth manifolds intersecting transversely along $\mathbf{P}(N)$. If α is a smooth form on W, the form $k^*\alpha$ is unambiguously defined on $W_\infty-\mathbf{P}(N)$, and defines an integrable current on W_∞. Furthermore, as a current on W, $k^*(\alpha)\delta_{W_\infty}$ is exactly the product of the currents α and δ_{W_∞}.

For $0 < \mathrm{Re}(s) < 1$, let $\zeta_{\tilde{\xi}}^{\infty}(s)$ be the current on W_∞ defined by

$$\zeta_{\tilde{\xi}}^{\infty}(s) := \frac{1}{\Gamma(s)} \int_0^\infty u^s [k^* \mathrm{Tr}_s [N_H \exp(-\tilde{A}_u^2)] + \varphi^* [(\mathrm{td})^{-1\prime}(N, g^N) \mathrm{ch}(\eta, g_\eta)] \delta_X] \frac{du}{u}.$$

Since the forms $\mathrm{Tr}_s[N_H \exp(-\tilde{A}_u^2)]$ decay exponentially as $u \to +\infty$ on compact subsets of $W_\infty - X$, and in particular near $\mathbf{P}(N)$, the finiteness theorem given in chapter I.8.5 shows that the above process is well-defined. Furthermore, this function of s extends to a meromorphic function, which is holomorphic at $s = 0$. So we may introduce the following

Lemma. (a) With the same notation as above,

$$\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}}) \delta_{W_\infty} = [2\pi \mathrm{i}] \zeta_{\tilde{\xi}}^{\infty\prime}(0)\, \delta_{W_\infty}.$$

(b) $\zeta_{\tilde{\xi}}^{\infty\prime}(0) = 0$ on $W_\infty - P$.

Proof of the Lemma. (a) We show this fact as follows: First, replace on both sides the integration from 0 to ∞ by integration from 0 to a finite T, and then equality is an obvious consequence of the previous considerations. Finally, let $T \to +\infty$. Since the truncated integrals approximate $\mathrm{ch}_{\mathrm{BC}}(\eta, g_\eta; I, g^I; \tilde{\xi}, g^{\tilde{\xi}})\, \delta_{W_\infty}$ in $\mathcal{D}'_{N'^*_{\mathbb{R}}}(W)$, and the multiplication by δ_{W_∞} is a continuous map of $\mathcal{D}'_{N'^*_{\mathbb{R}}}(W)$ into $\mathcal{D}'(W)$, we have the above equality.

(b) This follows since on $W_\infty - P$, the complex $(\tilde{\xi}, \tilde{\nu})$ splits as a hermitian complex,

$$\tilde{A}_u^2 = (\nabla^{\tilde{\xi}})^2 + u I_{\tilde{\xi}}.$$

Now we continue the proof of the claim or (1) of the theorem. The support of the current $\zeta_{\tilde{\xi}}^{\infty\prime}(0)$ is contained in P. More precisely, the restriction of $\zeta_{\tilde{\xi}}^{\infty\prime}(0)$ to P is exactly the singular current associated with the complex $\tilde{\xi}|_P$ of hermitian vector bundles. This provides a resolution of the direct image $s_*\eta$ of η by the immersion $s : Y \to P$.

Now consider $\zeta_{\tilde{\xi}}^{\infty\prime}(0)$ as a current on P. By the exact sequence

$$0 \to \pi_P^* \mathcal{L}. \to f^* \tilde{\xi} \to K.(\varphi) \otimes \pi_P^* \eta \to 0,$$

we wee that on P, $\mathcal{L}$ is split acyclic even with the metrics. Then, by axiom 3, we know that

$$[2\pi \mathrm{i}] \zeta_{\tilde{\xi}}^{\infty\prime}(0) = \mathrm{ch}_{\mathrm{BC}}(K.(\varphi) \otimes \pi_P^* \eta,\, g^{\cdot}) - \sum_{j=0}^{m} (-1)^j \mathrm{ch}_{\mathrm{BC}}(A_j, \rho_{A_j})$$

in $P_X^P / P_X^{P,0}$. Here $g^{\cdot}$ denotes the induced metric.

By Theorem 3.d, we know that

$$[\mathrm{ch}_{\mathrm{BC}}(K.(\varphi)\otimes\pi_P^*\eta,\, g^\cdot) - \mathrm{td}^{-1}(H^*, g^{H^*})\,\sigma^*(e_{\mathrm{BC}}(H^*, g^{H^*}))]\mathrm{ch}(\eta, g^\eta)$$

lies in $\tilde{P}_X^{P,0}$, which is the claim. This completes the proof of (1) too.

(2) We now use the Chern-Weil theory to prove that $\phi_*[\mathrm{td}^{-1}(H^*, g^{H^*})\sigma^*(e_{\mathrm{BC}}(H^*, g^{H^*}))]$ is closed. Let Q be the bundle of unitary frames in N. Then Q is a $U(e)$-principal bundle, which we equip with the connection ∇_N. With the canonical metric on $\mathbf{C}^e$, there is a natural action of $U(e)$ on $\mathbf{P}(\mathbf{C}^e\oplus 1)$ as a group of holomorphic transformations and

$$P = \mathbf{P}(N\oplus 1) = Q\times_{U(e)}\mathbf{P}(\mathbf{C}^e\oplus 1).$$

On the 'fiber' we can form $\mathbf{P}(\mathbf{C}^e\oplus 1)$ the holomorphic hermitian vector bundle H_0, with a morphism $\varphi_0 : \mathcal{H}_0 \to \mathcal{O}_{\mathbf{P}(\mathbf{C}^e\oplus 1)}$, and hence, obtain the holomorphic hermitian Koszul chain complex $\wedge H_o = K(\varphi_0)$. The group $U(e)$ acts naturally on H_0 as a group of holomorphic unitary transformations, which preserves the map φ_0. So

$$H = Q\times_{U(e)} H_0,\quad K(\varphi) = Q\times_{U(e)} K(\varphi_0).$$

But the connection ∇_N induces a connection on the fibration $P\to X$. In particular, the curvature T of $P\to X$ is obtained by lifting the action of $(\nabla_N)^2$ on the fibers N to P. Hence T lifts a 2-form $\tilde{T}$ on X with values in the infinitesimal unitary transformation of H along the fibers. Let $\tilde{T}_H$ be the horizontal part of $\tilde{T}$ with respect to ∇_H. Then $\tilde{T}_H$ is a 2-form on X with values in the skew-adjoint endomorphism of H, so that $\tilde{T} = -\nabla_T^H + \tilde{T}_H$.

The connection ∇^N induces a splitting

$$T_{\mathbf{R}}P = \phi^* T_{\mathbf{R}}Y \oplus T_{\mathbf{R}}^V P,$$

and if R is the restriction of $(\nabla^H)^2$ to vectors of $T_{\mathbf{R}}^V P$, then $(\nabla^H)^2 = R + \tilde{T}_H$. Therefore, we have the follows.

(a) On $T_{\mathbf{R}}^V P$, $(\nabla^H)^2$ coincides with R;
(b) On the horizontal, $(\nabla^H)^2$ coincides with $\tilde{T}_H$;
(c) If $U\in\phi^*T_{\mathbf{R}}Y$ and $V\in T_{\mathbf{R}}^V P$, then $(\nabla^H)^2(U,V) = 0$.

We now make the following changes: In the Chern-Weil formula for $\mathrm{td}(H, g^H)$, and in the formula for $e_{\mathrm{BC}}(H, g^H)$, we replace $(\nabla^H)^2$ by $R_{\tilde{T}_H}$. We let u_0 be a unitary frame in N, which may also be thought as a linear isometry from $\mathbf{C}^e$ into N. The above discussion implies that for $A\in\mathcal{U}(e)$:= the Lie algebra of $U(e)$, there exists a smooth form $\omega(A)$ on $\mathbf{P}(\mathbf{C}^e\oplus 1)$ with the following properties:

(a) The map $A\mapsto\int_{\mathbf{P}(\mathbf{C}^e\oplus 1)}\omega(A)$ is ad-invariant;
(b) $\varphi_*[\mathrm{td}^{-1}(H^*, g^{H^*})\,\sigma^*(e_{\mathrm{BC}}(H^*, g^{H^*}))] = \int_{\mathbf{P}(\mathbf{C}^e\oplus 1)}\omega(u_0^{-1}(\nabla^N)^2 u_0)$.

Thus by the Chern-Weil theory, we know that $\phi_*[\mathrm{td}^{-1}(H^*, g^{H^*})\,\sigma^*(e_{\mathrm{BC}}(H^*, g^{H^*}))]$ is closed. The rest is rather simple.

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