EVALUATION FIBER SEQUENCE AND HOMOTOPY COMMUTATIVITY

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Abstract. We review the relation between the evaluation fiber sequence of mapping spaces and the various higher homotopy commutativities.

1. Introduction

Suppose that A is a non-degenerately based space and X a based space. We denote the space of continuous maps between A and X by $\mathrm{Map}(A, X)$. The subspace of based maps is denoted as $\mathrm{Map}_0(A, X)$. These spaces are often called *mapping spaces*. For a based map $\phi\colon A \to X$, we denote the path-components $\mathrm{Map}_0(A, X; \phi) \subset \mathrm{Map}_0(A, X)$ and $\mathrm{Map}(A, X; \phi) \subset \mathrm{Map}(A, X)$ consisting of maps freely homotopic to ϕ which are often considered to be based at the point ϕ. In order to guarantee the exponential law for any space, we will always assume that spaces are compactly generated.

Study of homotopy theory of mapping spaces is important in various situations. For example, see the survey by S. B. Smith [Smith10]. The study of them is often difficult since the geometry of them are very complicated in general. To connect with known objects, one of the easiest way is considering the *evaluation map*

$$\mathrm{ev}\colon \mathrm{Map}(A, X) \to X$$

such that $\mathrm{ev}(f) = f(*)$ or its associated homotopy fiber sequence

$$\cdots \to \Omega X \to \mathrm{Map}_0(A, X) \to \mathrm{Map}(A, X) \to X$$

called the *evaluation fiber sequence*. They are not only easy to define but also have rich homotopy theoretic structures. As an example among them, here we consider its relations with (higher) homotopy commutativity.

The author thank Doctors Sho Hasui and Takahito Naito for organizing the workshop and giving him an opportunity to talk there. This note is written considerably from the author's interest and may contain many incomplete explanations.

2. Recognition of homotopy fiber sequence

The "delooping" of a loop space in various sense has been considered. The classifying space of a topological group is the most well-known example. J. P. May [May73] and others studied when a space is delooped n-times. Such a type of result is now called a *recognition principle*. We review recognitions of maps in a homotopy fiber sequence from the viewpoint of topological groups.

Let $f\colon X \to Y$ be a based map. Taking homotopy fibers repeatedly, the map f generates the *homotopy fiber sequence*

$$\cdots \to \Omega F \xrightarrow{\Omega i} \Omega Y \xrightarrow{\Omega f} \Omega X \xrightarrow{\partial} F \xrightarrow{i} Y \xrightarrow{f} X,$$

where $i\colon F \to Y$ and $\partial\colon \Omega X \to F$ are the inclusion of homotopy fibers of f and i, respectively. We note that the map Ωf is a loop map and that ∂ is a fiber inclusion of a principal fibration. In this section, we study how to recover whole this sequence by ∂ or Ωf with some extra data.

First we set the notation. The *i-dimensional simplex* Δ^i is given as

$$\Delta^i = \{(t_0, \dots, t_i) \in [0,1]^{\times(i+1)} \mid t_0 + \cdots + t_i = 1\}.$$

There are the *boundary maps* $\partial_k \colon \Delta^{i-1} \to \Delta^i$ and the *degeneracy maps* $\sigma_k \colon \Delta^{i+1} \to \Delta^i$ for $k = 0, 1, \dots, i$ such that

$$\partial_k(t_0, \dots, t_i) = (t_0, \dots, t_{k-1}, 0, t_k, \dots, t_i),$$
$$\sigma_k(t_0, \dots, t_{i+1}) = (t_0, \dots, t_{k-1}, t_k + t_{k+1}, t_{k+2}, \dots, t_{i+1}).$$

Unless otherwise stated, we assume that every topological monoid is based at its identity element.

Definition 2.1. For a topological monoid G, let X be a right G-space and Y a left G-space. The *bar construction* $B(X, G, Y)$ of the triple (X, G, Y) is defined by

$$B(X, G, Y) = \left(\coprod_{i=0}^{\infty} \Delta^i \times X \times G^{\times i} \times Y\right)\Big/ \sim,$$

where the identification $\sim$ is described as

$$(\partial_k \mathbf{t}; x, g_1, \dots, g_i, y) \sim \begin{cases} (\mathbf{t}; xg_1, g_2, \dots, g_i, y) & k = 0 \\ (\mathbf{t}; x, g_1, \dots, g_k g_{k+1}, \dots, g_i, y) & 0 < k < i \\ (\mathbf{t}; x, g_1, \dots, g_{i-1}, g_i y) & k = i, \end{cases}$$
$$(\mathbf{t}; x, g_1, \dots, g_{k-1}, *, g_{k+1}, \dots, g_i) \sim (\sigma_k \mathbf{t}; x, g_1, \dots, g_{k-1}, g_{k+1}, \dots, g_i).$$

The image of $\coprod_{i=0}^{n} \Delta^i \times X \times G^{\times i} \times Y$ in $B(X, G, Y)$ is denoted by $B_n(X, G, Y)$.

Lemma 2.2. *Let G be a topological monoid and X a right G-space. Then, the obvious inclusion $X \to X \times G = B_0(X, G, G) \to B(X, G, G)$ is a homotopy equivalence and its homotopy inverse $B(X, G, G) \to X$ is given by*

$$[\mathbf{t}; x, g_1, \dots, g_i, g_{i+1}] \mapsto xg_1 \cdots g_i g_{i+1}.$$

Proof. Take a deformation $H \colon [0,1] \times B(X, G, G) \to B(X, G, G)$ of $B(X, G, G)$ into X as

$$H(s, [\mathbf{t}; x, g_1, \dots, g_i, g_{i+1}]) = [(1-s)\mathbf{t}, s; x, g_1, \dots, g_i, g_{i+1}, *].$$

□

For the proof of the next theorem, see [May75, Theorem 8.2].

Theorem 2.3. *Let G be a non-degenerately based topological group. Then, for a right G-space X and a left G-space Y, the projections*

$$B(X, G, Y) \to B(*, G, Y) \quad \text{and} \quad B(X, G, Y) \to B(X, G, *)$$

are fiber bundles with fiber X and Y, respectively. In particular, $B(X, G, G) \to B(X, G, *)$ is a principal G-bundle.

Suppose that a topological group G is non-degenerately based. Let us denote $EG = B(*, G, G)$ and $BG = B(*, G, *)$. Since EG is contractible by Lemma 2.2, the principal G-bundle $EG \to BG$ is universal. Moreover, the square

$$\begin{array}{ccc} B(X, G, G) & \longrightarrow & B(*, G, G) \\ \downarrow & & \downarrow \\ B(X, G, *) & \longrightarrow & B(*, G, *) \end{array}$$

is pullback and thus the map $B(X, G, *) \to B(*, G, *) = BG$ induced by the G-equivariant map $X \to *$ is the classifying map of the principal bundle $B(X, G, G) \to B(X, G, *)$.

Corollary 2.4. *Let G be a non-degenerately based topological group and X a right G-space. Then, the following sequence is a homotopy fiber sequence:*

$$G \to X \to B(X, G, *) \to BG,$$

where the left arrow is given by $g \mapsto g*$ for the basepoint $* \in X$.

Corollary 2.5. *Let G and H be non-degenerately based topological groups. For a homomorphism $f : G \to H$, the following sequence is a homotopy fiber sequence:*

$$G \xrightarrow{f} H \to B(H, G, *) \to BG \xrightarrow{Bf} BH.$$

Proof. The left four term is a homotopy fiber sequence by Corollary 2.4. Moreover, the square

$$\begin{array}{ccc} B(H, G, *) & \xrightarrow{B(\mathrm{id}, f, *)} & B(H, H, *) \\ \downarrow & & \downarrow \\ B(*, G, *) & \xrightarrow[B(*, f, *)]{} & B(*, H, *) \end{array}$$

is a homotopy pullback. Then the right three term is also a homotopy fiber sequence. □

Corollary 2.6. *Let G and H be non-degenerately based topological groups, where H is a CW complex. Suppose that a homomorphism $G \to H$ is surjective and has the homotopy lifting property. Denote the kernel by K. Then the sequence*

$$K \to G \to H \to BK \to BG \to BH$$

is a homotopy fiber sequence, where the right two maps are induced by the homomorphisms $K \to G$ and $G \to H$, and $H \to BK$ is given as

$$H \xleftarrow{\simeq} B(G, K, *) \to BK.$$

Proof. By the commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & B(G, K, K) & \longrightarrow & B(G, K, *) \\ \| & & \downarrow \simeq & & \downarrow \\ K & \longrightarrow & G & \longrightarrow & H \end{array}$$

where each row is a homotopy fiber sequence, the map $B(G, K, *) \to H$ is a weak equivalence. Consider the commutative diagram

$$\begin{array}{ccccc} B(G, K, *) & \longrightarrow & BK & \longrightarrow & BG \\ \simeq \downarrow & & \downarrow & & \| \\ B(H, G, G) & \longrightarrow & B(H, G, *) & \longrightarrow & BG \end{array}$$

of which the rows are homotopy fiber sequences. Then the map $BK \to B(H, G, *)$ is a homotopy equivalence. Combining these equivalences, we obtain the desired homotopy fiber sequence. □

Example 2.7. The space $B_nG = B_n(*, G, *)$ is called the *n-projective space* of G.

- For a topological monoid G, the "projective line" B_1G is naturally homeomorphic to the reduced suspension ΣG.

- According as $G = \mathrm{O}(1), \mathrm{U}(1), \mathrm{Sp}(1)$, the n-th projective space B_nG is homeomorphic to $\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n$, respectively.

Lemma 2.8. *Let G be a non-degenerately based topological group. Then the composite*

$$\zeta\colon G \xrightarrow{E} \Omega\Sigma G \cong \Omega B_1 G \to \Omega BG$$

of the suspension E and the loop of the inclusion is a weak homotopy equivalence. Moreover, the map ζ is an H-map.

Proof. Consider a contracting homotopy $H : [0,1] \times EG \to BG$

$$H(s, [\mathbf{t}; *, g_1, \ldots, g_i, g_{i+1}]) = [(1-s)\mathbf{t}, s; *, g_1, \ldots, g_i, g_{i+1}, *].$$

This induces a map $EG \to PBG$ covering the identity on BG. Then the restriction $G \to \Omega BG$ on the fiber is the considered map above. Since $G \to EG \to BG$ is a fiber bundle and EG is contractible, it is a weak homotopy equivalence.

Let $g_1, g_2 \in G$. Then we can compute the composite of $\zeta(g_1)$ and $\zeta(g_2)$ as

$$\zeta(g_1)\zeta(g_2)(t) = \begin{cases} [1-2t, 2t, 0; *, g_1, g_2, *] & 0 \le t \le 1/2 \\ [0, 2-2s, 2s-1; *, g_1, g_2, *] & 1/2 \le t \le 1. \end{cases}$$

Obviously, this path is canonically homotopic to

$$\zeta(g_1 g_2)(t) = [1-t, 0, t; *, g_1, g_2, *].$$

Therefore, ζ is an H-map. □

The multiplication on a based loop space ΩX is not a group unless the path-component of the base-point is just a point. But ΩX have the good an identity element, associativity and homotopy inversion in an appropriate sense. Then, naively, we expect the existence of some "group model" for ΩX. This can be carried out with use of the *Kan's simplicial loop group*. For details, see [Neisendorfer10, Section 6.8].

Proposition 2.9. *For a path-connected based space X, there is a principal bundle $G_X \to T_X \to \bar{X}$ functorial about X such that $G_X, T_X, \bar{X}$ are CW complex, there is a natural weak equivalence $\bar{X} \to X$ and T_X is contractible.*

This means that the based loop space ΩX has a group model which is a CW complex. More precisely, there is a commutative diagram

$$\begin{array}{ccccc} G_X & \longrightarrow & EG_X & \longrightarrow & BG_X \\ \downarrow & & \downarrow & & \downarrow \\ \Omega X & \longrightarrow & PX & \longrightarrow & X \end{array}$$

such that all the vertical arrows are weak homotopy equivalences and the left vertical map is an H-map by Lemma 2.8 (in fact, is an A_∞-map).

Let $f\colon X \to Y$ be a based map. Then, by Proposition 2.9, the homotopy fiber sequence associated to f is equivalent to the sequence

$$\cdots \to G_X \xrightarrow{G_f} G_Y \to B(G_Y, G_X, *) \to BG_X \xrightarrow{BG_f} BG_Y.$$

In this sense, the maps in a homotopy fiber sequence is recognized as follows:

$$\begin{array}{l} \Omega X \\ \big\downarrow \ \text{homomorphism of topological groups} \\ \Omega Y \\ \big\downarrow \ \text{action of } \Omega Y \text{ on } F \\ F \\ \big\downarrow \\ X \\ \big\downarrow \ \text{based map} \\ Y. \end{array}$$

Conversely, each of these structures recovers the whole homotopy fiber sequence by Corollary 2.4 and 2.5.

3. Connecting map of evaluation fiber sequence

Suppose that G is a non-degenerately based topological group. Let us denote the conjugation by $\alpha(g)(x) = gxg^{-1}$ and the left translation by $\lambda(g)(x) = gx$. The conjugation induces a bundle map $E\alpha(g) = B(*, \alpha(g), \lambda(g)) \colon EG \to EG$ covering $B\alpha(g) = B(*, \alpha(g), *) \colon BG \to BG$. Then G naturally acts on BG.

For a non-degenerately based space A, take a based map $\phi \colon A \to BG$ and the induced principal bundle $P = \phi^* EG \to A$. The mapping space $\mathrm{Map}_0(A, BG; \phi)$ is equipped with the right action by G induced by the inverse conjugation on BG as well. The following theorem is essentially due to Kishimoto–Kono [KK10, Section 6]. More detailed proof can be found in [Tsutaya, Section 7].

Theorem 3.1. *Suppose that G is a non-degenerately based topological group and A is a based CW complex. Take a based map $\phi \colon A \to BG$. Then the evaluation fiber sequence*

$$\Omega BG \to \mathrm{Map}_0(A, BG; \phi) \to \mathrm{Map}(A, BG; \phi) \xrightarrow{\mathrm{ev}} BG$$

is weakly equivalent to the homotopy fiber sequence

$$G \to \mathrm{Map}_0(A, BG; \phi) \to \mathrm{Map}_0(A, BG; \phi) \times_G EG \to BG$$

induced by the conjugation action on $\mathrm{Map}_0(A, BG; \phi)$ *by G. More precisely, for some CW complex M, there is a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Map}_0(A, BG; \phi) & \longrightarrow & \mathrm{Map}(A, BG; \phi) & \xrightarrow{\mathrm{ev}} & BG \\ \big\uparrow & & \big\uparrow & & \big\Vert \\ M \times_{BG} EG & \longrightarrow & M & \longrightarrow & BG \\ \big\downarrow & & \big\downarrow & & \big\Vert \\ \mathrm{Map}_0(A, BG; \phi) & \longrightarrow & B(\mathrm{Map}_0(A, BG; \phi), G, *) & \longrightarrow & BG \end{array}$$

such that all the vertical arrows are weak equivalences.

Remark 3.2. In the above statement, the mapping spaces $\mathrm{Map}(A, BG; \phi)$ and $\mathrm{Map}_0(A, BG; \phi)$ are not necessarily homotopy equivalent to CW complexes and then one cannot expect the existence of a direct weak equivalence between $\mathrm{Map}(A, BG; \phi)$ and $B(\mathrm{Map}_0(A, BG; \phi), G, *)$. This is the reason why the weak equivalence between them is given indirectly. For the existence of the direct weak equivalence, it is sufficient to suppose that G is a CW complex and A is a finite CW complex.

Lemma 3.3. *Let G be a non-degenerately based topological group and A a CW complex. Take a based map $\phi\colon \Sigma A \to BG$. Then there is a natural weak homotopy equivalence*

$$\mathrm{Map}_0(A, G) \to \mathrm{Map}_0(\Sigma A, BG)$$

which is G-equivariant with respect to the conjugation actions on G and BG.

Proof. Note that the suspension $E\colon G \to \Omega\Sigma G$ and the inclusion $\Sigma G \to BG$ are G-equivariant. Then, by Lemma 2.8, the map $\zeta\colon G \to \Omega BG$ is a G-equivariant weak homotopy equivalence. Combining the G-equivariant homeomorphism

$$\mathrm{Map}_0(\Sigma A, BG) \cong \mathrm{Map}_0(A, \Omega BG),$$

we obtain the desired result. □

Corollary 3.4. *Let G be a non-degenerately based topological group and A a CW complex. Take a based map $\phi\colon \Sigma A \to BG$. Then for the connecting map $\partial^*\colon \Omega BG \to \mathrm{Map}_0(\Sigma A, BG; \phi)$, there is a natural commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\delta} & \mathrm{Map}_0(A, G) \\ \downarrow & & \downarrow \\ \Omega BG & \xrightarrow[\partial^*]{} & \mathrm{Map}_0(\Sigma A, BG), \end{array}$$

where the vertical arrows are weak homotopy equivalences and $\delta(g) = \alpha(g^{-1}) \circ \phi$.

Example 3.5 (free loop space)**.** The evaluation fiber sequence

$$\Omega BG \to LBG \to BG$$

on the free loop space LBG is equivalent to the fiber bundle

$$G \to \mathrm{ad}\, EG \to BG,$$

where the central term is the associated bundle $\mathrm{ad}\, EG = EG \times_G G$ with respect to the conjugation on G. The bundle $\mathrm{ad}\, EG$ is a fiberwise topological group.

This fact follows from Lemma 3.3 for $A = S^0$ and Theorem 3.1.

4. Homotopy commutativity I: mapping space as centralizer

Let Γ be a discrete group. Then, by the fiber sequence

$$\Gamma \to E\Gamma \to B\Gamma,$$

we obtain that $\pi_i(B\Gamma) = 0$ for $i \neq 1$. By using Lemma 2.8, we fix the identification

$$\pi_1(B\Gamma) \cong \pi_0(\Omega B\Gamma) \cong \Gamma.$$

We investigate the mapping spaces $\mathrm{Map}(X, B\Gamma)$ and $\mathrm{Map}_0(X, B\Gamma)$ for a connected CW complex X and a discrete group Γ. Their homotopy types had been determined by D. H. Gottlieb [Gottlieb69a]. By standard obstruction argument, we obtain that $\pi_i(\mathrm{Map}(X, B\Gamma)) = 0$ for $i \geq 2$ and $\pi_i(\mathrm{Map}_0(X, B\Gamma)) = 0$ for $i \geq 1$ with respect to an arbitrary basepoint.

Lemma 4.1. *Let X be a connected CW complex and Γ a discrete group. Then, the fundamental group functor induces a bijection*

$$\pi_1 \colon [X, B\Gamma]_0 \to \mathrm{Hom}(\pi_1(X), \pi_1(B\Gamma)) \cong \mathrm{Hom}(\pi_1(X), \Gamma).$$

Moreover, this map is Γ-equivariant with respect to the conjugation on $B\Gamma$ and Γ.

Proof. Being bijective follows from standard obstruction argument using the van Kampen's theorem. This map is Γ-equivariant since the identification $\Gamma \cong \pi_0(\Gamma) \cong \pi_0(\Omega B\Gamma) \cong \pi_1(B\Gamma)$ is Γ-equivariant. □

Corollary 4.2. *Let X be a connected CW complex and Γ a discrete group. Then there is a natural bijection*

$$[X, B\Gamma] \cong \mathrm{Hom}(\pi_1(X), \Gamma)/\Gamma,$$

where the quotient on the right hand side is taken by the conjugation.

Proof. Since X and $B\Gamma$ are connected CW complexes, the map

$$[X, B\Gamma]_0 \to [X, B\Gamma]$$

is surjective. Then, by Lemma 4.1, we have bijections

$$[X, B\Gamma] \cong [X, B\Gamma]/\pi_1(B\Gamma) \cong [X, B\Gamma]/\Gamma \cong \mathrm{Hom}(\pi_1(X), \Gamma)/\Gamma.$$

□

Now, all we have to determine is the fundamental group of each component of $\mathrm{Map}(X, B\Gamma)$.

Proposition 4.3. *Let X be a connected CW complex, Γ a discrete group and $f \colon X \to B\Gamma$ a based map. Then the fundamental group of $\mathrm{Map}(X, B\Gamma; f)$ is computed as*

$$\pi_1(\mathrm{Map}(X, B\Gamma; f)) = Z_\Gamma(f_*(\pi_1(X))),$$

where $Z_\Gamma(\Lambda)$ denotes the centralizer of a subgroup $\Lambda \subset \Gamma$.

Proof. By the evaluation fiber sequence, we have the following exact sequence of sets:

$$0 \to \pi_1(\mathrm{Map}(X, B\Gamma; f)) \to \Gamma \to \mathrm{Hom}(\pi_1(X), \Gamma),$$

where the right map is the composition $\gamma \mapsto \alpha_{\gamma^{-1}} \circ f_*$ with the conjugation $\alpha_{\gamma^{-1}}$. Note that $\alpha_{\gamma^{-1}} \circ f_* = f_*$ if and only if $\gamma \in Z_\Gamma(f_*(\pi_1(X)))$. Thus we can compute $\pi_1(\mathrm{Map}(X, B\Gamma; f))$ as above. □

Theorem 4.4 (Gottlieb)**.** *Let X be a connected CW complex, Γ a discrete group and $f \colon X \to B\Gamma$ a based map. Then the mapping space $\mathrm{Map}(X, B\Gamma; f)$ has the weak homotopy type of the classifying space $BZ_\Gamma(f_*(\pi_1(X)))$. In particular, $\mathrm{Map}(B\Gamma, B\Gamma; \mathrm{id})$ has the weak homotopy type of the classifying space $BZ(\Gamma)$ of the center $Z(\Gamma)$.*

If G is a Lie group, the mapping space $\mathrm{Map}(X, BG; \phi)$ is complicated. But a certain variant of the above theorem is proved by W. G. Dwyer and C. Wilkerson [DW95]. Note that the multiplication map $Z(G) \times G \to G$ induces the classifying map $BZ(G) \times BG \cong B(Z(G) \times G) \to BG$ and its adjoint $BZ(G) \to \mathrm{Map}(BG, BG; \mathrm{id})$.

Theorem 4.5 (Dwyer–Wilkerson)**.** *Let G be a compcat Lie group. Then the above map*

$$BZ(G) \to \mathrm{Map}(BG, BG; \mathrm{id})$$

induces an isomorphism on homology with any finite coefficient.

Remark 4.6. This map does not induce an isomorphism on rational homology in general. For example, if $G = \mathrm{SU}(2)$, there are homotopy equivalences $BZ(\mathrm{SU}(2))_{(0)} \simeq *$ and $\mathrm{Map}(B\,\mathrm{SU}(2), B\,\mathrm{SU}(2); \mathrm{id})_{(0)} \simeq B\,\mathrm{SU}(2)_{(0)}$.

5. Homotopy commutativity II: Whitehead product and Samelson product

The *Whitehead product* on homotopy groups of spheres is introduced by J. H. C. Whitehead [Whitehead41]. M. Arkowitz [Arkowitz62] generalized it for maps from a general suspension to a base space.

Let $\alpha\colon \Sigma A \to X$ and $\beta\colon \Sigma B \to X$ be maps from the reduced suspensions ΣA and ΣB to a based space X. The map $w : \Sigma(A \times B) \to X$ defined by

$$w = (\alpha \circ \Sigma p_1) + (\beta \circ \Sigma p_2) - (\alpha \circ \Sigma p_1) - (\beta \circ \Sigma p_2)$$

determines the map $[\alpha, \beta] : \Sigma(A \wedge B) \to X$ uniquely up to homotopy such that, for the quotient map $q : A \times B \to A \wedge B$, $[\alpha, \beta] \circ \Sigma q = w$ in $[\Sigma(A \times B), X]_0$. The uniqueness is verified by the splitting Puppe sequence

$$0 \to [\Sigma(A \wedge B), X]_0 \xrightarrow{(\Sigma q)^*} [\Sigma(A \times B), X]_0 \to [\Sigma A \vee \Sigma B, X]_0 \to 0.$$

The map $[\alpha, \beta]$ or its homotopy class is called the *Whitehead product* of α and β.

The next theorem is due to M. Arkowitz [Arkowitz62, Section 4].

Theorem 5.1. *Let A and B be based spaces. Denote the inclusions $\iota_A\colon \Sigma A \to \Sigma A \vee \Sigma B$ and $\iota_B\colon \Sigma B \to \Sigma A \vee \Sigma B$. Then the mapping cone inclusion of $[\iota_A, \iota_B]\colon \Sigma(A \wedge B) \to \Sigma A \vee \Sigma B$ is equivalent to the obvious inclusion $\Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B$.*

Let $\alpha\colon A \to \Omega X$ and $\beta\colon B \to \Omega X$ be maps from the based spaces A and B to the based loop space ΩX. Like the Whitehead product, the map $s\colon A \times B \to \Omega X$

$$w = (\alpha \circ p_1) + (\beta \circ p_2) - (\alpha \circ p_1) - (\beta \circ p_2)$$

determines the map $\langle \alpha, \beta \rangle\colon A \wedge B \to \Omega X$ uniquely up to homotopy such that $\langle \alpha, \beta \rangle \circ q = s$ in $[A \times B, \Omega X]_0$. The map $\langle \alpha, \beta \rangle$ or its homotopy class is called the *Samelson product* of α and β.

Example 5.2. Note that SU(2) is homeomorphic to the 3-dimensional sphere S^3. The Samelson product $\langle \mathrm{id}_{S^3}, \mathrm{id}_{S^3} \rangle$ of two copies of the identity $\mathrm{id}_{S^3}\colon S^3 \to S^3$ with respect to the loop structure $S^3 \simeq \Omega B\,\mathrm{SU}(2)$ is known to have order 12. This generates the homotopy group $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$.

Two maps corresponding to each other by the adjunction

$$[\Sigma A, X]_0 \cong [A, \Omega X]_0$$

are said to be mutually *adjoint*. The next proposition says that the Samelson product is adjoint to the Whitehead product in some sense.

Proposition 5.3. *Let $\alpha\colon \Sigma A \to X$ and $\beta\colon \Sigma B \to X$ be maps from the reduced suspensions ΣA and ΣB to a based space X. Denote the adjoint maps of α and β by $\alpha' : A \to \Omega X$ and $\beta' : B \to \Omega X$. Then, the Samelson product $\langle \alpha', \beta' \rangle$ is adjoint to the Whitehead product $[\alpha, \beta]$.*

Proof. The adjoint α' is defined by $\alpha'(a)(t) = \alpha(t, a)$. Then the desired adjointness immediately follows from definitions. □

Suppose that X is a non-degenerately based space and A and B are CW complexes. Then, taking the group model $f\colon G_X \xrightarrow{\simeq} \Omega X$, note that the Samelson product $[\alpha, \beta]$ of $\alpha \in [A, \Omega X]$ and $\beta \in [B, \Omega X]$ is equal to the image of the commutator $g_*(\tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1}\tilde{\beta}^{-1})$, where $g_*(\tilde{\alpha}) = \alpha$ and $g_*(\tilde{\beta}) = \beta$.

Let A and X be based CW complexes and $\phi\colon \Sigma A \to X$ a based map. Let us consider the following evaluation fiber sequence:

$$\cdots \to \Omega X \xrightarrow{\partial^*} \mathrm{Map}_0(\Sigma A, X; \phi) \to \mathrm{Map}(\Sigma A, X; \phi) \xrightarrow{\mathrm{ev}} X.$$

The next theorem is due to G. E. Lang [Lang73].

Theorem 5.4. *Under the above setting, the adjoint $\Sigma\Omega X \wedge A \to X$ of the composite*

$$\Omega X \xrightarrow{\partial} \mathrm{Map}_0(\Sigma A, X; \phi) \xrightarrow{-\phi} \mathrm{Map}_0(\Sigma A, X; *)$$

is homotopic to the Whitehead product $[-e, \phi]$ of the evaluation $e \colon \Sigma\Omega X \to X$ and ϕ.

Proof. Taking the group model $G_X \simeq \Omega X$, the corresponding map $G_X \to \mathrm{Map}_0(A, G_X; *)$ is the commutator map $g \mapsto (a \mapsto g^{-1}\phi g \phi^{-1})$ by Corollary 3.4. Then the adjoint $\Omega X \wedge A \to \Omega X$ is the Samelson product $[-\mathrm{id}_{G_X}, \phi']$, where ϕ' is the adjoint of ϕ. By Proposition 5.3, the adjoint is the Whitehead product $[-e, \phi]$. □

An H-space X having the homotopy inversion is said to be *homotopy commutative* if the commutator map

$$X \times X \to X, \qquad (x, y) \to ((xy)x^{-1})y^{-1}$$

is null-homotopic.

Since the Whitehead product is the adjoint of the Samelson product, if ΩX is homotopy commutative, then all the generalized Whitehead products in X vanish. This implies the following corollary.

Corollary 5.5. *Under the setting of Theorem 5.4, if ΩX is homotopy commutative, then the connecting map $\Omega X \to \mathrm{Map}_0(A, X; \phi)$ is null-homotopic. In particular, when X is a CW complex, ΩX is homotopy commutative if and only if the connecting map $\Omega X \to \mathrm{Map}_0(\Sigma\Omega X, X; e)$ for $A = \Sigma\Omega X$ is null-homotopic.*

Remark 5.6. If X is a CW complex, then $\Sigma\Omega X$ has the homotopy type of a CW complex.

6. Homotopy commutativity III: Higher homotopy commutativity

In this section, we outline the results on higher homotopy commutativity by the author [Tsutaya]. To state it, it is convenient to consider grouplike topological monoids while we have considered only topological groups so far. A topological monoid G is said to be *grouplike* if the induced monoid $\pi_0(G)$ is a group.

In the previous section, we saw that the Samelson product defined by commutator map is the adjoint of the Whitehead product. There are many notion of higher homotopy commutativity. We collect various higher homotopy commutativities and describe them as a generalization of the Whitehead product. Though each of them have the description in terms of higher homotopy, we do not explain them in such a way.

The simplest one is defined by F. D. Williams [Williams69], where the following description is due to Saümell [Saümell95].

Definition 6.1. Let G be a non-degenerately based grouplike topological monoid. Then G is said to be a *Williams C_n-space* if the n-fold wedge sum

$$\iota_1^{\vee n} \colon (\Sigma G)^{\vee n} \to BG$$

of the inclusion $\iota_1 \colon \Sigma G \cong B_1 G \to BG$ extends to the map $(\Sigma G)^{\times n} \to BG$ from the product $(\Sigma G)^{\times n}$.

The obstructions to extending the map $\iota_1^{\vee n}$ is called the *higher Whitehead products*. This definition requires that all the i-th Whitehead products for $i \leq n$ on BG vanish.

A stronger homotopy commutativity was defined by M. Sugawara [Sugawara60] before Williams. His definition is generalized by C. A. McGibbon [McGibbon89].

Definition 6.2. Let G be a non-degenerately based grouplike topological monoid. Then G is said to be a *Sugawara C^n-space* if the wedge sum

$$\iota_n \vee \iota_n \colon B_nG \vee B_nG \to BG$$

of the inclusion $\iota_n \colon B_nG \to BG$ extends over the union of products

$$\bigcup_{i+j=n} B_iG \times B_jG.$$

Equivalently, G is a Sugawara C^n-space if the multiplication map $G \times G \to G$ is an A_n-map in the sense of Stasheff [Stasheff63].

Further higher homotopy commutativities are considered by Hemmi–Kawamoto [HK11] and by Kishimoto–Kono [KK10].

Definition 6.3. Let G be a non-degenerately based grouplike topological monoid.

(i) G is said to be a *$C_k(n)$-space* if the wedge sum $\iota_k \vee \iota_n$ extends over the union of products $B_iG \times B_jG$ for $i + j = n$ and $i \leq k$.
(ii) G is said to be a *$C(k, \ell)$-space* if the wedge sum $\iota_k \vee \iota_\ell$ extends over the product $B_kG \times B_\ell G$.

For $1 \leq n \leq \infty$ and $1 \leq k \leq n$, it is not difficult to see the implications

$$\text{Sugawara } C^n\text{-space} \Rightarrow C_k(n)\text{-space} \Rightarrow C(k, n-k)\text{-space} \Rightarrow \text{Williams } C_n\text{-space}.$$

In particular, we have the equivalences

$$\text{Sugawara } C^2\text{-space} \Leftrightarrow \text{Williams } C_2\text{-space} \Leftrightarrow \text{homotopy commutative}$$

and

$$\text{Sugawara } C^\infty\text{-space} \Leftrightarrow \text{the classifying space is an } H\text{-space}.$$

Remark 6.4. A topological group G is said to be a *double loop space* if the classifying space BG is homotopy equivalent to a based loop space ΩX for some space X. A double loop space has much better homotopy commutativity than Sugawara C^∞-spaces since the classifying space BG of a Sugawara C^∞-space G is just an H-space.

In relation with free loop spaces, J. Aguadé [Aguadé87] proved the next proposition.

Proposition 6.5 (Aguadé)**.** *Let X be a connected CW complex. Then the free loop fibration*

$$\Omega X \to LX \to X$$

is trivial if and only if the based Moore loop space $\Omega^{\mathrm{M}}X$ is a $C_1(\infty)$-space.

Jim Stasheff considered the intermediate conditions between a loop map and a based map with no extra structure.

Definition 6.6 (Stasheff)**.** A based map $f \colon G \to G'$ between non-degenerately based grouplike topological monoids is said to be an *A_n-map* if the adjoint

$$f' \colon B_1G \cong \Sigma G \to BG'$$

of f extends over B_nG.

The author [Tsutaya, Lemma 7.3] proved that there is a G-equivariant weak equivalence

$$\mathcal{A}_n(G, G; \mathrm{id}) \to \mathrm{Map}_0(B_nG, BG; \iota_n)$$

from a grouplike topological monoid equipped with G-action induced from a homomorphism $G \to \mathcal{A}_n(G, G; \mathrm{id})$. This implies that the evaluation fiber sequence $\mathrm{Map}_0(B_nG, BG; \iota_n) \to BG$ extends to the right by Corollary 2.5 since the map $G \to \mathrm{Map}_0(B_nG, BG; \iota_n)$ is the connecting map. For higher homotopy commutativity, he proved that the following.

Theorem 6.7 (T). *Let G be a non-degenerately based grouplike topological monoid which is a CW complex. Then G is a $C(k,\ell)$-space if and only if the homomorphism $G \to \mathcal{A}_\ell(G, G; \mathrm{id})$ is null-homotopic as an A_k-map.*

Remark 6.8. The author does not know $\mathcal{A}_n(G, G; \mathrm{id})$ is non-degenerately based or not. Then we should replace it by an appropriate non-degenerately based approximation.

7. Another application: Gottlieb groups

The 1st Gottlieb group was introduced by Jiang [Jiang64] and applied to the Nielsen fixed point theory. Later, Gottlieb defined the higher Gottlieb groups in [Gottlieb69b].

Definition 7.1. Let X be a based space. The n-th *Gottlieb group* $G_n(X) \subset \pi_n(X)$ is the image under the homomorphism

$$\mathrm{ev}_*\colon \pi_n(\mathrm{Map}(X, X; \mathrm{id}_X)) \to \pi_n(X).$$

Proposition 7.2. *Let X be a non-degenerately based space and $\alpha \in \pi_n(X)$. Then the following three conditions are equivalent:*

(i) $\alpha \in G_n(X)$,

(ii) the map $(\alpha, \mathrm{id})\colon S^n \vee X \to X$ extends to a map $S^n \times X \to X$,

(iii) in the homotopy exact sequence of the evaluation fiber sequence $\mathrm{Map}_0(X, X; \mathrm{id}) \to \mathrm{Map}(X, X; \mathrm{id}) \to X$, $\partial\alpha = 0$ in $\pi_{n-1}(\mathrm{Map}_0(X, X; \mathrm{id}))$.

Proposition 7.3. *Let X be a based CW complex. Then the following properties hold.*

(i) The group $G_1(X)$ is contained in the center of $\pi_1(X)$.

(ii) For the universal covering $p\colon \tilde{X} \to X$ and $\alpha \in \pi_1(X)$, $\alpha \in G_1(X)$ if and only if the covering transformation $\tilde{X} \to \tilde{X}$ corresponding to the loop α is (basepoint-freely) homotopic to the identity $\mathrm{id}\colon \tilde{X} \to \tilde{X}$.

Proof. (i) Consider a commutative diagram

$$\begin{array}{ccc}
\Omega X & \xrightarrow{\partial} & \mathrm{Map}_0(X, X; \mathrm{id}) \\
\| & & \downarrow{\scriptstyle e^*} \\
\Omega X & \xrightarrow{\partial} & \mathrm{Map}_0(\Sigma\Omega X, X; \mathrm{ev}) \\
\| & & \downarrow{\scriptstyle \cong} \\
\Omega X & \xrightarrow{\partial} & \mathrm{Map}_0(\Omega X, \Omega X; \mathrm{id}).
\end{array}$$

Then, the bottom horizontal arrow is the adjoint of the conjugation $\Omega X \times \Omega X \to \Omega X$. This implies that the kernel of $\partial\colon \pi_1(X) \to \pi_0(\mathrm{Map}_0(\Sigma\Omega X, X; \mathrm{ev}))$ is just the center of $\pi_1(X)$. Thus $G_1(X)$ is contained in the center of $\pi_1(X)$.

(ii) Every extension $S^1 \times X \to X$ of (α, id) lifts to the desired homotopy of maps $\tilde{X} \to \tilde{X}$. □

Example 7.4. It is obvious that if X is an H-space, then $G_n(X) = \pi_n(X)$ for any n.

Gottlieb [Gottlieb65] applied $G_1(X)$ as follows.

Theorem 7.5. *Let X be a compact connected polyhedron, to which the Lefschetz fixed point theorem can be applied. Then, if the Euler characteristic of X is not zero, we have $G_1(X) = 0$.*

Outline of the proof. We use the Nielsen theory. All of the fixed points of $\mathrm{id}\colon X \to X$ is contained in the class of the identity $\mathrm{id}\colon \tilde{X} \to \tilde{X}$. Then the Lefschetz number is equal to the fixed point index of this class. By the Lefschetz fixed point theorem, it is equal to the Euler characteristic of X. For any $\alpha \in G_1(X)$, the class of the corresponding covering map is freely homotopic to the identity $\mathrm{id}\colon \tilde{X} \to \tilde{X}$. As is well-known, this implies that their fixed point indices are the same. Thus we conclude that $\alpha = 0$ or the Euler characteristic is zero. □

Jiang applied $G_1(X)$ to the Nielsen fixed point theory. For details, see [Jiang83].

8. Another application: Universal fibration

Let X be a based CW complex. Let us denote the subspaces of homotopy equivalences by $\mathrm{Eq}_0(X) \subset \mathrm{Map}_0(X, X)$ and $\mathrm{Eq}(X) \subset \mathrm{Map}(X, X)$, which are considered to be based at the identity map. Since the map $\mathrm{Eq}(X) \to X$ comes from the action of $\mathrm{Eq}(X)$ on X, the evaluation fiber sequence extends to the right with appropriate replacement of spaces by Corollary 2.4:

$$\Omega X \to \mathrm{Eq}_0(X) \to \mathrm{Eq}(X) \xrightarrow{\mathrm{ev}} X \to B\,\mathrm{Eq}_0(X) \to B\,\mathrm{Eq}(X).$$

Gottlieb [Gottlieb73] proved that the last three terms classify the fibration with homotopy fiber X.

Theorem 8.1. *Suppose that X is a based CW complex. Then, for any map $E \to B$ between based CW complexes with homotopy fiber X, there exists a homotopy pullback square*

$$\begin{array}{ccc} E & \longrightarrow & B\,\mathrm{Eq}_0(X) \\ \downarrow & & \downarrow \\ B & \longrightarrow & B\,\mathrm{Eq}(X), \end{array}$$

where a map $B \to B\,\mathrm{Eq}(X)$ admitting such a homotopy pullback is unique up to (basepoint-free) homotopy.

By this theorem, we have a commutative diagram

$$\begin{array}{ccc} \pi_n(X) & \longrightarrow & \pi_n(E) \\ \| & & \downarrow \\ \pi_n(X) & \longrightarrow & \pi_{n-1}(\mathrm{Map}_0(X, X; \mathrm{id})). \end{array}$$

This means that the connecting map $\pi_n(X) \to \pi_{n-1}(\mathrm{Map}_0(X, X; \mathrm{id}))$ of the evaluation fiber sequence is universal among the inclusions of homotopy fibers in some sense.

Corollary 8.2. *Suppose that X is a based CW complex. Then, the n-th Gottlieb group $G_n(X)$ is characterized as the union of the kernel of the induced map $\pi_n(X) \to \pi_n(E)$ with respect to any fibration $X \to E \to B$.*

As a consequence of this corollary, if $\alpha \in \pi_n(X)$ is not contained in the Gottlieb group $G_n(X)$, then the image of α under the map $\pi_n(X) \to \pi_n(E)$ never vanishes for any fibration $X \to E \to B$.

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