Vortices and Polynomials

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This lecture is based on a paper with the same title written for the memorial issue of *Fluid Dynamics Research* (vol. **39**, p.5, 2007) dedicated to Professor Isao Imai. From the standpoint of fluid mechanics we are interested in studying the relative equilibria of interacting point vortices. Some motivational remarks on why this problem is of interest will first be given (see also the figure below). It leads rather quickly to the consideration of a system of algebraic equations in N complex variables, $z_1, \ldots, z_N$, which are the positions of the N vortices. In most situations the strengths of the vortices are given from other physical considerations. At the present stage of development of the subject the best results concern the case when the vortex strengths are ± 1 in appropriate units. We shall restrict attention to this case. One is then led to consider a *generating polynomial* of the form $P(z) = (z - z_1) \ldots (z - z_N)$ of degree N with roots at the vortex positions. When there are two species of vortices of both positive and negative circulation, one considers a generating polynomial $P(z)$ for the positive vortices and another polynomial $Q(z)$ for the negative vortices. The problem of interest is the interplay between these polynomials and vortex configuration.

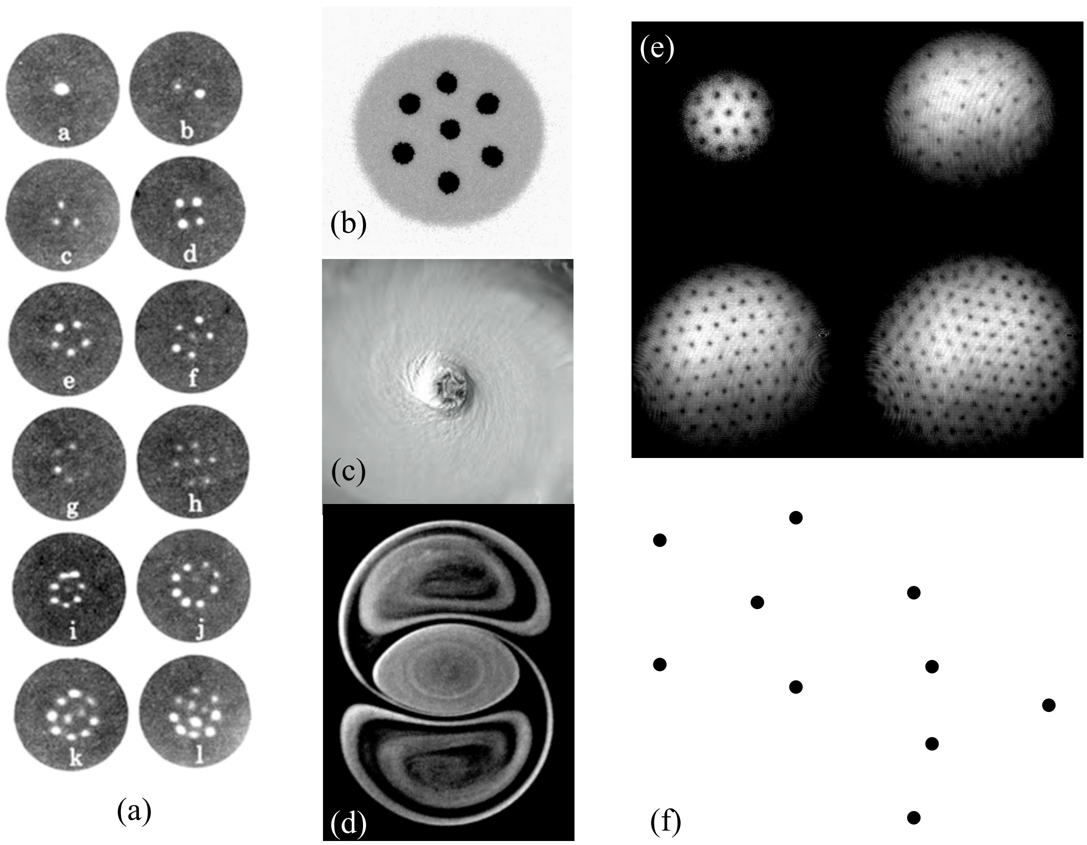
In the case of vortices of mixed signs the connection between polynomials and vortex patterns solves the problem completely. This was shown by Bartmann in 1983 a few years before his untimely death. He showed that stationary equilibria of n vortices of strength $+1$ and m vortices of strength -1 are only possible if n and m are consecutive triangular numbers, $m = p(p-1)/2$, $n = p(p+1)/2$, $p = 2, 3, \ldots$. Formally, the case of a single vortex, $p = 1$, comes under this heading as well. The case $p = 2$ of a single vortex at the center of an equilateral triangle of opposite vortices is well known. Bartmann showed that if a polynomial $P(z)$ is associated with the positive vortices, and a polynomial $Q(z)$ with the negative vortices, then P and Q would be successive so-called *Adler-Moser polynomials*. Since these polynomials may be computed recursively, the problem of stationary configurations of a system of point vortices of strengths ± 1 is solved.

The symmetrical case of identical point vortices would seem to be the simplest to solve, but has withstood several attempts at solution. In a short paper from 1885 Stieltjes showed that if N vortices were all arranged on a line, the positions necessary for a rotating relative equilibrium of identical vortices are the zeros of the N'th Hermite polynomial. This solution has been rediscovered several times. It highlights the unexpected connection between point patterns corresponding to a relative equilibrium and functions from classical mathematical physics.

Placing vortices at the corners of a regular polygon, with or without one at the center, leads to relative equilibria, as has been understood for many years. J. J. Thomson, best known for this discovery of the electron, wrote a treatise on vortex motion in 1882 in which a famous stability result was enunciated and proved (with some corrections to come later): The regular N-gon is stable to infinitesimal perturbations for $N \leq 6$ and unstable for $N \geq 8$. It is marginally stable for $N = 7$. Returning to the issue of existence of relative equilibria, we may show that if N identical vortices are on a circle, and in a relative equilibrium, then they must form a regular N-gon. This seems easiest to do using the generating polynomial.

The regular polygon solutions can be generalized to nested polygons with or without a vortex at the center. This was done by Havelock in 1931 for two rings, and by van Buren and the speaker in 2005 for three rings. A rather rich family of solutions emerges. Numerical exploration shows, however, that there are many additional solutions with lower symmetry. In fact, in 1998 Vainchtein and the speaker found the first examples of entirely asymmetric patterns for eight or more vortices. We have no analytical understanding of these at present.

A numerical exploration of the space of relative equilibria of N identical vortices was initiated by Campbell and Ziff in a Los Alamos report from 1979. They focused on patterns that were not linearly unstable. For eight vortices, for example, they cite just one pattern, the centered, regular heptagon. We have found at least 19 relative equilibria for 8 vortices. These are all unstable except for the centered, regular heptagon. Nevertheless, the structure of these patterns sheds light on the phase space of the 8-vortex problem.



Examples of the patterns of interacting points considered in this lecture: (a) Vortices in rotating superfluid He-II; (b) electron density pattern in a plasma; (c) multiple vortices in the eye of a hurricane (typhoon); (d) the vortex tripole; (e) vortex patterns in a Bose-Einstein condensate; (f) recently found relative equilibria of 5 identical point vortices.

Recently, it has become clear that the generating polynomial can be used to calculate the energy of a vortex pattern in a much more efficient way than by calculating all the distances of the vortices pairwise. The connection between energy and the characteristic polynomial arises through the discriminant. This will also be explained.

In summary, patterns of points corresponding to relative equilibria of point vortices reveal interesting connections to the classical theory of polynomials and to some of the special polynomials encountered in mathematical physics. While some of these connections have been revealed, we believe there is much more that awaits elucidation.

Marvelous Mathematical Models in Mechanics

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According to wikipedia a mathematical model is “the use of mathematical language to describe the behavior of a system”. In the physical sciences most of our theories – even the most basic – may be viewed as mathematical models of some set of natural phenomena. Mathematical models come in many forms, from the crude to the refined. At the end of the day, the ones that survive are the simple ones that capture the essence of a phenomenon in a non-trivial way. There is a quote attributed to Einstein that “A scientific theory should be as simple as possible, but no simpler.” This is seen as a modern version of Occam’s razor. The best mathematical models – the ones I am referring to as “marvelous” – have this quality. They are simple, usually elegant in their statement. They capture the essence of the phenomenon being modeled. And, ideally, the model will give rise to interesting mathematical results and will raise fundamental mathematical issues.

Mathematical models come in many shapes and sizes. Some are deterministic, some stochastic. The variables employed in the model may be continuous or discrete or a mix of continuous and discrete. Models in which everything is explicit are called “white box”. Models, such as neural nets that essentially construct themselves first, and only later make predictions, are called “black box”. In fluid mechanics, the field from which I will predominantly select my models, most are deterministic, continuous and “white-box”.

The examples selected are (i) point vortices, (ii) chaotic advection, (iii) bubbles and foams, (iv) coagulation, and (v) L-systems. Obviously, I want to convince you that they are, indeed, marvelous! *Point vortices*, which are really a set of ODEs that capture essential aspects of the two-dimensional Euler equation, a PDE, have been a great love of my scientific modeling life, and I hope I can make them part of yours as well. Modeling flows in terms of point vortices has been an active area of research since Helmholtz introduced the idea in this seminal paper of 1858. Today point vortex models are closely realized in the patterns seen in superfluids such as Bose-Einstein condensates.

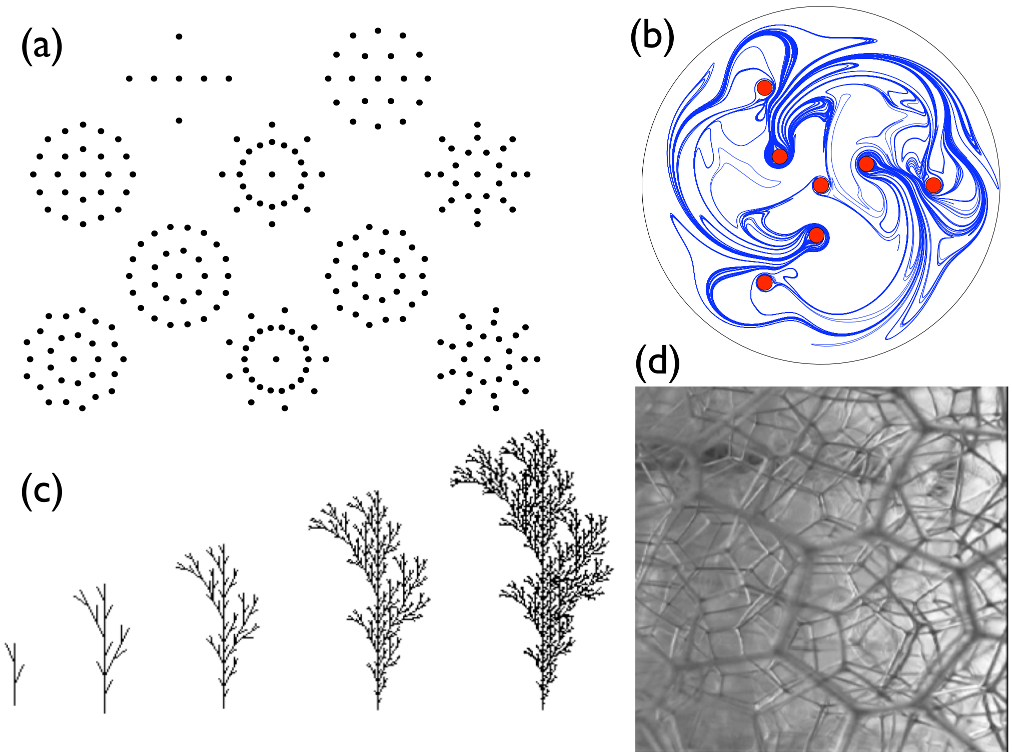
The Lagrangian focus of point vortices, and of vortex dynamics in general, leads naturally to exploring fluid mechanics from a particle point of view. The equations of motion of a passive particle in a flow are a set of three ODEs. Even for flows that the fluid dynamicist considers “simple” these equations may be non-integrable. This leads to the phenomenon of *chaotic advection*, introduced 25 years ago, which has spawned a large and growing literature. The idea of chaotic advection is today applied to flows on all length scales from geophysical fluid dynamics to microfluidics.

Next, I consider the mathematical modeling of *soap bubbles and foams* and, particularly, the modeling of a dry foam in the limit where there is little liquid trapped in the bubble surfaces of the foam. We shall explore some simple geometrical results for small clusters of bubbles and also consider the evolution of multi-bubble systems, what we call foams or froths.

I shall switch gears again and try to interest you in a class of models of coagulation phenomena of which the *Smoluchowski equation* is the best known. This integro-differential equations which has appeared in a multitude of fields exhibits scaling solutions, similar to those seen in turbulence theory. In some cases a rigorous derivation of the scaling exponent may be given.

Finally, I shall describe a set of geometrical models of complex structures that I believe hold great promise for the future. These are the so-called *L-systems*, originated by the biologist Lindenmeyer, which allow fractal objects to be constructed in a very intuitive way.

As I describe these models I will also try to indicate opportunities for research and how you might get started on these problems should you have an interest in doing so.



Examples of the models considered in this lecture: (a) Relative equilibria of point vortices; (b) chaotic advection via topological chaos (courtesy of Finn & Thiffeault); (c) L-systems; (d) three-dimensional foam.

There are many, many other models that are used and studied extensively and I cannot do a comprehensive survey. Furthermore, I am not familiar with the details of many of these models, so I have confined myself to some of the mathematical models on which I have worked.

Some of the most famous models have names. The model that is supposed to explain how the universe works and how all the elementary particles fit together is known as the "Standard Model" to high energy physicists. In phase transformation theory we encounter the "Ising model" and its many variations, including the "Heisenberg model", the "Baxter model", the "Spherical model", and on and on. "Diffusion limited aggregation" or DLA has become a famous model to explain a variety of pattern formation processes. And I am sure you have heard about "self-organized criticality", which is supposed to arise in a large variety of models that "live" on the edge of incipient instability. The "Bohr model of the atom" was a forerunner to quantum mechanics. In turbulence what is known as the "closure problem" for the set of equations describing auto-correlations of the velocity field has led to a large number of models, one of the earliest of which is known as "the $k - \epsilon$ model". The "Big Bang model" is supposed to explain how it all began. And so on.

In summary, we may say that scientists have been creating mathematical models of physical phenomena for a long time. Such models are useful for gaining basic scientific understanding and for putting the physical world to work in invention and design, which is what we usually call engineering. Some models have taken on a life of their own. We remark that creating models for the biological world is an ongoing challenge.