Symmetry Characterizations

for Homogeneous Siegel Domains

Related to Laplace–Beltrami Operators

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Siegel Domains — Introduction —

Introduced by Pjatetskii-Shapiro (1959),
holomorphically equivalent to bounded domains

⋆ Study of homogeneous bounded domains (HBD) by É. Cartan
[Abh. Math. Sem. Univ. Hamburg, **11** (1935)]

- HBD in $\mathbb{C}^2$ and $\mathbb{C}^3$ are all symmetric.

Assumptions on the automorphism group: ($\Longrightarrow$ symmetry).

A. Borel, Koszul	semisimple	(1954, 1955)
Hano	unimodular	(1957)

Note: Cartan did *not* make the conjecture that all HBDs are symmetric. What Cartan actually wrote is:
"..., il semble que là, comme dans beaucoups d'autres problèmes, il faille s'appuyer sur une idée nouvelle."

Example of non-symmetric (type II) Siegel domain in $\mathbb{C}^4$ (1959)
Gindikin wrote: [Israel Math. Conf. Proc.]
"It is funny to remember now, how suspiciously we listened for the first time to the proof that this domain is nonsymmetric."

Example of non-selfdual homogeneous convex cone in $\mathbb{R}^5$
by Vinberg (1960)
$\rightsquigarrow$ $\mathbb{C}^5$ contains a non-symmetric type I Siegel domain
(= tube domain)

Natural Question. How do we characterize symmetric Siegel domains (among homogeneous Siegel domains)?

Siegel Domains — Definition —

V : a real vector space

$\cup$

Ω : a *regular* open convex cone

($\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ contains *no* entire line)

$W := V_{\mathbb{C}}$ $(w \mapsto w^*$: conjugation w.r.t. $V)$

U : another *complex* vector space

$Q : U \times U \to W$, Hermitian sesquilinear Ω-positive

i.e., $\begin{cases} Q(u', u) = Q(u, u')^* \\ Q(u, u) \in \overline{\Omega} \setminus \{0\} \ (0 \neq \forall u \in U) \end{cases}$

$$D := \{(u, w) \in U \times W \, ; \, w + w^* - Q(u, u) \in \Omega\}$$

Siegel domain (of type II)

- If $U = \{0\}$, then $D = \Omega + iV$.

(tube domain or type I domain)

Assume that D is homogeneous

i.e., $\mathrm{Hol}(D) \curvearrowright D$ transitively

Symmetry Characterizations

D: a homogeneous Siegel domain

Recall that

D is *symmetric*

$\underset{\text{def}}{\Longleftrightarrow}$ $\forall z \in D,\ \exists \sigma_z \in \mathrm{Hol}(D)$ s.t.

$$\begin{cases} \sigma_z^2 = \text{identity}, \\ z \text{ is an isolated fixed point of } \sigma_z. \end{cases}$$

Satake, Dorfmeister (both late '70s)

$\cdots$ In terms of defining data

(I will touch on this later in this talk.)

D'Atri (1979) $\cdots$ Diff. Geometric (curvature cond.)

[DDZ] D'Atri, Dorfmeister and Y. Zhao (1985)

$\cdots$ Study of isotropy representation

One of DDZ's results is:

$\mathbf{D}(D)^{\mathrm{Hol}(D)^\circ}$ is commutative $\iff$ D is symm.

Today's talk

$\mathscr{L}$: Laplace–Beltrami operator
(w.r.t. a standard metric of D)

Theorem A. [N, 2001]
$\mathscr{L}$ commutes with the Berezin transforms
$\Longleftrightarrow$ D is symmetric and
the metric considered is Bergman
(up to const. multiple > 0).

Theorem B. [N, preprint]
The Poisson–Hua kernel is annihilated by $\mathscr{L}$
$\Longleftrightarrow$ D is symmetric and
the metric considered is Bergman
(up to const. multiple > 0).

Remark. If one takes the Bergman metric from the beginning in Theorem B, then the theorem is due to
Hua–Look ('59), Korányi ('65) for $\Leftarrow$
Xu ('79) for $\Rightarrow$
However, I think very few people traced Xu's proof (required to understand his own theory of N-Siegel domains, and to read some of his papers written in Chinese without English translation).

Pjatetskii-Shapiro algebras – normal j-algebras –

$\exists G$: split solvable $\curvearrowright D$ simply transitively

$\mathfrak{g} := \mathrm{Lie}(G)$ has a structure of normal j-algebra.
(Pjatetskii-Shapiro algebra)

$\exists J$: integrable almost complex structure on $\mathfrak{g}$,
$\exists \omega$: admissible linear form on $\mathfrak{g}$, *i.e.*,
$\langle x | y \rangle_\omega := \langle [Jx, y], \omega \rangle$ defines a J-invariant inner product on $\mathfrak{g}$.

Example (Koszul '55). Koszul form.

$$\langle x, \beta \rangle := \mathrm{tr}\big(\mathrm{ad}(Jx) - J\,\mathrm{ad}(x)\big) \quad (x \in \mathfrak{g}).$$

β is admissible

- In fact, $\langle x | y \rangle_\beta$ is the real part of the Hermitian inner product on $\mathfrak{g} \equiv T_e(D)$ defined by the Bergman metric on $D \approx G$ (up to a positive scalar multiple).

Structure of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n} \qquad \begin{cases} \mathfrak{a} : \text{ abelian,} \\ \mathfrak{n} : \text{ sum of } \mathfrak{a}\text{-root spaces (positive roots only)} \end{cases}$$

Always contains a product of $ax+b$ algebra:

$\exists H_1, \ldots, H_r$: a basis of $\mathfrak{a}$ $(r := \operatorname{rank} \mathfrak{g})$ s.t.

if $E_j := -JH_j \in \mathfrak{n}$, then $[H_j, E_k] = \delta_{jk} E_k$.

Possible forms of roots:

$$\tfrac{1}{2}(\alpha_k \pm \alpha_j)\ (j<k), \quad \alpha_k, \quad \tfrac{1}{2}\alpha_k$$

$\alpha_1, \ldots, \alpha_r$: basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$.

- $\mathfrak{g}_{\alpha_k} = \mathbb{R} E_k$ $(\forall k)$.
- $\mathfrak{g}_\alpha$ are mutually orthogonal w.r.t. $\langle \cdot | \cdot \rangle_\omega$ ($\forall \omega$: adm.)

$E_k^* \in \mathfrak{g}^*$: $\langle E_k, E_k^* \rangle = 1$ and $=0$ on $\mathfrak{a}$ and $\mathfrak{g}_\alpha$ $(\alpha \neq \alpha_k)$.

- Admissible linear forms are $\mathfrak{a}^* \oplus \{0\} \oplus \sum_{k=1}^{r} \mathbb{R}_{>0} E_k^*$.

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we put $E_\mathbf{s}^* := \sum_{k=1}^{r} s_k E_k^* \in \mathfrak{g}^*$.

If $s_1 > 0, \ldots, s_r > 0$ (we'll write $\mathbf{s} > 0$), then

$\langle x | y \rangle_\mathbf{s} := \langle [Jx, y], E_\mathbf{s}^* \rangle$ is a J-inv. inner product on $\mathfrak{g}$

$\rightsquigarrow$ left invariant Riemannian metric on G

$\rightsquigarrow$ $\mathscr{L}_\mathbf{s}$: the corresponding L-B operator on G.

Berezin transforms

κ : the Bergman kernel of D

the Berezin kernel

$$A_\lambda(z_1,z_2) := \left(\frac{|\kappa(z_1,z_2)|^2}{\kappa(z_1,z_1)\kappa(z_2,z_2)}\right)^\lambda \quad (z_j \in D;\ \lambda \in \mathbb{R})$$

- A_λ is G-invariant: $\quad A_\lambda(g\cdot z_1,\ g\cdot z_2) = A_\lambda(z_1,z_2)$.

Since $D \approx G$, we work on G:

$a_\lambda(g) := A_\lambda(g\cdot \mathrm{e}, \mathrm{e}) \quad (g \in G,\ \mathrm{e} \in D$: fixed ref. pt.)

- $a_\lambda \in L^1(G)$ if $\lambda > \lambda_0$ ($0 < \lambda_0 < 1$: explicitly calculated).

(non-vanishing condition for Hilbert spaces of holomorphic functions on D, in which κ^λ is the reproducing kernel.)

Berezin transform

$$B_\lambda f(x) := \int_G f(y) a_\lambda(y^{-1}x)\, dy = f * a_\lambda(x)$$

$B_\lambda \in \mathbf{B}(L^2(G))$: selfadjoint, positive.

Recall $\beta \in \mathfrak{g}$: Koszul form. $\boxed{\beta|_{\mathfrak{n}} = E^*_{\mathbf{c}}|_{\mathfrak{n}}}$ with $\mathbf{c} > 0$.

Theorem A. $\lambda > \lambda_0$: fixed.
B_λ commutes with $\mathscr{L}_{\mathbf{s}}$
$\iff$ D is symmetric and $\mathbf{s} = \gamma\mathbf{c}$ with $\gamma > 0$.

Poisson–Hua kernel

$S(z_1, z_2)$: the Szegö kernel of D

(= reprod. kernel of the Hardy space)

Hardy space

Hilbert space of holomorphic functions F on D s.t.

$$\sup_{t\in\Omega} \int_U dm(u) \int_V \left|F\left(u, t + \tfrac{1}{2}Q(u,u) + ix\right)\right|^2 dx < \infty$$

Σ : the Shilov boundary of D

$\Sigma = \left\{(u,w) \in U \times W \,;\, 2\mathrm{Re}\, w = Q(u,u)\right\}$

$S(z,\zeta)$ for $z \in D$ and $\zeta \in \Sigma$ still has a meaning.

$$P(z,\zeta) := \frac{|S(z,\zeta)|^2}{S(z,z)} \quad (z \in D,\ \zeta \in \Sigma) :$$

the Poisson kernel of D

$$P^G_\zeta(g) := P(g \cdot \mathrm{e},\ \zeta) \quad (g \in G).$$

Theorem B. $\mathscr{L}_{\mathbf{s}} P^G_\zeta = 0$ for $\forall \zeta \in \Sigma$
$\iff$ D is symmetric and $\mathbf{s} = \gamma \mathbf{c}$ with $\gamma > 0$.

Geometric backgrounds

Geometric reason that Theorems A and B are true ?

Connection with a geometry of bounded models of homogeneous Siegel domains

geometry $\leftrightsquigarrow$ geometric norm equality

- Validity of norm equality
  $\Longleftrightarrow$ Symmetry of the domain

Specialists' folklore

There is *no* (most) canonical bounded model for non-(quasi)symmetric Siegel domains.

My standpoint

Appropriate bounded model varies with problems one treats.

- Canonical bounded model for symmetric Siegel domains

  ······ Harish-Chandra model

  of a non-cpt Hermitian symmetric space

  (Open unit ball of a positive Hermitian JTS w.r.t the spectral norm)

- Canonical bounded model for quasisymmetric Siegel domains

  ······ by Dorfmeister (1980)

  Image of a Siegel domain under the Cayley transform naturally defined in terms of Jordan algebra structure

- For general homogeneous Siegel domains

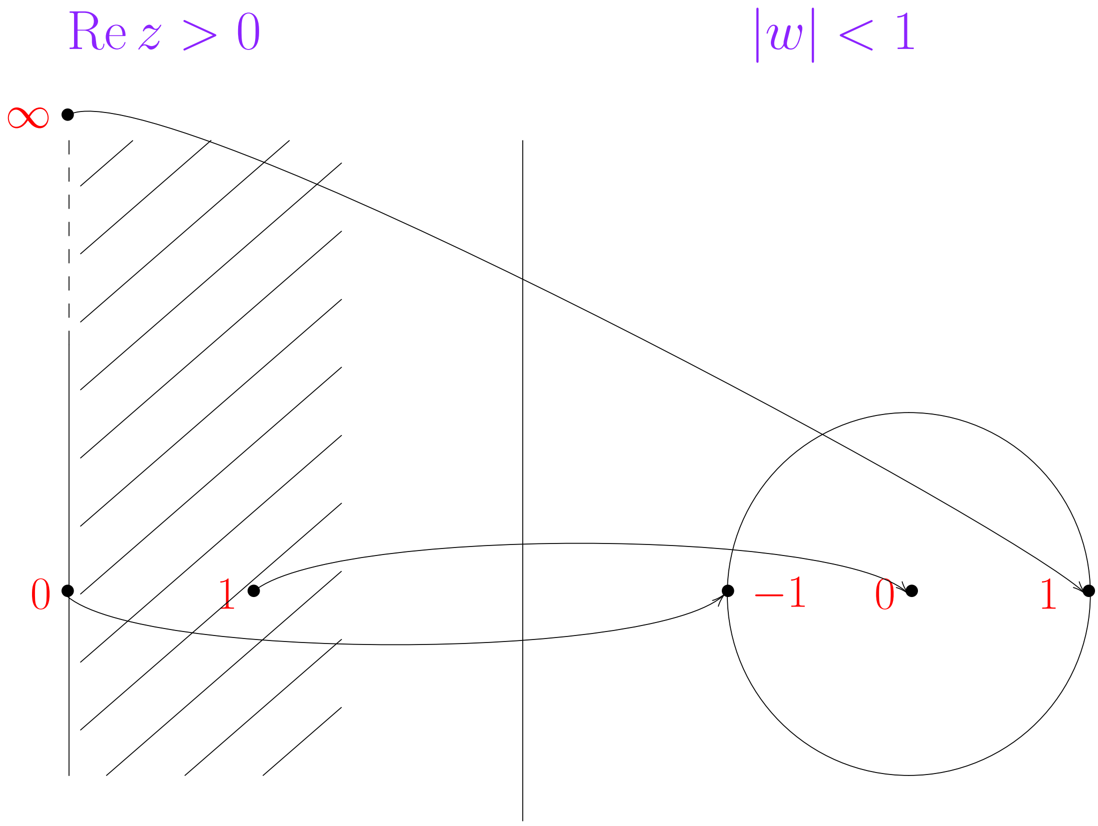
We can consider

- Cayley transf. assoc. to the Szegö kernel
- Cayley transf. assoc. to the Bergman kernel
- Cayley transf. assoc. to the char. ftn of the cone

etc. . .

More generally, we can define Cayley transforms associated to the admissible linear forms $E^*_{\mathbf{s}}$ $(\mathbf{s} > 0)$.

Cayley transform



$$z = \frac{1+w}{1-w} = -1 + \frac{2}{1-w} \qquad w = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$$

If one puts in a complex semisimple Jordan algebra

$$z = \frac{e+w}{e-w}, \qquad w = \frac{z-e}{z+e},$$

then the above figure is the case for symmetric tube domains.

- In general, if one can define something like $(z+1)^{-1}$ (denominator), one has a Cayley transform by $1-2(z+1)^{-1}$ for tube domains.

Pseudoinverse assoc. to $E^*_{\mathbf{s}}$

$\exists H \subset G$: s.t. $H \curvearrowright \Omega$ simply transitively

$E \in \Omega$ (canonically fixed base point)

Then $H \approx \Omega$ (diffeo) by $h \mapsto hE$.

- Note $G = N \rtimes A$, $H = N_0 \rtimes A$ with $A := \exp \mathfrak{a}$

For $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{R}^r$, put $\alpha_{\mathbf{s}} := \sum_{j=1}^{r} s_j \alpha_j \in \mathfrak{a}^*$

($\alpha_1, \dots, \alpha_r$: basis of $\mathfrak{a}^*$ dual to $H_1, \dots, H_r$).

Then, $\langle x, \alpha_{-\mathbf{s}} \rangle = \langle Jx, E^*_{\mathbf{s}} \rangle$ $(\forall x \in \mathfrak{a})$.

$\chi_{\mathbf{s}}(\exp x) := \exp\langle x, \alpha_{\mathbf{s}} \rangle$ $(x \in \mathfrak{a})$:

character of A, hence of H.

$\rightsquigarrow$ function on Ω by $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ $(h \in H)$

- $\Delta_{\mathbf{s}}$ extends to a holomorphic function on $\Omega + iV$ as the Laplace transform of the Riesz distribution on the dual cone Ω^* (Gindikin, Ishi (2000)), where

$$\Omega^* := \{\xi \in V^* ; \langle x, \xi \rangle > 0 \ \forall x \in \overline{\Omega} \setminus \{0\}\}.$$

For each $x \in \Omega$, define $\mathscr{I}_{\mathbf{s}}(x) \in V^*$ by

$$\langle v, \mathscr{I}_{\mathbf{s}}(x)\rangle := -D_v \log \Delta_{-\mathbf{s}}(x) \quad (v \in V).$$

$$\left(D_v f(x) := \tfrac{d}{dt} f(x+tv)\big|_{t=0}\right)$$

- $\mathscr{I}_{\mathbf{s}}(\lambda x) = \lambda^{-1}\mathscr{I}_{\mathbf{s}}(x) \quad (\lambda > 0)$

Proposition. Suppose $E_{\mathbf{s}}^*$ is admissible.

(1) $\mathscr{I}_{\mathbf{s}}(x) \in \Omega^*$ and $\mathscr{I}_{\mathbf{s}} : \Omega \twoheadrightarrow \Omega^*$ is bijective.

(2) $\mathscr{I}_{\mathbf{s}}$ extends analytically to a rational map $W \to W^*$.

(3) One also has an explicit formula for $\mathscr{I}_{\mathbf{s}}^{-1} : \Omega^* \twoheadrightarrow \Omega$, which continues analytically to a rational map $W^* \to W$.
Thus $\mathscr{I}_{\mathbf{s}}$ is birational.

(4) $\mathscr{I}_{\mathbf{s}} : \Omega + iV \twoheadrightarrow \mathscr{I}_{\mathbf{s}}(\Omega + iV)$ is biholo.

Remark. Bergman kernel and Szegö kernel are of the form (up to positive const.)
$\eta(z_1, z_2) = \Delta_{-\mathbf{s}}(w_1 + w_2^* - Q(u_1, u_2))$ $(z_j = (u_j, w_j))$, and the char. ftn of Ω is $\Delta_{-\mathbf{s}}$ for some $\mathbf{s} > 0$ (up to positive const.).

- In general $\mathscr{I}_{\mathbf{s}}(\Omega + iV) \not\subset \Omega^* + iV^*$.

Cayley transform

One has $E_{\mathbf{s}}^* = \mathscr{I}_{\mathbf{s}}(E) \in \Omega^*$.

$C_{\mathbf{s}}(w) := E_{\mathbf{s}}^* - 2\,\mathscr{I}_{\mathbf{s}}(w+E)$ for tube domains

$$\mathscr{C}_{\mathbf{s}}(u,w) := \underbrace{2\,\langle Q(u,\cdot), \mathscr{I}_{\mathbf{s}}(w+E)\rangle}_{\in U^\dagger} \oplus \underbrace{C_{\mathbf{s}}(w)}_{\in W^*}$$

$U^\dagger$: the space of antilinear forms on U

Proposition.
(1) $\mathscr{C}_{\mathbf{s}} : D \twoheadrightarrow \mathscr{C}_{\mathbf{s}}(D)$ is birat. and biholomorphic.
(2) $\mathscr{C}_{\mathbf{s}}^{-1}$ can be written explicitly.

Theorem [N]. $\mathscr{C}_{\mathbf{s}}(D)$ is bounded (in $U^\dagger \oplus W^*$).

Remark. For general $\mathbf{s} > 0$, $\mathscr{C}_{\mathbf{s}}(D)$ for symmetric D is *not* the standard Harish-Chandra model of a non-compact Hermitian symmetric space
(can be even non-convex, for example).

Norm equality I

$\langle x|y\rangle_{\mathbf{s}}$: J-inv. inner prod. on $\mathfrak{g}$

$\rightsquigarrow$ Upon $G \equiv D$ by $g \mapsto g \cdot \mathrm{e}$, we have Hermitian inner prod. on $T_{\mathrm{e}}(D) \equiv U \oplus W$

$\rightsquigarrow$ Hermitian inner product $(\cdot|\cdot)_{\mathbf{s}}$ and norm $\|\cdot\|_{\mathbf{s}}$ on the "dual' vector space $U^{\dagger} \oplus W^{*}$.

Take $\Psi_{\mathbf{s}} \in \mathfrak{g}$ so that $\mathrm{tr\,ad}\,(x) = \langle x|\Psi_{\mathbf{s}}\rangle_{\mathbf{s}}$ $(\forall x \in \mathfrak{g})$.
Then $\Psi_{\mathbf{s}} \in \mathfrak{a}$.

Recall that $\beta|_{\mathfrak{n}} = E_{\mathbf{c}}^{*}|_{\mathfrak{n}}$ for some $\mathbf{c} > 0$, so $\Delta_{-\mathbf{c}}(w_1 + w_2^{*} - Q(u_1, u_2))$ is the Bergman kernel of D (up to pos. const.).

Proposition. For any $g \in G$

$$\mathscr{L}_{\mathbf{s}} a_{\lambda}(g) = \lambda a_{\lambda}(g)\big(-\lambda\|\mathscr{C}_{\mathbf{c}}(g\cdot\mathrm{e})\|_{\mathbf{s}}^{2} + \langle\Psi_{\mathbf{s}}, \alpha_{\mathbf{c}}\rangle\big).$$

Observations. (1) $a_{\lambda}(g) = a_{\lambda}(g^{-1})$ for $\forall g \in G$.
(2) B_{λ} commutes with $\mathscr{L}_{\mathbf{s}}$
$\iff \mathscr{L}_{\mathbf{s}} a_{\lambda}(g) = \mathscr{L}_{\mathbf{s}} a_{\lambda}(g^{-1})$ for $\forall g \in G$.

Theorem. [N, 2001]
$\|\mathscr{C}_{\mathbf{c}}(g\cdot \mathrm{e})\|_{\mathbf{s}} = \|\mathscr{C}_{\mathbf{c}}(g^{-1}\cdot \mathrm{e})\|_{\mathbf{s}}$ for $\forall g \in G$
$\iff$ D is symmetric and $\mathbf{s} = \gamma\mathbf{c}$ with $\gamma > 0$.

Since $\mathscr{C}_{\mathbf{c}}(\mathrm{e}) = 0$, the Theorem can be rephrased as:

Theorem.
$\|h\cdot 0\|_{\mathbf{s}} = \|h^{-1}\cdot 0\|_{\mathbf{s}}$ for $\forall h \in \mathscr{C}_{\mathbf{c}}\circ G\circ \mathscr{C}_{\mathbf{c}}^{-1} \iff$
$\mathscr{D} := \mathscr{C}_{\mathbf{c}}(D)$ is symmetric and $\mathbf{s} = \gamma\mathbf{c}$ with $\gamma > 0$.

If D is symmetric, $\mathscr{D}$ is essentially the Harish-Chandra model of a non-cpt Hermitian symmetric space.

$\mathsf{G} := \mathrm{Hol}(\mathscr{D})^{\circ}$: semisimple Lie group
$\mathsf{K} := \mathrm{Stab}_{\mathsf{G}}(0)$: maximal cpt subgroup of G.

Using $\mathsf{G} = \mathsf{KAK}$ with $\mathsf{A} := \mathscr{C}_{\mathbf{c}}\circ A\circ \mathscr{C}_{\mathbf{c}}^{-1}$, one can prove easily that $\|h\cdot 0\|_{\mathbf{c}} = \|h^{-1}\cdot 0\|_{\mathbf{c}}$ for any $h \in \mathsf{G}$.

The case of unit disk $\mathbb{D} \subset \mathbb{C}$

$$\mathsf{G} = SU(1,1) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \; ; \; |\alpha|^2 - |\beta|^2 = 1 \right\}$$

with $\quad g \cdot z = \dfrac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \qquad (z \in \mathbb{D}).$

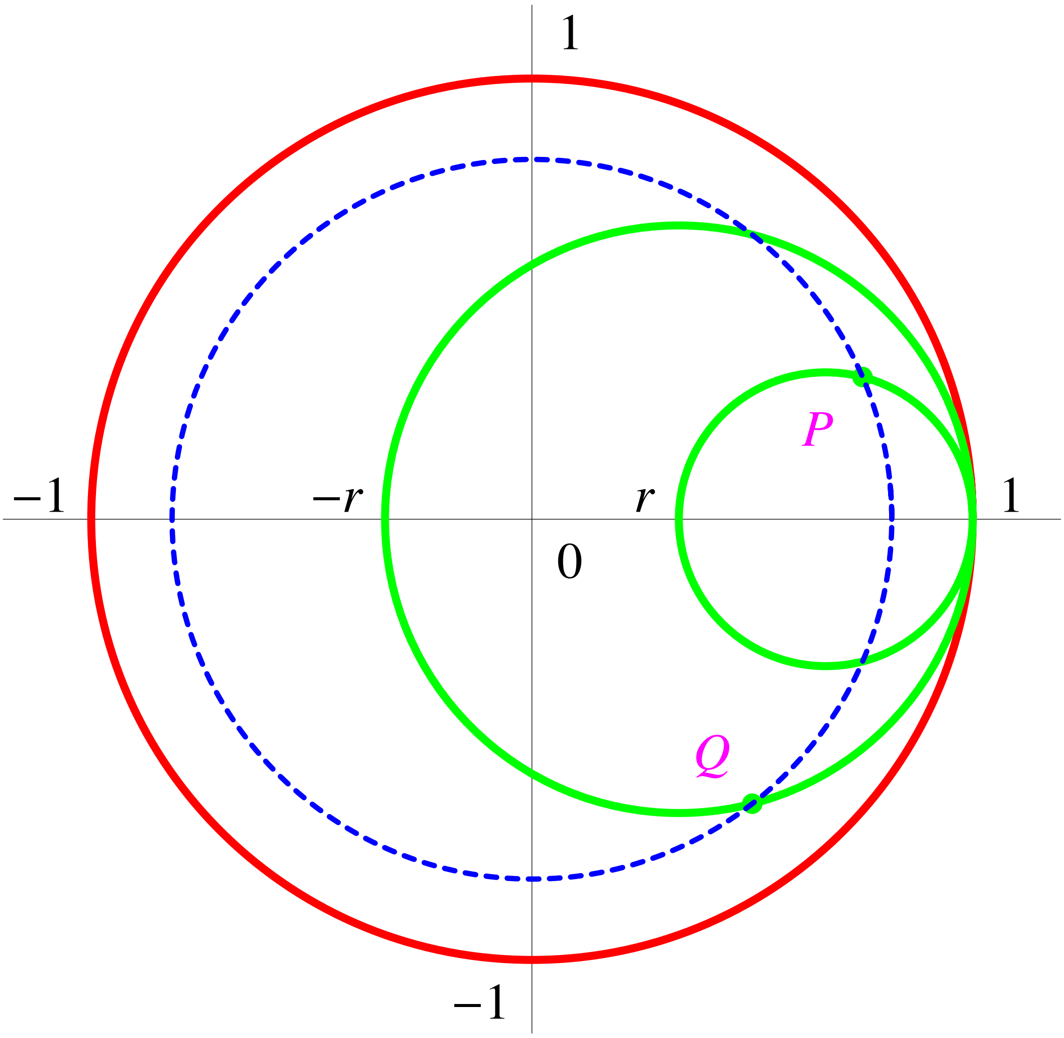
$$\begin{cases} g \cdot 0 = \dfrac{\beta}{\overline{\alpha}} \\ g^{-1} \cdot 0 = -\dfrac{\beta}{\alpha} \end{cases} \Longrightarrow |g \cdot 0| = |g^{-1} \cdot 0|.$$

However, if one stays within the Iwasawa solvable subgroup, we have an interesting picture.

$$\mathsf{A} := \left\{ a_t := \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \; ; \; t \in \mathbb{R} \right\},$$

$$\mathsf{N} := \left\{ n_\xi := \begin{pmatrix} 1 - \frac{i}{2}\xi & \frac{i}{2}\xi \\ -\frac{i}{2}\xi & 1 + \frac{i}{2}\xi \end{pmatrix} \; ; \; \xi \in \mathbb{R} \right\}.$$

Then $\mathscr{C}_{\mathbf{c}} \circ G \circ \mathscr{C}_{\mathbf{c}}^{-1} = \mathsf{NA}$.



$r := a_t \cdot 0 = \tanh(t/2)$

P : $n_\xi a_t \cdot 0 = n_\xi \cdot r \in \mathsf{N} \cdot r$:
horocycle emanating from $1 \in \partial\mathbb{D}$ cutting $\mathbb{R}$ at r.

Q : $(n_\xi a_t)^{-1} \cdot 0 = n_{-e^{-t}\xi} a_{-t} \cdot 0 = n_{-e^{-t}\xi} \cdot (-r) \in \mathsf{N} \cdot (-r)$:
horocycle emanating from $1 \in \partial\mathbb{D}$ cutting $\mathbb{R}$ at $-r$.

Norm equality II

Take $\mathbf{b} > 0$ so that $\Delta_{-\mathbf{b}}(w_1 + w_2^* - Q(u_1, u_2))$ is the Szegö kernel of D (up to positive const.).

Proposition.

$$\mathscr{L}_{\mathbf{s}} P_\zeta^G(e) = (-\|\mathscr{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 + \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}} \rangle) P_\zeta^G(e).$$

Remark. By $P(g \cdot z,\ \zeta) = \chi_{-\mathbf{b}}(g) P(z,\ g^{-1} \cdot \zeta)$ $(g \in G)$,

$$\mathscr{L}_{\mathbf{s}} P_\zeta^G = 0\ \forall \zeta \in \Sigma \iff \mathscr{L}_{\mathbf{s}} P_\zeta^G(e) = 0\ \forall \zeta \in \Sigma.$$

Theorem [N].

$\|\mathscr{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 = \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}} \rangle$ for $\forall \zeta \in \Sigma$

$\iff$ D is symmetric and $\mathbf{s} = \gamma \mathbf{b}$ with $\gamma > 0$.

In this case we also have $\mathbf{s} = \gamma' \mathbf{c}$ with $\gamma' > 0$.

Recall $\mathbf{c} > 0$ is taken so that $\beta|_{\mathfrak{n}} = E_{\mathbf{c}}^*|_{\mathfrak{n}}$, where β is the Koszul form.

Validity of NE for symmetric D ($\mathbf{s} = \mathbf{c}$)

D : symmetric $\Longrightarrow$ $\mathscr{D} := \mathscr{C}_{\mathbf{c}}(D)$ is the Harish-Chandra model of a Hermitian symmetric space

In particular, $\mathscr{D}$ is circular (Note $\mathscr{C}_{\mathbf{c}}(\mathsf{e}) = 0$).

$\mathsf{G} := \mathrm{Hol}(\mathscr{D})^{\circ}$: semisimple Lie gr. (with trivial center)

$\mathsf{K} := \mathrm{Stab}_{\mathsf{G}}(0)$: maximal cpt subgr. of G

Circularity of $\mathscr{D}$ ($\Longrightarrow$ K is linear)
+ K-invariance of the Bergman metric
$\Longrightarrow$ $\mathsf{K} \subset$ Unitary group

$$\begin{cases} \mathscr{C}_{\mathbf{c}} : \Sigma \ni 0 \mapsto -E_{\mathbf{c}}^{*}, \\ \text{Shilov boundary } \Sigma_{\mathscr{D}} \text{ of } \mathscr{D} = \mathsf{K} \cdot (-E_{\mathbf{c}}^{*}). \end{cases}$$

Since $\Sigma_{\mathscr{D}}$ is also a G-orbit $\Sigma_{\mathscr{D}} = \mathsf{G} \cdot (-E_{\mathbf{c}}^{*})$ and since Σ is an orbit of a nilpotent subgroup of $G \subset \mathrm{Hol}(D)^{\circ}$, we get

$$\begin{aligned} \mathscr{C}_{\mathbf{c}}(\Sigma) &\subset \mathsf{G} \cdot (-E_{\mathbf{c}}^{*}) = \Sigma_{\mathscr{D}} \\ &= \mathsf{K} \cdot (-E_{\mathbf{c}}^{*}) \\ &\subset \{z \,;\, \|z\|_{\mathbf{c}} = \|E_{\mathbf{c}}^{*}\|_{\mathbf{c}}\}. \end{aligned}$$

We see easily that $\|E_{\mathbf{c}}^{*}\|_{\mathbf{c}}^{2} = \langle \Psi_{\mathbf{c}}, \alpha_{\mathbf{b}} \rangle$ in this case (because $\mathbf{b}$ is a multiple of $\mathbf{c}$).

Norm equality $\implies$ symmetry of D

Assumption :

(i) $\|\mathscr{C}_{\mathbf{c}}(g\cdot \mathrm{e})\|_{\mathbf{s}} = \|\mathscr{C}_{\mathbf{c}}(g^{-1}\cdot \mathrm{e})\|_{\mathbf{s}}$ for $\forall g \in G$.
or
(ii) $\|\mathscr{C}_{\mathbf{b}}(\zeta)\|_{\mathbf{s}}^2 = \langle \Psi_{\mathbf{s}}, \alpha_{\mathbf{b}}\rangle$ for $\forall \zeta \in \Sigma$.

What we do is substitute specific $g \in G$ in (i) (resp. $\zeta \in \Sigma$ in (ii)) and extract informations.

(1) Reduction to a quasisymmetric domain

κ : the Bergman kernel of D
Recall that $\kappa(z_1,z_2) = \Delta_{-\mathbf{c}}(w_1 + w_2^* - Q(u_1,u_2))$
(up to positive const.).

If $x, y \in V$, define $\langle x\,|\,y\rangle_\kappa := D_x D_y \log \Delta_{-\mathbf{c}}(E)$.

Definition. $D = D(\Omega, Q)$ is *quasisymmetric*

$\underset{\text{def}}{\iff}$ Ω is selfdual w.r.t. $\langle\cdot\,|\,\cdot\rangle_\kappa$.

Define a non-associative product xy in V by

$$\langle xy\,|\,z\rangle_\kappa = -\tfrac{1}{2} D_x D_y D_z \log \Delta_{-\mathbf{c}}(E).$$

Prop. (Dorfmeister-D'Atri-Dotti-Vinberg)

D is quasisymmetric $\iff$ product xy is Jordan.

In this case, V is a Euclidean Jordan algebra.

My tool is the following

Proposition. (D'Atri-Dotti) D : irreducible.

D is quasisymmetric

$$\iff \begin{cases} (1) \ \dim \mathfrak{g}_{(\alpha_k+\alpha_j)/2} \text{ is indep. of } j, k, \\ (2) \ \dim \mathfrak{g}_{\alpha_k/2} \text{ is indep. of } k. \end{cases}$$

Extend $\langle \cdot | \cdot \rangle_\kappa$ to a $\mathbb{C}$-bilinear form on $W \times W$.

$(u_1 | u_2)_\kappa := \langle Q(u_1, u_2) | E \rangle_\kappa$
defines a Hermitian inner product on U.

For each $w \in W$, define $\varphi(w) \in \mathrm{End}_{\mathbb{C}}(U)$ by

$$(\varphi(w)u_1 | u_2)_\kappa = \langle Q(u_1, u_2) | w \rangle_\kappa.$$

Clearly $\varphi(E) =$ identity operator on U.

Proposition. (Dorfmeister). D is quasisymm.

$\Longrightarrow$ $w \mapsto \varphi(w)$ is a Jordan $*$-repre. of $W = V_{\mathbb{C}}$

$$\begin{cases} \varphi(w^*) = \varphi(w)^*, \\ \varphi(w_1 w_2) = \frac{1}{2}\big(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1)\big). \end{cases}$$

(2) Reduction : quasisymm $\Longrightarrow$ symm

Quasisymmetric Siegel domain

$\leftrightarrow$ $\begin{cases} \text{Euclidean Jordan algebra } V \text{ and} \\ \text{Jordan } *\text{-representation } \varphi \text{ of } W = V_{\mathbb{C}}. \end{cases}$

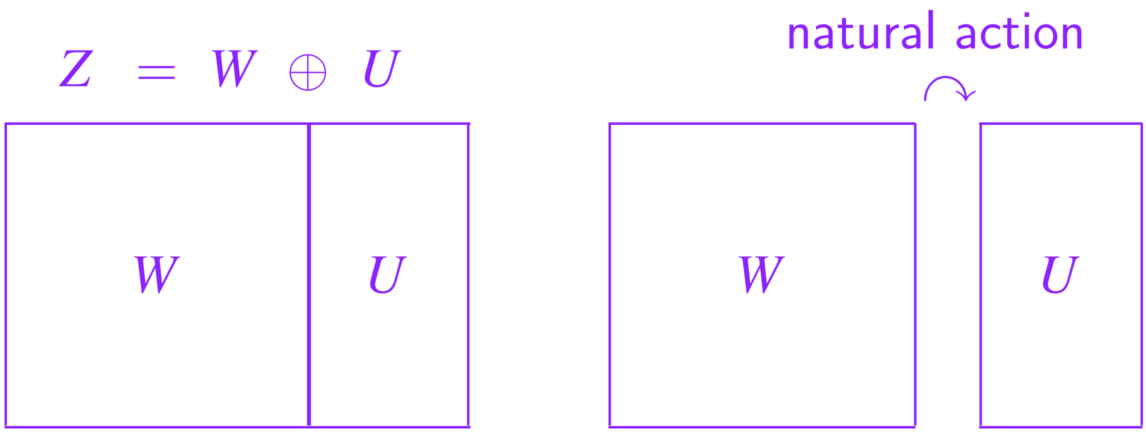
Symmetric Siegel domain

$\leftrightarrow$ Positive Hermitian JTS

The following strange formula fills the gap:

$$\varphi(w)\varphi(Q(u,u'))u = \varphi(Q(\varphi(w)u,u'))u,$$

where $u, u' \in U$ and $w \in W$.



complex semisimple Jordan algebra $W = V_{\mathbb{C}}$ with V Euclidean JA

Jordan algebra $*$-repre. of W

> **Proposition.** (Satake) Quasisymm. D is symm. $\iff$ V and φ come from a positive Hermitian JTS this way.

Definition of triple product: $z_j = (u_j, w_j)$ $(j = 1, 2, 3)$, $\{z_1, z_2, z_3\} := (u, w)$, where

$$u := \tfrac{1}{2}\varphi(w_3)\varphi(w_2^*)u_1 + \tfrac{1}{2}\varphi(w_1)\varphi(w_2^*)u_3 + \tfrac{1}{2}\varphi(Q(u_1, u_2))u_3 + \tfrac{1}{2}\varphi(Q(u_3, u_2))u_1,$$

$$w := (w_1 w_2^*)w_3 + w_1(w_2^* w_3) - w_2^*(w_1 w_3) + \tfrac{1}{2}Q(u_1, \varphi(w_3^*)u_2) + \tfrac{1}{2}Q(u_3, \varphi(w_1^*)u_2).$$

Proposition. (Dorfmeister)

Irreducible quasisymmetric D is symmetric

$\iff$ $\exists f_1, \ldots, f_r$: Jordan frame of V s.t.
with $U_k := \varphi(f_k)U$ we have

$$\varphi(Q(u_1, u_2))u_1 = 0$$

for $\forall u_1 \in U_1$ and $\forall u_2 \in U_2$.

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