Geometric Connection of the Poisson Kernel with a Cayley Transform for Homogeneous Siegel Domains

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Motivation of this work

D : a Homogeneous Siegel domain

Σ : the Shilov boundary of D

$P(z,\zeta)$ $(z \in D,\ \zeta \in \Sigma)$:

the Poisson kernel of D defined à la Hua

$\mathcal{L}$: the Laplace–Beltrami operator of D

(with respect to the Bergman kernel)

Theorem (Hua-Look ('59), Korányi ('65), Xu ('79))
$\mathcal{L}P(\cdot,\zeta) = 0\ \forall\zeta \in \Sigma \iff D$: symm.

D : symmetric

$$\underset{\text{def}}{\iff} \forall z \in D,\ \exists \sigma_z \in \mathrm{Hol}(D) \text{ s.t.}$$

$$\begin{cases} \sigma_z^2 = \text{identity}, \\ z \text{ is an isolated fixed point of } \sigma_z. \end{cases}$$

[$\Leftarrow$] well known

- Hua-Look : direct and case-by-case computation for 4 classical domains

- Korányi : stronger result for general *symmetric* domains

  $P(\cdot, \zeta)$ is annihilated by *any* $\mathrm{Hol}(D)^\circ$-invariant differential operator without const. term

  ($\mathrm{Hol}(D)^\circ$ is semisimple for symmetric D)

[$\Rightarrow$] less known

- Lu Ru-Qian : An example of non-symmetric Siegel domain for which $P(\cdot, \zeta)$ is *not* killed by $\mathcal{L}$
  (Chinese Math. Acta, **7** (1965))

- Xu Yichao : though the proof is hardly traceable at least for me

(1) Needs to understand his own theory of "N-Siegel domains",

(2) Some of cited papers of his are written in Chinese

not available in English.

The purpose of this talk (my contribution)

Wants to know a geometric reason that the theorem is true

($\longrightarrow$ geometric relationship with a Cayley transform)

- Connection with a geometric property of a bounded model of homogeneous Siegel domains

Validity of some norm equality $\iff$ Symmetry of the domain

Specialists' folklore

There is *no* canonical bounded model for non-(quasi)symmetric Siegel domains.

- Canonical bounded model for symmetric Siegel domains

  $\cdots\cdots$ Harish-Chandra model

  of a Hermitian symmetric space

  $\left(\begin{array}{l}\text{Open unit ball of a positive Hermitian JTS} \\ \text{w.r.t the spectral norm}\end{array}\right)$

- Canonical bounded model for quasisymmetric Siegel domains

  $\cdots\cdots$ by Dorfmeister (1980)

  Image of a Siegel domain under the Cayley transform

  naturally defined in terms of Jordan algebra structure

  (requires a proof for the bddness of the image, of course)

• For general homogeneous Siegel domains

We can consider

- Cayley transf. assoc. with the Szegö kernel

  (N, today's talk)
- Cayley transf. assoc. with the Bergman kernel

  (N, JLT, 2001)
- Cayley transf. assoc. with the char. ftn of the cone

  (R. Penney, 1996)

etc

♣ More generally, one can define a family of Cayley transform parametrized by admissible linear forms (N, preprint, 2001).

Siegel Domains

V : a real vector space

$\cup$

Ω : a regular open convex cone

($\underset{\text{def}}{\Longleftrightarrow}$ contains *no* entire line)

$W := V_{\mathbb{C}}$ ($w \mapsto w^*$: conjugation w.r.t. V)

U : another *complex* vector space

$Q : U \times U \to W$, Hermitian sesquilinear Ω-positive

i.e., $\begin{cases} Q(u', u) = Q(u, u')^* \\ Q(u, u) \in \overline{\Omega} \setminus \{0\} \ (0 \neq \forall u \in U) \end{cases}$

$$\boxed{D := \{(u, w) \in U \times W \ ; \ w + w^* - Q(u, u) \in \Omega\}}$$

Siegel domain (of type II)

Assume that D is homogeneous

i.e., $\mathrm{Hol}(D) \curvearrowright D$ transitively

- If $U = \{0\}$, then $D = \Omega + iV$.
  (tube domain or type I domain)

$\exists G$: split solvable $\curvearrowright D$ simply transitively

$\mathfrak{g} := \mathrm{Lie}(G)$ has a structure of normal j-algebra.

(Pjatetskii-Shapiro)

$\exists J$: integrable almost complex structure on $\mathfrak{g}$

$\exists \omega$: admissible linear form on $\mathfrak{g}$, *i.e.*,

$\langle\, x \,|\, y \,\rangle_\omega := \langle [Jx, y], \omega \rangle$ defines a J-invariant inner product on $\mathfrak{g}$.

Example (Koszul '55). Koszul form.

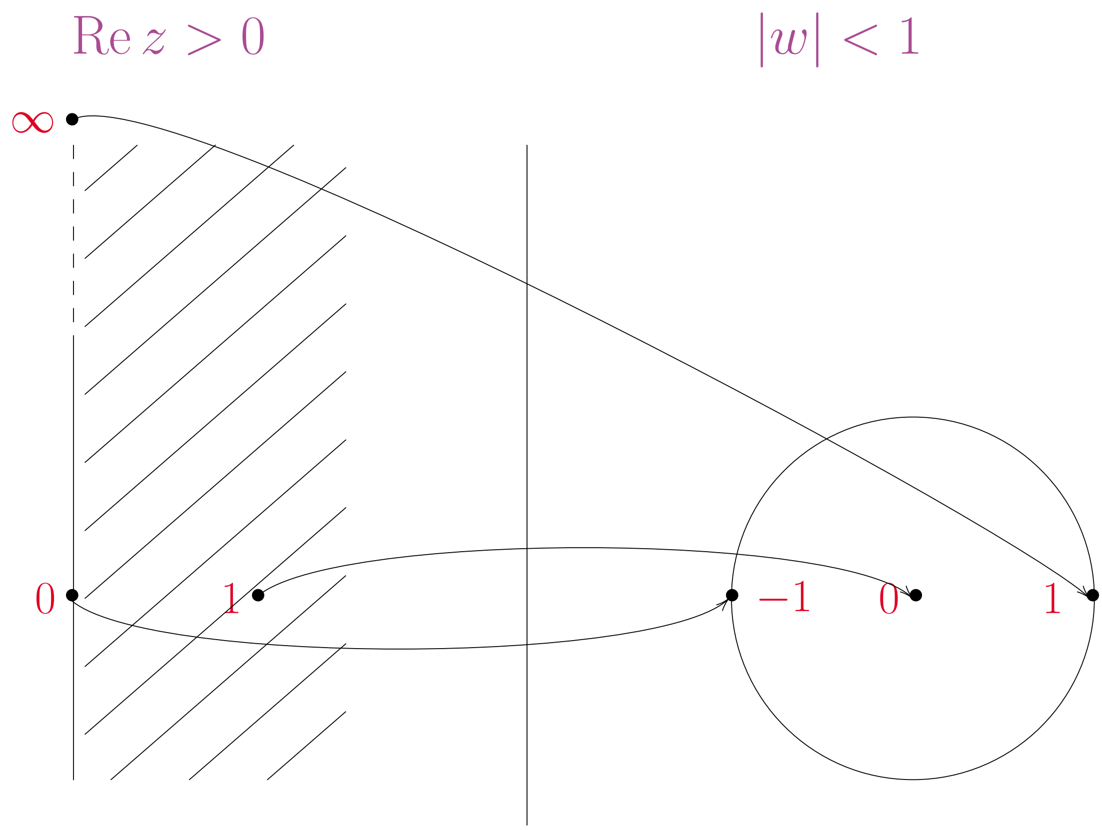
$$\langle x, \beta \rangle := \mathrm{tr}\big(\mathrm{ad}(Jx) - J\,\mathrm{ad}(x)\big) \quad (x \in \mathfrak{g}).$$

β is admissible

- In fact, $\langle\, x \,|\, y \,\rangle_\beta$ is the real part of the Hermitian inner product defined by the Bergman metric on $D \approx G$ (up to a positive scalar multiple).

Throughout this talk we take $\omega = \beta$.

Cayley transform



$$z = \frac{1+w}{1-w} = -1 + \frac{2}{1-w} \qquad w = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$$

If one puts in a complex semisimple Jordan algebra

$$z = \frac{e+w}{e-w}, \qquad w = \frac{z-e}{z+e},$$

then the above figure is the case for symmetric tube domains.

- In general, if one can define something like $(z+1)^{-1}$ (denominator), one has a Cayley transform by $1 - 2(z+1)^{-1}$ for tube domains.

Pseudoinverse assoc. with the Szegö kernel

S : the Szegö kernel of D

(= reprod. kernel of the Hardy space)

Hardy space $H^2(D)$

holomorphic functions F on D such that

$$\sup_{t\in\Omega}\int_U\int_V \left|F\left(u, t+\tfrac{1}{2}Q(u,u)+ix\right)\right|^2 dx\,dm(u) < \infty$$

$\exists\eta$: holomorphic on $\Omega + iV$ such that

$$S(z_1, z_2) = \eta\big(w_1 + w_2^* - Q(u_1, u_2)\big)$$

$$(z_j = (u_j, w_j) \in D)$$

In more detail

$\exists H \subset G$: s.t. $H \curvearrowright \Omega$ simply transitively

$E \in \Omega$ (base point; virtual identity matrix)

Then $H \approx \Omega$ (diffeo) by $h \mapsto hE$.

For each $\chi : H \longrightarrow \mathbb{R}_+^\times$ one dim. repre.

define Δ_χ on Ω by

$$\Delta_\chi(hE) := \chi(h) \quad (h \in H)$$

- Δ_χ extends to a holomorphic function on $\Omega + iV$ as the Laplace transform of the Riesz distribution on the dual cone Ω^* (Gindikin, Ishi (J. Math. Soc. Japan, 2000)), where

$$\Omega^* := \{\xi \in V^* ; \langle x, \xi \rangle > 0 \ \forall x \in \overline{\Omega} \setminus \{0\}\}.$$

- $\exists \chi, \quad \exists c > 0$ s.t. $\eta = c\Delta_\chi$

For each $x \in \Omega$, define $\mathcal{I}(x) \in V^*$ by

$$\langle v, \mathcal{I}(x) \rangle := -D_v \log \eta(x)$$

$$\left(D_v f(x) := \frac{d}{dt} f(x+tv)\Big|_{t=0}\right)$$

- $\mathcal{I}(\lambda x) = \lambda^{-1} \mathcal{I}(x) \qquad (\lambda > 0)$

Prop. (1) $\mathcal{I}(x) \in \Omega^*$ and $\mathcal{I} : \Omega \twoheadrightarrow \Omega^*$ is bijec.

(2) $\mathcal{I}$ extends analytically to a rational map $W \to W^*$.

(3) One also has an explicit formula for $\mathcal{I}^{-1} : \Omega^* \twoheadrightarrow \Omega$, which continues analytically to a rational map $W^* \to W$.
Thus $\mathcal{I}$ is birational.

(4) $\mathcal{I} : \Omega + iV \twoheadrightarrow \mathcal{I}(\Omega + iV)$ is biholomorphic.

Remark. If $\chi : H \to \mathbb{R}_+^\times$ is defined in a natural way by an admissible linear form, then the above proposition holds for $\mathcal{I} = \mathcal{I}_\chi$ [N, preprint 2001].

Cayley transform

$E^* := \mathcal{I}(E) \in \Omega^*$.

$C(w) := E^* - 2\mathcal{I}(w+E)$ for tube domains

$$\mathcal{C}(u,w) := \underbrace{2\,\langle Q(u,\cdot), \mathcal{I}(w+E)\rangle}_{\in U^\dagger} \oplus \underbrace{C(w)}_{\in W^*}$$

$U^\dagger$: the space of antilinear forms on U

Prop. (1) $\mathcal{C} : D \twoheadrightarrow \mathcal{C}(D)$ is birational and biholomorphic.
(2) $\mathcal{C}^{-1}$ can be written explicitly.

Theorem [N]. $\mathcal{C}(D)$ is bounded (in $U^\dagger \oplus W^*$).

Remark. (1) C_χ and $\mathcal{C}_\chi$ can be defined similarly from $\mathcal{I}_\chi$. One can prove that $\mathcal{C}_\chi(D)$ is bounded [N].
(2) For general χ, $\mathcal{C}_\chi(D)$ for symmetric D is *not* the standard Harish-Chandra model of a Hermitian symmetric space (no circularity).

Norm equality

e $:= (0, E) \in D$: base point

the Bergman metric of the Siegel domain D

$\rightsquigarrow$ Hermitian inner prod. on $T_{\mathsf{e}}(D) \equiv U \oplus W$

$\rightsquigarrow$ Hermitian inner prod $(\cdot \mid \cdot)$ and norm $\| \cdot \|$ on the dual vector space $U^\dagger \oplus W^*$.

Σ : the Shilov boundary of D

$\Sigma = \{(u, w) \in U \times W\, ;\, 2\,\mathrm{Re}\, w = Q(u, u)\}$

$\langle x, \beta \rangle = \mathrm{tr}\,(\mathrm{ad}\,(Jx) - J\,\mathrm{ad}\,(x))$: Koszul form

$\rightsquigarrow$ inner prod. $\langle\, x \mid y \,\rangle_\beta = \langle [Jx, y], \beta \rangle$ on $\mathfrak{g}$

- $\Psi \in \mathfrak{g}$ for which $\mathrm{tr}\,\mathrm{ad}\,(x) = \langle\, x \mid \Psi \,\rangle_\beta \;\; (\forall x \in \mathfrak{g})$

$S(z_1, z_2) = \eta(w_1 + w_2^* - Q(u_1, u_2))$ with $\eta = c\,\Delta_\chi$

$\Delta_\chi(hE) = \chi(h) = e^{-\langle \log h, \alpha \rangle} \; (\alpha \in \mathfrak{h}^* \subset \mathfrak{g}^*)$.

Theorem [N].

$\|\mathcal{C}(\zeta)\|^2 = \langle \Psi, \alpha \rangle$ for $\forall \zeta \in \Sigma \iff D$ is symm.

$\langle\, x \,|\, y \,\rangle_\beta$ inner prod. on $\mathfrak{g}$

$\rightsquigarrow$ left invariant Riemannian metric on G

$\rightsquigarrow$ Laplace–Beltrami operator $\mathcal{L}_\beta$ on G

$G \approx D$ (diffeo) by $g \mapsto g \cdot$ e.

Upon $G \equiv D$, we have $\mathcal{L}_\beta = c'\mathcal{L}$ ($c' > 0$),

($\mathcal{L}$: Laplace–Beltrami operator $\leftrightsquigarrow$ the Bergman metric of D)

<u>Prop (Urakawa '79).</u> $\mathcal{L}_\beta = -\Lambda + \Psi.$

- $\Lambda := X_1^2 + \cdots + X_{\dim \mathfrak{g}}^2 \in U(\mathfrak{g})$,
- $\{X_1, \ldots, X_{\dim \mathfrak{g}}\}$ is an ONB of $\mathfrak{g}$ w.r.t. $\langle \cdot \,|\, \cdot \rangle_\beta$
  (Λ is independent of choice of ONB.)
- $\langle \cdot \,|\, \Psi \,\rangle_\beta = \operatorname{tr} \operatorname{ad}(\cdot)$,
- Elements of $U(\mathfrak{g})$ are regarded as left invariant differential operators on G — thus if $X \in \mathfrak{g}$,

$$X f(x) = \frac{d}{dt} f(x \exp tX)\Big|_{t=0}.$$

Poisson kernel

$S(z_1, z_2)$: the Szegö kernel of the Siegel domain D

We know

$$S(z_1, z_2) = \eta(w_1 + w_2^* - Q(u_1, u_2)), \quad \eta = c\,\Delta_\chi.$$

$S(z, \zeta)$ for $z \in D$ and $\zeta \in \Sigma$ has a meaning.

$$P(z, \zeta) := \frac{|S(z, \zeta)|^2}{S(z, z)} \quad (z \in D,\ \zeta \in \Sigma) :$$

the Poisson kernel of D

$$P_\zeta^G(g) := P(g \cdot \mathrm{e},\ \zeta) \quad (g \in G).$$

<u>Theorem [N].</u>

$$\mathcal{L}_\beta P_\zeta^G(e) = (-\|\mathcal{C}(\zeta)\|^2 + \langle \Psi, \alpha \rangle) P_\zeta^G(e),$$

where α is related to χ by $\chi(\exp T) = e^{-\langle T, \alpha \rangle}$.

<u>Remark</u>. By $P(g \cdot z,\ \zeta) = \chi(g) P(z,\ g^{-1} \cdot \zeta)$ $(g \in G)$,

$$\mathcal{L}_\beta P_\zeta^G = 0 \ \ \forall \zeta \in \Sigma \iff \mathcal{L}_\beta P_\zeta^G(e) = 0 \ \ \forall \zeta \in \Sigma.$$

<u>Theorem</u>.

$\mathcal{L}_\beta P_\zeta^G = 0$ for $\forall \zeta \in \Sigma \iff D$ is symmetric.

Validity of the norm equality for symmetric D

D : symmetric $\Longrightarrow$ $\mathcal{D} := \mathcal{C}(D)$ is the Harish-Chandra model of a Hermitian symmetric space

In particular, $\mathcal{D}$ is circular (Note $\mathcal{C}(\mathrm{e}) = 0$).

$\mathsf{G} := \mathrm{Hol}(\mathcal{D})^\circ$: semisimple Lie gr. (with trivial center)

$\mathsf{K} := \mathrm{Stab}_{\mathsf{G}}(0)$: maximal cpt subgr. of G

Circularity of $\mathcal{D}$ $\Longrightarrow$ $\mathsf{K} \subset$ Unitary group

$$\begin{cases} \mathcal{C} : \Sigma \ni 0 \mapsto -E^*, \\ \text{Shilov boundary } \Sigma_{\mathcal{D}} \text{ of } \mathcal{D} = \mathsf{K} \cdot (-E^*). \end{cases}$$

Since $\Sigma_{\mathcal{D}}$ is also a G-orbit $\Sigma_{\mathcal{D}} = \mathsf{G} \cdot (-E^*)$ and since Σ is an orbit of a nilpotent subgroup of $G \subset \mathrm{Hol}(D)^\circ$, we get

$$\begin{aligned} \mathcal{C}(\Sigma) \subset \mathsf{G} \cdot (-E^*) &= \Sigma_{\mathcal{D}} \\ &= \mathsf{K} \cdot (-E^*) \\ &\subset \{z \,;\ \|z\| = \|E^*\|\}. \end{aligned}$$

We see easily that $\|E^*\|^2 = \langle \Psi, \alpha \rangle$ in this case.

Norm equality $\Longrightarrow$ symmetry of D

Assumption : $\|\mathcal{C}(\zeta)\|^2 = \langle \Psi, \alpha \rangle$ for $\forall \zeta \in \Sigma$.

(1) Reduction to a quasisymmetric domain

κ : the Bergman kernel of D

$$\left(\begin{array}{l} \kappa(z_1, z_2) = \eta_0(w_1 + w_2^* - Q(u_1, u_2)), \\ \exists \chi_0 : H \to \mathbb{R}_+^\times, \quad \exists c_0 > 0 \ \text{ s.t. } \ \eta_0 = c_0 \Delta_{\chi_0}, \\ \Delta_{\chi_0}(hE) = \chi_0(h) \text{:} \quad \Delta_{\chi_0} \rightsquigarrow \text{hol. ftn on } \Omega + iV \end{array} \right.$$

$\langle\, x \,|\, y \,\rangle_\kappa := D_x D_y \log \Delta_{\chi_0}(E)$: inner prod. of V

Def. $D = D(\Omega, Q)$ is *quasisymmetric* $\underset{\text{def}}{\Longleftrightarrow}$ Ω is selfdual w.r.t. $\langle \cdot \,|\, \cdot \rangle_\kappa$.

Define a non-associative prod. xy in V by

$$\langle\, xy \,|\, z \,\rangle_\kappa = -\tfrac{1}{2} D_x D_y D_z \log \Delta_{\chi_0}(E).$$

Prop. (Dorfmeister, D'Atri, Dotti Miatello, Vinberg). D is quasisymmetric $\Longleftrightarrow$ prod. xy is Jordan.

In this case, V is a Euclidean Jordan algebra.

$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n}$

$\mathfrak{a}$: abelian, $\mathfrak{n}$: sum of $\mathfrak{a}$-root spaces

(positive roots only)

Possible forms of roots:

$\frac{1}{2}(\alpha_k \pm \alpha_k)\ (j < k), \quad \alpha_k, \quad \frac{1}{2}\alpha_k$

Always $\dim \mathfrak{g}_{\alpha_k} = 1\ (\forall k)$.

Prop. (D'Atri and Dotti Miatello '83; D : irred.)

D is quasisymmetric

$$\iff \begin{cases} (1) & \dim \mathfrak{g}_{(\alpha_k+\alpha_j)/2} \text{ is indep. of } j, k, \\ (2) & \dim \mathfrak{g}_{\alpha_k/2} \text{ is indep. of } k. \end{cases}$$

Extend $\langle \cdot \mid \cdot \rangle_\kappa$ to a $\mathbb{C}$-bilinear form on $W \times W$.

$(u_1 \mid u_2)_\kappa := \langle Q(u_1, u_2) \mid E \rangle_\kappa$

defines a Hermitian inner product on U.

For each $w \in W$, define $\varphi(w) \in \mathrm{End}_{\mathbb{C}}(U)$ by

$$(\varphi(w)u_1 \mid u_2)_\kappa = \langle Q(u_1, u_2) \mid w \rangle_\kappa.$$

Clearly $\varphi(E) =$ identity operator on U.

Prop. (Dorfmeister). D is quasisymmetric

$\Longrightarrow$ $w \mapsto \varphi(w)$ is a Jordan $*$-repre. of $W = V_{\mathbb{C}}$

$$\begin{cases} \varphi(w^*) = \varphi(w)^*, \\ \varphi(w_1 w_2) = \frac{1}{2}\big(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1)\big). \end{cases}$$

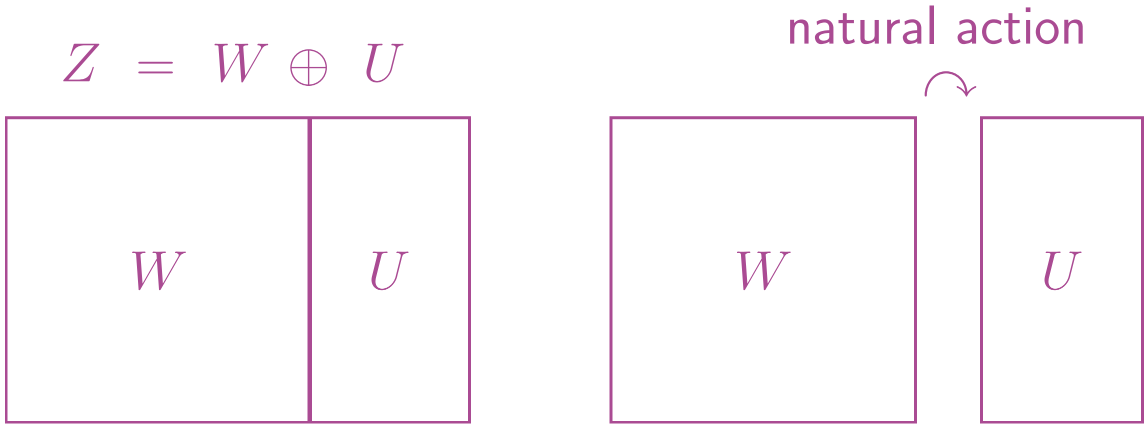
(2) Reduction : quasisymm $\Longrightarrow$ symm

Quasisymmetric Siegel domain

$\leftrightarrow$ $\begin{cases} \text{Euclidean Jordan algebra } V \text{ and} \\ \text{Jordan } *\text{-representation } \varphi \text{ of } W = V_{\mathbb{C}}. \end{cases}$

Symmetric Siegel domain

$\leftrightarrow$ Positive Hermitian JTS



complex semisimple
Jordan algebra
$W = V_{\mathbb{C}}$
with V Euclidean JA

Jordan algebra $*$-repre. of W

Prop. (Satake).

Quasisymmetric D is symmetric

$\iff$ V and φ come from a positive Hermitian JTS this way.

Prop. (Dorfmeister).

Irreducible quasisymmetric D is symmetric

$\iff$ $\exists f_1, \ldots, f_r$: Jordan frame of V s.t.
with $U_k := \varphi(f_k)U$ we have

$$\varphi(Q(u_1, u_2))u_1 = 0$$

for $\forall u_1 \in U_1$ and $\forall u_2 \in U_2$.

In a similar way

<u>Theorem [N; Diff. Geom. Appl., 15-1 (2001)]</u>.
Berezin transforms on D commute with the Laplace–Beltrami operator
$\iff$ D is symmetric.

<u>Related norm equality</u>

$\mathcal{C}_B$: Cayley transf. assoc. with the Bergman kernel.

<u>Theorem [N; Transform. Groups, 6-3 (2001)]</u>.
$\|\mathcal{C}_B(g \cdot \mathrm{e})\| = \|\mathcal{C}_B(g^{-1} \cdot \mathrm{e})\|$ holds for $\forall g \in G$
$\iff$ D is symmetric.