Homogeneous Siegel Domains

— Analysis and Geometry —

Takaaki NOMURA

Kyushu University

於：釜慶大学

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• Homogeneous bounded domains

The case of $\mathbb{C}$:

Riemann mapping theorem implies:

biholomorphically equivalent to $\mathbb{D}$ (open unit disk).

The Lie group

$$SU(1,1) := \left\{ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \; ; \; \begin{matrix} \alpha, \beta \in \mathbb{C}, \\ |\alpha|^2 - |\beta|^2 = 1 \end{matrix} \right\}$$

acts on $\mathbb{D}$ by linear fractional transformations:

$$g \cdot z = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \qquad (z \in \mathbb{D}).$$

⋆ The action is *transitive* — thus $\mathbb{D}$ is homogeneous.

⋆ $\mathbb{D}$ is *symmetric*: $\sigma : \mathbb{D} \ni z \mapsto -z \in \mathbb{D}$ $(\alpha = i,\ \beta = 0)$ satisfies

(1) $\sigma^2 = \mathrm{Identity}$,

(2) 0 is an isolated (in fact unique) fixed point of σ.

By homogeneity you can show: for $\forall z \in \mathbb{D}$, $\exists \sigma_z$ such that

(1) $\sigma_z^2 = \mathrm{Identity}$,

(2) z is an isolated fixed point of σ_z.

<u>The cases $\mathbb{C}^2$, $\mathbb{C}^3$:</u>

E. Cartan's work (1935):

$\star$ Any homogeneous bounded domain in $\mathbb{C}^2$ or $\mathbb{C}^3$ is symmetric.

Cartan's problem: What happens in $\mathbb{C}^n$ $(n \geqq 4)$.

Cartan's conjecture:

To find non-symmetric domains, some new idea is necessary.

D: a (bounded) domain.

$\mathrm{Hol}(D) := \{$holomorphic automorphisms of $D\}$.

(finite dim. Lie group by cpt open topology if $D \approx$ bdd domain)

D is *homogenous* $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Hol}(D)$ acts on D transitively.

D is *symmetric* $\overset{\text{def}}{\Longleftrightarrow}$ for $\forall z \in \mathbb{D}$, $\exists \sigma_z \in \mathrm{Hol}(D)$ such that

(1) $\sigma_z^2 = \mathrm{Identity}$,

(2) z is an isolated fixed point of σ_z.

Cartan's problem: What happens in $\mathbb{C}^n$ $(n \geqq 4)$?

Cartan's conjecture:

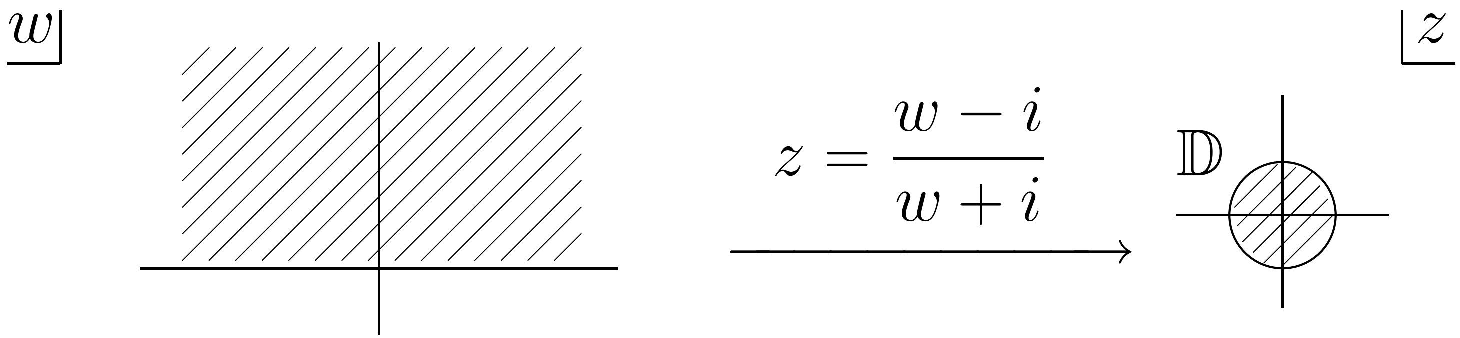
To find non-symmetric homogeneous bounded domains, some new idea is necessary.

Piatetski-Shapiro (1959)

found non-symmetric homogeneous Siegel domains in $\mathbb{C}^4$ and $\mathbb{C}^5$.

⋆ Siegel domain ≈ bounded domain (biholomorphically)

⋆ Later P.-S. showed that in $\dim \geqq 7$, there are continuum cardinality of inequivalent homogeneous Siegel domains.



$\Downarrow$ generalization to higher dimensions

$$\text{Siegel domains} \xrightarrow{\text{Cayley transform}} \text{bounded domains}$$

Siegel domains. — Definition —

V : a real vector space ($\dim V < \infty$)

$\cup$

Ω : a *proper* ($\overset{\text{def}}{\iff}$ contains *no* entire line) open convex cone

$W := V_{\mathbb{C}}$ ($w \mapsto w^*$: conjugation w.r.t. V)

U : another complex vector space ($\dim U < \infty$)

$Q : U \times U \to W$, Hermitian sesquilinear Ω-positive

i.e., $$\begin{cases} Q(u', u) = Q(u, u')^* \\ Q(u, u) \in \overline{\Omega} \setminus \{0\} \quad (0 \neq \forall u \in U) \end{cases}$$

Siegel domain (of type II)

$$D := \{(u, w) \in U \times W \ ; \ w + w^* - Q(u, u) \in \Omega\}$$

- $U = \{0\}$ is allowed. In this case $D = \Omega + iV$.

  (**tube domain** or **type I domain**)

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  (tube domain or type I domain)

Assume that D is homogeneous

i.e., $\mathrm{Hol}(D) \curvearrowright D$ transitively.

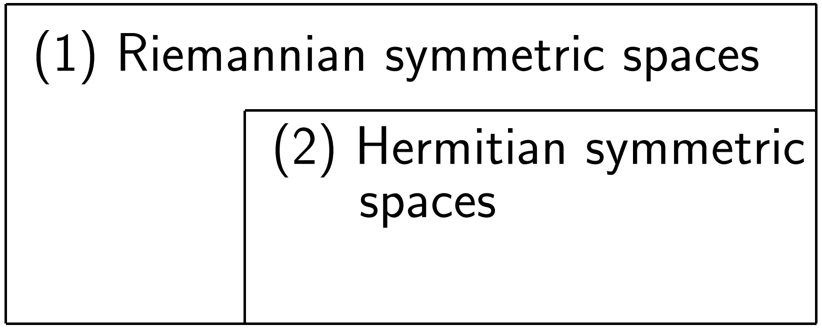
Then Ω is also homogeneous: *i.e.*,

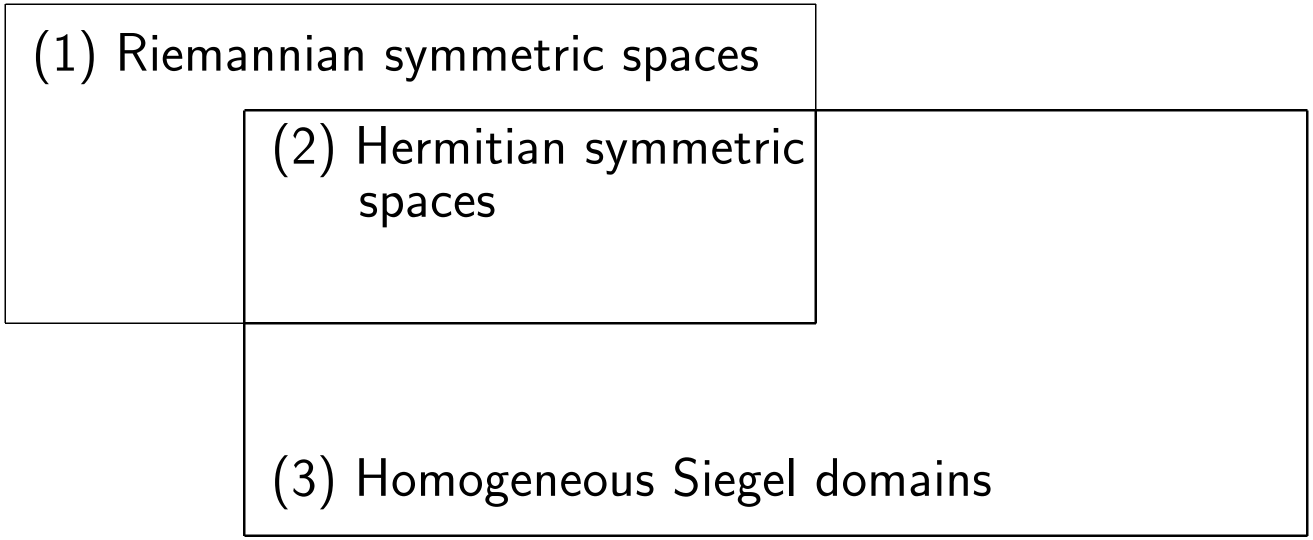
$G(\Omega) := \{g \in GL(V) \; ; \; g(\Omega) = \Omega\}$

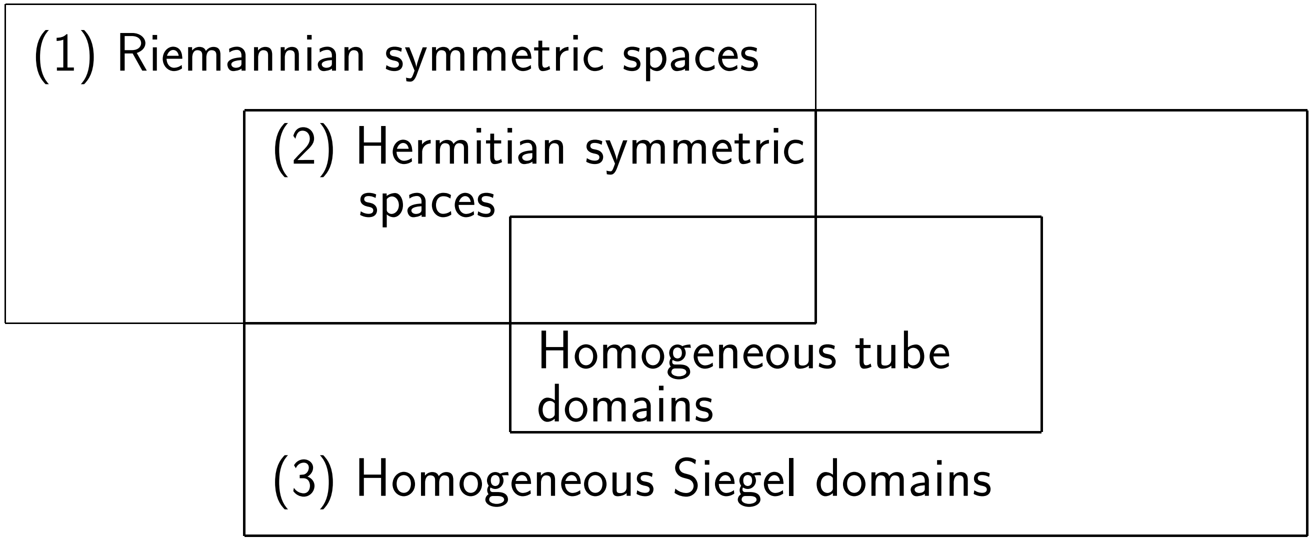
(linear automorphism group of Ω)

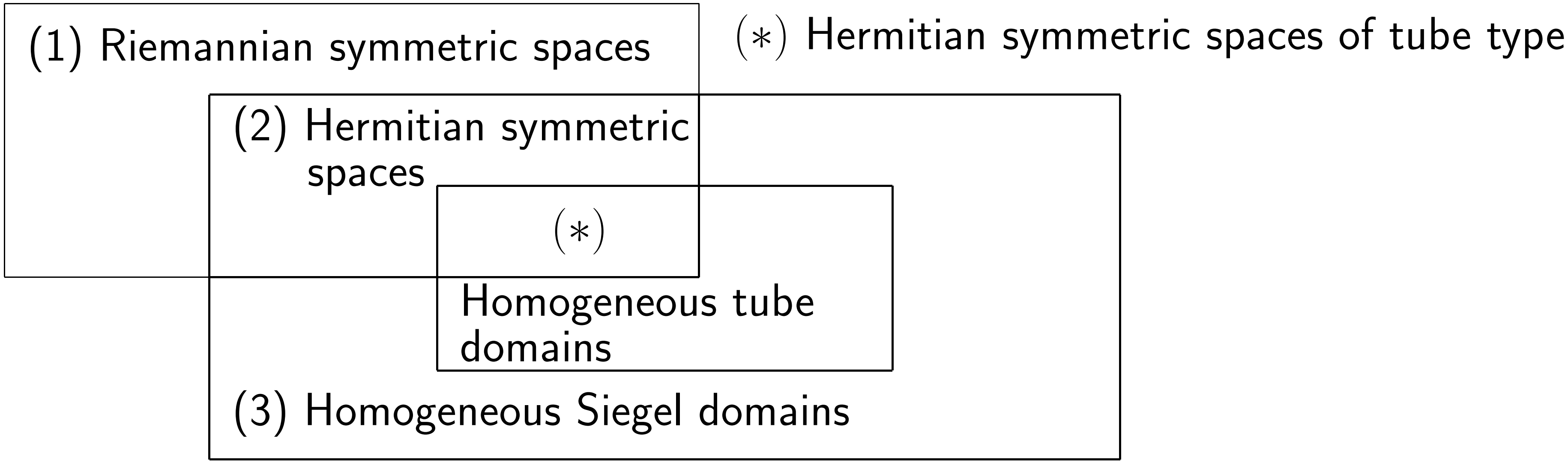
$G(\Omega) \curvearrowright \Omega$ transitively.

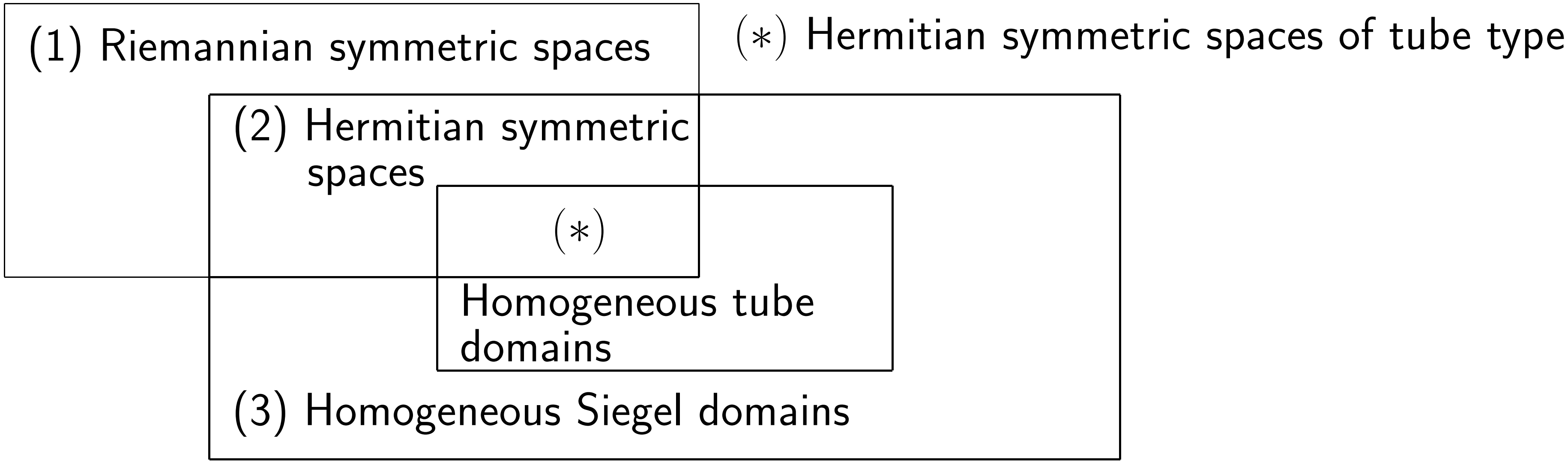
(1) Riemannian symmetric spaces


(1) Riemannian symmetric spaces
(2) Hermitian symmetric spaces


(1) Riemannian symmetric spaces
(2) Hermitian symmetric spaces
(3) Homogeneous Siegel domains


(1) Riemannian symmetric spaces
(2) Hermitian symmetric spaces
Homogeneous tube domains
(3) Homogeneous Siegel domains


(1) Riemannian symmetric spaces
(∗) Hermitian symmetric spaces of tube type
(2) Hermitian symmetric spaces
(∗)
Homogeneous tube domains
(3) Homogeneous Siegel domains



- List of irreducibles for (1) and (2).
- Harish-Chandra model for (2):

  open unit ball for certain norm (in general not Hibertian; *spectral norm* in a certain triple product system)
- Cayley transform given by Korányi–Wolf (1965) inside (2):

  symmetric Siegel domain $\longrightarrow$ Harish-Chandra model

Characterization theorems of symmetric domains

- By means of transitive group:
  D is symmetric if D is a homogeneous space of a semisimple Lie group (Borel 1954, Koszul 1955), of a unimodular Lie group (Hano 1957) $\cdots$ old results before Piatetski-Shapiro
- By means of defining data of Siegel domains:
  Satake (book published in 1980),
  Dorfmeister (Habilitationsschrift 1978)
- By means of a curvature condition: D'Atri–Dotti 1983
- By means of the eigenvalues of the curvature operator:
  Azukawa 1985
- By means of a discrete subgroup acting properly on D: Vey 1970
- By means of some equi-dimensionality of root subspaces:
  D'Atri–Dotti (1983)

Characterization theorems of symmetric domains

• Some results of D'Atri–Dorfmeister–Zhao 1985:

The following (1) $\sim$ (4) are equivalent: ($\mathbf{G} := \mathrm{Hol}(D)^\circ$)

(1) D is symmetric,

(2) The almost complex structure is represented by an operator of the infinitesimal isotropy representation,

(3) The only $\mathbf{G}$-invariant vector field on D is a trivial one.

(4) The algebra $\mathbf{D}(D)^{\mathbf{G}}$ of $\mathbf{G}$-invariant differential operator on D is commutative.

(2) is a well-known fact in Hermitian symmetric spaces.

(4) is well known in analysis on Riemannian symmetric spaces.

In fact, if D is Riemannian symmetric, $\mathbf{D}(D)^{\mathbf{G}}$ is isomorphic to a polynomial algebra with the number of generators equal to $\mathrm{rank}(D)$.

Characterization theorems of symmetric domains

Berezin transform B: $\mathbf{G}$-invariant positive bounded selfadjoint operator on $L^2(D)$ (w.r.t. $\mathrm{Hol}(D)$-invariant measure)

$\star$ Homogeneous Kähler metric on D
$\rightsquigarrow$ Laplace–Beltrami operator $\mathcal{L}$ on D.

Theorem 1 (N. 2001). *B commutes with $\mathcal{L}$*
$\iff$ *D is symmetric and the metric considered is Bergman (up to a positive number multiple).*

Remark. Spectral decomposition of B for symmetric D can be obtained explicitly by Helgason's geometric Fourier analysis on Riemannian symmetric spaces (Berezin 1978 for classical domains, Unterberger–Upmeier 1994 for general domains).
However for non-symmetric D, we have no result for the moment.

Theorem 2 (N. 2003). *The Poisson–Hua kernel is $\mathcal{L}$–harmonic (killed by $\mathcal{L}$) $\iff$ D is symmetric and the metric considered is Bergman (up to a positive number multiple).*

Remark. $[\Longleftarrow]$ has been proved by Hua–Look (1959) for classical domains, by Korányi (1965) for general domains.
$[\Longrightarrow]$ with Bergman metric is first proved by Xu (1979).

For every Kähler metric h on a homogeneous Siegel domain D, one can define a Cayley transform $\mathcal{C}_h$ of D.

Theorem 3 (N. 2003). *$\mathcal{C}_h(D)$ is bounded.*

Theorem 1 (N. 2001). *B commutes with $\mathcal{L}$*
$\iff$ *D is symmetric and the metric considered is Bergman (up to a positive number multiple).*

Theorem 2 (N. 2003). *The Poisson–Hua kernel is $\mathcal{L}$–harmonic (killed by $\mathcal{L}$) $\iff$ D is symmetric and the metric considered is Bergman (up to a positive number multiple).*

Remark. One can understand Theorems 1 and 2 by the shape of $\mathcal{C}_h(D)$ (Theorem 1) or of the Shilov boundary of $\mathcal{C}_h(D)$ (Theorem 2).

Theorem 4 (C. Kai 2007). *$\mathcal{C}_h(D)$ is a convex set*
$\iff$ *D is symmetric and the metric considered is Bergman (up to a positive number multiple).*