Homogeneous Convex Cones and Basic Relative Invariants

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- Homogeneous convex cones provide
  many examples of non-reductive prehomogeneous vector spaces

  $\rightsquigarrow$ important to know the basic relative invariants

- Applications to statistics
  (from positive-definite matrices to general convex cones)

- Matrix realizations of interesting homogeneous convex cones

  By Vinberg (1963), homogeneous cones are sets of matrices of the form TT^*, where T's are regular upper triangular matrices from some non-associative algebras.

  ◦ beautiful in theory but hard to handle in practice

Homogeneous convex cones

V: a real vector space $(\dim V < \infty)$
$V \supset \Omega$: a regular open convex cone (containing no entire line)
$GL(\Omega) := \{g \in GL(V) \,;\, g(\Omega) = \Omega\}$: the linear automorphism group of Ω
(a Lie group as a closed subgroup of $GL(V)$)

Ω is **homogeneous** $\overset{\text{def}}{\iff}$ $GL(\Omega) \curvearrowright \Omega$ is transitive.

Vinberg (1963)

homogeneous (regular affine) convex domain
$\rightleftarrows$ algebraic structure of the ambient vector space
($\equiv$ tangent space of a reference point)

Algebras associated to homogeneous convex domains (Vinberg 1963)

Definition

V is a real VS with a bilinear product $x \triangle y = L(x)y$.

V is a **clan** $\overset{\text{def}}{\Longleftrightarrow}$

(1) $[L(x), L(y)] = L(x \triangle y - y \triangle x)$ (**l**eft symmetric **a**lgebra),

(2) $\exists s \in V^*$ s.t. $s(x \triangle y)$ is an onner pruduct of V (**c**ompact),

(3) Each $L(x)$ has only real eigenvalues (**n**ormal).

- clans with unit element $\longleftrightarrow$ homogeneous convex cones.
- **homogeneous convex cones $\Longrightarrow$ clans**

$\exists H$: a split solvable subgroup of $GL(\Omega)$ s.t. $H \curvearrowright \Omega$ simply transitively.

$\rightsquigarrow$ Fixing $E \in \Omega$, we have $H \approx HE = \Omega$ (diffeo)

$\rightsquigarrow$ $\mathfrak{h} := \mathrm{Lie}(H) \cong T_E(\Omega) \equiv V$ (linear isomorphism obtained by differentiation)

$\rightsquigarrow$ $\forall x \in V$, $\exists 1 T \in \mathfrak{h}$ s.t. $TE = x$.

$\rightsquigarrow$ Writing $T = L(x)$, we define a product $\triangle$ by $x \triangle y := L(x)y$.
E is a unit element.

- the **dual cone** Ω^* of Ω (w.r.t $\langle\cdot\,|\,\cdot\rangle$)

$$\overset{\text{def}}{\iff} \quad \Omega^* := \{ y \in V \; ; \; \langle x \,|\, y \rangle > 0 \quad (\forall x \in \overline{\Omega} \setminus \{0\}) \}$$

- Ω is **selfdual** $\overset{\text{def}}{\iff}$ $\exists\langle\cdot\,|\,\cdot\rangle$ s.t. $\Omega^* = \Omega$
- **symmetric cone** $\overset{\text{def}}{\iff}$ homogeneous selfdual open convex cone

- symmetric cone $\Omega \rightleftarrows$ Euclidean Jordan algebra V: $\quad \Omega = \mathrm{Int}\{x^2 \; ; \; x \in V\}$.

- V: a vector space with bilinear product xy.

  V is a **Jordan algbera** $\overset{\text{def}}{\iff}$ (1) $xy = yx$, $\quad$ (2) $x^2(xy) = x(x^2y)$.
- real Jordan algebra with unit element e_0 is **Euclidean**

  $\overset{\text{def}}{\iff}$ $\exists\langle\cdot\,|\,\cdot\rangle$ (associative inner product) s.t. $\langle xy \,|\, z \rangle = \langle x \,|\, yz \rangle \quad (\forall x, y)$.

- symmetric cone is irreducible $\iff$ corresponding EJA is simple

Example. $V = \mathrm{Sym}(r, \mathbb{R})$

- Jordan product $\circ$ is given by $x \circ y := \frac{1}{2}(xy + yx)$.
- clan product $\triangle$ is given by $x \triangle y = \underline{x}\, y + y\, {}^t(\underline{x})$,

where for $x = (x_{ij}) \in \mathrm{Sym}(r, \mathbb{R})$, we set

$$\underline{x} := \begin{pmatrix} \frac{1}{2}x_{11} & & & 0 \\ x_{21} & \frac{1}{2}x_{22} & & \\ \vdots & \ddots & \ddots & \\ x_{r1} & \cdots & x_{r,r-1} & \frac{1}{2}x_{rr} \end{pmatrix}.$$

Note $x = \underline{x} + {}^t(\underline{x})$.

classification
of irreducible symmetric cones $\Longleftrightarrow$ of simple EJA

- $\Omega = \mathrm{Sym}(r, \mathbb{R})^{++} \subset V = \mathrm{Sym}(r, \mathbb{R})$
- $\Omega = \mathrm{Herm}(r, \mathbb{C})^{++} \subset V = \mathrm{Herm}(r, \mathbb{C})$
- $\Omega = \mathrm{Herm}(r, \mathbb{H})^{++} \subset V = \mathrm{Herm}(r, \mathbb{H})$
- $\Omega = \mathrm{Herm}(3, \mathbb{O})^{++} \subset V = \mathrm{Herm}(3, \mathbb{O})$
- $\Omega = \Lambda_n$ (n-dim. Lorentz cone) $\subset V = \mathbb{R}^n$: linear part of Clifford algebra

- Non-symm. homogeneous open convex cones (HOCC) appear from dimension 5.
- In dim. ≥ 11, $\exists$mutually linearly inequivalent HOCC with a continuous parameter.
- In dim. $\leq$ 10, only finitely many irreducible HOCC exist up to linearly equiv.
  — Classification by Kaneyuki–Tsuji ('74)
  — concrete realizations up to 7-dim.
- Methods to realize general HOCC by real symmetric matrices
  (1) By Ishi
  (2) By Yamasaki–N. (more direct than (1); preprint just finished a few days ago)
  (2) obtains realizations of $8, 9, 10$-dim. HOCC left unrealized by K.–T.

Basic relative invariants

Ω: HOCC $\subset V$, $GL(\Omega)$: the linear automorphism group of Ω.
$\exists H$: a split solvable $\subset GL(\Omega)$ s.t. $H \curvearrowright \Omega$ simply transitively

- a function f on Ω is **relatively invariant** (w.r.t. H)

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists\chi$: 1-dim. rep of H s.t. $f(hx) = \chi(h)f(x)$ $(h \in H,\ x \in \Omega)$.

Theorem [Ishi 2001]

$\exists \Delta_1, \ldots, \Delta_r$ $(r := \mathrm{rank}(\Omega))$: irreducible relat. inv. polynomial functions on V s.t. any relat. inv. polynomial function P on V is uniquely written as

$$P(x) = c\,\Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \quad (c = \mathrm{const.},\ (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geqq 0}).$$

- $\Delta_1(x), \ldots, \Delta_r(x)$: **the basic relative invariants associated to Ω**

Example. When $V = \mathrm{Sym}(r, \mathbb{R})$,
$\Delta_k(x)$ is the k-th principal minor of $x \in V$ taken from the upper-left corner
(also can be taken from the lower-right corner)

- general EJA: Fix a Jordan frame $c_1, \ldots, c_r$
(complete system of orthogonal primitive idempotents)

$\rightsquigarrow$ JA principal minors $\Delta_1(x), \ldots, \Delta_r(x)$ are the basic relative invariants.
In $V = \mathrm{Sym}(r, \mathbb{R})$,
$c_k := E_{kk}$ $(k = 1, \ldots, r) \implies \Delta_k(x)$ is from the upper-left corner
$c_k := E_{r-k+1, r-k+1}$ $(k = 1, \ldots, r) \implies \Delta_k(x)$ is from the lower-right corner.

In general, suppose HOCC $\Omega \subset V$ with clan structure of V.

Theorem [Ishi-N. 2008]

$R(x)y := y \triangle x$: the right multiplication operator by x in V

$\implies$ the irreducible factors of $\mathrm{Det}\, R(x)$ coincide with $\Delta_1(x), \ldots, \Delta_r(x)$.

Problem

Let us put $\operatorname{Det} R(x) = \Delta_1(x)^{n_1} \Delta_2(x)^{n_2} \cdots \Delta_r(x)^{n_r}$. Then express the positive integers $n_1, \ldots, n_r$ in terms of the constants related to the clan V.

$\boldsymbol{n} := (n_1, \ldots, n_r)$ is called the **basic index** of V.

Example. If V is a simple JA, we have $\boldsymbol{n} = (d, \ldots, d, 1)$,
where $d :=$ common dim. of the "off-diagonals" V_{kj} $(j < k)$.

$\mathrm{Sym}(r, \mathbb{R})$: $d = 1$,

$\mathrm{Herm}(r, \mathbb{K})$ $(\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O})$: $d = \dim_{\mathbb{R}} \mathbb{K}$ (only $r = 3$ occurs when $\mathbb{K} = \mathbb{O}$)

if $\Omega = \Lambda_n$, the Lorentz cone in $\mathbb{R}^n$ $(n \geq 3)$, then $r = 2$, $d = n - 2$.

- For general clan V, the result is due to H. Nakashima (preprint, 2013).

$$\boxed{\boldsymbol{n} = \boldsymbol{m}\sigma^{-1}}$$

$$V = \begin{pmatrix} \mathbb{R} & V_{21} & \cdots & V_{r-1,1} & V_{r1} \\ V_{21} & \mathbb{R} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ V_{r-1,1} & & & \mathbb{R} & V_{r,r-1} \\ V_{r1} & \cdots & \cdots & V_{r,r-1} & \mathbb{R} \end{pmatrix}$$: the normal decomposition of V.

Let $m_k := 1 + \sum_{l>k} \dim V_{lk}$, and put $\boldsymbol{m} := (m_1, \ldots, m_r)$.

σ is the **multiplier matrix** of V

$\overset{\text{def}}{\iff}$ $r \times r$-matrix obtained by arranging the parameters of the 1-dim. rep. corresponding to $\Delta_1(x), \ldots, \Delta_r(x)$.

- If V is a simple EJA, then $\sigma = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}$.
- In general, σ is a unipotent matrix with non-negative interger entries.

Defining a clan from representations of a EJA

V is a EJA with unit element e_0, and E is a real vector space with $\langle \cdot \mid \cdot \rangle_E$.

Definition

A linear map $\varphi : V \to \mathrm{End}(E)$ is a **selfadjoint representation** of V

$$\overset{\text{def}}{\iff} \begin{cases} (1)\ \varphi(x) \in \mathrm{Sym}(E) \quad \text{for } \forall x \in V, \\ (2)\ \varphi(xy) = \frac{1}{2}\big(\varphi(x)\varphi(y) + \varphi(y)\varphi(x)\big), \quad \varphi(e_0) = I \ \text{ if } \varphi \neq 0. \end{cases}$$

- $V = \mathrm{Herm}(3, \mathbb{O}) \implies \varphi = 0$
- $V = \mathrm{Herm}(r, \mathbb{K})$ $(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$
  $\implies E = \mathrm{Mat}(r \times p, \mathbb{K})$, $\varphi(x)\xi = x\xi$ $(x \in V,\ \xi \in E)$
- V: Lorentzian $\implies V = \mathbb{R}e_0 \oplus W$, where (W, B) is a Euclidean VS.

  JA representation of $V \rightleftarrows$ Clifford algebra representation of $\mathrm{Cl}(W)$
  ($\mathrm{Cl}(W)$: Clifford algebra with $w^2 = B(w, w)$)

  In fact, $V \hookrightarrow \mathrm{Cl}(W)$

$c_1, \ldots, c_r$: Jordan frame of V. Then $V = \begin{pmatrix} \mathbb{R}c_1 & V_{21} & \cdots & V_{r-1,1} & V_{r1} \\ V_{21} & \mathbb{R}c_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ V_{r-1,1} & & & \mathbb{R}c_{r-1} & V_{r,r-1} \\ V_{r1} & \cdots & \cdots & V_{r,r-1} & \mathbb{R}c_r \end{pmatrix}$

(φ, E) is a selfadjoint representation of V with $\dim E > 0$.

$\rightsquigarrow \varphi(c_1), \ldots, \varphi(c_r)$ are complete system of orthogonal projections of equal rank.

- the **lower triangular part** $\underline{\varphi}(x)$ of $\varphi(x)$ is defined as

$$\underline{\varphi}(x) := \frac{1}{2}\sum_i \lambda_i \varphi(c_i) + \sum_{j<k} \varphi(c_k)\varphi(x_{kj})\varphi(c_j) \quad \Big(x = \sum_i \lambda_i c_i + \sum_{j<k} x_{kj}\Big).$$

Then, $\underline{\varphi}(x) + \underline{\varphi}(x)^* = \varphi(x)$.

Proposition

φ is also a clan representation of V:

$$\varphi(x \triangle y) = \underline{\varphi}(x)\varphi(y) + \varphi(y)\underline{\varphi}(x)^* \qquad (x, y \in V).$$

- **the symmetric bilinear map** $Q : E \times E \to V$ **associated to** φ:

$$\langle \varphi(x)\xi \,|\, \eta \rangle_E = \langle Q(\xi, \eta) \,|\, x \rangle \qquad (x \in V,\ \xi, \eta \in E).$$

- Define a product $\triangle$ in $V_E := E \oplus V$ by

$$(\xi + x) \triangle (\eta + y) := \underline{\varphi}(x)\eta + (Q(\xi, \eta) + x \triangle y) \qquad (x, y \in V,\ \ \xi, \eta \in E).$$

Theorem

$(V_E, \triangle)$ is a clan, and as an admissible linear form we take

$$s'(\xi + x) := \mathrm{Tr}\, L(x) \qquad (\xi \in E,\ x \in V).$$

- V_E does not have unit element.

  $\because)$ If $\eta_0 + y_0$ is a unit element, then taking $0 \neq \xi \in E$, we have a contradiction

$$\xi + 0 = (\xi + 0) \triangle (\eta_0 + y_0) = 0 + Q(\xi, \eta_0).$$

- The homogeneous convex domain corresponding to V_E is the following real Siegel domain defined by

$$D(\Omega, Q) = \left\{ \xi + x \ ;\ x - \tfrac{1}{2}Q(\xi, \xi) \in \Omega \right\}.$$

Adjoining the unit element e to V_E, we have $V_E^0 := \mathbb{R}e \oplus V_E$.

Put $u := e - e_0$ (recall e_0 is the unit element of V), we have $V_E^0 = \mathbb{R}u \oplus E \oplus V$. The product is written as

$$(\lambda u + \xi + x) \triangle (\mu u + \eta + y) = (\lambda\mu)u + (\mu\xi + \tfrac{1}{2}\lambda\eta + \underline{\varphi}(x)\eta) + (Q(\xi,\eta) + x \triangle y)$$

$$(\lambda, \mu \in \mathbb{R},\ \xi, \eta \in E \text{ and } x, y \in V).$$

- V_E^0 may be imaged as

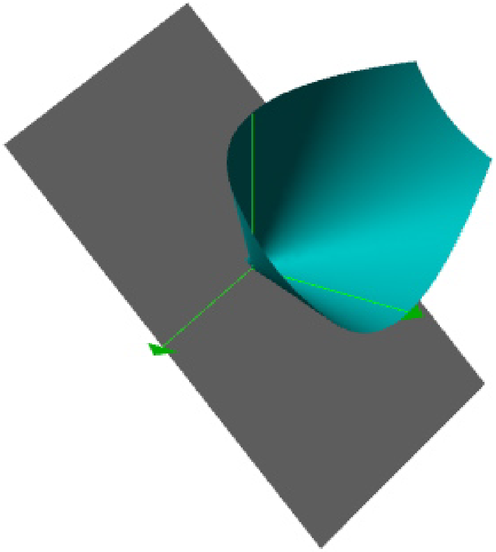
$$V_E^0 = \left(\begin{array}{ccc|c} \lambda & & & \\ & \ddots & & {}^tE \\ & & \lambda & \\ \hline & E & & V \end{array}\right).$$

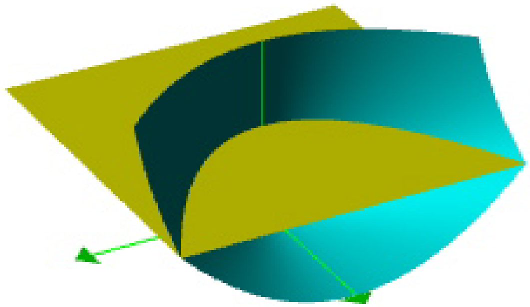
- Let Ω^0 be the HOCC corresponding to V_E^0.

- Description of $\boldsymbol{\Omega^0}$

$$\Omega^0 = \left\{ \lambda u + \xi + x \in V_E^0 \,;\, \lambda > 0,\ \lambda x - \tfrac{1}{2} Q(\xi, \xi) \in \Omega \right\}.$$

If you cut Ω^0 by the hyperplane $\lambda = 1$, then the Siegel domain $D(\Omega, Q)$ appears as the cross-section.





Basic relative invariants associated to Ω^0

Let V be a EJA, and $\varphi : V \to \mathrm{Sym}(E)$ a selfadjoint representation of V.

Definition

φ is regular $\overset{\text{def}}{\Longleftrightarrow}$ $\exists \xi_0 \in E$ s.t. $Q(\xi_0, \xi_0) = e_0$ (the unit element of V).

In what follows let $V = \mathrm{Herm}(r, \mathbb{K})$ $(r \geq 3;\ \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$. Then

$$E = \mathrm{Mat}(r \times p,\ \mathbb{K}),\quad \varphi(x)\xi = x\xi\ (x \in V,\ \xi \in E),\quad Q(\xi, \eta) = \tfrac{1}{2}(\xi\eta^* + \eta\xi^*)$$

Fact: φ is regular $\Longleftrightarrow$ $p \geq r$ (*i.e.*, $E = \square$ or $E = \square$).

$V_E^0 = \mathbb{R}u \oplus E \oplus V \ni \lambda u + \xi + x =: v$, $\Delta_k(x)$: k-th principal minor (upper-left)

Theorem

If φ is regular, then the basic relative invariants associated to Ω^0 are

$$\Delta_0^0(v) = \lambda, \quad \Delta_j^0(v) = \Delta_j(\lambda x - \tfrac{1}{2}\xi\xi^*)\ (j = 1, \ldots, r).$$

- If φ is not regular ($p < r$), then $\Delta_j(\xi\xi^*) = 0$ ($j = p+1, \dots r$)

$\Longrightarrow$ For such j, the polynomial $\Delta_j(\lambda x - \frac{1}{2}\xi\xi^*)$ is not irreducible

$\Longrightarrow$ at least should be $\lambda^{-(j-p)}\Delta_j(\lambda x - \frac{1}{2}\xi\xi^*) \longleftarrow$ I do not like this.

Theorem

If $p < r$, then the basic relative invariants associated to Ω^0 are

$$\begin{cases} \Delta_0^0(v) = \lambda, & \\ \Delta_j^0(v) = \Delta_j(\lambda x - \frac{1}{2}\xi\xi^*) & (j = 1, \dots, p), \\ \Delta_j^0(v) = \det^{(p+j)} \begin{pmatrix} \lambda I_p & \frac{1}{\sqrt{2}}\xi^* \\ \frac{1}{\sqrt{2}}\xi & x \end{pmatrix} & (j = p+1, \dots, r). \end{cases}$$

Here, $\det^{(p+j)} X$ is the upper-left $(p+j)$-th principal minor of X.
Moreover if $\mathbb{K} = \mathbb{H}$, it should be taken as the Jordan algebra determinant.

Dual cone $(\Omega^0)^*$, and the associated basic relative invariants

Introduce an inner product in $V_E^0 = \mathbb{R}u \oplus E \oplus V$ by

$$\langle\, \lambda u + \xi + x \,|\, \lambda' u + \xi' + x' \,\rangle^0 = \lambda\lambda' + \langle\, \xi \,|\, \xi' \,\rangle_E + \langle\, x \,|\, x' \,\rangle.$$

Let $(\Omega^0)^*$ be the dual cone of Ω^0 w.r.t this inner product:

$$(\Omega^0)^* := \big\{ v \in V_E^0 \,;\, \langle\, v \,|\, v' \,\rangle^0 > 0 \ \ \forall v' \in \overline{(\Omega^0)} \setminus \{0\} \big\}.$$

- $v \bigtriangledown v' = {}^t L^0(v) v'$ defines a clan structure in V_E^0
  ($L^0(v)$ is the left-multiplication operator in V_E^0).

Proposition

$$(\Omega^0)^* = \big\{ v = \lambda u + \xi + x \,;\, x \in \Omega,\ \lambda > \tfrac{1}{2} \langle\, \varphi(x)^{-1} \xi \,|\, \xi \,\rangle_E \big\}.$$

Remark. The proposition says that $(\Omega^0)^*$ coincides with what Rothaus ('66) called *the extension of Ω by the representation φ*.

$\Delta_1^*(x), \ldots, \Delta_r^*(x)$: JA principal minors associated to $c_r, \ldots, c_1$
(we have reversed the order of the original Jordan frame)

- In the case $\mathrm{Sym}(r, \mathbb{R})$, we just take the lower-right princial minors.

Theorem

The basic relative invarants $P_j(v)$ associated to $(\Omega^0)^*$ are

$$P_j(\lambda u + \xi + x) = \Delta_j^*(x) \qquad (j = 1, \ldots, r),$$
$$P_{r+1}(\lambda u + \xi + x) = \lambda \det x - \tfrac{1}{2}\langle\, \varphi({}^{\mathrm{co}}x)\xi \,|\, \xi \,\rangle.$$

- If $x \in V$ is invertible, then ${}^{\mathrm{co}}x := (\det x)x^{-1}$.
- In general ${}^{\mathrm{co}}x$ is a polynomial map of degree $r-1$ that is defined through the JA version of the Cayley–Hamilton theorem.
- $\deg P_j = j$ $(j = 1, \ldots, r,\ r+1)$.

The previous theorem systematically provides examples of HOCC such that the degrees of the asociated basic relative invariants are

$$1, 2, \ldots, r = \mathrm{rank}(\Omega)$$

even for non-symmetric cones.
This generalizes an example given in Ishi–N. [2008].

Problem

Let Ω be a HOCC of rank r

Then Ω is a symmeric cone
$\iff$ the degrees of the basic relative invariants associated to Ω, and the degrees of the basic relative invariants associted to Ω^* are both $1, 2, \ldots, r$.

T. Yamasaki wrote up a paper very recently in the affirmative.
(I'm currently checking his first draft ...)

- Another project: Starting with a clan rep. instead of JA rep.
  H. Nakashima, preprint (submitted 1 month ago).