Irreducible homogeneous non-symmetric cones linearly isomorphic to the dual cones

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§1. Introduction.

In this note we present irreducible homogeneous non-symmetric open convex cones of rank 3 that are linearly isomorphic to the dual cones. As noted in [4, p. 343], if the irreducibility condition is dropped, then we have an easy example $\Omega \oplus \Omega^*$ of non-symmetric cone that is linearly isomorphic to its dual cone. One motivation of trying to find a concrete example with the property in question stems from the proof of non-selfduality of the Vinberg cone given in [2, Exercise 10, p. 21]. There one actually proves a stronger fact that the Vinberg cone is never linearly isomorphic to its dual cone, and the non-selfduality follows as a corollary. Here one wonders if there is a concretely described non-symmetric cone which is linearly isomorphic to its dual cone. In fact selfduality of a cone requires a positive definite operator which gives a linear isomorphism between the cone and its dual as in the following lemma.

Lemma 1.1. *Let Ω be an open convex cone in a real inner product vector space $(V, \langle \cdot \,|\, \cdot \rangle)$. Denote by Ω^* the dual cone of Ω defined by*

$$\Omega^* := \big\{ y \in V \ ;\ \langle\, x \,|\, y \,\rangle > 0 \quad \text{for all } x \in \overline{\Omega} \setminus \{0\} \big\}.$$

Then, Ω is selfdual if and only if there is a positive definite operator T on V such that $\Omega^* = T(\Omega)$.

Proof. Suppose $\Omega^* = T(\Omega)$ for some positive definite T. Then we define a new inner product in V by $\langle\, x \,|\, y \,\rangle_T := \langle\, Tx \,|\, y \,\rangle$. Let

$$\Omega_T := \{ y \in V \ ;\ \langle\, x \,|\, y \,\rangle_T > 0 \quad (\forall x \in \overline{\Omega} \setminus \{0\}) \}.$$

It is clear that $\Omega_T = T^{-1}(\Omega^*) = \Omega$. Therefore Ω is selfdual. Conversely, suppose that Ω is selfdual with respect to some inner product. Represent this inner product as the form $\langle\, Tx \,|\, y \,\rangle$ with a positive definite operator T. Then we have $\Omega = \Omega_T = T^{-1}(\Omega^*)$. □

§2. Description of the cones.

Let $\boldsymbol{e}$ be the column m-vector with the first entry equal to 1 and the others 0:

$$\boldsymbol{e} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m.$$

Writing I_m for the m-th order identity matrix, we denote by V the vector space of matrices x of the $(m+2)$-th order such that

$$x := \left(\begin{array}{c|cc} x_{11}I_m & x_{21}\boldsymbol{e} & \boldsymbol{\xi} \\ \hline x_{21}{}^t\boldsymbol{e} & x_{22} & x_{32} \\ {}^t\boldsymbol{\xi} & x_{32} & x_{33} \end{array}\right), \tag{2.1}$$

where $x_{ij} \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^m$. We note that $V \subset \mathrm{Sym}(m+2, \mathbb{R})$. Let Ω be the cone of positive definite ones in V:

$$\Omega := \{x \in V \ ; \ x \text{ is positive definite}\}.$$

If $m = 1$, we have evidently $V = \mathrm{Sym}(3, \mathbb{R})$, so that we assume $m \geqq 2$ in what follows.

Let us show that Ω is homogeneous. To do so, we consider the following two subgroups A and N of $GL(m+2, \mathbb{R})$:

$$A := \left\{ a := \left(\begin{array}{c|cc} a_1 I_m & 0 & 0 \\ \hline 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{array}\right) \ ; \ a_j > 0 \quad (j = 1, 2, 3) \right\},$$

$$N := \left\{ n := \left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline n_{21}{}^t\boldsymbol{e} & 1 & 0 \\ {}^t\boldsymbol{n} & n_{32} & 1 \end{array}\right) \ ; \ n_{21}, n_{32} \in \mathbb{R}, \ \boldsymbol{n} \in \mathbb{R}^m \right\}.$$

It is clear that A normalizes N and we consider the semidirect product group $H := N \rtimes A$. We actually show that the action of H on Ω given by $H \times \Omega \ni (h, x) \mapsto \rho(h)x := hx{}^th$ is simply transitive. Thus given $x \in \Omega$, we just look for unique $a \in A$ and $n \in N$ so that $na{}^tn = x$. In view of $na{}^tn = \rho(na^{1/2})I_{m+2}$, we will obtain the desired simple transitivity. In order to describe the unique solution, we introduce the following polynomial functions Δ_j $(j = 1, 2, 3)$ on V: for x of the form (2.1)

$$\Delta_1(x) := x_{11},$$

$$\Delta_2(x) := x_{11}x_{22} - x_{21}^2,$$

$$\Delta_3(x) := (x_{11}x_{22} - x_{21}^2)(x_{11}x_{33} - \|\boldsymbol{\xi}\|^2) - (x_{11}x_{32} - x_{21}\xi_1)^2,$$

where ξ_1 is the first entry of the vector $\boldsymbol{\xi} \in \mathbb{R}^m$ appearing in the expression (2.1) of x. Now the solution to the equation $na\,{}^t n = x$ is uniquely given by

$$a_1 = \Delta_1(x), \qquad a_2 = \frac{\Delta_2(x)}{\Delta_1(x)}, \qquad a_3 = \frac{\Delta_3(x)}{\Delta_1(x)\Delta_2(x)},$$
$$\boldsymbol{n} = \frac{\boldsymbol{\xi}}{\Delta_1(x)}, \qquad n_{21} = \frac{x_{21}}{\Delta_1(x)}, \qquad n_{32} = \frac{x_{11}x_{32} - x_{21}\xi_1}{\Delta_2(x)}.$$

Therefore H acts on Ω simply transitively. Note that proceeding as in [3, Theorem 2.2], we see that $\Delta_j(x)$ $(j = 1, 2, 3)$ are irreducible polynomials[1].

Let us describe Ω in another way. Using an elementary determinant formula

$$\det\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = (\det A)\big(\det(D - CA^{-1}B)\big) \qquad (\text{if } \det A \neq 0),$$

we see that the principal minors $\delta_j(x)$ $(j = 1, \dots, m+2)$ of the matrix x in (2.1) computed by starting with $(1,1)$-entry are given by

$$\delta_j(x) = \begin{cases} \Delta_1(x)^j & (1 \leqq j \leqq m), \\ \Delta_1(x)^{m-1}\Delta_2(x) & (j = m+1), \\ \Delta_1(x)^{m-2}\Delta_3(x) & (j = m+2). \end{cases}$$

Hence we get the following description of Ω:

$$\Omega = \{x \in V\ ;\ \Delta_1(x) > 0,\ \Delta_2(x) > 0,\ \Delta_3(x) > 0\}.$$

§3. Dual cones.

Let us introduce an inner product in V by the formula

$$(3.1) \qquad \langle\, x \,|\, x' \,\rangle = x_{11}x'_{11} + x_{22}x'_{22} + x_{33}x'_{33} + 2(x_{21}x'_{21} + x_{32}x'_{32} + \boldsymbol{\xi} \cdot \boldsymbol{\xi}')$$

for $x, x' \in V$ as in (2.1), where $\boldsymbol{\xi} \cdot \boldsymbol{\xi}'$ stands for the standard inner product in $\mathbb{R}^m$. Let Ω^* denote the dual cone of Ω realized in V through the inner product (3.1):

$$\Omega^* := \big\{x' \in V\ ;\ \langle\, x \,|\, x' \,\rangle > 0 \quad \text{for any } x \in \overline{\Omega} \setminus \{0\}\big\}.$$

Let us define a linear operator m_0 on V by

$$m_0(x) = \left(\begin{array}{c|cc} x_{33}I_m & x_{32}\boldsymbol{e} & \boldsymbol{\xi} \\ \hline x_{32}\,{}^t\boldsymbol{e} & x_{22} & x_{21} \\ {}^t\boldsymbol{\xi} & x_{21} & x_{11} \end{array}\right) \qquad (\text{for } x \text{ as in } (2.1)).$$

It is obvious that m_0^2 is the identity operator, so that m_0 is an involution. Moreover m_0 is an isometry relative to the inner product (3.1). Put $\sigma(h) := m_0\rho(h)m_0$ for

[1]However, $\Delta_3(x)$ is reducible if $m = 1$, the case we have excluded.

$h \in H$. Clearly σ defines a representation of H on V. Observe that the group H is described as the set of all

$$h := \left(\begin{array}{c|cc} h_1 I_m & 0 & 0 \\ \hline h_{21}{}^t\boldsymbol{e} & h_2 & 0 \\ {}^t\boldsymbol{h} & h_{32} & h_3 \end{array}\right) \tag{3.2}$$

with $h_j > 0$ $(j = 1, 2, 3)$, $h_{21} \in \mathbb{R}$, $h_{32} \in \mathbb{R}$ and $\boldsymbol{h} \in \mathbb{R}^m$. Then we define an involution $h \mapsto \check{h}$ in H by

$$\check{h} := \left(\begin{array}{c|cc} h_3 I_m & 0 & 0 \\ \hline h_{32}{}^t\boldsymbol{e} & h_2 & 0 \\ {}^t\boldsymbol{h} & h_{21} & h_1 \end{array}\right) \qquad \text{for } h \text{ as in (3.2).}$$

By direct computation we see that this involution is an anti-automorphism.

Lemma 3.1. *$\langle \rho(h)x \,|\, y \rangle = \langle x \,|\, \sigma(\check{h})y \rangle$ for any $x, y \in V$ and $h \in H$.*

Proof. Since m_0 is an involutive isometry, what we have to prove is

$$\langle \rho(h)x \,|\, y \rangle = \langle m_0(x) \,|\, \rho(\check{h})m_0(y) \rangle. \tag{3.3}$$

We note that

$$\left(\begin{array}{c|cc} h_1 I_m & 0 & 0 \\ \hline h_{21}{}^t\boldsymbol{e} & h_2 & 0 \\ {}^t\boldsymbol{h} & h_{32} & h_3 \end{array}\right) = \left(\begin{array}{c|cc} h_1 I_m & 0 & 0 \\ \hline h_{21}{}^t\boldsymbol{e} & 1 & 0 \\ {}^t\boldsymbol{h} & 0 & 1 \end{array}\right)\left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & h_2 & 0 \\ 0 & h_{32} & 1 \end{array}\right)\left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & h_3 \end{array}\right)$$

and that the first and the second terms on the right hand side still decompose as follows:

$$\left(\begin{array}{c|cc} h_1 I_m & 0 & 0 \\ \hline h_{21}{}^t\boldsymbol{e} & 1 & 0 \\ {}^t\boldsymbol{h} & 0 & 1 \end{array}\right) = \left(\begin{array}{c|cc} h_1 I_m & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline h_{21}{}^t\boldsymbol{e} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & 1 & 0 \\ {}^t\boldsymbol{h} & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & h_2 & 0 \\ 0 & h_{32} & 1 \end{array}\right) = \left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & h_2 & 0 \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & h_{32} & 1 \end{array}\right).$$

Since $h \mapsto \check{h}$ is an anti-automorphism, it is enough to prove (3.3) for each of these pieces. We omit the details of simple computations. □

Theorem 3.2. *One has $\Omega^* = m_0(\Omega)$.*

Proof. Any $x \in \overline{\Omega}$ is a positive semidefinite matrix, so that $\langle x \,|\, I_{m+2} \rangle = x_{11} + x_{22} + x_{33} > 0$ if $x \neq 0$. Thus $I_{m+2} \in \Omega^*$. By Lemma 3.1, this implies $\Omega^* = \sigma(H)I_{m+2}$. Since

$$\sigma(H)I_{m+2} = (m_0\rho(H)m_0)(I_{m+2}) = m_0(\rho(H)I_{m+2}) = m_0(\Omega),$$

we get $\Omega^* = m_0(\Omega)$. □

Asano's criterion [1, Theorem 4] says that our cone Ω is irreducible, and Vinberg's criterion [5, Proposition 3, p. 73] together with the classification of irreducible symmetric cones tells us that Ω is not symmetric. Hence our cone is an irreducible non-symmetric cone that is linearly isomorphic to Ω^*.

References

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