階数2の対称錐上のある定積分

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2011年12月25日

V：階数 2 の Euclid 型 Jordan 代数.
E_1, E_2：Jordan 枠 $\rightsquigarrow$ $V = \mathbb{R}E_1 \oplus V_{21} \oplus \mathbb{R}E_2$：Peirce 分解

$$V = \begin{array}{|c|c|} \hline \mathbb{R}E_1 & V_{21} \\ \hline V_{21} & \mathbb{R}E_2 \\ \hline \end{array}$$

以下 $n := \dim V_{21}$

$\langle \cdot | \cdot \rangle$：$V$ の trace 内積，$\| \cdot \|$：trace 内積から得られるノルム，従って $\|E_1\| = \|E_2\| = 1$.
V の対称錐 Ω は

$$\Omega = \left\{ x = x_1E_1 + x_{21} + x_2E_2 \ ; \ x_1 > 0,\ x_1x_2 - \tfrac{1}{2}\|x_{21}\|^2 > 0 \right\}.$$

(注意：Jordan 代数としての determinant は $\det x = x_1x_2 - \frac{1}{2}\|x_{21}\|^2$ である.)

$W := \mathbb{R}(E_1 - E_2) \oplus V_{21}$ とし，$B\big[\alpha(E_1 - E_2) + v\big] = \alpha^2 + \frac{1}{2}\|v\|^2$ とおくと

$$\Omega = \left\{ \lambda(E_1 + E_2) + w \in \mathbb{R}(E_1 + E_2) + W \ ; \ \lambda^2 - B[w] > 0 \right\}$$

($n + 2$ 次元の Lorentz 錐)

$d\mu$：Ω 上の $G(\Omega)$ 不変測度，$x \in \Omega$.

$$I(x) := \int_\Omega e^{-\langle x \,|\, y \rangle} \, y_1^\alpha \, y_2^\beta \left(y_1 y_2 - \tfrac{1}{2}\|y_{21}\|^2\right)^\gamma d\mu(y) \qquad (y = y_1 E_1 + y_{21} + y_2 E_2).$$

定理：$\operatorname{Re}\gamma > \frac{1}{2}n$，$\operatorname{Re}(\alpha+\gamma) > 0$，$\operatorname{Re}(\beta+\gamma) > 0$ のとき，積分は絶対収束して

$$I(x) = \frac{(2\pi)^{\frac{1}{2}n}}{x_1^{\alpha+\gamma} x_2^{\beta+\gamma}} \Gamma(\gamma - \tfrac{1}{2}n) \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\gamma)} F\left(\alpha+\gamma, \beta+\gamma, \gamma; \frac{\|x_{21}\|^2}{2\,x_1 x_2}\right).$$

ただし，$F(a,b,c;x)$ は Gauss の超幾何函数.

$\beta = 0$ のとき，$F\left(\alpha+\gamma, \gamma, \gamma; \frac{\|x_{21}\|^2}{2\,x_1 x_2}\right) = \left(1 - \frac{\|x_{21}\|^2}{2\,x_1 x_2}\right)^{-(\alpha+\gamma)}$ であるので

$$\begin{aligned} I(x) &= \frac{(2\pi)^{\frac{1}{2}n}}{x_1^{\alpha+\gamma} x_2^{\gamma}} \Gamma(\gamma - \tfrac{1}{2}n)\Gamma(\alpha+\gamma) \left(1 - \frac{\|x_{21}\|^2}{2\,x_1 x_2}\right)^{-(\alpha+\gamma)} \\ &= (2\pi)^{\frac{1}{2}n} \Gamma(\gamma - \tfrac{1}{2}n)\Gamma(\alpha+\gamma)\, x_2^\alpha \Delta_2(x)^{-(\alpha+\gamma)}. \qquad (\Delta_2(x) := \det x). \end{aligned}$$

$I(x) = (2\pi)^{\frac{1}{2}n}\,\Gamma(\gamma - \frac{1}{2}n)\Gamma(\alpha+\gamma)\,x_2^{\alpha}\Delta_2(x)^{-(\alpha+\gamma)}$ において，$\alpha = s_1 - s_2$，$\gamma = s_2$ とおけば

$$I(x) = (2\pi)^{\frac{1}{2}n}\,\Gamma(s_1)\Gamma(s_2 - \tfrac{1}{2}n)x_2^{s_1-s_2}\Delta_2(x)^{-s_1}$$

ここで，$\mathbf{s} = (s_1, s_2)$ に対して $\mathbf{s}^* = (s_2, s_1)$ とおくと，$-\mathbf{s}^* = (-s_2, -s_1)$ であって

$$x_2^{s_2-s_1}\Delta_2(x)^{-s_1} = \Delta^*_{-\mathbf{s}^*}(x) = \Delta_{\mathbf{s}}(x^{-1}).$$

ゆえに既知の結果 $I(x) = (2\pi)^{\frac{1}{2}n}\,\Gamma(s_1)\Gamma(s_2 - \frac{1}{2}n)\Delta_{\mathbf{s}}(x^{-1})$ を得る.

$\alpha = 0$ のときも同様.

証明の方針：　Brute force

$d\mu(y) = \dfrac{dy_1 dy_2 dy_{21}}{(y_1y_2 - \frac{1}{2}\|y_{21}\|^2)^{1+\frac{1}{2}n}}$ であるから

$$I(x) = \int_0^\infty e^{-x_1y_1} y_1^{\alpha}\, dy_1 \int_0^\infty e^{-x_2y_2} y_2^{\beta}\, dy_2 \int_{y_1y_2>\frac{1}{2}\|y_{21}\|^2} e^{-\langle\, x_{21} \,|\, y_{21} \,\rangle} \big(y_1y_2 - \tfrac{1}{2}\|y_{21}\|^2\big)^{\gamma-1-\frac{1}{2}n}\, dy_{21}.$$

さらに $y_{21} = \sqrt{2\,y_1y_2}\, v$ とおくと，

$$I(x) = 2^{\frac{1}{2}n} \int_0^\infty e^{-x_1y_1} y_1^{\alpha+\gamma-1}\, dy_1 \int_0^\infty e^{-x_2y_2} y_2^{\beta+\gamma-1}\, dy_2 \int_{\|v\|<1} e^{-\sqrt{2y_1y_2}\langle\, x_{21} \,|\, v \,\rangle} (1 - \|v\|^2)^{\gamma-1-\frac{1}{2}n}\, dv.$$

ここで，$u \in V_{21}$ に対して

$$J(u) := \int_{\|v\|<1} e^{-\langle\, u \,|\, v \,\rangle} (1 - \|v\|^2)^{\gamma-1-\frac{1}{2}n}\, dv.$$

- $J(ku) = J(u)$ ($\forall k \in SO(V_{21})$) より，$J(u) = J(\|u\|\boldsymbol{e}_n)$ ($\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ は V_{21} の ONB).

補題：$u \in V_{21}$ のとき

$$J(u) = \pi^{\frac{1}{2}n}\,\Gamma(\gamma - \tfrac{1}{2}n) \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\gamma)} \left(\frac{\|u\|}{2}\right)^{2m}.$$

証明の方針：

$$\begin{cases} v_1 = r\sin\theta_{n-1}\cdots\sin\theta_3\sin\theta_2\sin\theta_1, \\ v_2 = r\sin\theta_{n-1}\cdots\sin\theta_3\sin\theta_2\cos\theta_1, \\ v_3 = r\sin\theta_{n-1}\cdots\sin\theta_3\cos\theta_2, \\ \vdots \\ v_k = r\sin\theta_{n-1}\cdots\sin\theta_k\cos\theta_{k-1}, \\ \vdots \\ v_{n-1} = r\sin\theta_{n-1}\cos\theta_{n-2}, \\ v_n = r\cos\theta_{n-1} \end{cases}$$

ただし，$r > 0$，$0 \leqq \theta_1 < 2\pi$，$0 \leqq \theta_j < \pi\ (j = 2, \ldots, n-1)$.

Jacobian は $\dfrac{D(v_1, \ldots, v_{n-1}, v_n)}{D(\theta_1, \ldots, \theta_{n-1}, r)} = r^{n-1}(\sin\theta_{n-1})^{n-2}\ldots\sin\theta_2$

$$
\begin{aligned}
J(u) = J(\|u\|\boldsymbol{e}_n) = \int_0^1 r^{n-1}(1-r^2)^{\gamma-1-\frac{1}{2}n}\,dr \int_0^{2\pi} d\theta_1 \\
\times \int_0^{\pi} \sin\theta_2\,d\theta_2 \cdots \int_0^{\pi} (\sin\theta_{n-2})^{n-3}\,d\theta_{n-2} \int_0^{\pi} (\sin\theta_{n-1})^{n-2} e^{-\|u\| r \cos\theta_{n-1}}\,d\theta_{n-1} \\
= 2\pi \left(\prod_{k=1}^{n-3} \int_0^{\pi} (\sin\theta)^k\,d\theta \right) \int_0^1 r^{n-1}(1-r^2)^{\gamma-1-\frac{1}{2}n}\,dr \int_0^{\pi} (\sin\varphi)^{n-2} e^{-\|u\| r\cos\varphi}\,d\varphi.
\end{aligned}
$$

ここで

$$
J_1 := \int_0^{\pi} (\sin\varphi)^{n-2} e^{-\|u\| r\cos\varphi}\,d\varphi = 2\int_0^{\pi/2} (\sin\varphi)^{n-2}\cosh(\|u\| r\cos\varphi)\,d\varphi
$$

とおくと，cosh のべき級数展開を代入して

$$
\begin{aligned}
J_1 &= 2\sum_{m=0}^{\infty} \frac{\|u\|^{2m} r^{2m}}{(2m)!} \int_0^{\pi/2} (\cos\varphi)^{2m}(\sin\varphi)^{n-2}\,d\varphi \\
&= \sum_{m=0}^{\infty} \frac{\|u\|^{2m} r^{2m}}{(2m)!} B\Big(m+\frac{1}{2},\, \frac{n-1}{2}\Big) \\
&= \sum_{m=0}^{\infty} \frac{\|u\|^{2m} r^{2m}}{(2m)!} \frac{\Gamma\big(m+\frac{1}{2}\big)\Gamma\big(\frac{n-1}{2}\big)}{\Gamma\big(m+\frac{n}{2}\big)}. \quad //
\end{aligned}
$$

さて補題において，$u=\sqrt{2\,y_1y_2}\,x_{21}$ とおくと

$$J\big(\sqrt{2\,y_1y_2}\,x_{21}\big)=\pi^{n/2}\Gamma(\gamma-\tfrac{1}{2})\sum_{m=0}^{\infty}\frac{y_1^m y_2^m}{m!\,\Gamma(m+\gamma)}\left(\frac{\|x_{21}\|^2}{2}\right)^m .$$

よって

$$\begin{aligned}
I(x)&=(2\pi)^{\frac{1}{2}n}\Gamma(\gamma-\tfrac{1}{2}n)\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\gamma)}\left(\frac{\|x_{21}\|^2}{2}\right)^m\\
&\qquad\times\int_0^\infty e^{-x_1y_1}y_1^{\alpha+m+\gamma-1}\,dy_1\int_0^\infty e^{-x_2y_2}y_2^{\beta+m+\gamma-1}\,dy_2\\
&=\frac{(2\pi)^{\frac{1}{2}n}}{x_1^{\alpha+\gamma}x_2^{\beta+\gamma}}\,\Gamma(\gamma-\tfrac{1}{2}n)\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+m+\gamma)\Gamma(\beta+m+\gamma)}{m!\,\Gamma(m+\gamma)}\;\left(\frac{\|x_{21}\|^2}{2\,x_1x_2}\right)^m\\
&=\frac{(2\pi)^{\frac{1}{2}n}}{x_1^{\alpha+\gamma}x_2^{\beta+\gamma}}\,\Gamma(\gamma-\tfrac{1}{2}n)\frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\,F\Big(\alpha+\gamma,\beta+\gamma,\gamma;\,\frac{\|x_{21}\|^2}{2\,x_1x_2}\Big).
\end{aligned}$$