On Wiener functionals of order 2 associated with soliton solutions of the KdV equation

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Abstract

The eigenvalues and eigenvectors of the Hilbert-Schmidt operators corresponding to the Wiener functionals of order 2, which gives a rise of soliton solutions of the KdV equation, are determined. Two explicit expressions of the stochastic oscillatory integral with such Wiener functional as phase function is given; one is of infinite product type and the other is of Lévy’s formula type. As an application, the asymptotic behavior of the stochastic oscillatory integral will be discussed.

Key words: Wiener functional of order 2, Hilbert-Schmidt operator, stochastic oscillatory integral, the Malliavin calculus, the KdV equation

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1 Introduction

Let $x > 0$, $\mathcal{W}^n$ be the $n$-dimensional Wiener space over $[0, x]$, i.e., the space of all continuous functions $w : [0, x] \to \mathbb{R}^n$ with $w(0) = 0$, and $P$ be the Wiener measure on $\mathcal{W}^n$. A Wiener functional $q : \mathcal{W}^n \to \mathbb{R}$ is said to be of order 2 if $\nabla^3 q = 0$, $\nabla$ being the Malliavin gradient. The study of stochastic oscillatory integrals with phase Wiener functional $q$ of order 2

$$
\int_{\mathcal{W}^n} e^{i\zeta q} \, dP,
$$

where $\zeta \in \mathbb{C}$ has sufficiently small real part, goes back to 1940’s, and there are a lot of exact evaluations of such Wiener integrals. For example, see

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Recently, N. Ikeda and the author showed ([7]) that the Wiener integral in (1) is applicable to the study of reflectionless potentials and \( n \)-soliton solutions of the Korteweg-de Vries (KdV) equation. In this paper, we investigate the Wiener functional \( q \) of order 2 of the form

\[
q = -\frac{a^2}{2} \int_0^x (c, \xi_p(y))^2 \, dy + \frac{1}{2} \langle \beta \xi_p(x), \xi_p(x) \rangle,
\]

where \( a > 0 \), \( c \in \mathbb{R}^n \), \( \xi_p(y) \) is the \( \mathbb{R}^n \)-valued Ornstein-Uhlenbeck process with parameter \( p \in \mathbb{R}^n \) starting from 0, \( \beta \) is a symmetric \( n \times n \) matrix, and \( \langle \ , \ \rangle \) denotes the inner product in \( \mathbb{R}^n \). For details, see Section 2. The Wiener functionals applied in [7] to the stochastic representation of reflectionless potentials and soliton solutions of the KdV equations is exactly of this form. Also see Section 5. We shall determine the eigenvalues \( \{a_n\} \) of the Hilbert-Schmidt operator \( A = \nabla^2 q \) and the associated normalized eigenvectors \( \{h_n\} \). We shall then obtain the infinite product expression

\[
\int_{\mathcal{W}_n} e^{\zeta q} \, dP = \left( \prod_{j=1}^{\infty} (1 - \zeta a_j) \right)^{-1/2}
\]

for \( \zeta \in \mathbb{C} \) with sufficiently small real part. See Section 2. Such an infinite product expression is fundamental and plays a key role in the recent study of stochastic oscillatory integrals. For example, see [6,10,12].

The infinite product expression (3) suggests another way to obtain eigenvalues of \( A \). Namely if we have another expression of \( (\int_{\mathcal{W}_n} e^{\zeta q} \, dP)^{-2} \), then eigenvalues \( \zeta \) which does not vanish are obtained as reciprocal of zeros of the function \( \zeta \mapsto (\int_{\mathcal{W}_n} e^{\zeta q} \, dP)^{-2} \). Following the idea of [5], we shall decompose \( A \) into a sum of a Volterra operator and an operator with finite dimensional range, and establish another exact expression of the stochastic oscillatory integral. As applications, we shall revisit the characterization of eigenvalues of \( A \) and observe the asymptotic behavior of the stochastic oscillatory integral with phase Wiener function \( q \) of order 2 and amplitude function \( \psi \).

In Section 2, we shall specify quadratic Wiener functionals which we are interested in and compute the associated Hilbert-Schmidt operators. Their eigenvalues and eigenvectors will be obtained in Section 3. In Section 4, following the idea developed in [5], we shall establish an exact expression of \( \int_{\mathcal{W}_n} e^{\zeta q} \, dP \) by decomposing the Hilbert-Schmidt operator into a sum of a Volterra operator and an operator with finite dimensional range. In the same section, the expression will be applied to the study of the asymptotic behavior of stochastic oscillatory integrals. Section 5 will be devoted to applications of our result to the Wiener integrals associated with reflectionless potentials and soliton solutions of the KdV equation.
2 Preliminaries

We first fix the notation which we shall use throughout the paper. The \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) is thought of to be the vector space of column vectors \(t(a_1, \ldots, a_n)\), where the superscript \(t\) indicates the transpose. Let \(p = (p_1, \ldots, p_n) \in \mathbb{R}^n\) and \(c = (c_1, \ldots, c_n) \in \mathbb{R}^n \setminus \{0\}\). Set \(D = D_p = \text{diag}[p_1, \ldots, p_n]\). Define the \(n\)-dimensional Ornstein-Uhlenbeck process \(\xi_p(y) = (\xi_1^p(y), \ldots, \xi_n^p(y))\) associated with \(D\) by the stochastic differential equation

\[
d \xi_p(y) = dw(y) + D \xi_p(y) dy, \quad \xi_p(0) = 0,
\]
where \(dw(y)\) stands for the Itô integral with respect to the Wiener process \(\{w(y)\}_{y \in [0, x]}\) on \(W^n\). Let \(\mathbb{R}^{n \times n}\) be the space of all real \(n \times n\) matrices, and for \(M \in \mathbb{R}^{n \times n}\) we denote by \(e^M\) the exponential of \(M\); \(e^M = \sum_{k=0}^\infty (1/k!)M^k\). Note that

\[
\xi_p(y) = e^{yD} \int_0^y e^{-zD} dw(z), \quad y \in [0, x].
\]

(4)

Denoting the matrix \((c_i c_j)_{1 \leq i, j \leq n}\) by \(c \otimes c\), for \(\zeta \in \mathbb{C}\), we define the complex \(n \times n\) matrix \(B(\zeta)\) by

\[
B(\zeta) = D^2 + \zeta c \otimes c,
\]

(5)

which plays a key role in the later sections. Let \(a > 0\) and \(\beta\) be a symmetric \(n \times n\) matrix. Throughout this paper, we consider the Wiener functional

\[
q = -\frac{a^2}{2} \int_0^x (c, \xi_p(y))^2 dy + \frac{1}{2} \langle \beta \xi_p(x), \xi_p(x) \rangle.
\]

(6)

Such a Wiener functional appeared in [7] to investigate reflectionless potentials and soliton solutions of the KdV equation. For details, see [7] and Section 5.

We next investigate the Hilbert-Schmidt operator associated with \(q\). We denote by \(H\) the Cameron-Martin subspace of \(W^n\); \(H\) is the real Hilbert space of all absolutely continuous \(h \in W^n\) with \(L^2\)-derivative \(h'\). The inner product in \(H\) is given by

\[
\langle h, k \rangle_H = \int_0^x \langle h'(y), k'(y) \rangle dy, \quad h, k \in H.
\]

Lemma 1 Let \(\nu\) be a finite measure on \([(0, x], B([0, x]))\), \(S \in \mathbb{R}^{n \times n}\) be symmetric and non-negative definite, and \(f \in C([0, x]^2, \mathbb{R}^{n \times n})\), where \(B([0, x])\) is
the Borel σ-field of \([0, x]\). Define

\[
Q = \int_0^x \left\langle S \int_0^x f(y, z) \, dw(z), \int_0^x f(y, z) \, dw(z) \right\rangle \nu(\,dy).
\]

Then, i) \(Q\) is infinitely differentiable in the sense of the Malliavin calculus, ii) the operator \(L = \nabla^2 Q : H \to H\) is deterministic and of trace class, and iii) \(Q - (1/2)\text{tr} L \in \mathcal{C}_2\), where \(\mathcal{C}_k\) is the space of Wiener chaos of order \(k\). Moreover, if we denote by \(\{b_j\}_{j=1}^{\infty}\) the eigenvalues of \(L\) counted repeatedly according to the multiplicity, then (3) holds with \(q = Q\) and \(a_j = b_j, j = 1, 2, \ldots,\) for \(\zeta \in \mathbb{C}\) with sufficiently small real part.

**PROOF.** It is easily seen that \(Q\) is infinitely differentiable. Observe that

\[
\langle Lh, k \rangle_H = 2 \int_0^x \left\langle S \int_0^x f(y, z) h'(z) \, dz, \int_0^x f(y, z) k'(z) \, dz \right\rangle \nu(\,dy).
\]

In particular, \(L\) is deterministic. Moreover, \(L\) is non-negative definite and

\[
0 \leq \text{tr} L = 2 \int_0^x \int_0^x \text{tr}\{f(y, z) S f(y, z)\} \, dz \, \nu(\,dy) < \infty.
\]

Thus \(L\) is of trace class.

Since

\[
\left\langle S \int_0^x f(y, z) \, dw(z), \int_0^x f(y, z) \, dw(z) \right\rangle - \int_0^x \text{tr}\{f(y, z) S f(y, z)\} \, dz \in \mathcal{C}_2,
\]

we have that \(Q - (1/2)\text{tr} L \in \mathcal{C}_2\).

To see the last assertion, note that \(\nabla^2 \{(\nabla^*)^2 L\} = 2L\), where \(\nabla^*\) is the dual operator of \(\nabla\) and \(L\) is thought of as a constant Wiener functional. Hence

\[
Q - \frac{1}{2}\text{tr} L = \frac{1}{2}(\nabla^*)^2 L.
\]

Then it holds that

\[
\int_{\mathcal{W}^n} e^{\zeta Q} \, dP = \int_0^x \exp\left[\frac{\zeta}{2}(\nabla^*)^2 L + \frac{\zeta}{2} \text{tr} L\right] \, dP = (\det_2(I - \zeta L))^{-1/2} e^{(\zeta/2)\text{tr} L}
\]

\[
= (\det(I - \zeta L))^{-1/2} = \left(\prod_{j=1}^{\infty} (1 - \zeta b_j)\right)^{-1/2}
\]

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for \( \zeta \in \mathbb{C} \) with sufficiently small real part, where \( \det_2 \) denotes the Carleman-Fredholm determinant (cf. [6,10]). Thus (3) has been verified.

Due to (4) and Lemma 1, \( q \) in (6) satisfies that i) \( q \in \mathcal{C}_0 \oplus \mathcal{C}_2 \), ii) the symmetric Hilbert-Schmidt operator \( A : H \to H \) defined by

\[
A = \nabla^2 q \tag{7}
\]

is of trace class, and iii) for \( \zeta \in \mathbb{C} \) with sufficiently small real part, it holds that

\[
\int_{\mathcal{W}_n} e^{\zeta q} \, dP = \left\{ \det(I - \zeta A) \right\}^{-1/2} = \left( \prod_{j=1}^{\infty} (1 - \zeta a_j) \right)^{-1/2}, \tag{8}
\]

where \( a_1, a_2, \ldots \) are eigenvalues of \( A \) counted repeatedly according to the multiplicity.

We shall close this section by giving an explicit expression of \( A \);

**Lemma 2** Define \( A : H \to H \) by (7). For each \( h \in H \), it holds that

\[
(\mathbf{Ah})'(y) = -e^{-yD} \int_{y}^{x} e^{zD}(a^2 \mathbf{c} \otimes \mathbf{c})\xi(z; h) \, dz + e^{(x-y)D} \beta \chi(x; h), \quad y \in [0, x],
\]

where

\[
\xi(y; h) = e^{yD} \int_{0}^{y} e^{-zD} h'(z) \, dz \in \mathbb{R}^n, \quad y \in [0, x].
\]

**PROOF.** Due to (4), it holds that

\[
\left\langle \nabla (\xi^i(y)), h \right\rangle_H = \xi^i(y; h), \quad i = 1, \ldots, n, \ h \in H, \ y \in [0, x],
\]

where \( \xi(y; h) = ^t(\xi^1(y; h), \ldots, \xi^n(y; h)) \). Hence

\[
\left\langle \mathbf{Ah}, k \right\rangle_H = \int_{0}^{x} \left\langle (-a^2 \mathbf{c} \otimes \mathbf{c})\xi(y; h), \xi(y; k) \right\rangle \, dy + \left\langle \beta \chi(x; h), \xi(x; k) \right\rangle
\]

\[
= -\int_{0}^{x} \left\langle e^{yD}(a^2 \mathbf{c} \otimes \mathbf{c})\xi(y; h), \int_{0}^{y} e^{-zD} k'(z) \, dz \right\rangle \, dy
\]

\[
+ \left\langle e^{xD} \beta \chi(x; h), \int_{0}^{x} e^{-yD} k'(y) \, dy \right\rangle.
\]
Applying the Fubini theorem, we obtain the desired identity.

3 Eigenvalues and eigenvectors

Let $A$ be as in Section 2. In this section, we compute its eigenvalues and eigenvectors.

Let

$$\mathcal{P}(c) = \{g \in L^2([0, x]; \mathbb{R}^n) : \langle g, c \rangle = 0 \text{ a.e.} \}.$$ 

If $d_1, \ldots, d_{n-1}$ are linearly independent and perpendicular to $c$, then

$$\mathcal{P}(c) = \left\{ \sum_{j=1}^{n-1} f_j d_j : f_j \in L^2([0, x]; \mathbb{R}), \ j = 1, \ldots, n-1 \right\}.$$ 

For $M \in \mathbb{R}^{n \times n}$, we define

$$s(y; M) = \sum_{j=0}^{\infty} \frac{y^{2j+1}}{(2j+1)!} M^j, \quad c(y; M) = \sum_{j=0}^{\infty} \frac{y^{2j}}{(2j)!} M^j.$$ (9)

Our goal of this section is

Theorem 3 It holds that

$$\ker(A) = \left\{ h \in H : h' = g - D \int_0^x g(z) \, dz \text{ for some } g \in \mathcal{P}(c) \right\}.$$ (10)

In particular, $\ker(A) = \{0\}$ if $n = 1$. Moreover, $\lambda \neq 0$ is an eigenvalue of $A$ if and only if

$$\det [c(x; B(a^2/\lambda) - \{(1/\lambda)\beta + D\} s(x; B(a^2/\lambda))] = 0,$$ (11)

where $B(\zeta)$ is the matrix defined in (5). In this case, the corresponding eigenvector $h \in H$ is given by

$$h'(y) = \{c(y; B(a^2/\lambda)) - Ds(y; B(a^2/\lambda))\} u$$

for some $u \in \ker [c(x; B(a^2/\lambda)) - \{(1/\lambda)\beta + D\} s(x; B(a^2/\lambda))] \setminus \{0\}$.

PROOF. Note that

$$\xi'(\cdot; h) = h' + D\xi(\cdot; h),$$ (12)
where $\xi'(y; h) = (d/dy)(\xi(y; h))$. Let $\lambda \in \mathbb{R}$ and $h \in H \setminus \{0\}$. By Lemma 2 and (12), we see that the identity $Ah = \lambda h$ is equivalent to that

$$
- \int_y^x e^{zD} (a^2 c \otimes c) \xi(z; h) \, dz + e^{zD} \beta \xi(x; h) = \lambda e^{yD} \{\xi'(y; h) - D\xi(y; h)\}, \quad y \in [0, x].
$$

(13)

We first show the characterization (10) of $\ker(A)$. To do this, suppose that $\lambda = 0$. Then (13) holds if and only if

$$
\beta \xi(x; h) = 0 \quad \text{and} \quad \langle c, \xi(y; h) \rangle = 0, \quad y \in [0, x].
$$

(14)

Let $K$ be the set described in the right hand side of the identity (10). If $h \in \ker(A)$, then by (14), $\xi'(y; h) \in \mathcal{P}(c)$ and $\xi(x; h) \in \ker(\beta)$. Due to (12),

$$
h'(y) = \xi'(y; h) - D \int_0^y \xi'(z; h) \, dz.
$$

Thus $h \in K$ and hence $\ker(A) \subset K$. Conversely let $h \in K$. For $g \in \mathcal{P}(c)$ such that $h' = g - D \int_0^y g(z) \, dz$ and $\int_0^x g(z) \, dz \in \ker(\beta)$, we have that

$$
\xi(y; h) = \int_0^y g(z) \, dz.
$$

This implies that (14) holds, and hence $h \in \ker(A)$. Thus $K \subset \ker(A)$. The identity (10) has been verified.

If $n = 1$, then $c = (c_1)$, $c_1 \neq 0$, and hence $\mathcal{P}(c) = \{0\}$. Thus the second assertion follows.

To see the last assertion, let $\lambda \neq 0$. Notice that (13) is equivalent to

$$
\xi''(y; h) - B(a^2/\lambda) \xi(y; h) = 0, \quad y \in [0, x],
$$

(15)

with the terminal condition that

$$
\beta \xi(x; h) = \lambda \{\xi'(x; h) - D\xi(x; h)\}.
$$

(16)

Since $\xi(0; h) = 0$, (15) is equivalent to that

$$
\xi(y; h) = s(y; B(a^2/\lambda)) u, \quad y \in [0, x], \quad \text{for some } u \in \mathbb{R}^n.
$$
Plugging this into (16), we obtain that
\[ c(x; B(a^2/\lambda))u = \{(1/\lambda)\beta + D\}s(x; B(a^2/\lambda))u. \] (17)

It is easily seen that \( s(\cdot; B(a^2/\lambda))u \equiv 0 \) if and only if \( u = 0 \), and that \( \xi(\cdot; h) \equiv 0 \) if and only if \( h = 0 \). Thus \( \lambda \neq 0 \) is an eigenvalue of \( A \) if and only if there exists \( u \in \mathbb{R}^n \setminus \{0\} \) fulfilling (17), i.e. (11) holds, and then a corresponding eigenvector \( h \) satisfies that \( \xi(y; h) = s(y; B(a^2/\lambda))u \). By (12), this identity implies that \( h' = \{c(\cdot; B(a^2/\lambda)) - Ds(\cdot; B(a^2/\lambda))\}u \).

4 Stochastic oscillatory integrals

We continue to consider the same \( q \) and \( A \) as defined in Section 2. By virtue of (8), the non-zero eigenvalues of \( A \) can be computed by zeros of the entire function obtained as the holomorphic extension of the mapping \( \zeta \mapsto \left( f_{W_n} e^{\zeta q} \, dP \right)^{-2} \). Thus an alternative explicit expression of \( f_{W_n} e^{\zeta q} \, dP \) has its own interest. Moreover, the exact expression is applicable to the study of the asymptotic behavior of stochastic oscillatory integral of the form \( f_{W_n} e^{\lambda \psi} \, dP \) as \( \lambda \in \mathbb{R} \) tends to infinity, where \( i \) is the imaginary unit.

4.1 Exact expression

In this subsection, we show that

**Theorem 4** It holds that

\[ \int_{W_n} e^{\zeta q} \, dP = \left\{ e^{x^\text{tr}D} \det[c(x; B(a^2 \zeta)) - (\zeta \beta + D)s(x; B(a^2 \zeta))] \right\}^{-1/2} \] (18)

for \( \zeta \in \mathbb{C} \) with sufficiently small real part, where \( s(x; B(a^2 \zeta)) \) and \( c(x; B(a^2 \zeta)) \) are defined as in (9). In particular, \( \lambda \neq 0 \) is an eigenvalue of \( A \) if and only if (11) holds.

Before proceeding to the proof, we recall the method achieved by Ikeda-Kusuoka-Manabe ([5]) to compute stochastic oscillatory integrals. Let \( U : H \rightarrow H \) be a symmetric Hilbert-Schmidt operator which admits a decomposition that

\[ U = U_V + U_F, \]
where \( U_V : H \to H \) is a Volterra operator and \( U_F : H \to H \) is an operator with finite dimensional range. Then it holds that

\[
\int_{W^n} e^{\langle \zeta/2 \rangle (\nabla^*)^2 U} \, dP = \{ \det(I - \zeta U_F(I - \zeta U_V)^{-1}) \}^{-1/2} e^{-\langle \zeta/2 \rangle tr U_F}.
\]

If, in addition, \( U \) is of trace class, then

\[
\int_{W^n} e^{\langle \zeta/2 \rangle (\nabla^*)^2 U + tr U} \, dP = \{ \det(I - \zeta U_F(I - \zeta U_V)^{-1}) \}^{-1/2}.
\]

(19)

It should be mentioned that the above \( \det(\cdot \cdot \cdot) \) is a determinant of finite dimensional matrix, because the range of \( U_F \) is of finite dimension.

As for the decomposition of \( A \), we have that

**Lemma 5** Define \( A_V, A_F : H \to H \) by

\[
(A_V h)'(y) = e^{-yD} \int_0^y e^{zD}(a^2 c \otimes c)\xi(z;h) \, dz
\]

\[
(A_F h)'(y) = -e^{-yD} \int_0^y e^{zD}(a^2 c \otimes c)\xi(z;h) \, dz + e^{(x-y)D} \beta \xi(x;h).
\]

Let \( e_1, \ldots, e_n \) be the standard orthonormal basis of \( \mathbb{R}^n \), and define \( \varphi_i \in H \), \( i = 1, \ldots, n \), by \( \varphi_i'(y) = e^{-yD} e_i \). Then, i) \( A = A_V + A_F \), ii) \( A_V \) is a Volterra operator, and iii) the range of \( A_F \) is contained in the subspace of \( H \) spanned by \( \varphi_1, \ldots, \varphi_n \).

When \( \beta = 0 \), such a decomposition as above has been obtained by Hara-Ikeda [4], while theirs are of slightly more complicated form.

**PROOF.** The assertion i) is an immediate consequence of Lemma 2. The assertions ii) and iii) follow from the very definition of \( A_V \) and \( A_F \).

**Proof of Theorem 4.** Due to the analyticity, it suffices to show the identity (18) for sufficiently small \( \lambda \in \mathbb{R} \). By Lemma 5, it holds that

\[
\det[I - \lambda A_F(I - \lambda A_V)^{-1}]
= \det[I_n - \left( \| \varphi_i \|_H^{-1} \| \varphi_j \|_H^{-1} \langle \varphi_i, \lambda A_F(I - \lambda A_V)^{-1} \varphi_j \rangle_H \right)]_{1 \leq i, j \leq n},
\]
where $I_n$ denotes the $n \times n$ unit matrix. Since $A$ is of trace class and $q = (1/2) \{(\nabla^*)^2 A + \text{tr} A\}$, by (19) with $U = A, U_V = A_V,$ and $U_F = A_F$, we have that

$$\int_{W^n} e^{\lambda q} \, dP$$

$$= \left\{ \det \left[ I_n - \left( \left\| \varphi_i \right\|^{-1}_H \left\| \varphi_j \right\|^{-1}_H \langle \varphi_i, \lambda A_F (I - \lambda A_V)^{-1} \varphi_j \rangle_H \right)_{1 \leq i,j \leq n} \right] \right\}^{-1/2}. \tag{20}$$

We shall compute the right hand side of (20). Let $k_j = (I - \lambda A_V)^{-1} \varphi_j \in H$, $j = 1, \ldots, n$. By the definition of $A_V$ and (12), the identity $(I - \lambda A_V)k_j = \varphi_j$ reads as

$$\xi'(y; k_j) - D\xi(y; k_j) - \lambda e^{-yD} \int_0^y e^{zD} (a^2 c \otimes c) \xi(z; k_j) \, dz = \varphi'_j(y) \tag{21}$$

for any $y \in [0, x]$. Since $\varphi''_j + D\varphi'_j = 0$, multiplying by $e^{yD}$ the both sides of (21) and then differentiating, we see that (21) is equivalent to that

$$\xi''(\cdot; k_j) - B(a^2 \lambda)\xi(\cdot; k_j) = 0, \quad \xi(0; k_j) = e_j.$$

Noticing that $\xi(0; k_j) = 0$, we have that

$$\xi(y; k_j) = s(y; B(a^2 \lambda))e_j.$$

Combining this with (21), we obtain that

$$\int_0^x e^{zD} (a^2 c \otimes c) \xi(z; k_j) \, dz = \lambda^{-1} e^{xD} \{c(x; B(a^2 \lambda)) - Ds(x; B(a^2 \lambda)) - e^{-xD}\} e_j,$$

which yields that

$$(\lambda A_F (I - \lambda A_V)^{-1} \varphi_j)'(y)$$

$$= - e^{(x-y)D} \{c(x; B(a^2 \lambda)) - (\lambda \beta + D)s(x; B(a^2 \lambda)) - e^{-xD}\} e_j.$$

For $M = (M_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$, observe that

$$\int_0^x e^{(x-y)D} M e_j dy = \sum_{k=1}^n e^{xp_k} M_{kj} \varphi_k,$$

and hence that

$$\left\langle \varphi_i, \int_0^x e^{(x-y)D} M e_j \, dy \right\rangle_H = \left\| \varphi_i \right\|^2_H e^{xp_i} M_{ij}.$$
Thus, if we denote by $Y = \text{diag}[\|\varphi_1\|_H, \ldots, \|\varphi_n\|_H]$, then

$$
\big(\|\varphi_i\|_H^{-1}\|\varphi_j\|_H^{-1}\langle \varphi_i, \lambda A_F(I - \lambda A_V)^{-1}\varphi_j \rangle_H\big)_{1 \leq i, j \leq n} = -Y e^{\sigma D}\{c(x; B(a^2\lambda)) - (\lambda \beta + D)s(x; B(a^2\lambda))\}Y^{-1} + I_n.
$$

In conjunction with (20), this implies the identity (18).

### 4.2 Asymptotic behaviour

We first give a review on analytic functions on $\mathcal{W}^n$. For a separable Hilbert space $E$, we denote by $D^{\infty, \infty}(E)$ the space of infinitely differentiable $E$-valued Wiener functionals in the sense of the Malliavin calculus, whose derivatives of all orders are $p$th integrable with respect to $P$ for any $p \in (1, \infty)$. For details, see [9,14]. We say $\psi \in D^{\infty, \infty}(\mathbb{R})$ is analytic ($\psi \in C^\omega$ in notation) if there is $p \in (1, \infty)$ such that

$$
\sum_{j=0}^{\infty} \frac{r^j}{j!} \|\nabla^j \psi\|_{L^p(H^{\otimes j})} < \infty \quad \text{for any } r > 0,
$$

where $H^{\otimes j}$ is the Hilbert space of Hilbert-Schmidt $j$-linear mappings on $H$ and $L^p(H^{\otimes j})$ is the space of $H^{\otimes j}$-valued $p$th integrable functions with respect to $P$. Choosing appropriate version of Malliavin gradients $\nabla^j \psi$’s, we have the expansion

$$
\psi(w + h) = \sum_{j=0}^{\infty} \frac{1}{j!} \langle \nabla^j \psi(w), h^{\otimes j} \rangle_{H^{\otimes j}} \quad \text{for every } w \in \mathcal{W}^n, h \in H, \tag{22}
$$

where $\langle \cdot, \cdot \rangle_{H^{\otimes j}}$ stands for the inner product on $H^{\otimes j}$. See [10,12]. In what follows, we always consider such nice versions of $\psi$ and $\nabla^j \psi$’s as above, and these versions will be used to evaluate $\psi$ and so on.

Write $\mathcal{W}^n \oplus iH$, $H \oplus iH$, and $\mathcal{W}^n \oplus i\mathcal{W}^n$ for $\mathcal{W}^n \times H$, $H \times H$, and $\mathcal{W}^n \times \mathcal{W}^n$, respectively. Then $\mathcal{W}^n \oplus i\mathcal{W}^n$ is a real Banach space with norm

$$
\|w + iw'\|_{\mathcal{W}^n \oplus i\mathcal{W}^n} = \|w\|_{\mathcal{W}^n} + \|w'\|_{\mathcal{W}^n}, \quad w, w' \in \mathcal{W}^n,
$$

where $\|\cdot\|_{\mathcal{W}^n}$ stands for the Banach norm on $\mathcal{W}^n$. For their elements $(w, h) \in \mathcal{W}^n \times H$, $(h, h') \in H \times H$, and $(w, w') \in \mathcal{W}^n \times \mathcal{W}^n$, we write $w + ih$, $h + ih'$, and $w + iw'$, respectively. Due to (22), $\psi$ extends to the function $\tilde{\psi}$ on $\mathcal{W}^n \oplus iH$, which we call the holomorphic prolongation of $\psi$, so that

$$
\tilde{\psi}(w + ih) = \sum_{j=0}^{\infty} \frac{j!}{j!} \langle \nabla^j \psi(w), h^{\otimes j} \rangle_{H^{\otimes j}}, \quad w \in \mathcal{W}^n, h \in H.
$$
Let $q$ be the Wiener functional of order 2 defined by (6) and $A = \nabla^2 q$. Denote by $\mathcal{W}_0^n$ the closure of $\ker(A)$ in $\mathcal{W}^n$, and by $\pi : \mathcal{W}^n \to \mathcal{W}_0^n$ the stochastic extension of the orthogonal projection $\pi : H \to \ker(A)$; if $\{k_j\}_{j=1}^\infty$ is an orthogonal basis of $\ker(A)$ in $H$, then $\pi = \sum_{j=1}^\infty (\nabla^* k_j) k_j$, which converges a.s. in $\mathcal{W}^n$. We are now ready to state our result on the asymptotic behavior of stochastic oscillatory integral.

**Theorem 6** Suppose that $\psi \in C^\omega$ satisfies that (i) $\sum_{j=0}^{\infty} (\tau^j/j)! \|\nabla^j \psi\|_{H^\infty}^2 \in L^1(P)$ for any $r > 0$, (ii) there is a measurable function $\psi_0 : \mathcal{W}_0^n \to \mathbb{R}$ such that $\tilde{\psi}(w + w') \to \psi_0(w)$ as $\|w' + \lambda H\|_{\mathcal{W}^n \oplus i\mathcal{W}^n} \to 0$ for any $w \in \mathcal{W}_0^n$, and (iii) there are $\lambda_0 \geq 0$ and $\delta > 0$ such that

$$\sup_{\lambda \geq \lambda_0} \int_{\mathcal{W}^n} |\tilde{\psi}((I - i\lambda A)^{-1/2} w)|^{1+\delta} P(dw) < \infty,$$

where $(I - i\lambda A)^{-1/2} w = w + \sum_{j=1}^\infty \{(1 - i\lambda a_j)^{-1/2} - 1\} (\nabla^* h_j)(w) h_j$, $\{a_j\}_{j=1}^\infty$ and $\{h_j\}_{j=1}^\infty$ being the eigenvalues of $A$ and the corresponding normalized eigenvectors. Then it holds that

$$\left\{ \det [c(x; B(i\lambda a^2)) - (i\lambda \beta + D)s(x; B(i\lambda a^2))] \right\}^{1/2} \int_{H^n} e^{i\lambda q} \psi \, dP \to e^{-(x/2)\text{tr} D} \int_{\mathcal{W}^n} \psi_0 \circ \pi \, dP \quad \text{as } \lambda \to \infty.$$

It was seen in [13] that the assumption (iii) is satisfied if there are $C \geq 0$ and $0 < \delta < 1/(4x)$ such that

$$|\tilde{\psi}(w + \lambda H)| \leq C(1 + \exp[\delta \|w + \lambda H\|_{\mathcal{W}^n \oplus i\mathcal{W}^n}^2]), \quad w \in \mathcal{W}^n, h \in H.$$

**PROOF.** Applying [10, Theorem 7.8], we can conclude that

$$\int_{\mathcal{W}^n} e^{i\lambda q} \psi \, dP = \int_{\mathcal{W}^n} e^{i\lambda q} \, dP \int_{\mathcal{W}^n} \tilde{\psi}((I - \lambda A)^{-1/2} w) P(dw). \quad (23)$$

Extend $\pi$ to the operator of $H \oplus iH$ to itself, say $\pi$ again, such that $\pi(h + iH) = \pi(h) \in H \subset H \oplus iH$. Since the operator $H \oplus iH \ni h + iH \mapsto (I - \lambda A)^{-1/2} h \in H \oplus iH$ converges to $\pi$ strongly as $\lambda \to \infty$, by virtue of [3, Corollary 5.1], we see that $\|(I - \lambda A)^{-1/2} - \pi\|_{\mathcal{W}^n \oplus i\mathcal{W}^n}$ does to 0 in probability. Hence, by the assumption (ii), $\tilde{\psi}(\{(I - \lambda A)^{-1/2} \})$ converges to $\psi_0 \circ \pi$ in probability. In conjunction with the assumption (iii), this yields that

$$\int_{\mathcal{W}^n} \tilde{\psi}((I - \lambda A)^{-1/2} w) P(dw) \to \int_{\mathcal{W}^n} \psi_0 \circ \pi \, dP \quad \text{as } \lambda \to \infty. \quad (24)$$
It follows from Theorem 4 that
\[
\int_{\mathcal{W}_n} e^{i\eta} \, dP = \left\{ e^{\text{tr} D} \det[c(x; B(i\lambda a^2)) - (i\lambda \beta + D)s(x; B(i\lambda a^2))] \right\}^{-1/2}.
\]

Plugging this and (24) into (23), we obtain the desired convergence.

5 Reflectionless potentials and \(n\)-solitons

In this section, we specify \(q\) in (6) so that \(\int_{\mathcal{W}_n} e^{i\eta} \, dP\) relates to reflectionless potentials and soliton solutions of the KdV equation. Throughout this section, we assume that \(p, c \in \mathbb{R}^n\) satisfy that  
\(p_i \neq p_j\) if \(i \neq j\) and \(c_i > 0\) for any \(i = 1, \ldots, n\).

5.1 Reflectionless potentials

We shall first consider the case where reflectionless potentials appear. The reflectionless potential with scattering data \(\eta_1, \ldots, \eta_n, m_1, \ldots, m_n > 0\) is by definition the function
\[
u(x) = -2 \frac{d^2}{dx^2} \log \det(I + G(x)),
\]
where
\[
G(x) = \left( \frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-(\eta_i + \eta_j)x} \right)_{1 \leq i, j \leq n}.
\]

It is well known that if \(u(x, t)\) is defined by (25) and (26) with \(m_j(t) = m_j \exp[-2\eta_j^2 t]\) instead of \(m_j\), \(1 \leq j \leq n\), then the function \(v(x, t) = -u(x, t)\) is an \(n\)-soliton solution of the KdV equation
\[
\partial_t v = \frac{3}{2} v \partial_x v + \frac{1}{4} \partial_x^3 v,
\]
where \(\partial_t = \partial/\partial t\) and \(\partial_x = \partial/\partial x\). For example, see [11].

Let \(\beta = 0\) in (6) and consider the Wiener functional
\[
q_x = -\frac{a^2}{2} \int_0^x \langle c, \xi_{p}(y) \rangle^2 \, dy.
\]
It has been shown by Ikeda and the author ([7, Theorem 2.1]) that we can find the scattering data $\eta_1, \ldots, \eta_n, m_1, \ldots, m_n$ uniquely determined by $p$ and $c$, and the corresponding reflectionless potential $u(x)$ is represented as

$$u(x) = 4 \frac{d^2}{dx^2} \log \left( \int_{\mathcal{W}^n} e^{q_x} \, dP \right).$$

Applying Theorems 3 and 4, we see that $q_x$ and its associated Hilbert-Schmidt operator $A_x = \nabla^2 q_x$ satisfy that i)

$$\ker(A_x) = \left\{ h \in H : h' = g - D \int_0^\bullet g(z) \, dz \text{ for some } g \in \mathcal{P}(c) \right\},$$

ii) $\lambda \neq 0$ is an eigenvalue of $A_x$ if and only if

$$\det[c(x; B(a^2/\lambda)) - Ds(x; B(a^2/\lambda))] = 0,$$

and then the corresponding eigenvector $h \in H$ is given by

$$h'(y) = \{c(y; B(a^2/\lambda)) - Ds(y; B(a^2/\lambda))\}u$$

for some $u \in \ker[c(x; B(a^2/\lambda)) - Ds(x; B(a^2/\lambda))] \setminus \{0\}$, and iii) it holds that

$$\int_{\mathcal{W}^n} e^{q_x} \, dP = \left\{ e^{x \text{tr}D} \det[c(x; B(a^2\zeta)) - Ds(x; B(a^2\zeta))] \right\}^{-1/2}$$

(28)

for $\zeta \in \mathbb{C}$ with sufficiently small real part.

In [7], $\int_{\mathcal{W}^n} e^{q_x} \, dP$ is represented in terms of the solution $\psi$ of the $n \times n$ matrix ordinary differential equation

$$\psi'' - B(a^2)\psi = 0, \quad \psi(0) = I_n, \quad \psi'(0) = -D.$$

In our notation, their representation is written as

$$\int_{\mathcal{W}^n} e^{q_x} \, dP = \left\{ e^{x \text{tr}D} \det[c(x; B(a^2)) - s(x; B(a^2))] \right\}^{-1/2}.$$  

(29)

Since $B(a^2)$ is symmetric, so are $c(x; B(a^2))$ and $s(x; B(a^2))$. Hence

$$\det[c(x; B(a^2)) - s(x; B(a^2))] = \det[c(x; B(a^2)) - Ds(x; B(a^2))].$$

Thus, continuing (29) holomorphically in $a^2$, we also arrive at (28).
5.2 \( n \)-Soliton solutions of the KdV equation

We now turn to the case of soliton solutions of the KdV equation. As was recalled in the previous subsection, we have the scattering data \( \eta_1, \ldots, \eta_n, m_1, \ldots, m_n \) determined by \( p \) and \( c \). Moreover, it was seen in [7] that there exists \( U \in O(n) \) such that \( B(a^2) = UR^2U^{-1}, \) where \( R = \text{diag}[\eta_1, \ldots, \eta_n] \). For \( y \geq 0 \) and \( t \in \mathbb{R} \), we define

\[
\phi(y, t) = U \left\{ \frac{\cosh(yR + tR^3)}{2} - \sinh(yR + tR^3)R^{-1}U^{-1}DU \right\} U^{-1}, \tag{30}
\]

where \( \cosh(X) = \frac{1}{2}(e^X + e^{-X}) \) and \( \sinh(X) = \frac{1}{2}(e^X - e^{-X}) \) for \( X \in \mathbb{R}^{n \times n} \). Then \( \det \phi(y, t) \neq 0 \) and \( \beta(y, t) = -((\partial_y \phi)\phi^{-1})(y, t) \) is symmetric for every \( (y, t) \in [0, x] \times \mathbb{R} \) ([7]). Setting \( \beta_t = \beta(0, t) \) and \( \beta = \beta_t - D \) in (6), we consider the Wiener functional

\[
q_{x, t} = -\frac{a^2}{2} \int_0^x \langle c, \xi_p(y) \rangle^2 dy + \frac{1}{2} \langle (\beta_t - D)\xi_p(x), \xi_p(x) \rangle.
\]

If we set

\[
v(x, t) = -4\partial_x^2 \log \left( \int_{W^n} e^{q_{x, t}} \, dP \right), \tag{31}
\]

then \( v \) is an \( n \)-soliton solution of the KdV equation (27). See [7].

Let \( A_{x, t} = \nabla^2 q_{x, t} \). Applying Theorems 3 and 4, we have that i)

\[
\ker(A_{x, t}) = \left\{ h \in H : h' = g - D \int_0^s g(z) \, dz \text{ for some } g \in \mathcal{P}(c) \right\},
\]

ii) \( \lambda \neq 0 \) is an eigenvalue of \( A_{x, t} \) if and only if

\[
\det[c(x; B(a^2/\lambda)) - \{(1/\lambda)(\beta_t - D) + D\} s(x; B(a^2/\lambda))] = 0, \tag{32}
\]

and then the corresponding eigenvector \( h \in H \) is given by

\[
h'(y) = \{c(y; B(a^2/\lambda)) - Ds(y; B(a^2/\lambda))\} u
\]

for some \( u \in \ker[c(x; B(a^2/\lambda)) - \{(1/\lambda)(\beta_t - D)D\} s(x; B(a^2/\lambda))] \setminus \{0\} \), and iii) it holds that

\[
\int_{W^n} e^{\xi_{q_{x, t}}} \, dP
\]
\[ e^{xuD} \det[c(x; B(a^2\zeta)) - (\zeta(\beta_t + D) + D)s(x; B(a^2\zeta))]^{-1/2} \]  \hspace{1cm} (33)

for \( \zeta \in \mathbb{C} \) with sufficiently small real part.

Moreover, if we denote by \( \{\lambda_j(x, t)\} \) the zeros of the identity (32) counted repeatedly according to the order, then it holds that

\[
\int_{W^n} e^{q(x,t)} \, dP = \left( \prod_{j=1}^{\infty} (1 - \lambda_j(x, t)) \right)^{-1/2},
\]

which gives an infinite product representation of the \( n \)-soliton solution of the KdV equation.

5.3 1-soliton solution of the KdV equation

We continue the notation in the previous subsection. Restricting ourselves to the case where \( n = 1 \), we obtain more concrete expressions. Namely, let \( n = 1 \).

Then \( n \times n \)-matrices and \( n \)-vectors are all scalars; for example, \( p = p, c = c, \xi_p(y) = \xi_p(y) \) are all real numbers. Since

\[
a^2 \int_0^x (c, \xi_p(y))^2 \, dy = a^2 c^2 \int_0^x \xi_p(y)^2 \, dy,
\]

without loss of generality, we may and will assume that \( c = 1 \). We set

\[
\Omega_{p,a} = \mathbb{C} \setminus \{ \zeta \in \mathbb{C} : \text{Re} \zeta = -p^2/a^2, \text{Im} \zeta < 0 \},
\]

\[
\eta(\zeta) = (p^2 + \zeta a^2)^{1/2}, \quad \zeta \in \Omega_{p,a}, \quad \text{and} \quad \eta = \eta(1),
\]

where the branch of \( \zeta^{1/2} \) so that \( 1^{1/2} = 1 \) has been used. It then holds that

\[
B(a^2\zeta) = \eta(\zeta)^2, \quad \zeta \in \Omega_{p,a}.
\]

The \( \eta_1 \) used in (30) coincides with \( \eta \) ([7]), and hence the function \( \phi(y, t) \) in (30) is represented as

\[
\phi(y, t) = \frac{\eta - p}{2\eta} e^{\eta y + \eta^3 t} \left( 1 + \frac{\eta + p}{\eta - p} e^{-2(\eta y + \eta^3 t)} \right).
\]

Since \( |p| < \eta \), we directly see that that \( \phi(y, t) > 0 \) for any \( y \geq 0 \) and \( t \in \mathbb{R} \). Moreover, it holds that

\[
\beta_t - D = -\frac{a^2 \sinh(\eta^3 t)}{\eta \cosh(\eta^3 t) - p \sinh(\eta^3 t)}.
\]
Then the Wiener functional \( q_{x,t} \) of our interest is rewritten as

\[
q_{x,t} = -\frac{a^2}{2} \int_0^x \xi_p(y)^2 \, dy - \frac{1}{2} \frac{a^2 \sinh(\eta^3 t)}{\eta \cosh(\eta^3 t) - p \sinh(\eta^3 t)} \xi_p(x)^2.
\]

The associated \( v(x,t) \) defined by (31) is a 1-soliton solution of the KdV equation (27).

Since \( n = 1 \),

\[ \ker(A_{x,t}) = \{0\}. \]

If we set

\[ \gamma_t = \frac{\sinh(\eta^3 t)}{\eta \cosh(\eta^3 t) - p \sinh(\eta^3 t)} \quad \text{and} \quad \delta_t = -\frac{p \eta \cosh(\eta^3 t)}{\eta \cosh(\eta^3 t) - p \sinh(\eta^3 t)}, \]

then

\[ \zeta(\beta_t - D) + D = -\gamma_t B(a^2 \zeta) - \delta_t. \] (35)

In conjunction with (34), this implies that, if \( \lambda \neq 0 \), then

\[
c(x; B(a^2/\lambda)) - \{(1/\lambda)(\beta_t - D) + D\} s(x; B(a^2/\lambda)) = \cosh(x\eta(1/\lambda)) + \{\gamma_t \eta(1/\lambda) + (\delta_t/\eta(1/\lambda))\} \sinh(x\eta(1/\lambda)).
\]

Thus, by Theorem 3, we see that \( \lambda \in \mathbb{R} \) is an eigenvalue of \( A_{x,t} \) if and only if \( \lambda \neq 0 \) and solves the identity that

\[ \cosh(x\eta(1/\lambda)) + \{\gamma_t \eta(1/\lambda) + (\delta_t/\eta(1/\lambda))\} \sinh(x\eta(1/\lambda)) = 0. \] (36)

Plugging (35) into (33), we see that

\[
\int_{W_1^1} e^{\zeta q_{x,t}} \, dP = \left\{ e^{px} \left[ \cosh(x\eta(\zeta)) + \{\gamma_t \eta(\zeta) + (\delta_t/\eta(\zeta))\} \sinh(x\eta(\zeta)) \right] \right\}^{-1/2}
\]

for \( \zeta \in \mathbb{C} \) with sufficiently small real part.

Let \( \psi \in C^\omega \) and \( \psi_0 \) are same as in Theorem 6. Since \( \ker(A) = \{0\}, W_1^0 = \{0\}, \) and hence \( \psi_0 \circ \pi = \psi_0(0) \). Note that \( \text{Re} \eta(i\lambda) \to \infty \) and

\[ \cosh(x\eta(i\lambda)) e^{-x\eta(i\lambda)}, \sinh(x\eta(i\lambda)) e^{-x\eta(i\lambda)} \to \frac{1}{2} \quad \text{as} \ \lambda \to \infty. \]
Then, for $t \neq 0$,
\[
\left(\frac{\gamma_t}{2} \eta(i \lambda)e^{x\eta(i \lambda)}\right)^{1/2} \int_{\mathcal{W}^1} e^{i\lambda q, t} \psi dP \rightarrow e^{-px/2} \psi_0 (0) \quad \text{as } \lambda \rightarrow \infty,
\]
and for $t = 0$,
\[
\frac{e^{x\eta(i \lambda)/2}}{\sqrt{2}} \int_{\mathcal{W}^1} e^{i\lambda q, 0} \psi dP \rightarrow e^{-px/2} \psi_0 (0) \quad \text{as } \lambda \rightarrow \infty.
\]

References


