調和振動子の微分に付随する２次ウィナー汎関数について

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1. Introduction & statements of result

$(\mathcal{W}^n, P)$; n-dim Wie. sp, $\mathcal{H}^n$; C-M subsp

$Q_A := (\nabla^*)^2 A$ ($A : \mathcal{H}^n \to \mathcal{H}^n$: HS op)

∇: Malliavin 微分, $E[\langle \nabla^* F, G\rangle] = E[\langle F, \nabla G\rangle]$

$$\int_{\mathcal{W}^n} \exp(zQ_A/2)dP = \{\det_2(I - zA)\}^{-1/2}.$$

$\nabla^3\phi = 0,\ E[\nabla\phi] = 0,\ E[\phi] = 0 \overset{\text{iff}}{\Leftrightarrow} \phi = Q_A/2,\ A = \nabla^2 q$

$\det_2(I - zA) = ?$

$n = 1,\ Q_A = \int_0^T w(t)^2 dt \Rightarrow \cos(\sqrt{z}T)$

$n = 2,\ Q_A = \int_0^T \{w^1 dw^2 - w^2 dw^1\} \Rightarrow \cos^2(zT/2)$

$n = 1,\ Q_A = \int_0^T (w(t) - \overline{w})^2 dt \Rightarrow \sin(\sqrt{\lambda}T)/(\sqrt{\lambda}T)$

$\mathcal{W} = \mathcal{W}^1,\ \mathcal{H} = \mathcal{H}^1$

$$q = \int_0^T \left(\int_t^T w(s)ds\right)^2 dt$$

1. New exact formula
2. 調和振動子 $q_0 = \int_0^T w(s)^2 ds$;
   Cameron-Martin 1940's

$\int_{\mathcal{W}} \exp(zq_0/2)dP \leftrightsquigarrow \frac{1}{2}(d/dx)^2 + \frac{z}{2}|x|^2$

$q = \|\nabla q_0\|_{\mathcal{H}}^2/4 \rightsquigarrow$ 停留点

$\because)\ \langle \nabla q_0, h\rangle_{\mathcal{H}} = \lim_{\varepsilon\to 0}\{q_0(w + \varepsilon h) - q_0(w)\}/\varepsilon$
$= 2\int_0^T w(t)h(t)dt = 2\int_0^T \left(\int_t^T w(s)ds\right)\dot{h}(t)dt$

THEOREM (i) 十分小な $\lambda > 0$ に対し

$$\int_{\mathcal{W}} \exp[(\lambda/2)q]\,dP = \left\{\frac{1}{\cosh(\lambda^{1/4}T)\ \cos(\lambda^{1/4}T)}\right\}^{1/2},$$

$$\int_{\mathcal{W}} \exp[(\lambda/2)q]\delta_0(w(T))\,dP$$

$$= \frac{1}{\sqrt{2\pi T}}\left\{\frac{\lambda^{1/4}T}{\sin(\lambda^{1/4}T)\cosh(\lambda^{1/4}T) + \sinh(\lambda^{1/4}T)\cos(\lambda^{1/4}T)}\right\}^{1/2}$$

(ii) $P(q/2 \in dx) = p_T(x)\chi_{[0,\infty)}(x)dx$. ただし

$$p_T(x) = \frac{4}{\pi T^4}\int_0^{\pi/2} \frac{\theta(u;x)}{\sqrt{\cos u}}du,$$

$$\theta(u;x) = \sum_{k=-\infty}^{\infty} (-1)^{k-1} \frac{\{u + (2k+1)\pi\}^3 e^{-x\{u+(2k+1)\pi\}^4/T^4}}{\sqrt{\cosh(u + (2k+1)\pi)}}.$$

- $q = Q_A + \mathrm{tr}\, A$, $\mathrm{tr}\, A = \frac{T^4}{6}$. ただし $A : \mathcal{H} \to \mathcal{H}$:

$$Ah(t) = \int_0^t \int_s^T \int_0^u \int_v^T h(x)dxdvduds$$

$\because$) $q = \int_0^T (\int_t^T w(s)ds)^2 dt = \int_0^T \int_t^T \int_t^T w(s)w(u)dsdudt$

$w(s)w(u) - s \wedge u \in \mathfrak{C}_2$ (2 次 Wiener Chaos)

$q - \int_0^T \int_t^T \int_t^T s \wedge u \, dsdudt \in \mathfrak{C}_2$

$\phi \in \mathfrak{C}_2 \Rightarrow \phi = Q_{\nabla^2 \phi}/2 = Q_{\nabla^2 \phi/2}$

$$\langle \nabla^2 q, h \otimes k \rangle_{\mathcal{H}^{\otimes 2}} = 2 \int_0^T \big(\int_t^T h(s)ds\big)\big(\int_t^T k(s)ds\big)dt = 2\langle Ah, k \rangle_{\mathcal{H}}$$

- $U = U_V + U_F : \mathcal{H} \to \mathcal{H}$ (U_V:Vol, $\dim R(U_F) < \infty$)

(i) $$\int_{\mathcal{W}} \exp\big[(\lambda/2)Q_U\big]dP = \Big\{\det\big(I - \lambda U_F (I - \lambda U_V)^{-1}\big)\Big\}^{-1/2} e^{-(\lambda/2)\mathrm{tr}\, U_F}$$

(ii) $E = \mathrm{span}\{\eta_1, \ldots, \eta_d\} \subset R(U_F)$, $\eta = (\nabla^* \eta_j)_{1 \le j \le d}$

$$\int_{\mathcal{W}} \exp[(\lambda/2)Q_U]\delta_0(\eta)\, dP = \frac{\{\det(I - \lambda U_1^\natural (I - \lambda U_V)^{-1})\}^{-1/2} e^{-(\lambda/2)\mathrm{tr}\, U_F}}{\sqrt{(2\pi)^d \det C(\eta)}}$$

$U_1^\natural = -\pi_E U_V + (I - \pi_E)U_F$, $C(\eta) = \big(\langle \eta_i, \eta_j \rangle_{\mathcal{H}}\big)_{1 \le i,j \le d}$

$$\det{}_2(I + C)(I + D) = \det(I + C)\det{}_2(I + D)e^{-\mathrm{tr} C(I+D)}$$

- $\mathcal{I}h(t) := \int_0^t h(s)ds$, $A_V = \mathcal{I}^4$, $A_F = A - A_V$.

(i) $A = A_V + A_F$, (ii) A_V: Volterra,

(iii) $R(A_F) = \{a\eta_1 + b\eta_2 \,|\, a, b \in \mathbb{R}\}$, ($\eta_j(t) = t^{2j-1}$),

(iv) $(I - \lambda A_V)^{-1}h(t)$

$$= \frac{1}{2}\int_0^t \dot{h}(s)\big\{\cosh(\lambda^{1/4}(t-s)) + \cos(\lambda^{1/4}(t-s))\big\}ds$$

$\because$) $(I - \lambda A_V)g = h$, $f = \mathcal{I}^4 g$. $f^{(4)} - \lambda f = h$

$$\frac{d}{dt}\begin{pmatrix} f \\ f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix} = B_\lambda \begin{pmatrix} f \\ f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ h \end{pmatrix}, \quad B_\lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix}$$

$$A_F h = \{\tfrac{T^2}{2}\mathcal{I}h(T) - \mathcal{I}^3 h(T)\}\eta_1 - \tfrac{1}{6}\mathcal{I}h(T)\eta_2$$

$$A_1^\natural h = \{-\tfrac{1}{T}\mathcal{I}^4 h(T) + \tfrac{T^2}{6}\mathcal{I}h(T)\big\}\eta_1 - \tfrac{1}{6}\mathcal{I}h(T)\eta_2$$

$\mathrm{span}\{\eta_1, \eta_2\} = R(A_F)$ 上 ($E = \mathbb{R}\eta_1$)

$$I - \lambda A_F (I - \lambda A_V)^{-1} = \begin{pmatrix} \frac{T^2\lambda^{1/2}}{4}(\alpha_\lambda - \beta_\lambda) + \frac{1}{2}(\alpha_\lambda + \beta_\lambda) & \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) \\ -\frac{3T^2}{2}(\alpha_\lambda + \beta_\lambda) + 3\lambda^{-1/2}(\alpha_\lambda - \beta_\lambda) & \frac{1}{2}(\alpha_\lambda + \beta_\lambda) \end{pmatrix}$$

$$I - \lambda A_1^\natural (I - \lambda A_V)^{-1} = \begin{pmatrix} \frac{\lambda^{-1/4}}{T}(\sigma_\lambda + \tau_\lambda) - \frac{T^2\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) & \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) \\ \frac{\lambda^{-3/4}}{T}(\sigma_\lambda - \tau_\lambda) - \frac{T^2}{2}(\alpha_\lambda + \beta_\lambda) & \frac{1}{2}(\alpha_\lambda + \beta_\lambda) \end{pmatrix}$$

$\alpha_\lambda = \cosh(\lambda^{1/4}T)$, $\beta_\lambda = \cos(\lambda^{1/4}T)$,
$\sigma_\lambda = \sinh(\lambda^{1/4}T)$, $\tau_\lambda = \sin(\lambda^{1/4}T)$

$$z = re^{i\theta}\ (r \ge 0,\ -\tfrac{3}{2}\pi \le \theta < \tfrac{1}{2}\pi),\ \sqrt{z} = r^{1/2}e^{i\theta/2}$$

$$G(z) = \begin{cases} \sqrt{\cos z}, & \text{if } a)\ |\mathrm{Re}\, z| < \frac{\pi}{2},\ \text{or} \\ & \quad b)\ \mathrm{Im}\, z > 0,\ \mathrm{Re}\, z \in [-\frac{\pi}{2}, \frac{3\pi}{2})_{4\pi}\ \text{or} \\ & \quad c)\ \mathrm{Im}\, z < 0,\ \mathrm{Re}\, z \in [-\frac{3\pi}{2}, \frac{\pi}{2})_{4\pi} \\ -\sqrt{\cos z}, & \text{if } a)\ \mathrm{Im}\, z > 0,\ \mathrm{Re}\, z \in [\frac{3\pi}{2}, \frac{7\pi}{2})_{4\pi}\ \text{or} \\ & \quad b)\ \mathrm{Im}\, z < 0,\ \mathrm{Re}\, z \in [\frac{\pi}{2}, \frac{5\pi}{2})_{4\pi} \end{cases}$$

$\cos(u+iv)$:

	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
$v > 0$	IV	III	II	I
$v < 0$	I	II	III	IV

$$\{\cosh z \cos z\}^{1/2} = G(z)G(iz),$$
$$z \in D_0 \equiv \mathbb{C} \setminus \{\xi, i\xi \mid \xi \in \mathbb{R},\ |\xi| \ge \pi/2\}$$

$$\exists D \supset \left\{re^{i\theta} \mid r \ge 0,\ \theta \in \bigcup_{k=0}^{3}\left[\frac{\pi}{8} + \frac{k\pi}{2}, \frac{3\pi}{8} + \frac{k\pi}{2}\right]\right\},$$

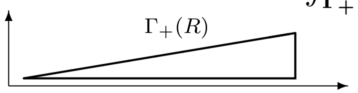
$$\int_{\mathcal{W}} \exp[z^4 q/2]dP = \frac{1}{\{\cosh(zT)\cos(zT)\}^{1/2}},\ z \in D$$

$$p_T(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixt}I(t)dt\ \left(I(t) = \int_{\mathcal{W}} \exp[itq/2]dP\right)$$

$$\Gamma_{\pm}(R):\ 0 \rightsquigarrow Re^{\pm i\pi/8};\ \gamma_{\pm}(t) = t^{1/4}e^{\pm i\pi/8},\ t \in [0, R^4]$$

$$\int_{\Gamma_{\pm}(R)} f(z^4)z^3 dz = \pm\frac{i}{4}\int_0^{R^4} f(\pm it)dt$$

$$2\pi p_T(x) = \lim_{R\to\infty}\left\{4i\int_{\Gamma_-(R)} \frac{z^3 e^{-xz^4}}{\{\cosh(zT)\cos(zT)\}^{1/2}}dz - 4i\int_{\Gamma_+(R)} \frac{z^3 e^{-xz^4}}{\{\cosh(zT)\cos(zT)\}^{1/2}}dz\right\}$$



$$\lim_{R\to\infty}\int_{\Gamma_{\pm}(R)} \frac{z^3 e^{-xz^4}}{\{\cosh(zT)\cos(zT)\}^{1/2}}dz = \int_0^{\infty} \frac{u^3 e^{-xu^4}}{\lim_{h\downarrow 0}\{\cosh(uT \pm ih)\cos(uT \pm ih)\}^{1/2}}du$$

$$\lim_{h\downarrow 0}\{\cosh(uT \pm ih)\cos(uT \pm ih)\}^{1/2}$$
$$= \begin{cases} \sqrt{\cosh(uT)\cos(uT)}, & \text{if } uT \in [-\pi \pm \frac{\pi}{2}, \pi \pm \frac{\pi}{2})_{4\pi} \\ -\sqrt{\cosh(uT)\cos(uT)}, & \text{if } uT \in [\pi \pm \frac{\pi}{2}, 3\pi \pm \frac{\pi}{2})_{4\pi} \end{cases}$$

$$2\pi p_T(x) = 8i\sum_{k=0}^{\infty}\int_{\{(\pi/2)+2k\pi\}/T}^{\{(3\pi/2)+2k\pi\}/T} \frac{(-1)^{k-1}u^3 e^{-xu^4}}{\sqrt{\cosh(uT)\cos(uT)}}du$$

$$\int_{\{(\pi/2)+2k\pi\}/T}^{\{(3\pi/2)+2k\pi\}/T} \cdots du$$
$$= \frac{1}{iT^4}\int_0^{\pi/2} \frac{\{v + (2k+1)\pi\}^3 e^{-x\{v+(2k+1)\pi\}^4/T^4}}{\sqrt{\cosh\{v + (2k+1)\pi\}\cos v}}dv - \frac{1}{iT^4}\int_0^{\pi/2} \frac{\{v - (2k+1)\pi\}^3 e^{-x\{v-(2k+1)\pi\}^4/T^4}}{\sqrt{\cosh\{v - (2k+1)\pi\}\cos v}}dv$$