Geometric aspects of black hole horizons

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Abstract

In this review, we introduce the following two facts:

1. In a certain situation, the black hole event horizon can be regarded as a minimal surface on a spacelike hypersurface.
2. For a particular class of black hole solutions in higher-dimensional spacetime, the event horizon becomes a surface with constant mean curvature embedded in background spacetime (which can be flat spacetime).

We will review basics of the theory of general relativity and black holes solutions, and then explain more precise meaning and implications of the above claims.

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1 Introduction: special and general relativity

We use the natural unit in which the speed of light is equal to the unity ($c = 1$), but keep it in the following unless otherwise noted just for reference. We consider four-dimensional asymptotically flat spacetime, and later on generalize the argument to higher dimensions in section 3.

Most part of this review is based on references [1, 2, 3] and standard textbooks of general relativity such as [4, 5]. Section 3 is based on [6].

We define the symbols for symmetrization and anti-symmetrization of the indices of a tensor $T_{\mu\nu}$ as

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right), \qquad T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right). \tag{1}$$

1.1 Special relativity

The special theory of relativity is a theory of physics constructed based on the principle of relativity claiming that "the laws of physics are invariant in all inertial (non-accelerating) systems" and that "the speed of light in vacuum is constant for all (inertial) observers". We can construct this theory by using a scale and a clock that measures the distances in space and time directions and studying the propagation of light with them.

The distance between two points in spacetime is called spacetime interval ds, which is calculated by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{2}$$

where $\mu, \nu = 0, 1, 2, 3$, $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ and $\eta_{\mu\nu} = \mathrm{Diag}(-c, 1, 1, 1)$ the metric of the flat (Minkowski) spacetime. The special relativity assumes that this interval is invariant even when we switch from the rest frame to an inertial frame that is moving at a constant velocity relative to the rest frame. This switching of the frame is equivalent to applying the Lorentz transformation to the coordinates, hence the special relativity studies physical phenomena that are invariant under Lorentz transformation.

In this theory, the physical quantities are expressed as tensors (scalar, vector, 2-tensor, ...). For example, the velocity of a particle is expressed as a vector v^μ and the electromagnetic field strength is expressed by an anti-symmetric 2-tensor $F_{\mu\nu}$. Various phenomena such as the time delay due to the large velocity and the conversion between electric and magnetic fields can be derived by applying the Lorentz transformation to these tensor quantities.

1.2 General relativity

The general theory of relativity, the general relativity (GR) is a theory of physics constructed based on the principle of general relativity claiming that "the laws of physics are invariant in any (including non-inertial) systems" and the equivalence principle stating that "the gravitational force is equivalent the pseudo-force in accelerated system". Instead of explicitly introducing the gravity as a force, this theory assumes that the spacetime may be curved and the gravitational force acting on a free-fall observer is generated by the curvature of the spacetime.

We consider four-dimensional Lorentzian spacetime. We describe the (generally curved) spacetime by means of the metric tensor $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$, which is a symmetric 2-tensor with Lorentzian signature related to the spacetime interval as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{3}$$

The metric admits the unique derivative operator ∇_μ for which the metric is constant $\nabla_\rho g_{\mu\nu} = 0$ and called the covariant derivative. We can always find a coordinate transformation with which the covariant derivative coincides with the partial derivative. In this new coordinates the metric (3) reduces to the flat metric (2). Physically, the coordinate frame corresponding to these coordinates correspond to the free-falling frame (local inertial frame) in which the gravitation is canceled locally by the acceleration of the coordinate frame.

In general the metric does not coincide with (2), in which case the spacetime is curved. The spacetime curvature is quantified by the Riemann tensor $R_{\mu\nu\rho\sigma}$, which is defined by

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, v^\rho = R_{\mu\nu}{}^{\rho}{}_{\sigma} v^\sigma, \tag{4}$$

where v^μ is an arbitrary vector. The indices are raised and lowered by the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ satisfying $g_{\mu\rho}g^{\rho\nu} = \delta_\mu^\nu$ (e.g. $R_{\mu\nu}{}^{\rho}{}_{\sigma} = R_{\mu\nu\rho'\sigma}g^{\rho'\rho}$). Indeed, the Riemann tensor is given in terms of the metric as

$$R_{\mu\nu\rho\sigma} = -\frac{1}{2}\left[\partial_\mu\partial_\rho g_{\nu\sigma} + \partial_\nu\partial_\sigma g_{\mu\rho} - (\mu \leftrightarrow \nu)\right] + (\text{terms involving } g, \partial g). \tag{5}$$

In GR, the spacetime metric, not only the physical objects (particles and fields) in the spacetime, becomes dynamical. More precisely, the components of the metric tensor depend on time, and their time evolution is governed by the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{6}$$

where G is the Newton constant, $R_{\mu\nu} \equiv g^{\rho\sigma}R_{\mu\rho\nu\sigma}$ is the Ricci tensor and $T_{\mu\nu}$ is a symmetric 2-tensor (called the stress-energy tensor) describing physical matter present in the spacetime.

One way to see that this equation is a wave equation is to consider a situation where the spacetime is sufficiently close to a flat one (2). Decomposing the metric $g_{\mu\nu}$ to the flat metric $\eta_{\mu\nu}$ and deviation from it:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{7}$$

the Einstein equation (6) at the order linear in $h_{\mu\nu}$ reduces to[1]

$$\eta^{\rho\sigma}\partial_\rho\partial_\sigma\psi_{\mu\nu} = \left(-c^{-2}\partial_t^2 + \Delta\right)\psi_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{8}$$

where

$$\psi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}, \qquad h \equiv \eta^{\mu\nu}h_{\mu\nu}, \qquad \Delta \equiv \eta^{ij}\partial_{x^i}\partial_{x^j} \quad (i, j = 1, 2, 3). \tag{9}$$

Equation (8) is a wave equation for the metric perturbation $h_{\mu\nu}$, and implies that the wave at the initial moment propagates at the speed of light.

In this review, we focus on the vacuum spacetime $T_{\mu\nu} = 0$. In this case the Einstein equation simplifies as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \quad \Leftrightarrow \quad R_{\mu\nu} = 0. \tag{10}$$

Spacetime satisfying this equation is called Ricci-flat.

2 Black hole spacetime

2.1 Schwarzschild solution

One trivial example of the Ricci-flat spacetime is the flat metric $\eta_{\mu\nu}$ given by Eq. (2). A simple but nontrivial solution of this equation is the so-called Schwarzschild solution given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)c^2dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{\mathrm{II}}^2, \qquad f(r) = 1 - \frac{r_g}{r}, \tag{11}$$

where $d\Omega_{\mathrm{II}}^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on S^2, and r_g is a constant called the gravitational radius or Schwarzschild radius. Although this metric is Ricci-flat, it is curved in the sense that the Riemann tensor does not vanish. It also has the curvature singularity at $r = 0$ where the coordinate invariant made of curvature diverges $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \propto 1/r^4$.

[1] We need to choose the coordinates in which $\partial_\mu\psi^\mu_\nu$ vanishes to derive Eq. (8).

2.2 Basics of causality

As Eq. (8) implies, waves of the spacetime metric propagates at the light speed $c = 1$. More precisely, the characteristic hypersurface (physically the boundary of the causal region or the surface of the fastest propagation allowed in the theory) is given by a null hypersurface with respect to the metric $\eta_{\mu\nu}$ appearing in Eq. (8), that is, a tangent vector V^μ of such a hypersurface obeys $\eta_{\mu\nu}V^\mu V^\nu = 0$.

When the background spacetime is curved, waves of the metric and also light propagates along a hypersurface whose tangent vector V^μ satisfies $g_{\mu\nu}V^\mu V^\nu = 0$. A vector satisfying this is called a null vector, and the hypersurface tangent to it is called a null hypersurface. It is confirmed that any physical particles found so far does not propagate faster than the light speed. Hence a null hypersurface gives a boundary of causal contact. The null hypersurface emanating from a spacetime point forms a cone of three-dimensional hypersurface, which is usually called a light cone.

Keeping this in mind, we may classify a vector V^μ according to the sign of its norm as

$$g_{\mu\nu}V^\mu V^\nu \begin{cases} < 0 & \text{(timelike)} \\ = 0 & \text{(null or lightlike)} \\ > 0 & \text{(spacelike)} \end{cases} . \tag{12}$$

Timelike and spacelike vectors stemming on the tip of a light cone reside inside and outside of the light cone, respectively. Two spacetime point can be causally connected if they can be connected by a timelike or null curve, while not when they are spacelike separated.

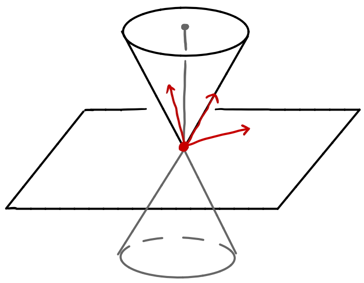


Figure 1: A light cone and timelike, null and spacelike vectors.

Also a free-falling observer and light moves along a timelike or null geodesic curves that satisfy

$$T^\mu \nabla_\mu T^\nu = 0, \tag{13}$$

where T^μ is the tangent of the curve. This equation is called the geodesic equation. In flat spacetime, a timelike and null geodesics is simply given by a straight line in timelike and null directions, respectively.

2.3 Causal structure of Schwarzschild spacetime

In the flat spacetime (2), light ray just propagate along a null direction and reach the infinity. In a curved spacetime, the light ray bends according to the background spacetime curvature and then the causal structure of the spacetime may become nontrivial. Let us study the causal structure of the Schwarzschild spacetime (11) below and introduce the event horizon associated with it.

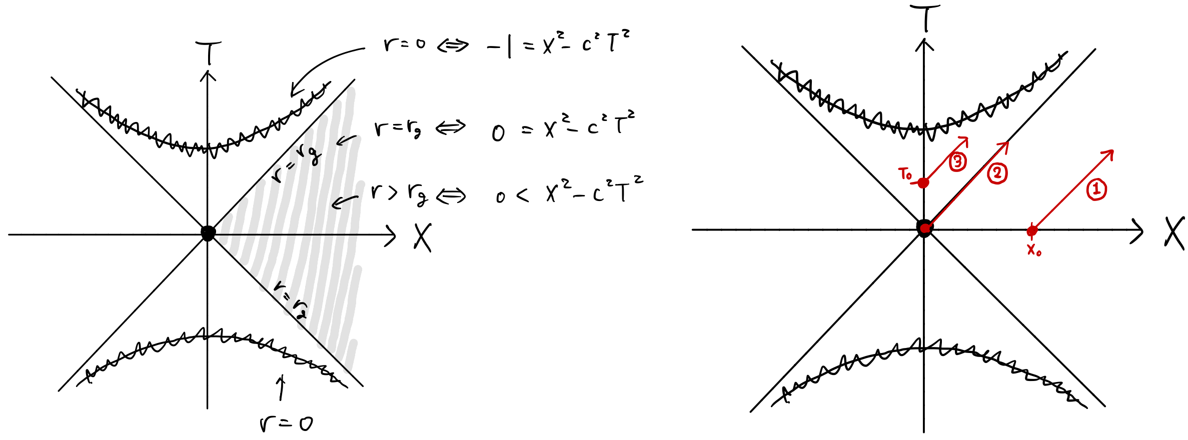
The metric of the Schwarzschild solution is singular at $r = r_g$. It is not a curvature singularity but merely a coordinate singularity, and can be resolved by switching to new coordinates appropriately. Skipping all of the details, let us introduce new coordinates T, X implicitly by the following relationship (where $c = 1$):

$$\left(\frac{r}{r_g} - 1\right) e^{r/r_g} = X^2 - c^2T^2, \qquad \frac{ct}{r_g} = \ln\left(\frac{cT + X}{X - cT}\right) = 2\tanh^{-1}\left(\frac{cT}{X}\right). \tag{14}$$

The gravitational radius $r = r_g$ corresponds to a curve $X^2 - c^2T^2 = 0$. In the (T, X) coordinates (called the Kruskal coordinates), the spacetime metric (11) becomes manifestly regular at $r = r_g$ and is given by

$$ds^2 = -f(r)c^2dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{\mathrm{II}}^2 = \frac{4r_g^3 e^{-r/r_g}}{r}\left(-c^2dT^2 + dX^2\right) + r^2 d\Omega_{\mathrm{II}}^2. \tag{15}$$

Since this spacetime is spherically symmetric, its causal structure can be found by studying the null lines along which radial light rays propagate.



(a) Kruskal coordinates

(b) Light rays

Figure 2: The Schwarzschild spacetime in the Kruskal coordinates and the three light rays studied below.

Let us consider the following three kinds of light rays.

1. $X = X_0 + cT$ $(X_0 > 0)$:

   The radial position $r(T)$ of this light ray obeys

   $$\left(\frac{r}{r_g} - 1\right) e^{r/r_g} = X^2 - c^2T^2 = {X_0}^2 + 2cX_0T.$$

   $T \to \infty \Rightarrow r \to \infty$, hence this light ray escapes to the infinity in the asymptotic future.

2. $X = cT$:

   $$\left(\frac{r}{r_g} - 1\right) e^{r/r_g} = X^2 - c^2T^2 = 0 \quad \Leftrightarrow \quad r = r_g,$$

   hence this light ray stays at $r = r_g$ forever, and particularly it cannot escape to the infinity $r \to \infty$.

3. $X = c(T - T_0)$ $(0 < T_0 < c^{-1})$:

   $$\left(\frac{r}{r_g} - 1\right) e^{r/r_g} = X^2 - c^2T^2 = c^2T_0(T_0 - 2T).$$

   After starting from $(T, X) = (T_0, 0)$, the light ray hits $X^2 - c^2T^2 = -1 \quad \Leftrightarrow \quad r = 0$ at $T = \frac{1}{2}\left(T_0 + \frac{1}{c^2T_0}\right)$. Hence the light ray in this region dives into the curvature singularity within finite time T although it is propagating outward (toward X direction).

2.4 Event horizon

As shown by the analysis above, $r = r_g$ marks the border of outer communication by means of radial outgoing light rays. In this sense the two-sphere at $r = r_g$ is called the event horizon. Events on and inside the event horizon is not invisible from the exterior observer.

More generally, a black hole is defined as follows.

> A *black hole* (BH) is the spacetime region that is not connected to the (null) infinity by causal curves. The boundary of this region is called an *event horizon* (EH).

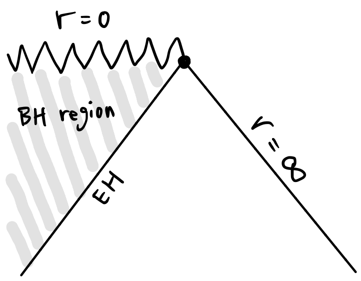


Figure 3: A schematic of an event horizon.

This definition implies the following.

- An EH is a null hypersurface.
- Black hole is (not necessarily but) typically generated by strong gravity that prohibit light escaping to infinity. Particularly it is not a star, but a region in which any signals including light are incarcerated.
- EH is determined according to the *global* structure of the spacetime, that is, we need to know the whole spacetime including the future and light propagation therein to find an EH.

Some other basic properties of a black hole are the following (not to be explained in this review):

- Singularity theorem: Causal geodesics inside BH hits singularity within finite time parameter.
- Black hole area theorem: $dA_{\rm BH}(t)/dt \geq 0$ if the null energy condition is satisfied.
- Positive mass theorem: In asymptotically flat spacetime, the ADM energy is not negative as long as the dominant energy condition is satisfied. The energy becomes zero only for the Minkowski spacetime.
- Uniqueness theorem: In an asymptotically flat vacuum spacetime, static black hole solution is uniquely given by the Schwarzschild solution.

2.5 Apparent horizon

One problem to find an event horizon is that it cannot be determined locally and need to know the global structure of the spacetime. Then it is desirable to have an alternative to the EH that can be determined locally. Motivated by this, we introduce the *apparent horizon* (AH) which can be used for such purpose.

To define AH, we first introduce a congruence of null geodesics and its expansion. Let S a two-dimensional spacelike surface on a three-dimensional spacelike hypersurface Σ (say a time=constant surface). Then consider the ingoing null vector n_-^μ and outgoing null vector n_+^μ emanating from S. Using timelike and spacelike unit vectors t^μ and r^μ such that

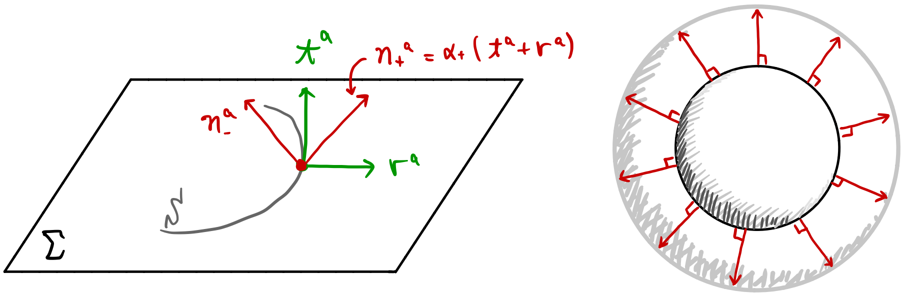
$$-t^\mu t_\mu = 1 = r^\mu r_\mu, \qquad t^\mu r_\mu = 0, \tag{16}$$

$n_\pm^\mu$ are given by

$$n_\pm^\mu = \alpha_\pm \left(t^\mu \pm r^\mu\right), \tag{17}$$

where $\alpha_\pm$ are functions. We assume that t^μ and r^μ are orthogonal and tangent to Σ, respectively. The induced metric on S, $h_{\mu\nu}$, is given by

$$h_{\mu\nu} = g_{\mu\nu} + t_\mu t_\nu - r_\mu r_\nu. \tag{18}$$



(a) Surfaces and vectors

(b) Expansion for S^2

Figure 4: Panel (a): surfaces and vectors used in the derivation. Panel (b): expansion of the null geodesic congruence emanating from S^2.

Then we define the *expansion* of the null geodesic congruence by

$$\begin{aligned}
\hat{\theta} &= h^{\mu\nu}\nabla_\mu n_\nu^+ = \alpha_+ h^{\mu\nu}\nabla_\mu \left(t_\nu + r_\nu\right) \\
&= \alpha_+ \left(h^{\mu\nu}\nabla_\mu t_\nu + h^{\mu\nu}\nabla_\mu r_\nu\right) \\
&= \alpha_+ \left[\left(g^{\mu\nu} + t^\mu t^\nu - r^\mu r^\nu\right)\nabla_\mu t_\nu + h^{\mu\nu}\nabla_\mu r_\nu\right] \\
&= \alpha_+ \left(K - K_{\mu\nu} r^\mu r^\nu + k_\mu^\mu\right),
\end{aligned} \tag{19}$$

where $K_{\mu\nu} \equiv (g_{\mu\rho} + t_\mu t_\rho)(g_{\nu\sigma} + t_\nu t_\sigma)\nabla^\rho t^\sigma$ and $k_{\mu\nu} \equiv h_{\mu\rho} h_{\nu\sigma}\nabla^\rho r^\sigma$ are the extrinsic curvature tensors of Σ and S, respectively. The expansion $\hat{\theta}$ quantifies the divergence of the null geodesic congruence along its propagation in the outgoing null direction. Also it is related to the area increase rate of S as $\dot{A}_S = \int_S \hat{\theta} dS$.

For later use let us derive also the evolution equation of the expansion. We first assume that n_+^μ obeys the geodesic equation $n_+^\mu \nabla_\mu n_+^\nu = 0$, and quantify the deformation of the null geodesic congruence by

$$\tilde{B}_{\mu\nu} = h_\mu{}^\rho h_\nu{}^\sigma \nabla_\sigma n_\rho^+ = \frac{1}{2}\hat{\theta} h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{20}$$

where $\sigma_{\mu\nu} \equiv \tilde{B}_{(\mu\nu)} - \frac{1}{2}\hat{\theta} h_{\mu\nu}$ and $\omega_{\mu\nu} \equiv \tilde{B}_{[\mu\nu]}$ are the shear and the rotation of the null geodesic congruence, respectively. Then the derivative of the expansion along n_+^μ is given by

$$\begin{aligned} n_+^\mu \nabla_\mu \hat{\theta}_+ &= n_+^\mu \nabla_\mu \left(h^{\mu\nu} \nabla_\mu n_\nu^+ \right) \\ &= n_+^\mu \nabla_\mu \nabla_\nu n_+^\nu \\ &= n_+^\mu \left(\nabla_\nu \nabla_\mu n_+^\nu - R_{\mu\nu} n_+^\nu \right) \\ &= -\left(\nabla_\mu n_+^\nu \right) \left(\nabla_\nu n_+^\mu \right) - R_{\mu\nu} n_+^\mu n_+^\nu \\ &= -\tilde{B}_\mu{}^\nu \tilde{B}_\nu{}^\mu - R_{\mu\nu} n_+^\mu n_+^\nu \\ &= -\frac{1}{2}\hat{\theta}^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu} n_+^\mu n_+^\nu . \end{aligned} \tag{21}$$

Hence we obtain

$$\frac{d\hat{\theta}}{d\lambda} = -\frac{1}{2}\hat{\theta}^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu} n_+^\mu n_+^\nu , \tag{22}$$

where we parameterized the null geodesic by assuming $n_+^\mu = (d/d\lambda)^\mu$. This equation is called the Raychaudhuri equation. By Frobenius theorem $\omega_{\mu\nu} = 0$ initially, and it can be shown that it is zero if it is initially zero. Then, rewriting $R_{\mu\nu}$ by the energy-momentum tensor, we find

$$\frac{d\hat{\theta}}{d\lambda} = -\frac{1}{2}\hat{\theta}^2 + \omega_{\mu\nu}\omega^{\mu\nu} - 8\pi G T_{\mu\nu} n_+^\mu n_+^\nu \leq -\frac{1}{2}\hat{\theta}^2 , \tag{23}$$

where we assumed the strong energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0$ for any null vector k^μ. Equation (23) guarantees that $\hat{\theta} \leq 0$ along a null geodesic if it is non-positive initially, and it diverges at finite λ ($\theta \to -\infty$ at $\lambda \leq -1/2\hat{\theta}_0$).

Now we define a *trapped surface* and the *apparent horizon* as follows.

> An *outer trapped surface* on Σ is a closed two-dimensional surface s.t. for outgoing null geodesics orthogonal to S, $\hat{\theta} < 0$ everywhere on S. We call the portion of Σ that contains trapped surfaces a *trapped region* $\mathcal{T}$. Then the *apparent horizon* (AH) is defined as the boundary of the trapped region $\partial\mathcal{T}$.

An AH satisfies the following property.

1. $\hat{\theta} = 0$ everywhere on AH. Hence it is called also a marginally trapped surface.
2. When an AH exists, an EH always exists outside AH. They coincide with each other when the spacetime is stationary (because horizon area is invariant in time and it implies $\hat{\theta} = 0$).
3. (Union of the time evolution of) AH is a spacelike hypersurface.
4. The topology of an AH is S^2 in four-dimensional spacetime if the dominant energy condition is satisfied.
5. On a time-symmetric hypersurface Σ, an AH becomes a minimal surface embedded in Σ.

- **The property 2** follows from the fact that the expansion is non-positive on and inside the AH (as guaranteed by the Raychaudhuri equation (23)), hence the light ray within AH cannot reach the infinity at which the expansion must be positive. implies that an AH is a good approximation of an EH when the spacetime is close to a stationary one. Since AH can be found based only on the local structure of the spacetime, it is widely used in studies on general relativity such as the numerical relativity.

- For **the property 4**, let us introduce a simple proof under some assumptions. We assume that the spacetime metric is given by

$$ds^2 = -2e^{-2f}dudv + h_{AB}dx^A dx^B \tag{24}$$

and take $n_+^\mu = (\partial_u)^\mu$ and $n_-^\mu = (\partial_v)^\mu$. The (rescaled) expansion for these null vectors are defined as

$$\hat{\theta}_\pm = h^{AB}\nabla_A n_B^\pm. \tag{25}$$

Then, the uv component of the Einstein equation is given by

$$R_{\mu\nu}n_+^\mu n_-^\mu + \frac{1}{2}e^{2f}R = 8\pi G T_{\mu\nu}n_+^\mu n_-^\mu, \tag{26}$$

in which the left-hand side is equal to $\frac{1}{2}e^{2f}h^{AB}R_{AB}$. Decomposing the Ricci tensor $R_{\mu\nu}$ into the two-dimensional part ${}^{(2)}R_{AB}$ and the other parts that depend on f, Eq. (26) can be reduced to

$${}^{(2)}R = -2e^{-2f}\hat{\theta}_+\hat{\theta}_- + 2\mathcal{D}^2 f + 2(\mathcal{D}f)^2 - 2e^{-2f}\partial_v\hat{\theta}_+ + 16\pi G e^{-2f}T_{\mu\nu}n_+^\mu n_-^\nu, \tag{27}$$

where $\mathcal{D}_A$ is the covariant derivative with respect to h_{AB}. Integrating this equation on the AH, we have

$$\int_{AH} {}^{(2)}R\, dS = \int dS\left[2(\mathcal{D}f)^2 - 2e^{-2f}\partial_v\hat{\theta}_+ + 16\pi G e^{-2f}T_{\mu\nu}n_+^\mu n_-^\nu\right]. \tag{28}$$

The right-hand side is positive since $\partial_v\hat{\theta}_+ < 0$ follows from the definition of the AH (at which $\hat{\theta} = 0$ and $\hat{\theta}_+ < 0$ inside), and the third term is guaranteed to be positive if we assume (a slightly extended version of) the dominant energy condition. Also the left-hand side is equal to $8\pi(1-g)$ (g: genus of AH) by Gauss-Bonnet theorem. Then Eq. (26) gives

$$1 - g > 0, \tag{29}$$

which implies that $g = 0$, that is, the topology of AH must be S^2.

- About **the property 5**, a time-symmetric hypersurface is defined as a three-dimensional space-like hypersurface on which the extrinsic curvature vanishes ($K_{\mu\nu} = 0$) everywhere.[2] Then the defining equation (19) of the AH reduces to $k_\mu^\mu = 0$, that is, the mean curvature is S is vanishing and then it is a minimal surface on Σ.

  In the case of the Schwarzschild solution, any $t =$ constant hypersurface is time-symmetric. Let us focus on the $T = 0$ hypersurface in Eq. (15) for an example. The induced metric on this hypersurface is given by

$$d\ell^2 = \frac{4r_g^3 e^{-r/r_g}}{r}dX^2 + r^2 d\Omega_{\mathrm{II}}^2, \qquad \left(\frac{r}{r_g} - 1\right)e^{r/r_g} = X^2. \tag{30}$$

  On this three-dimensional space, a minimal surface is found at $r = r_g$ ($X = 0$), which is just at the event horizon as it should be.

[2] Note that the extrinsic curvature is expressed also as $K_{\mu\nu} = \frac{1}{2}\partial_t\tilde{g}_{\mu\nu}$, where t is related to t^μ as $t_\mu = (dt)_\mu$ and $\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} + t^\mu t^\nu$ is the induced metric on Σ. $\partial_t\tilde{g}_{\mu\nu} = 0$ everywhere on a time-symmetric surface Σ, hence the spacetime is momentarily static at Σ, and the time evolution to the future and to the past from Σ are symmetric with each other.

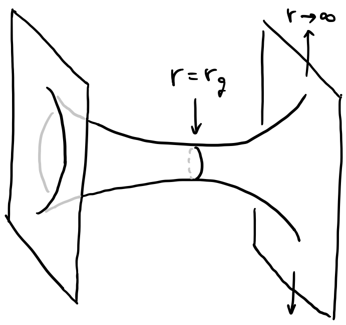


Figure 5: Geometry on the $T = 0$ surface of the Schwarzschild spacetime. S^2 at $r = r_g$, where the EH is present, is a minimal surface.

3 Black holes in higher-dimensional spacetime

In this section we consider the general relativity in higher-dimensional spacetime, and show that the event horizon of black hole of certain type becomes an analogue of constant mean curvature surface embedded in background spacetime, which we can freely prescribe. For example, we can take the flat spacetime and consider black holes in it, in which case the event horizon becomes a version of constant mean curvature surface in flat spacetime.

3.1 Black objects in higher-dimensional spacetime

In D-dimensional vacuum spacetime, the Einstein equation is given by the same equation as that in four-dimensional spacetime (10).

In four-dimensional asymptotically flat spacetime, there is the black hole uniqueness theorem stating that a stationary black hole in must have an event horizon with S^2 topology. In D-dimensional spacetime with $D > 4$, we have an analogue of this theorem when the spacetime is asymptotically flat and static, in which case a black hole solution is the higher-dimensional Schwarzschild solution with S^{D-2} horizon. However, in more general situations (stationary, not asymptotically flat, nonzero cosmological constant, ...) the uniqueness of the black hole solutions is lost and there can be variety of black objects with nontrivial horizon topology, e.g. $S^{D-3} \times \mathbb{R}$, $S^1 \times S^{D-3}$ etc..

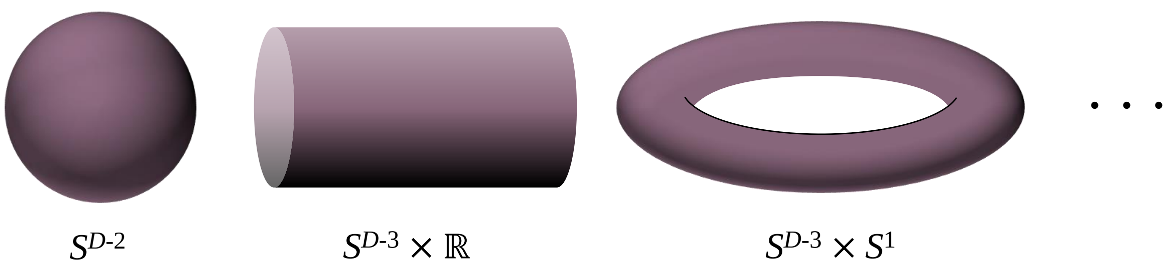


Figure 6: Examples of black objects in D-dimensional spacetime.

3.2 Large-D limit ($1/D$ expansion) of general relativity

In higher-dimensional spacetime there are various solutions describing black hole solutions, and it becomes more and more difficult to solve the Einstein equation to find all those solutions. Hence it is desirable to have a method that facilitate studies on higher-dimensional gravity and black object solutions.

An example of such method to analyze the higher-dimensional gravity is the Large-D limit of general relativity [7], which focuses on particular class of black hole solutions and expand the Einstein equation with respect to $1/D \ll 1$. In some cases this expansion simplifies the equation drastically and enables us to study black objects in higher-dimensional spacetime analytically and/or simple numerical analysis.

3.3 Basic idea

The basic idea behind the large-D limit is the separation of scales between $r_g \gg r_g/D$ and decoupling between the near-horizon region and the far region where the gravitation due to the black hole becomes very weak.

Such decoupling is best understood by focusing on the behavior of the metric function $f(r)$ of the Schwarzschild solution and comparing that in four dimensions to that in higher dimensions. $f(r)$ in four-dimension is given by Eq. (11), while in higher dimensions it becomes

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{\text{D-2}}, \qquad f(r) = 1 - \left(\frac{r_g}{r}\right)^{D-3}, \tag{31}$$

where $d\Omega^2_{\text{D-2}}$ is the line element on S^{D-2}. Physically, $\Phi(r) \equiv f(r) - 1 = -\left(r_g/r\right)^{D-3}$ corresponds to the gravitational potential generated by the black hole. We focus on the behavior of $\Phi(r)$ outside the event horizon. When $D \gg 1$, $\Phi(r)$ develops a sharp peak near $r \gtrsim r_g$, and the peak is localized in the region $r_g < r \lesssim r_g + r_0/D$.[3]

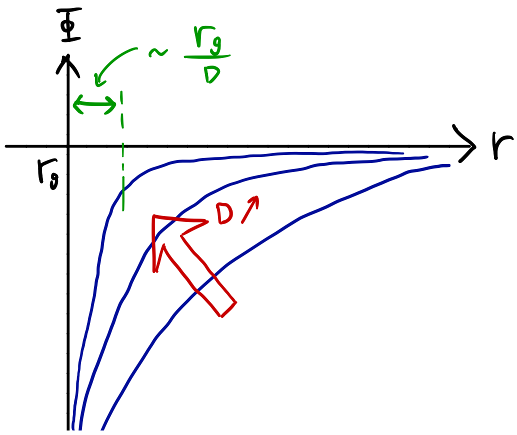


Figure 7: r and D dependence of the gravitational potential Φ.

Having this in mind, we can separate the horizon exterior region into the following two parts:

- Far region $r - r_g \gg r_g/D$:

  The spacetime in this region is close to the flat spacetime since $|\Phi(r)| = \left(r_g/r\right)^{D-3} \lesssim D^{-D}$ rapidly approaches zero.

- Near-horizon region $r - r_g \ll r_g$:

  Gravitation becomes nontrivial in this region particularly when $r - r_g \lesssim r_g/D$ where $\phi(r) \gtrsim 1$.

[3] This length scale can be inferred from $\Phi(r)/\Phi'(r) = -r_g/(D-3)$.

Then we can solve the Einstein equation in the near-horizon region where we can regard r_g/D as a small parameter, and then connect the solution to that in the far region where $g_{\mu\nu} - \eta_{\mu\nu}$ becomes small. Particularly, we can derive the effective equation governing the shape and dynamics of the black hole horizon by expanding the Einstein equation w.r.t. $1/D$ and solving it order by order.

3.4 Effective theory of black holes

The effective equation that governs the horizon dynamics can be derived as follows [6]. In the large D limit, the spacetime develops a very sharp feature near the horizon. Hence we take a coordinate adapted to it as

$$ds^2 = N^2(\rho, x)\frac{d\rho^2}{(D-1)^2} + g_{\mu\nu}(\rho, x)dx^\mu dx^\nu, \tag{32}$$

and let us focus on black objects whose metric is given by

$$ds^2 = N^2\frac{d\rho^2}{(D-1)^2} - V^2(\rho, z)dt^2 + g_{ab}(\rho, z)dz^a dz^b + \mathsf{R}^2(\rho, z)d\Omega_{n+1}^2, \tag{33}$$

where $a, b = 1, \ldots, p$, $n = D - p - 3$ and $d\Omega_{n+1}^2$ is the line element on S^{n+1}.

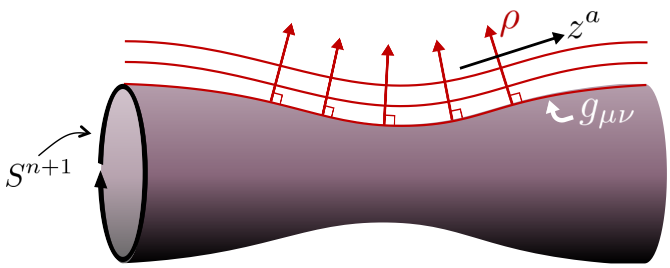


Figure 8: Coordinates adapted to the horizon in the large-D limit.

Then, the Einstein equation can be decomposed into the following components:

$$\rho\rho: \quad K^2 - K^\mu{}_\nu K^\nu{}_\mu = R \tag{34}$$

$$\rho\mu: \quad \nabla_\nu K^\nu{}_\mu - \nabla_\mu K = 0 \tag{35}$$

$$\mu\nu: \quad \frac{D-1}{N}\partial_\rho K^\mu{}_\nu + KK^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{N}\nabla^\mu\nabla_\nu N. \tag{36}$$

where $K_{\mu\nu}$ is the extrinsic curvature of ρ-constant surfaces:

$$K_{\mu\nu} = \frac{D-1}{2N}\frac{\partial g_{\mu\nu}}{\partial\rho}. \tag{37}$$

The $\mu\nu$ components (36) can be regarded as a second-order differential equation for $g_{\mu\nu}$, and they can be integrated w.r.t. ρ to find the function forms of $K_{\mu\nu}$ and $g_{\mu\nu}$. The integration constants can be fixed by imposing the $\rho\rho$ component (34). Then the $\rho\mu$ components require that certain combination of the metric function is constant. Its explicit form is given by

$$\sqrt{-g_{tt}}K\Big|_{\text{horizon}} = 2\kappa, \tag{38}$$

where κ is a constant called the surface gravity of the black hole horizon. Equation (38) formally takes form of the soap-film equation where the surface has a constant mean curvature, though a ρ-constant surface contains t direction and not Euclidean. Equation (38) also determines ρ-constant surfaces are embedded to the spacetime of the far region. For example, we can set the spacetime in the far region to be D-dimensional Minkowski spacetime. Then the black hole horizon will be given by a hypersurface with constant mean curvature obeying Eq. (38) in flat spacetime.

4 Future problems

- Classification of black hole solutions
- Properties of black hole solutions (stability, end point of instability growth, ...)
- Solutions with matter fields (Einstein-Vlasov system, ...)
- Astrophysical applications (gravitational wave, ...)
- Applications to string theory (D-branes $\sim$ black holes in supergravity, ...)
- Gravitation theories other than the general relativity (modified gravity theories)

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