

Invertibility conditions for field transformations with derivatives: toward extensions of disformal transformation with higher derivatives

Norihiro Tanahashi

Department of Physics, Chuo University, 1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan

April 27, 2022

Abstract

We discuss a field transformation from fields ψ_a to other fields ϕ_i that involves derivatives, $\phi_i = \bar{\phi}_i(\psi_a, \partial_\alpha \psi_a, \dots; x^\mu)$, and derive conditions for this transformation to be invertible, primarily focusing on the simplest case that the transformation maps between a pair of two fields and involves up to their first derivatives. General field transformation of this type changes number of degrees of freedom, hence for the transformation to be invertible, it must satisfy certain degeneracy conditions so that additional degrees of freedom do not appear. Our derivation of necessary and sufficient conditions for invertible transformation is based on the method of characteristics, which is used to count the number of independent solutions of a given differential equation. As applications of the invertibility conditions, we show some non-trivial examples of the invertible field transformations with derivatives, and also give a rigorous proof that a simple extension of the disformal transformation involving a second derivative of the scalar field is not invertible. We also discuss the invertibility conditions for more general class of disformal transformations with higher derivatives and their implications.

1 Introduction

Field transformations are quite ubiquitous in all of the fields of physics and mathematics. The reason is that by using suitable fields (variables), one can often get a better and more intuitive insight into physical phenomena, find a different form of equations of motion which may allow to obtain solutions more easily, and so on. For these purposes, the (local) invertibility of a field transformation is essential because, otherwise, physics would not be the same after the field transformation. In case that a field transformation does not involve the derivatives, its local invertibility can be judged by the well-known inverse function theorem. On the other hand, when a field transformation does involve derivatives, it is clear that the invertibility conditions become much more complicated. If one regards such a field transformation as differential equations for old variables (fields), one can naively expect the presence of integration constants associated with the derivatives, which breaks one-to-one correspondence between old and new variables. Thus, apparently, one might arrive at a conclusion that no invertible transformation with derivatives exists. But, of course, this is not true in general. If one assumes specific degeneracy of the derivative terms in a field transformation which prohibits the appearance of associated integration constants, one can have an invertible field transformation with the derivatives. Indeed, in our letter [1] we have explicitly given necessary and sufficient conditions for the invertibility of a field transformation involving two-fields and first derivatives.¹

The purpose of this paper is twofold. First, we give the full and complete proof of necessary and sufficient conditions for the invertibility of field transformations with derivatives. Here we fill some gaps of our proof that has been omitted in our letter [1]. More importantly, we provide the full proof for sufficient condition, while in the letter we gave it only within the perturbative regime. The second goal of this paper is to prove the no-go theorem of disformal transformation of the metric that involves second derivatives of a scalar field. We are strongly motivated by pursuing the extensions of conformal and disformal transformations, which are often used in gravity, cosmology, and many other fields (see e.g. Ref. [3] for classification of new

¹See e.g. Ref. [2] for earlier discussions on invertibility conditions of field transformations.

theories generated from a simpler theory [4] using disformal transformations). A disformal transformation involving the first derivative of a scalar field is a natural extension of a conformal transformation [5]. The next natural question is whether one can further extend a disformal transformation to that involving the second derivatives of a field or even its higher derivatives. As a useful application of our invertibility conditions, we explicitly prove that there is no invertible disformal transformation involving the second derivatives given by $\tilde{g} = C(\chi, X)g + D(\chi, X)\nabla\chi\nabla\chi + E(\chi, X)\nabla\nabla\chi$ ($X \equiv (\nabla\chi)^2$) with $E \neq 0$. To the best of our knowledge, this is the first rigorous proof of the absence of such a transformation.

The organization of this paper is as follows. In section 2, we give a complete derivation of the necessary conditions. The results in sections 2.1 apply to field transformations involving arbitrary number of fields and their first derivatives. To show the explicit form of the invertibility conditions, in section 2.2 and 2.3 we focus on the field transformation between two fields and their first derivatives. In the section 3, the complete proof of the sufficient conditions is given. As an application of our result, in section 4 we construct some examples of invertible field transformation by solving the invertibility conditions. As another application, in section 5 the no-go theorem for a class of disformal transformations of the metric with second derivatives is proven, and also the invertibility conditions for the generalized disformal transformations with higher derivatives are proposed based on the behavior of the transformations on the FRW spacetime background. Section 6 is devoted to conclusions and discussion. In Appendices A and B, we show details of some calculations in section 2. Appendix C gives some technical details of the derivation in section 4.

Notations. Here we summarize the convention for indices used in this work.

$$\partial_{\mu_1\mu_2\dots\mu_n} := \partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}, \quad (1)$$

$$C^{(\mu_1\mu_2)} = \frac{1}{2}(C^{\mu_1\mu_2} + C^{\mu_2\mu_1}), \quad (2)$$

$$C^{[\mu_1\mu_2]} = \frac{1}{2}(C^{\mu_1\mu_2} - C^{\mu_2\mu_1}). \quad (3)$$

$C^{(\mu_1\mu_2\dots\mu_n)}$ and $C^{[\mu_1\mu_2\dots\mu_n]}$ are defined similarly by permutation (with a factor $(n!)^{-1}$). We introduce the Levi-Civita symbol $\epsilon^{i_1\dots i_n i_{n+1}\dots i_N}$, which satisfies the following identities;

$$\begin{aligned} & \epsilon^{i_1\dots i_n i_{n+1}\dots i_N} \epsilon_{j_1\dots j_n j_{n+1}\dots j_N} M_{i_1}^{j_1} \dots M_{i_n}^{j_n} M_{i_{n+1}}^{j_{n+1}} \\ &= \epsilon^{i_1\dots i_N} \epsilon_{j_1\dots j_N} \delta_{[l_1}^{j_1} \dots \delta_{l_n}^{j_n} \delta_{l_{n+1}}^{j_{n+1}}] M_{i_1}^{l_1} \dots M_{i_{n+1}}^{l_{n+1}} \\ &= \frac{1}{(n+1)!(N-n-1)!} \epsilon^{i_1\dots i_N} \epsilon_{j_1\dots j_N} \epsilon^{j_1\dots j_n k l_{n+2}\dots l_N} \epsilon_{l_1\dots l_N} M_{i_1}^{l_1} \dots M_{i_{n+1}}^{l_{n+1}} \\ &= \frac{n!(N-n)!}{(n+1)!(N-n-1)!} \epsilon^{i_1\dots i_N} \epsilon_{l_1\dots l_N} M_{i_1}^{l_1} \dots M_{i_{n+1}}^{l_{n+1}} \delta_{[j_{n+1}}^k \delta_{j_{n+2}}^{l_{n+2}} \dots \delta_{j_N}^{l_N} \\ &= \frac{N-n}{n+1} \epsilon^{i_1\dots i_N} \epsilon_{l_1\dots l_N} M_{i_1}^{l_1} \dots M_{i_{n+1}}^{l_{n+1}} \delta_{[j_{n+1}}^k \delta_{j_{n+2}}^{l_{n+2}} \dots \delta_{j_N}^{l_N}]. \end{aligned} \quad (4)$$

Throughout this work, we do not distinguish the lower and upper indices for the field space indices a, b, \dots and i, j, \dots , while in some parts upper/lower indices are used for clarity of the notation.

2 Derivation of the necessary conditions

In this section we present a complete derivation of necessary conditions for invertibility of the transformation $\phi_i = \bar{\phi}_i(\psi_a, \partial_\alpha\psi_a, x^\mu)$ that transforms fields ϕ_i into fields ψ_a .² We divide the derivation of the invertibility conditions into two parts. In the section 2.1, we derive necessary conditions for the invertibility. An invertible transformation preserves the number of degree of freedom, while a transformation with field derivatives typically generates additional degrees of freedom. To formulate such an idea mathematically,

²Here we concentrate on the case with first-order derivatives. However, our idea of the use of characteristics and degeneracy applies to the case with arbitrary order derivatives (and arbitrary number of fields) in the same way.

we employ the method of characteristics for a differential equation to count the number of propagating modes [6].

In section 2.1.1 we explain our approach to derive the necessary conditions for the invertibility based on the method of characteristics. In this approach, the transformation equation (5) given below is converted to a set of differential equations that relates old variables to new ones, and the number of independent solutions for these equations is related to the number of degrees of freedom. To establish the invertibility, the transformation must satisfy certain degeneracy conditions to remove unnecessary additional modes originating from the derivatives in the transformation equation. Such degeneracy conditions must be imposed at each order of the aforementioned differential equations, which are summarized in sections 2.1.2 and 2.1.3. This procedure should be applied iteratively until the number of independent solutions are reduced appropriately, and then we may impose the non-degeneracy conditions to ensure the number of degree of freedom is not changed by the transformation. We summarize this procedure in section 2.1.4.

The procedure in 2.1 applies to field transformation for general number of fields. For illustration of our method, we focus on the transformation between two fields in section 2.2 and 2.3. The necessary conditions for the invertibility in the two-field case are derived in section 2.2 based on the general method introduced in the previous sections. The expressions of these conditions are rather complicated, and actually they can be simplified by solving a part of the conditions explicitly. Based on such an idea, the expressions of the necessary conditions are simplified in sections 2.3.1, 2.3.2 and 2.3.3. After simplification, the necessary conditions for the invertibility of a two-field transformations are summarized in Eqs. (50), (68) and (73).

After deriving the necessary conditions in this section, in section 3 we show that these conditions are actually sufficient to guarantee the invertibility.

2.1 Method of characteristics as the key to derive the necessary conditions

2.1.1 Invertibility and number of degeneracies

Let us consider a field transformation from ψ_a to ϕ_i given by

$$\phi_i = \bar{\phi}_i(\psi_a, \partial_\alpha \psi_a, x^\mu), \quad (5)$$

where $\bar{\phi}_i$ is a function of ψ_a , $\partial_\alpha \psi_a$ and x^μ . We suppose that the numbers of fields before and after transformation are the same, that is, we assume that both a and i run from 1 to N , where N is the number of field ψ_a .

If the transformation (5) is invertible, it does not change physical properties of theories before and after the transformation, and particularly the causal structure should be invariant. If the transformation changes the number of characteristic hypersurfaces, the causal structure is changed correspondingly. Therefore, for the invertibility, the transformation should not change the number of characteristic hypersurfaces. This gives the necessary conditions for invertibility.

To examine if the transformation (5) changes the number of characteristic hypersurfaces, we employ the method of characteristics for partial differential equations. This method can be applied only to quasi-linear differential equations, while the transformation equation (5) is nonlinear in $\partial_\alpha \psi_i$ in general. To convert Eq. (5) to a quasi-linear partial differential equation, we act a differential operator $K_{bi}^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n}$ on it to obtain

$$\begin{aligned} & K_{bi}^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n} \phi_i \\ &= K_{bi}^{(\mu_1 \dots \mu_n)} A_{ia}^{\mu_{n+1}} \partial_{\mu_1 \dots \mu_{n+1}} \psi_a \\ &+ \left(K_{bi}^{(\mu_1 \dots \mu_n)} B_{ia} + n K_{bi}^{(\alpha \mu_1 \dots \mu_{n-1})} \partial_\alpha A_{ia}^{\mu_n} \right) \partial_{\mu_1 \dots \mu_n} \psi_a \\ &+ \left(n K_{bi}^{(\alpha \mu_1 \dots \mu_{n-1})} \partial_\alpha B_{ia} + \frac{n(n-1)}{2} K_{bi}^{(\alpha_1 \alpha_2 \mu_1 \dots \mu_{n-2})} \partial_{\alpha_1 \alpha_2} A_{ia}^{\mu_{n-1}} \right) \partial_{\mu_1 \dots \mu_{n-1}} \psi_a + \dots \\ &+ \left(\binom{n}{k} K_{bi}^{(\alpha_1 \dots \alpha_k \mu_1 \dots \mu_{n-k})} \partial_{\alpha_1 \dots \alpha_k} B_{ia} + \binom{n}{k+1} K_{bi}^{(\alpha_1 \dots \alpha_{k+1} \mu_1 \dots \mu_{n-k-1})} \partial_{\alpha_1 \dots \alpha_{k+1}} A_{ia}^{\mu_{n-k}} \right) \partial_{\mu_1 \dots \mu_{n-k}} \psi_a \\ &+ \dots + \mathcal{O} \left(\partial^{\lfloor \frac{n}{2} \rfloor + 1} \psi_a, \partial^{\lfloor \frac{n}{2} \rfloor} \psi_a, \dots \right), \end{aligned} \quad (6)$$

where A_{ia}^α, B_{ia} are $N \times N$ matrices defined by

$$A_{ia}^\alpha := \frac{\partial \bar{\phi}_i}{\partial (\partial_\alpha \psi_a)}, \quad B_{ia} := \frac{\partial \bar{\phi}_i}{\partial \psi_a}, \quad (7)$$

and $\binom{n}{k}$ is the binomial coefficient for $k \leq n - \lfloor \frac{n}{2} \rfloor - 2$. In this expression, $\partial^{\lfloor \frac{n}{2} \rfloor + 2} \psi_a$ and higher derivatives of ψ_a appear linearly, while the $\mathcal{O}(\partial^{\lfloor \frac{n}{2} \rfloor + 1} \psi_a, \partial^{\lfloor \frac{n}{2} \rfloor} \psi_a, \dots)$ part is a nonlinear function of lower-order derivatives of ψ_a . We show the derivation of the formula (6) in Appendix A.

The operation of the differential operator $K_{bi}^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n}$ introduces additional characteristic hypersurfaces to those of Eq. (5). Hence, the characteristics of ψ_a , determined by the right-hand side of Eq. (6), are comprised of the original ones inherent to the field transformation equation (5) and the additional ones generated by the operation of $K_{bi}^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n}$.

Now we can analyse the characteristics of quasi-linear differential equation (6), regarding this equation as partial differential equations on ψ_a . We have N differential equations with $(n+1)$ -th order derivatives of ψ_a . Therefore, they generically give $N \times (n+1)$ integration constants in a solution ψ_a , which corresponds $N \times (n+1)$ characteristics. However, if the field transformation (5) is invertible, there are no characteristics inherent to (5), which implies that Eq. (6) has only the additional characteristics generated by the operation of $K_{bi}^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n}$. The number of these additional characteristics is $N \times n$, hence there is a mismatch between the number of derivatives in the equations, $N \times (n+1)$, and that of the characteristics required by the invertibility, $N \times n$.

Such a mismatch can be resolved if the structure of the highest-order derivative part is degenerate. Since the difference between them is N , the invertibility requires N degrees of degeneracies. We derive the conditions giving such N degeneracies below.

2.1.2 Degeneracy condition at leading order

The characteristic equation for Eq. (6) is given by

$$\det \left(K_{bi}^{(\mu_1 \dots \mu_n)} A_{ia}^{\mu_{n+1}} \xi_{\mu_1} \dots \xi_{\mu_{n+1}} \right) = \det \left(K_{bi}^{(\mu_1 \dots \mu_n)} \xi_{\mu_1} \dots \xi_{\mu_n} \right) \det \left(A_{ia}^{\mu_{n+1}} \xi_{\mu_{n+1}} \right) = 0, \quad (8)$$

where ξ_μ is a vector which is not tangent to would-be characteristics hypersurfaces. If $\det(A_{ia}^\mu \xi_\mu)$ does not vanish identically for any ξ_μ , the characteristic equation (8) implies that there are $N \times (n+1)$ characteristics, and then the transformation (5) is not invertible as explained above. Hence, for the invertibility $A_{ia}^\mu \xi_\mu$ must be degenerate for any ξ_μ , that is,

$$\det(A_{ia}^\mu \xi_\mu) = 0 \quad \text{for any } \xi_\mu. \quad (9)$$

This condition is equivalent to

$$\sum_{a \in S_n} \text{sgn}(a) A_{1a_1}^{\alpha_1} \dots A_{Na_N}^{\alpha_N} = \frac{1}{N!} \epsilon^{i_1 \dots i_N} \epsilon^{a_1 \dots a_N} A_{i_1 a_1}^{\alpha_1} \dots A_{i_N a_N}^{\alpha_N} = 0, \quad (10)$$

where N is the number of fields and the sum is computed over the set S_n of all permutations $a = \{a_1, \dots, a_N\}$ of $\{1, \dots, N\}$. Here, $\text{sgn}(a)$ denotes the signature of a permutation a , which is $+1$ whenever the reordering a can be achieved by successively interchanging two entries an even number of times, and -1 whenever it can be achieved by an odd number of such interchanges.

2.1.3 Degeneracy condition at subleading order

The condition (10) indicates the degeneracy of the highest-order derivatives of Eq. (6). If the number of degeneracy is N (*i.e.* the dimension of kernel $A_{ia}^\mu \xi_\mu$ is N), it implies that $A_{ia}^\mu = 0$, and thus transformation is independent of $\partial_\mu \psi_a$. It is a trivial case; we can directly use the implicit function theorem to Eq. (5). Therefore, we consider non-trivial cases where the number of degeneracy of (10) is less than N . Here, we

discuss the case where the number of degeneracy of (10) is 1 for simplicity. Generic cases may be discussed in a similar manner.

For later use, we introduce the adjugate matrix of A_{ia}^μ , which is the transposed cofactor matrix, as

$$\begin{aligned}\bar{A}_{ai}^{\alpha_1 \dots \alpha_{N-1}} &:= {}^t \bar{A}_{ia}^{\alpha_1 \dots \alpha_{N-1}} = (-1)^{i+a} \sum_{\bar{a} \in \bar{S}_n} \text{sgn}(\bar{a}) A_{i_1 \bar{a}_1}^{(\alpha_1)} \dots A_{i_{N-1} \bar{a}_{N-1}}^{(\alpha_{N-1})} \\ &= \frac{1}{(N-1)!} \epsilon_i^{i_1 \dots i_{N-1}} \epsilon_a^{a_1 \dots a_{N-1}} A_{i_1 a_1}^{\alpha_1} \dots A_{i_{N-1} a_{N-1}}^{\alpha_{N-1}},\end{aligned}\quad (11)$$

where $\{\bar{i}_1, \dots, \bar{i}_{N-1}\} = \{1, \dots, i-1, i+1, \dots, N\}$ and the sum is computed over the set \bar{S}_n of all permutations $\bar{a} = \{\bar{a}_1, \dots, \bar{a}_{N-1}\}$ of $\{1, \dots, a-1, a+1, \dots, N\}$. This adjugate matrix satisfies

$$\bar{A}_{ai}^{(\alpha_1 \dots \alpha_{N-1})} A_{ib}^{\alpha_N} = 0, \quad A_{ia}^{(\alpha_1)} \bar{A}_{aj}^{\alpha_2 \dots \alpha_N} = 0. \quad (12)$$

As commented above, we suppose that the dimension of kernel of $A_{ia}^\mu \xi_\mu$ is 1 for any ξ_μ . Then the adjugate matrix is a rank-1 matrix.

The fact that the highest derivative part of Eq. (6) is degenerate in one dimension implies that Eq. (6) contains one equation that involves only lower-order derivatives of ψ_a . This does not necessarily imply that one of the equations in (6) contains only lower derivative terms, instead, generically, by combining equations in (6) one should be able to find one lower-derivative equation. Let us extract this subleading equation. First, setting the operator $K_{bi}^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n}$ in Eq. (6) to $\tilde{K}_b^{(\mu_1 \dots \mu_m)} \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}} \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}}$ with $m = n - N + 1$, Eq. (6) yields

$$\begin{aligned}&\tilde{K}_b^{(\mu_1 \dots \mu_m)} \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}} \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \phi_i \\ &= \tilde{K}_b^{(\mu_1 \dots \mu_m)} \bar{A}_{bi}^{(\alpha_1 \dots \alpha_{N-1})} A_{ia}^{(\alpha_N)} \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_N} \psi_a \\ &\quad + \tilde{K}_b^{(\mu_1 \dots \mu_m)} \left[\bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}} B_{ia} + (N-1) \bar{A}_{bi}^{\beta \alpha_1 \dots \alpha_{N-2}} (\partial_\beta A_{ia}^{\alpha_{N-1}}) \right] \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \psi_a \\ &\quad + m \tilde{K}_b^{(\beta \mu_1 \dots \mu_{m-1})} \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}} (\partial_\beta A_{ia}^{\mu_m}) \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \psi_a + \mathcal{O}(\partial^{m+N-2} \psi).\end{aligned}\quad (13)$$

The first term of the right-hand side vanishes because of Eq. (12). Moreover, equation (12) gives

$$\bar{A}_{bi}^{(\alpha_1 \dots \alpha_{N-1})} (\partial_\beta A_{ia}^{\alpha_N}) = - (\partial_\beta \bar{A}_{bi}^{(\alpha_1 \dots \alpha_{N-1})}) A_{ia}^{\alpha_N}. \quad (14)$$

This equation shows that the last term on the right-hand side of Eq. (13) is written as

$$\begin{aligned}&m \tilde{K}_b^{(\beta \mu_1 \dots \mu_{m-1})} \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}} (\partial_\beta A_{ia}^{\mu_m}) \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \psi_a \\ &= -m \tilde{K}_b^{(\beta \mu_1 \dots \mu_{m-1})} (\partial_\beta \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}}) A_{ia}^{\mu_m} \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \psi_a.\end{aligned}\quad (15)$$

Next, Eq. (6) with $K_{bi}^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n}$ replaced by $m \tilde{K}_b^{(\beta \mu_1 \dots \mu_{m-1})} (\partial_\beta \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}}) \partial_{\mu_1 \dots \mu_{m-1} \alpha_1 \dots \alpha_{N-1}}$ gives

$$\begin{aligned}&m \tilde{K}_b^{(\beta \mu_1 \dots \mu_{m-1})} (\partial_\beta \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}}) \partial_{\mu_1 \dots \mu_{m-1} \alpha_1 \dots \alpha_{N-1}} \phi_i \\ &= m \tilde{K}_b^{(\beta \mu_1 \dots \mu_{m-1})} (\partial_\beta \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}}) A_{ia}^{\mu_m} \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \psi_a + \mathcal{O}(\partial^{m+N-2} \psi).\end{aligned}\quad (16)$$

Then, the last term on the right-hand side of Eq. (13) can be replaced with the derivatives of ϕ_i and $\mathcal{O}(\partial^{m+N-2} \psi)$ terms. Expanding Eq. (13) up to $\partial^{m+N-2} \psi$ and eliminating some $\partial^{m+N-1} \psi$ and $\partial^{m+N-2} \psi$ terms in favor of $\partial^{m+N-2} \phi$ and $\partial^{m+N-3} \phi$ terms as we did above, we finally obtain

$$\begin{aligned}&\tilde{K}_b^{(\mu_1 \dots \mu_m)} \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}} \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \phi_i + m \tilde{K}_b^{(\beta \mu_1 \dots \mu_{m-1})} (\partial_\beta \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}}) \partial_{\mu_1 \dots \mu_{m-1} \alpha_1 \dots \alpha_{N-1}} \phi_i \\ &\quad + \binom{m}{2} \tilde{K}_b^{(\beta \gamma \mu_1 \dots \mu_{m-2})} (\partial_{\beta \gamma} \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}}) \partial_{\mu_1 \dots \mu_{m-2} \alpha_1 \dots \alpha_{N-1}} \phi_i \\ &= \tilde{K}_b^{(\mu_1 \dots \mu_m)} \mathcal{A}_{2,ba}^{\alpha_1 \dots \alpha_{N-1}} \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \psi_a \\ &\quad + \left[\tilde{K}_b^{(\mu_1 \dots \mu_m)} \mathcal{B}_{2,ba}^{\alpha_1 \dots \alpha_{N-2}} + m \tilde{K}_b^{(\beta \mu_1 \dots \mu_{m-1})} (\partial_\beta \mathcal{A}_{2,ba}^{\mu_m \alpha_1 \dots \alpha_{N-2}}) \right] \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-2}} \psi_a \\ &\quad + \mathcal{O}(\partial^{m+N-3} \psi),\end{aligned}\quad (17)$$

where

$$\mathcal{A}_{2,ba}^{\alpha_1 \dots \alpha_{N-1}} := \bar{A}_{bi}^{\alpha_1 \dots \alpha_{N-1}} B_{ia} + (N-1) \bar{A}_{bi}^{\beta(\alpha_1 \dots \alpha_{N-2})} \left(\partial_\beta A_{ia}^{\alpha_{N-1}} \right), \quad (18)$$

$$\mathcal{B}_{2,ba}^{\alpha_1 \dots \alpha_{N-2}} := (N-1) \left[\bar{A}_{bi}^{\beta \alpha_1 \dots \alpha_{N-2}} (\partial_\beta B_{ia}) + \frac{N-2}{2} \bar{A}_{bi}^{\beta \gamma(\alpha_1 \dots \alpha_{N-3})} \left(\partial_{\beta \gamma} A_{ia}^{\alpha_{N-2}} \right) \right] \quad (19)$$

The right-hand side of Eq. (17) has one less derivative compared to that of Eq. (6), that is, it is the subleading equation. The combination of non-degenerate part of Eq. (6), which has $N-1$ equations, and Eq. (17) gives the structure of the characteristics in the subleading order.

So far we established the presence of one degeneracy, meanwhile to ensure invertibility, there must be N degeneracies. Hence the characteristics at this subleading order should be degenerate too. By assumption, the matrix $A_{ia}^\mu \xi_\mu$ has one degeneracy, that is, there exist only one eigenvector with zero eigenvalue, which is denoted $\psi_a^\perp(\xi)$. The other components of ψ_a are collectively defined as $\psi_a^\parallel(\xi)$. (Hereinafter, we omit (ξ) of $\psi_a^\perp(\xi)$ and $\psi_a^\parallel(\xi)$ for brevity.) The characteristic matrix for the highest derivative part of Eq. (17) and non-degenerate part of Eq. (6) is written as

$$\begin{array}{c} \text{non-degenerate part of Eq. (6)} \\ \text{Eq. (17)} \end{array} \begin{pmatrix} \psi^\parallel & \psi^\perp \\ \text{}^{(\text{nd})} A_{ia}^\mu \xi_\mu & 0 \\ K_a & K \end{pmatrix}. \quad (20)$$

where $(K_a, K) = \tilde{K}_b^{(\mu_1 \dots \mu_m)} \mathcal{A}_{2,ba}^{\alpha_1 \dots \alpha_{N-1}} \xi_{\mu_1} \dots \xi_{\mu_m} \xi_{\alpha_1} \dots \xi_{\alpha_{N-1}}$,³ and $\text{}^{(\text{nd})} A_{ia}^\mu \xi_\mu$ is a $(N-1) \times (N-1)$ matrix indicating only the non-degenerate components of $A_{ia}^\mu \xi_\mu$, that is, $\det(\text{}^{(\text{nd})} A_{ia}^\mu \xi_\mu) \neq 0$. This matrix determines the subleading characteristics, and it should be degenerate for the invertibility. Since $\text{}^{(\text{nd})} A_{ia}^\mu \xi_\mu$ is regular, the requirement of the degeneracy gives the condition that $K = 0$ for any $\tilde{K}_b^{(\mu_1 \dots \mu_m)}$, *i.e.* $\mathcal{A}_{2,ba}^{\alpha_1 \dots \alpha_{N-1}} \xi_{\alpha_1} \dots \xi_{\alpha_{N-1}} \psi_a^\perp$ vanishes for any ξ .

In order to show the condition explicitly, we express ψ^\perp in terms of A_{ia}^μ . It can be done with the adjugate matrix $\bar{A}_{ai}^{\mu_1 \dots \mu_{N-1}} \xi_{\mu_1} \dots \xi_{\mu_{N-1}}$. Equation (12) shows that $\bar{A}_{ai}^{\mu_1 \dots \mu_{N-1}} \xi_{\mu_1} \dots \xi_{\mu_{N-1}}$ is the projection matrix to the kernel of $A_{ia}^\mu \xi_\mu$, in which ψ^\perp lives by definition. Therefore, the subleading degeneracy condition is written as

$$\mathcal{A}_{2,ba}^{(\alpha_1 \dots \alpha_{N-1})} \bar{A}_{ai}^{\mu_1 \dots \mu_{N-1}} = 0. \quad (21)$$

2.1.4 Degeneracy and non-degeneracy conditions at lower orders

So far, two degeneracies were realized by imposing the degeneracy conditions (10) and (21). The condition (21) makes the matrix (20) degenerate, and then, the highest order derivative of the component parallel to ψ^\perp appears in a lower order equation, that is, the subsubleading order equation. The characteristic matrix is composed of the leading order equation for the ψ^\parallel components and of the subsubleading order equation for the ψ^\perp component. Now we have two degeneracy, and thus, for $N > 2$ this subsubleading characteristic matrix must give additional degeneracy for invertibility. Then, similar to the analysis of the subleading order, the degeneracy implies that the highest order derivative for ψ^\perp component in the subsubleading equation should vanish. This procedure is done iteratively until N degeneracies are established. After that, we impose the condition that the next-order characteristic matrix is not degenerate, in order for the transformation to be invertible. The last condition corresponds to that of the inverse function theorem (without derivatives).

In order to implement the procedure described above, it is useful to introduce a projection matrix $E_{N-1,ij}(\xi)$ to the $(N-1)$ -dimensional field space of ψ^\parallel , that is, $E_{N-1,ij}(\xi)$ is the identity matrix for the

³Using the projection tensor $E_{N-1,ab}$ onto the ψ^\parallel space defined by Eq. (26), the components of the vector (K_a, K) may be expressed more precisely as

$$\begin{aligned} K_a &\propto \tilde{K}_b^{(\mu_1 \dots \mu_m)} \mathcal{A}_{2,bc}^{\alpha_1 \dots \alpha_{N-1}} E_{N-1,cb} \xi_{\mu_1} \dots \xi_{\mu_m} \xi_{\alpha_1} \dots \xi_{\alpha_{N-1}}, \\ K &\propto \tilde{K}_b^{(\mu_1 \dots \mu_m)} \mathcal{A}_{2,bc}^{\alpha_1 \dots \alpha_{N-1}} \bar{A}_{ci}^{\nu_1 \dots \nu_{N-1}} \bar{A}_{ai}^{\nu_N \dots \nu_{2N-2}} \xi_{\mu_1} \dots \xi_{\mu_m} \xi_{\alpha_1} \dots \xi_{\alpha_{N-1}} \xi_{\nu_1} \dots \xi_{\nu_{2N-2}}. \end{aligned}$$

See section 2.1.4 for more details on this decomposition.

$(N-1)$ -dimensional field space and zero for the one-dimensional field space parallel to ψ^\perp . Let us express $E_{N-1,ij}(\xi)$ in terms of $A_{ia}^\mu \xi_\mu$. We consider a matrix

$$\tilde{A}_{ai}^{\mu_1 \dots \mu_{2N-3}} = \frac{1}{(N-1)!} \epsilon^{i_1 \dots i_{N-1}} \epsilon_{a_1 \dots a_{N-1}} \bar{A}_{a_1 i_1}^{(\mu_1 \dots \mu_{N-1}} A_{i_2 a_2}^{\mu_N} \dots A_{i_{N-1} a_{N-1}}^{\mu_{2N-3})}. \quad (22)$$

This matrix satisfies

$$\begin{aligned} (N-1) \tilde{A}_{ai}^{\mu_1 \dots \mu_{2N-3}} A_{ib}^{\mu_{2N-2}} \xi_{\mu_1} \dots \xi_{\mu_{2N-2}} \\ = (\bar{A}^{2, \mu_1 \dots \mu_{2N-2}} \delta_{ab} - \bar{A}_{ai}^{\mu_1 \dots \mu_{N-1}} \bar{A}_{bi}^{\mu_N \dots \mu_{2N-2}}) \xi_{\mu_1} \dots \xi_{\mu_{2N-2}} \\ = E_{N-1,ab}(\xi) \bar{A}^{2, \mu_1 \dots \mu_{2N-2}} \xi_{\mu_1} \dots \xi_{\mu_{2N-2}}, \end{aligned} \quad (23)$$

where

$$\bar{A}^{2, \mu_1 \dots \mu_{2N-2}} := \bar{A}_{ai}^{(\mu_1 \dots \mu_{N-1}} \bar{A}_{ai}^{\mu_N \dots \mu_{2N-2})} = \tilde{A}_{ai}^{(\mu_1 \dots \mu_{2N-3}} A_{ia}^{\mu_{2N-2})}. \quad (24)$$

In the calculation in Eq. (23), we use the fact that $\bar{A}_{ai}^{\mu_1 \dots \mu_{N-1}} \xi_{\mu_1} \dots \xi_{\mu_{N-1}}$ is rank-1 matrix, and then the components of $\bar{A}_{ai}^{\mu_1 \dots \mu_{N-1}} \bar{A}_{bi}^{\mu_N \dots \mu_{2N-2}} \xi_{\mu_1} \dots \xi_{\mu_{2N-2}}$ are zero except for the ψ^\perp - ψ^\perp component. The value of this nonzero component is shown to be $\bar{A}^{2, \mu_1 \dots \mu_{2N-2}} \xi_{\mu_1} \dots \xi_{\mu_{2N-2}}$ by direct calculations. Equation (23) divided by $\bar{A}^{2, \mu_1 \dots \mu_{2N-2}} \xi_{\mu_1} \dots \xi_{\mu_{2N-2}}$ gives $E_{N-1,ab}(\xi)$, which can be regarded as a projector to the $(N-1)$ -dimensional space of ψ^\parallel . Since Eq. (23) holds for any ξ^μ , it can be expressed equivalently as

$$\bar{A}_{ai}^{(\mu_1 \dots \mu_{N-1}} \bar{A}_{bi}^{\mu_N \dots \mu_{2N-2})} + (N-1) \tilde{A}_{ai}^{(\mu_1 \dots \mu_{2N-3}} A_{ib}^{\mu_{2N-2})} = \bar{A}^{2, (\mu_1 \dots \mu_{2N-2})} \delta_{ab}. \quad (25)$$

Let us demonstrate how to obtain the lower order equations iteratively with the projection tensor (23). On the right-hand side of Eq. (17), the leading term is the first term proportional to $\partial^{m+N-1} \psi_a$. The degeneracy condition for the subleading order (21) implies that the first term does not have the ψ^\perp component, *i.e.* it can be written as

$$\tilde{K}_b^{(\mu_1 \dots \mu_m)} \mathcal{A}_{2,ba}^{\alpha_1 \dots \alpha_{N-1}} \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \psi_a = \tilde{K}_b^{(\mu_1 \dots \mu_m)} \mathcal{A}_{2,bc}^{\alpha_1 \dots \alpha_{N-1}} E_{N-1,cd}(\partial) \partial_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_{N-1}} \psi_d. \quad (26)$$

Considering the case with $\tilde{K}_b^{\mu_1 \dots \mu_m} = \tilde{K}_b^{\mu_1 \dots \mu_{m'}} \bar{A}^{2, \gamma_1 \dots \gamma_{2N-2}}$, we can rewrite Eq. (26) as

$$\begin{aligned} \tilde{K}_b^{\mu_1 \dots \mu_{m'}} \bar{A}^{2, \gamma_1 \dots \gamma_{2N-2}} \mathcal{A}_{2,bc}^{\alpha_1 \dots \alpha_{N-1}} E_{N-1,ca}(\partial) \partial_{\mu_1 \dots \mu_{m'} \gamma_1 \dots \gamma_{2N-2} \alpha_1 \dots \alpha_{N-1}} \psi_k \\ = (N-1) \tilde{K}_b^{\mu_1 \dots \mu_{m'}} \mathcal{A}_{2,bc}^{\alpha_1 \dots \alpha_{N-1}} \tilde{A}_{ci}^{\gamma_1 \dots \gamma_{2N-3}} A_{ia}^{\gamma_{2N-2}} \partial_{\mu_1 \dots \mu_{m'} \alpha_1 \dots \alpha_{N-1} \gamma_1 \dots \gamma_{2N-2}} \psi_k \end{aligned} \quad (27)$$

where we used Eq. (23). Then, by subtracting Eq. (17) from Eq. (6) with its coefficient set to $K_{bi}^{\mu_1 \dots \mu_m} = (N-1) \tilde{K}_b^{\mu_1 \dots \mu_{m'}} \mathcal{A}_{2,bc}^{\alpha_1 \dots \alpha_{N-1}} \tilde{A}_{ci}^{\gamma_1 \dots \gamma_{2N-3}}$, the subleading ($\partial^{m'+3N-3} \psi$) term is canceled out and the subsubleading-order equation is obtained as a result.

The degeneracy condition can be obtained applying the procedure explained around (20) to the subsubleading equation, and the condition will be similar to the condition (21) for the subleading-order equation. Then, using the projector $E_{N-1,ab}$ again, we can construct the lower-order equation. Applying this procedure iteratively, we can derive N degeneracy conditions and the final non-degeneracy condition, which constitute the necessary conditions for the invertibility.

2.2 Necessary conditions in two-field case

Although we have already derived the leading and subleading degeneracy conditions in the previous subsection using the general procedure, it is instructive to follow a concrete example for understanding the method. For this purpose, based on the discussion in the previous section, we demonstrate how to derive the necessary conditions for the invertibility of a field transformation of two fields involving up to first derivative.⁴ That is, we consider the case where $N = 2$, $i = 1, 2$ and $a = 1, 2$. In this section, we will

⁴It can be shown that there is no invertible field transformation of one field involving first derivatives, because such a transformation inevitably increases the number of derivatives acting on the field and it generates additional degree of freedom. Hence, among (possibly invertible) transformations with up to first derivatives, the two field case is the simplest.

complete the iterations to derive all the degeneracy and the final non-degeneracy conditions. The obtained conditions will be simplified in section 2.3.

We apply the procedure explained in section 2.1 to the two-field case in this section. In this case, \bar{A}_{ai}^μ , $\mathcal{A}_{2,ab}^\mu$ and $\mathcal{B}_{2,ab}$ appeared in the previous section are written as

$$\bar{A}_{ai}^\mu = \epsilon_i^{i_1} \epsilon_a^{a_1} A_{i_1 a_1}^\mu, \quad \mathcal{A}_{2,ab}^\mu := \bar{A}_{ai}^\mu B_{ib} + \bar{A}_{ai}^\beta \partial_\beta A_{ib}^\mu, \quad \mathcal{B}_{2,ab} := \bar{A}_{ai}^\beta \partial_\beta B_{ib}, \quad (28)$$

and \tilde{A}_{ai}^μ is simplified as

$$\tilde{A}_{ai}^\mu = \epsilon_i^{i_1} \epsilon_a^{a_1} \bar{A}_{a_1 i_1}^\mu = \epsilon_i^{i_1} \epsilon_a^{a_1} \epsilon_{i_1}^{i_2} \epsilon_{a_1}^{a_2} A_{i_2 a_2}^\mu = A_{ia}^\mu. \quad (29)$$

Then the identity (25) becomes

$$\bar{A}_{ai}^{(\mu_1} \bar{A}_{bi}^{\mu_2)} + \tilde{A}_{ai}^{(\mu_1} A_{ib}^{\mu_2)} = \bar{A}_{ai}^{(\mu_1} \bar{A}_{bi}^{\mu_2)} + A_{ia}^{(\mu_1} A_{ib}^{\mu_2)} = \bar{A}^{2, \mu_1 \mu_2} \delta_{ab}. \quad (30)$$

We operate $K^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n}$ on the field transformation equation (5) to obtain

$$K^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n} \phi_i = K^{(\mu_1 \dots \mu_n)} A_{ia}^{\mu_{n+1}} \partial_{\mu_1 \dots \mu_{n+1}} \psi_a + \mathcal{O}(\partial^n \psi). \quad (31)$$

For (5) to be invertible, the coefficient of the highest-order derivative on the right-hand side Eq. (31) must be degenerate, which implies

$$\forall \xi_\mu, \quad \det(A_{ia}^\mu \xi_\mu) = 0 \quad \Leftrightarrow \quad \epsilon^{i_1 i_2} \epsilon^{a_1 a_2} A_{i_1 a_1}^{(\alpha_1} A_{i_2 a_2}^{\alpha_2)} = 0. \quad (32)$$

Following the previous section, we assume that the matrix $A_{ia}^\mu \xi_\mu$ is degenerate only in one dimension. Then the kernel of $A_{ia}^\mu \xi_\mu$ parallel to ψ_a^\perp and the field space in the other direction parallel to ψ_a^\parallel are one dimension each.

The degeneracy condition at the subleading order is obtained following the procedure in section 2.1.3. To construct the subleading equation from Eq. (5), we have to pick up the component of this equation corresponding to the kernel of the coefficient of the highest order derivative, A_{ia}^μ . It can be done by operating $\tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \partial_{\mu_1 \dots \mu_{n+1}}$ to the transformation equation (5) as

$$\begin{aligned} & \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \partial_{\mu_1 \dots \mu_{n+1}} \phi_i \\ &= \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} A_{ia}^\nu \partial_{\mu_1 \dots \mu_{n+1}} \psi_a \\ &+ \left(\tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} B_{ia} + n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}_{bi}^{\mu_n} \partial_\alpha A_{ia}^{\mu_{n+1}} + \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^\alpha \partial_\alpha A_{ia}^{\mu_{n+1}} \right) \partial_{\mu_1 \dots \mu_{n+1}} \psi_a \\ &+ \mathcal{O}(\partial^n \psi). \end{aligned} \quad (33)$$

The first term on the right-hand side vanishes thanks to Eq. (32). As shown in section 2.1.3, the second term in the brackets proportional to n can be canceled by adding $n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} (\partial_\alpha \bar{A}_{bi}^{\mu_n}) \partial_{\mu_1 \dots \mu_n} \phi_i$ to Eq. (33). Nevertheless, without such a cancellation, we can directly obtain the same subleading condition from Eq. (33) by rewriting it as

$$\begin{aligned} & \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \partial_{\mu_1 \dots \mu_{n+1}} \phi_i \\ &= \left(\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,ba}^{\mu_{n+1}} + n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}_{bi}^{\mu_n} \partial_\alpha A_{ia}^{\mu_{n+1}} \right) \partial_{\mu_1 \dots \mu_{n+1}} \psi_a + \mathcal{O}(\partial^n \psi). \end{aligned} \quad (34)$$

The subleading degeneracy condition is given by demanding that the coefficient of highest-order derivative for ψ^\perp vanishes. Noting that $\bar{A}_{ai}^\nu \xi_\nu$ works as a projector onto the ψ^\perp space, this condition is written as

$$\left(\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,ba}^{\mu_{n+1}} + n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}_{bi}^{\mu_n} \partial_\alpha A_{ia}^{\mu_{n+1}} \right) \bar{A}_{aj}^\nu \xi_{\mu_1} \dots \xi_{\mu_{n+1}} \xi_\nu = 0. \quad (35)$$

The second term vanishes, which is shown by using Eq. (12) and (14) as

$$\begin{aligned} & \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}_{bi}^{\mu_n} (\partial_\alpha A_{ia}^{\mu_{n+1}}) \bar{A}_{aj}^\nu \xi_{\mu_1} \dots \xi_{\mu_{n+1}} \xi_\nu \\ &= -\tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} (\partial_\alpha \bar{A}_{bi}^{\mu_n}) A_{ia}^{\mu_{n+1}} \bar{A}_{aj}^\nu \xi_{\mu_1} \dots \xi_{\mu_{n+1}} \xi_\nu = 0. \end{aligned} \quad (36)$$

Since $\tilde{K}^{(\mu_1 \dots \mu_n)}$ is arbitrary, the subleading condition is

$$\forall \xi_\mu, \quad \mathcal{A}_{2,ba}^\mu \bar{A}_{aj}^\nu \xi_\mu \xi_\nu = 0 \quad \Leftrightarrow \quad \mathcal{A}_{2,ba}^{(\mu} \bar{A}_{aj}^{\nu)} = \bar{A}_{bi}^{(\mu} B_{ia} \bar{A}_{aj}^{\nu)} + \bar{A}_{bi}^\beta \left(\partial_\beta A_{ia}^{(\mu} \bar{A}_{aj}^{\nu)} \right) = 0. \quad (37)$$

The non-degeneracy condition at the subsubleading order can be constructed following the procedure in section 2.1.4. In this procedure we need to see the structure of the coefficient of highest-order derivative term in Eq. (33) decomposing the variables into the ψ^\perp and ψ^\parallel space. For this purpose we use Eq. (30). Operating $\tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \mu_n + 2\mu_{n+3}} \partial_{\mu_1 \dots \mu_{n+3}}$ to Eq. (5), we have

$$\begin{aligned} & \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \mu_n + 2\mu_{n+3}} \partial_{\mu_1 \dots \mu_{n+3}} \phi_i \\ &= \left(\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,ba}^{\mu_{n+1}} \bar{A}^{2, \mu_n + 2\mu_{n+3}} + n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \mu_{n+1}} \bar{A}_{bi}^{\mu_{n+2}} \partial_\alpha A_{ia}^{\mu_{n+3}} \right. \\ & \quad \left. + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \mu_n + 1\alpha} \bar{A}_{bi}^{\mu_{n+2}} \partial_\alpha A_{ia}^{\mu_{n+3}} \right) \partial_{\mu_1 \dots \mu_{n+3}} \psi_a + \mathcal{O}(\partial^{n+2} \psi). \end{aligned} \quad (38)$$

Using Eq. (30) for the first term in brackets and Eq. (14) for the other two, we have

$$\begin{aligned} & \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \mu_n + 2\mu_{n+3}} \partial_{\mu_1 \dots \mu_{n+3}} \phi_i \\ &= \left(\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} - n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \mu_{n+1}} \partial_\alpha \bar{A}_{bi}^{\mu_{n+2}} \right. \\ & \quad \left. - 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \mu_n + 1\alpha} \partial_\alpha \bar{A}_{bi}^{\mu_{n+2}} \right) A_{ia}^{\mu_{n+3}} \partial_{\mu_1 \dots \mu_{n+3}} \psi_a + \mathcal{O}(\partial^{n+2} \psi). \end{aligned} \quad (39)$$

The $\partial^{n+3} \psi$ term in the right-hand side can be reproduced by applying the operator

$$\left[\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} - \left(n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \mu_{n+1}} + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \mu_n + 1\alpha} \right) \partial_\alpha \bar{A}_{bi}^{\mu_{n+2}} \right] \partial_{\mu_1 \dots \mu_{n+2}}$$

to Eq. (5). Hence, by subtracting it from Eq. (39), we obtain the subsubleading-order equation involving terms only up to $\partial^{n+2} \psi$. Expanding Eqs. (6) and Eq. (39) up to the $\partial^{n+2} \psi$ terms, we can show by a straightforward calculation that (see Appendix B for details)

$$\begin{aligned} & \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \mu_n + 2\mu_{n+3}} \partial_{\mu_1 \dots \mu_{n+3}} \phi_i \\ & - \left[\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} - \left(n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \mu_{n+1}} + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \mu_n + 1\alpha} \right) \partial_\alpha \bar{A}_{bi}^{\mu_{n+2}} \right] \partial_{\mu_1 \dots \mu_{n+2}} \phi_i \\ &= \left\{ -\frac{n(n-1)}{2} \tilde{K}^{(\alpha_1 \alpha_2 \mu_1 \dots \mu_{n-2})} \bar{A}^{2, \mu_n - 1\mu_n} \left(\partial_{\alpha_1 \alpha_2} \bar{A}_{bi}^{\mu_{n+1}} \right) A_{ia}^{\mu_{n+2}} \right. \\ & \quad \left. + n \tilde{K}^{(\alpha_1 \mu_1 \dots \mu_{n-1})} \left[-2 \bar{A}^{2, \mu_n \alpha_2} \left(\partial_{\alpha_1 \alpha_2} \bar{A}_{bi}^{\mu_{n+1}} \right) A_{ia}^{\mu_{n+2}} - \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} \partial_{\alpha_1} A_{ia}^{\mu_{n+2}} + \bar{A}^{2, \mu_n \mu_{n+1}} \partial_{\alpha_1} \mathcal{A}_{ba}^{\mu_{n+2}} \right] \right. \\ & \quad \left. + \tilde{K}^{(\mu_1 \dots \mu_n)} \left[-\bar{A}^{2, \alpha_1 \alpha_2} \left(\partial_{\alpha_1 \alpha_2} \bar{A}_{bi}^{\mu_{n+1}} \right) A_{ia}^{\mu_{n+2}} + 2 \bar{A}^{2, \mu_n + 1\alpha} \partial_\alpha \mathcal{A}_{2,ba}^{\mu_{n+2}} + \bar{A}^{2, \mu_n + 1\mu_{n+2}} \bar{A}_{bi}^\alpha \partial_\alpha B_{ia} \right. \right. \\ & \quad \left. \left. - 2 \mathcal{A}_{2,bc}^{(\alpha} \tilde{A}_{ci}^{\mu_{n+1})} \partial_\alpha A_{ia}^{\mu_{n+2}} - \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} B_{ia} \right] \right\} \partial_{\mu_1 \dots \mu_{n+2}} \psi_a + \mathcal{O}(\partial^{n+1} \psi_a). \end{aligned} \quad (40)$$

The coefficient of ψ^\perp component of the highest order derivative term $\partial^{n+2} \psi$ on the right-hand side of Eq. (40) determines the (non)degeneracy of the characteristics in the subsubleading order. This coefficient is obtained by contracting $\bar{A}_{ai}^{\mu_{n+3}}$ with the coefficient of $\partial^{n+2} \psi_a$ in Eq. (40). We can show that the terms proportional to $\tilde{K}^{(\alpha_1 \alpha_2 \mu_1 \dots \mu_{n-2})}$ and $\tilde{K}^{(\alpha_1 \mu_1 \dots \mu_{n-1})}$ become zero (see Appendix B). Among the terms proportional to $\tilde{K}^{(\mu_1 \dots \mu_n)}$, the first one ($\partial_{\alpha_1 \alpha_2} \bar{A}_{bi}^{\mu_{n+1}}$ term) vanishes by acting $\bar{A}_{ai}^{\mu_{n+3}}$, and the remaining term must not vanish for the ψ^\perp component to be non-degenerate in this order. Hence, the non-degeneracy condition is given by

$$\left(\bar{A}^{2, \mu_1 \mu_2} \mathcal{B}_{2,ab} - \mathcal{A}_{2,ac}^{\mu_1} \tilde{A}_{cj}^{\mu_2} B_{jb} + 2 \bar{A}^{2, \alpha \mu_1} \partial_\alpha \mathcal{A}_{2,ab}^{\mu_2} - 2 \mathcal{A}_{2,ac}^{(\alpha} \tilde{A}_{cj}^{\mu_1)} \partial_\alpha A_{jb}^{\mu_2} \right) \bar{A}_{bi}^{\mu_3} \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \neq 0 \quad (41)$$

for any ξ . Using Eq. (30), this can be also written as

$$\left(\bar{A}^{2, \mu_1 \mu_2} \mathcal{B}_{2,ab} \bar{A}_{bi}^{\mu_3} - \mathcal{A}_{2,ab}^{\mu_1} \tilde{A}_{bj}^{\mu_2} B_{jc} \bar{A}_{ci}^{\mu_3} + 2 \mathcal{A}_{2,ab}^{[\alpha} \tilde{A}_{bj}^{\mu_1} A_{jc}^{\mu_2]} \partial_\alpha \bar{A}_{ci}^{\mu_3} - \mathcal{A}_{2,ab}^{\mu_1} \bar{A}_{bj}^\alpha \bar{A}_{cj}^{\mu_2} \partial_\alpha \bar{A}_{ci}^{\mu_3} \right) \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \neq 0. \quad (42)$$

To summarize, the necessary conditions for the invertibility are given by Eqs. (32), (37) and (42).

2.3 Simplification of necessary conditions

We have derived above the necessary conditions for the invertibility (32), (37) and (42) in the two-field case with first-order derivative. We present a simpler form of these expressions in this section. The simplified form of the necessary conditions will be helpful in the proof that these conditions are also sufficient conditions shown in the next section. In addition, we will use these expressions for building examples of invertible transformations.

2.3.1 Simplification of condition (32)

First of all, we simplify condition (32). Equation (32) implies that the rank of $A_{ia}^\mu \xi_\mu$ is less than two. Since we have assumed that $A_{ia}^\mu \xi_\mu$ has only one eigenvector with zero eigenvalue, the rank of $A_{ia}^\mu \xi_\mu$ has to be unity for any ξ , that is,

$$\xi_0 A_{ia}^0 + \xi_1 A_{ia}^1 + \cdots + \xi_D A_{ia}^D \quad (43)$$

is a rank-one matrix for any $\xi_\mu = \{\xi_0, \xi_1, \dots, \xi_D\}$, where D is the spacetime dimension. Then, by setting $\xi_\mu = \delta_{\mu\nu}$ for each $1 \leq \nu \leq D$, we find that A_{ia}^ν is a rank-one matrix for any ν . Any rank-one 2×2 matrix can be written by a product of vectors, *i.e.* the matrices $A_{ia}^0, A_{ia}^1, \dots, A_{ia}^D$ are written as

$$A_{ia}^0 = V_i^0 U_a^0, \quad A_{ia}^1 = V_i^1 U_a^1, \quad \dots \quad A_{ia}^D = V_i^D U_a^D. \quad (44)$$

Below, we show by the induction that, if the rank of (43) is one for any ξ , A_{ia}^μ can be written as

$$A_{ia}^\mu = V_i U_a^\mu \quad \text{or} \quad A_{ia}^\mu = V_i^\mu U_a. \quad (45)$$

For $\mu \leq 0$ (that is, $\mu = 0$), A_{ia}^0 is written as $V_i^0 U_a^0$, and thus, regarding (V_i^0, U_a^0) as (V_i, U_a^μ) or (V_i^μ, U_a) for $\mu = 0$, Eq. (45) is satisfied. Then, what we need to show is that, for any integer k , Eq. (45) is satisfied for $\mu \leq k+1$ if it is satisfied for $\mu \leq k$. Since two choices of (45) are symmetric with respect to U_a and V_i , without loss of generality we assume the former ($A_{ia}^\mu = V_i U_a^\mu$) is satisfied for $\mu \leq k$. If $A_{ia}^{k+1} = 0$, $A_{ia}^\mu = V_i U_a^\mu$ for $\mu \leq k+1$ is trivially satisfied. Therefore, we consider the case where $A_{ia}^{k+1} = V_i^{k+1} U_a^{k+1} \neq 0$. If V_i^{k+1} is parallel to V_i , by rescaling $V_i^{k+1} \rightarrow c V_i^{k+1} = V_i$, $U_a^{k+1} \rightarrow U_a^{k+1}/c$ with a constant c , we can satisfy $A_{ia}^\mu = V_i U_a^\mu$ for $\mu \leq k+1$.

We show that if V_i^{k+1} is not parallel to V_i , all U_a^μ 's for $\mu \leq k+1$ become parallel, and thus A_{ia}^μ can be written as $V_i^\mu U_a$. We consider a vector $\xi_\mu = (\xi_0, \xi_1, \dots, \xi_{k+1}, 0, 0, \dots)$, where ξ_μ 's ($\mu \leq k+1$) are arbitrary. Since $A_{ia}^\mu \xi_\mu$ should have an eigenvector $e^a(\xi_\mu)$ with zero eigenvalue, we have

$$0 = A_{ia}^\mu \xi_\mu e^a = V_i \sum_{\mu=0}^k U_a^\mu e^a \xi_\mu + V_i^{k+1} U_a^{k+1} e^a \xi_{k+1}. \quad (46)$$

If V_i^{k+1} is not parallel to V_i , the above equation gives

$$\sum_{\mu=0}^k U_a^\mu \xi_\mu e^a = 0, \quad U_a^{k+1} \xi_{k+1} e^a = 0. \quad (47)$$

Now suppose that $\xi_{k+1} \neq 0$. In this case, the latter of these equations uniquely fixes e^a because the field space dimension is two. Then, because e^a is independent of ξ_μ ($\mu \leq k$) and also ξ_μ 's for $\mu \leq k$ are arbitrary, Eq. (47) implies

$$U_a^\mu e^a = 0 \quad (\mu \leq k+1). \quad (48)$$

This implies that all U_a^μ 's ($\mu \leq k+1$) are normal to e^a . Since e^a is non-zero and the space of index a is two-dimensional, all U_a^μ 's ($\mu \leq k+1$) are parallel to each other. Therefore, by rescaling U_a , A_{ia}^μ can be expressed in terms of the common U_a as

$$A_{ia}^\mu = C^\mu V_i U_a \quad (\mu \leq k), \quad A_{ia}^{k+1} = V_i^{k+1} U_a, \quad (49)$$

where C^μ is a normalization factor. Therefore, by redefining $\{C^\mu V_i^\mu, V_i^{k+1}\}$ ($\mu \leq k$) as $\{V_i, V_i^{k+1}\}$, A_{ia}^μ is written as $V_i^\mu U_a$ for $\mu \leq k+1$.

2.3.2 Further decomposition of A_{ia}^μ

We have seen that one of the invertibility conditions, Eq. (32), implies that A_{ia}^μ is decomposed as Eq. (45). Using the other conditions, we show that A_{ia}^μ can be decomposed further as

$$A_{ia}^\mu = a^\mu V_i U_a. \quad (50)$$

Let us show the proof in each case of Eq. (45).

i). $A_{ia}^\mu = V_i U_a^\mu$

In this case, if we can show $U_a^\mu = a^\mu U_a$, A_{ia}^μ is expressed as $a^\mu V_i U_a$. We define a vector normal n_i to V_j as

$$n_j := \epsilon_{ij} V_j. \quad (51)$$

We also define a vector m_a^μ with spacetime index μ as

$$m_a^\mu := \epsilon_{ab} U_b^\mu. \quad (52)$$

Then, \bar{A}_{ia}^μ and \tilde{A}_{ia}^μ are written as

$$\bar{A}_{ai}^\mu = m_a^\mu n_i, \quad \tilde{A}_{ai}^\mu = U_a^\mu V_i, \quad (53)$$

and condition (37) becomes

$$n_i B_{ia} m_a^\mu = 0 \quad (\text{for any } \mu). \quad (54)$$

Now, we show that U_a^μ has to be written as $a^\mu U_a$. It can be proven by contradiction; the assumption that two of U_a^μ 's are not parallel to each other (for instance, U_a^0 is not parallel to U_a^1) leads to a contradiction. Under this assumption, two of m_a^μ 's are not parallel correspondingly. Since the space spanned by index "a" is two-dimensional, Eq. (54) implies that

$$n_i B_{ia} = 0. \quad (55)$$

Then, the left hand side of (42) becomes

$$\begin{aligned} & \left[V_j^2 U_c^{\mu_1} U_c^{\mu_2} m_a^\beta n_k (\partial_\beta B_{kb}) m_b^{\mu_3} n_i - m_a^\beta U_b^{\mu_1} n_k (\partial_\beta V_k) U_b^{\mu_2} V_j B_{jc} m_c^{\mu_3} n_i \right. \\ & \quad - m_a^\beta U_b^{\mu_1} n_k (\partial_\beta V_k) m_b^\alpha n_j m_c^{\mu_2} n_j \partial_\alpha (m_c^{\mu_3} n_i) + m_a^\beta U_b^\alpha n_k (\partial_\beta V_k) U_b^{\mu_1} V_j U_c^{\mu_2} V_j \partial_\alpha (m_c^{\mu_3} n_i) \\ & \quad \left. - m_a^\beta U_b^{\mu_1} n_k (\partial_\beta V_k) U_b^{\mu_2} V_j U_c^\alpha V_j \partial_\alpha (m_c^{\mu_3} n_i) \right] \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \\ & = \left\{ V_j^2 U_c^{\mu_1} U_c^{\mu_2} m_a^\beta m_b^{\mu_3} n_i [n_k (\partial_\beta B_{kb}) + (\partial_\beta n_k) B_{kb}] \right. \\ & \quad - V_j^2 m_a^\beta n_k (\partial_\beta V_k) U_b^{\mu_1} U_b^{\mu_2} (U_c^{\mu_3} m_c^\alpha + U_c^\alpha m_c^{\mu_3}) (\partial_\alpha n_i) \\ & \quad \left. - V_j^2 m_a^\beta n_k (\partial_\beta V_k) (U_b^{\mu_1} m_b^\alpha m_c^{\mu_2} - U_b^\alpha U_b^{\mu_1} U_c^{\mu_2} + U_b^{\mu_1} U_b^{\mu_2} U_c^\alpha) (\partial_\alpha m_c^{\mu_3}) n_i \right\} \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \\ & = 0. \end{aligned} \quad (56)$$

Here we used Eq. (55) and

$$\begin{aligned} \mathcal{A}_{2,ab}^\mu &= m_a^\beta U_b^\mu n_k (\partial_\beta V_k), \quad \mathcal{B}_{2,ab} = m_a^\beta n_i \partial_\beta B_{ib}, \quad \bar{A}^{2,\mu_1\mu_2} = U_a^{\mu_1} U_a^{\mu_2} V_i V_i, \\ m_a^\mu m_a^\nu &= U_a^\mu U_a^\nu, \quad U_c^{(\mu_1} m_c^{\mu_2)} = 0, \quad V_k V_j + n_k n_j = V_i^2 \delta_{kj}, \quad m_b^{(\mu_1} m_c^{\mu_2)} + U_b^{(\mu_1} U_c^{\mu_2)} = U_a^{\mu_1} U_a^{\mu_2} \delta_{bc}, \\ n_k (\partial_\beta V_k) V_j B_{jc} &= -(\partial_\beta n_k) V_k V_j B_{jc} = -(\partial_\beta n_k) (V_k V_j + n_k n_j) B_{jc} = -V_k^2 (\partial_\beta n_j) B_{jc}, \\ n_k (\partial_\beta B_{kb}) + (\partial_\beta n_k) B_{kb} &= \partial_\beta (n_k B_{kb}) = 0, \\ U_b^{(\mu_1} m_b^\alpha m_c^{\mu_2)} - U_b^\alpha U_b^{(\mu_1} U_c^{\mu_2)} + U_b^{(\mu_1} U_b^{\mu_2)} U_c^\alpha &= -U_b^\alpha (m_b^{(\mu_1} m_c^{\mu_2)} + U_b^{\mu_1} U_c^{\mu_2}) + U_b^{\mu_1} U_b^{\mu_2} U_c^\alpha = 0. \end{aligned} \quad (57)$$

Equation (56) is inconsistent with the non-degeneracy condition (42), and thus all U_a^μ 's are parallel. Hence, U_a^μ can be written as $a^\mu U_a$.

ii). $A_{ia}^\mu = V_i^\mu U_a$

The proof in this case is parallel to that for the case $A_{ia}^\mu = V_i U_a^\mu$ shown above. Defining m_a and n_i^μ similarly as

$$m_a := \epsilon_{ij} U_a, \quad n_i^\mu := \epsilon_{ij} V_j^\mu, \quad (58)$$

the condition (37) becomes

$$n_i^\mu \left(B_{ia} - V_i^\beta \partial_\beta U_a \right) m_a = 0 \quad (\text{for any } \mu), \quad (59)$$

where we used

$$n_i^\beta V_i^\mu = \epsilon_{ij} V_j^\beta V_i^\mu = -V_i^\beta n_i^\mu. \quad (60)$$

As is the case $A_{ia}^\mu = V_i U_a^\mu$, we show $V_i^\mu = a^\mu V_i$ with proof by contradiction; we assume that two of V_i^μ 's are not parallel to each other. Then, two of n_i^μ 's are not parallel to each other, which gives a condition that

$$(B_{ia} - V_i^\beta \partial_\beta U_a) m_a = 0. \quad (61)$$

However, we can show that it is inconsistent with the non-degeneracy condition (42). The left-hand side of (42) is calculated as

$$\begin{aligned} & \left\{ U_c^2 V_k^{\mu_1} V_k^{\mu_2} m_a n_j^\beta (\partial_\beta B_{jb}) m_b n_i^{\mu_3} - m_a \left[n_j^{\mu_1} B_{jb} + n_j^\beta \partial_\beta (V_j^{\mu_1} U_b) \right] U_b V_k^{\mu_2} B_{kc} m_c n_i^{\mu_3} \right. \\ & \quad \left. + 2m_a \left[n_j^{[\alpha]} B_{jb} + n_j^\beta \partial_\beta (V_j^{[\alpha]} U_b) \right] U_b V_k^{\mu_1} V_k^{[\mu_2]} U_c \partial_\alpha (m_c n_i^{\mu_3}) \right\} \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \\ & = \left\{ m_a V_k^{\mu_1} V_k^{\mu_2} n_j^\beta [(\partial_\beta B_{jc}) U_b^2 m_c + B_{jb} U_b U_c \partial_\beta m_c + \partial_\beta (V_j^\alpha U_b) U_b U_c \partial_\alpha m_c] n_i^{\mu_3} \right. \\ & \quad \left. - m_a \left[n_j^{\mu_1} B_{jb} + n_j^\beta \partial_\beta (V_j^{\mu_1} U_b) \right] U_b V_k^{\mu_2} (B_{kc} - V_k^\alpha \partial_\alpha U_c) m_c n_i^{\mu_3} \right\} \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \\ & = m_a V_k^{\mu_1} V_k^{\mu_2} n_j^\beta \left[\partial_\beta (B_{jc} m_c) U_b^2 - B_{jb} m_b m_c \partial_\beta m_c + \partial_\beta (V_j^\alpha U_b) U_b U_c \partial_\alpha m_c \right] n_i^{\mu_3} \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \\ & = m_a V_k^{\mu_1} V_k^{\mu_2} n_j^\beta \left\{ \partial_\beta [V_j^\alpha (\partial_\alpha U_c) m_c] U_b^2 - V_j^\alpha (\partial_\alpha U_b) m_b m_c \partial_\beta m_c \right. \\ & \quad \left. + (\partial_\beta V_j^\alpha) U_b^2 U_c \partial_\alpha m_c + V_j^\alpha (\partial_\beta U_b) U_b U_c \partial_\alpha m_c \right\} n_i^{\mu_3} \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \\ & = m_a V_k^{\mu_1} V_k^{\mu_2} n_j^\beta \left\{ U_b^2 [(\partial_\beta V_j^\alpha) (\partial_\alpha U_c) m_c + V_j^\alpha (\partial_\alpha U_c) \partial_\beta m_c + (\partial_\beta V_j^\alpha) U_c \partial_\alpha m_c] \right. \\ & \quad \left. - V_j^\alpha (\partial_\alpha U_b) (U_b U_c + m_b m_c) \partial_\beta m_c \right\} n_i^{\mu_3} \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \\ & = 0, \end{aligned} \quad (62)$$

where we used Eqs. (60), (61) and

$$\begin{aligned} \mathcal{A}_{2,ab}^\mu &= m_a \left[n_i^\mu B_{ib} + n_i^\beta \partial_\beta (V_i^\mu U_b) \right], & \mathcal{B}_{2,ab} &= m_a n_i^\beta \partial_\beta B_{ib}, & \bar{A}^{2,\mu_1\mu_2} &= U_a U_b V_i^{\mu_1} V_i^{\mu_2}, \\ \mathcal{A}_{2,ab}^\mu m_b &= 0, & U_c \partial_\alpha m_c &= -(\partial_\alpha U_c) m_c, & V_j^\alpha n_j^\beta \partial_\alpha \beta U_c &= 0, & U_a U_b + m_a m_b &= U_c^2 \delta_{ab}, \\ n_j^\beta V_j^\alpha (\partial_\beta U_b) U_b U_c (\partial_\alpha m_c) &= -n_j^\beta V_j^\alpha (\partial_\alpha U_b) U_b U_c (\partial_\beta m_c). \end{aligned} \quad (63)$$

This is inconsistent with the non-degeneracy condition (42), and thus all n_i^μ 's are parallel. This implies that V_i^μ can be written as

$$V_j^\mu = a^\mu V_j. \quad (64)$$

2.3.3 Further simplification of conditions (37) and (42)

We have shown that, for the invertibility conditions to be satisfied, A_{ia}^μ should be written as

$$A_{ia}^\mu = a^\mu V_i U_a. \quad (65)$$

Without loss of generality, we can normalize V_i and U_a as $V_i V_i = 1 = U_a U_a$. We define unit vectors n_i and m_a that are normal to V_i and U_a respectively as

$$n_i := \epsilon_{ij} V_j, \quad m_a := \epsilon_{ab} U_b. \quad (66)$$

Since A_{ia}^μ is written with a^μ , V_i and U_a , the matrices \bar{A}_{ai}^μ , \tilde{A}_{ai}^μ , also $\bar{A}^{2,\mu\nu}$, $\mathcal{A}_{2,ab}^\mu$ and $\mathcal{B}_{2,ab}$ are written as

$$\begin{aligned} \bar{A}_{ai}^\mu &= a^\mu m_a n_i, & \tilde{A}_{ai}^\mu &= a^\mu U_a V_i, & \bar{A}^{2,\mu\nu} &= a^\mu a^\nu, \\ \mathcal{A}_{2,ab}^\mu &= a^\mu m_a n_i \left[B_{ib} + a^\beta (\partial_\beta V_i) U_b \right], & \mathcal{B}_{2,ab} &= m_a n_i a^\beta \partial_\beta B_{ib}. \end{aligned} \quad (67)$$

Substituting the above equations into the condition (37) we have

$$n_i B_{ia} m_a = 0. \quad (68)$$

The non-degeneracy condition (42) becomes

$$n_i a^\beta (\partial_\beta B_{ia}) m_a - n_i B_{ia} U_a V_j B_{jb} m_b - n_i a^\beta (\partial_\beta V_i) V_j B_{ja} m_a \neq 0. \quad (69)$$

The first term of the left-hand side can be transformed as

$$\begin{aligned} n_i a^\beta (\partial_\beta B_{ia}) m_a &= -a^\beta (\partial_\beta n_i) B_{ia} m_a - n_i B_{ia} a^\beta \partial_\beta m_a \\ &= -a^\beta (\partial_\beta n_i) B_{ia} m_a - n_i B_{ia} U_a U_b a^\beta \partial_\beta m_b \\ &= -a^\beta (\partial_\beta n_i) B_{ia} m_a + n_i B_{ia} U_a a^\beta (\partial_\beta U_b) m_b, \end{aligned} \quad (70)$$

where we use the fact that $U_a U_b + m_a m_b = \delta_{ab}$ and Eq. (68). The last term of (69) can be transformed as

$$-n_i a^\beta (\partial_\beta V_i) V_j B_{ja} m_a = a^\beta (\partial_\beta n_i) B_{ia} m_a, \quad (71)$$

where we use the fact that $V_i V_j + n_i n_j = \delta_{ij}$ and Eq. (68). Then, condition (69) can be written as

$$n_i B_{ia} U_a \left(V_j B_{jb} - a^\beta \partial_\beta U_b \right) m_b \neq 0. \quad (72)$$

This means that both of

$$n_i B_{ia} U_a \neq 0, \quad \left(V_j B_{jb} - a^\beta \partial_\beta U_b \right) m_b \neq 0 \quad (73)$$

should be satisfied. As a result, the invertibility conditions (32), (37) and (42) are equivalent to the simplified conditions (50), (68) and (73).

3 Sufficiency of the invertibility conditions

In the previous section, we derived the necessary conditions (50), (68) and (73) for invertibility of a field transformation involving two fields and up to first-order derivatives of them. In this section, we show that these conditions are also sufficient conditions, that is, (50), (68) and (73) are the necessary and sufficient conditions for the invertibility in the two-field case. As a preliminary step, in section 3.1, we introduce a notion of ‘‘partial invertibility’’ for a field transformation whose inverse transformation is uniquely determined for a part of variables. Then, we show in section 3.2 with the partial invertibility that (50), (68) and (73) are the necessary and sufficient conditions for invertibility.

3.1 Partial invertibility

We consider invertibility of a transformation $\psi_a \rightarrow \phi_i$. Suppose that the transformation can be described by

$$F_I(\psi_a, \partial_\mu \psi_a, \partial_\mu \partial_\nu \psi_a, \dots; \phi_i, \partial_\mu \phi_i, \partial_\mu \partial_\nu \phi_i, \dots; x^\mu) = 0, \quad (74)$$

where I runs from 1 to a constant \mathcal{N} . If \mathcal{N} is equal to the number of all the degrees of freedom in $\psi_a, \partial_\mu \psi_a, \partial_\mu \partial_\nu \psi_a, \dots$,⁵ which is denoted by \mathcal{N}_ψ , we can use the implicit function theorem by regarding $\psi_a, \partial_\mu \psi_a, \partial_\mu \partial_\nu \psi_a, \dots$ as independent variables at a point in spacetime. However, we may not need to have \mathcal{N}_ψ equations of the form of (74) to fix ψ_a uniquely in terms of ϕ_a , indeed, even if we have less than \mathcal{N}_ψ equations, *i.e.* $\mathcal{N} < \mathcal{N}_\psi$; it may be possible to prove the uniqueness of ψ_a if the equations have a special structure. Once ψ_a is fixed in this manner, its derivatives $\partial_\mu \psi_a, \partial_\mu \partial_\nu \psi_a, \dots$ are uniquely fixed by differentiating ψ_a repeatedly.

To illustrate the procedure explained above, we first discuss a generic case, where equations are for functions of variables x_a, y_b, z_c and w_d ,

$$F_I(x_a, y_b, z_c; w_d) = 0, \quad (75)$$

where x_a corresponds to ψ_a , while y_b and z_c correspond to some combinations of the derivatives $\partial_\mu \psi_a, \partial_\mu \partial_\nu \psi_a, \dots$.⁶ The difference between y_b and z_c is explained later. w_d corresponds to the other variables ϕ_i and their derivatives $\partial_\mu \phi_i, \partial_\mu \partial_\nu \phi_i, \dots$. If the number of equations is the same as sum of those of x_a, y_b and z_c , we can use the implicit function theorem for them, *i.e.* x_a, y_b and z_c can be fixed uniquely in terms of w_d , provided that the conditions for the implicit function theorem are satisfied. However, even if the number of equations is less than the sum of those of x_a, y_b and z_c , it may be possible to fix x_a in terms w_d uniquely if Eq. (75) have a certain structure, as explained below. Some of y_b and z_c may not be fixed uniquely in this process, but the uniqueness of x_a can be shown independently.

The variables x_a are fixed by stepwise application of the inverse function theorem provided that the function F_I has a structure given by Eq. (77), which will be introduced shortly. Suppose that we have $N + M$ equations of the form of Eq. (75), *i.e.* I runs from 1 to $N + M (= \mathcal{N})$, where N denotes the number of the fields x_a and M is a positive integer. The variation of (75) is

$$\begin{aligned} \delta F_I(x_a, y_b, z_c; w_d) &= \frac{\partial F_I}{\partial x_a}(x_a, y_b, z_c; w_d) \delta x_a + \frac{\partial F_I}{\partial y_b}(x_a, y_b, z_c; w_d) \delta y_b \\ &+ \frac{\partial F_I}{\partial z_c}(x_a, y_b, z_c; w_d) \delta z_c + \frac{\partial F_I}{\partial w_d}(x_a, y_b, z_c; w_d) \delta w_d. \end{aligned} \quad (76)$$

Here, we assume that δy_b and δz_c appear only as M independent combinations in any δF_I , *i.e.* δF_I is expressed as

$$\begin{aligned} \delta F_I(x_a, y_b, z_c; w_d) &= \frac{\partial F_I}{\partial x_a}(x_a, y_b, z_c; w_d) \delta x_a + \frac{\partial F_I}{\partial w_d}(x_a, y_b, z_c; w_d) \delta w_d \\ &+ \sum_{b=1}^M B_{Ib}(x_a, y_b, z_c; w_d) \left(\delta y_b + \tilde{B}_{bc}(x_a, y_b, z_c; w_d) \delta z_c \right). \end{aligned} \quad (77)$$

Without loss of generality, the principal components of M independent combinations are set to be y_b , which means that the rank of B_{Ib} is M . This makes the difference between y_b and z_c . It also fixes the number of y_b to be M . Then, picking up M equations from F_I 's such that y_b is fixed uniquely, we use the implicit function theorem to fix only y_b . Without loss of generality, we assume that these equations are given by F_I with $N + 1 \leq I \leq N + M$, that is, the square matrix B_{Ib} with $I = N + 1, \dots, N + M$ and $b = 1, \dots, M$

⁵For instance, if F_I depends on up to the second order derivative of ψ_a , the number of all the degrees of freedom of $(\psi_a, \partial_\mu \psi_a, \partial_\mu \partial_\nu \psi_a)$ is $N + N \times D + N \times D(D + 1)/2$, where N and D are the number of ψ_a fields and the dimension of spacetime, respectively.

⁶For our purpose to prove the invertibility for the two-field case with first-order derivative given by Eq. (5), x_a corresponds to ψ_a , y_b is a component of $\partial_\mu \psi_a$, and z_c correspond to the other components of the derivative $\partial_\mu \psi_a$ and $\partial_\mu \partial_\nu \psi_a$.

is regular. Since the number of these equations and that of y_b are the same, we can apply the implicit function theorem for y_b , and then, y_b is uniquely expressed in terms of x_a , z_c and w_d .

Now, y_b 's are uniquely written as functions of x_a , z_c and w_d . Substituting $y_b(x_a, z_c, w_d)$ into Eq. (75) for $N + 1 \leq I \leq N + M$, the derivative of Eq. (75) with respect to z_c is given by

$$\sum_{b=1}^M B_{Ib}(x_a, y_b, z_c; w_d) \left(\frac{\partial y_b}{\partial z_c} + \tilde{B}_{bc} \right) = 0. \quad (78)$$

Since B_{Ib} is assumed to be regular, this equation implies

$$\frac{\partial y_b}{\partial z_c} + \tilde{B}_{bc} = 0. \quad (79)$$

This equation fixes the z_c dependence of $y_b(x_a, z_c, w_d)$. Now the remaining equations can be written purely in terms of x_a , z_c and w_d as

$$F_I(x_a, y_b(x_a, z_c, w_d), z_c; w_d) = 0, \quad (80)$$

where I runs from 1 to N . Their variation is given by

$$\begin{aligned} \delta F_I(x_a, y_b(x_a, z_c, w_d), z_c; w_d) = & \left[\frac{\partial F_I}{\partial x_a}(x_a, y_b, z_c; w_d) \delta x_a + \frac{\partial F_I}{\partial w_d}(x_a, y_b, z_c; w_d) \delta w_d \right. \\ & \left. + \sum_{b=1}^M B_{Ib}(x_a, y_b, z_c; w_d) \left(\frac{\partial y_b}{\partial x_a} \delta x_a + \frac{\partial y_b}{\partial w_d} \delta w_d \right) \right]_{y_b=y_b(x_a, z_c, w_d)}. \end{aligned} \quad (81)$$

The δz_c term is canceled because of Eq. (79) in this equation. This implies that F_I 's for $I = 1, \dots, N$ are independent of z_c , *i.e.* they are functions of only x_a and w_d . Since the number of equations is the same as that of x_a , we can use the implicit function theorem; if

$$\det \left[\frac{\partial F_I}{\partial x_a}(x_a, y_b, z_c; w_d) + \sum_{b=1}^M B_{Ib}(x_a, y_b, z_c; w_d) \frac{\partial y_b}{\partial x_a} \right]_{y_b=y_b(x_a, z_c, w_d)} \neq 0 \quad (82)$$

is satisfied, x_a has a unique solution locally and it is written in terms of only w_d .

In the next section, we will see that the transformation (5) in the two field case behaves as Eq. (77) once the invertibility conditions (50), (68) and (73) are imposed, and then ψ_i is fixed in terms of ϕ_a uniquely as a consequence.

3.2 Sufficient conditions for invertibility of our transformation

The transformation equation (5) can be rewritten in terms of a function of ψ_a , $\partial_\mu \psi_a$ and ϕ_i as

$$F_i(\psi_a, \partial_\mu \psi_a; \phi_i; x^\mu) := \bar{\phi}_i(\psi_a, \partial_\mu \psi_a; x^\mu) - \phi_i = 0. \quad (83)$$

We analyze this equation in the two field case imposing the necessary conditions for the invertibility, Eqs. (50), (68) and (73). Operating $n_i \partial_\mu$ on Eq. (83), we have

$$G_\mu(\psi_a, \partial_\mu \psi_a; \phi_i, \partial_\mu \phi_i; x^\mu) := n_i \partial_\mu F_i = n_i B_{ib} U_b U_a \partial_\mu \psi_a - n_i C_{i\mu} - n_i \partial_\mu \phi_i = 0, \quad (84)$$

where B_{ib} , n_i and U_a are defined in section 2.3 and $C_{i\mu} := \partial \bar{\phi}_i / \partial x^\mu$. The variation of F_i becomes

$$n_i \delta F_i = (n_i B_{ib} U_b) U_a \delta \psi_a, \quad (85)$$

$$V_i \delta F_i = (V_i B_{ib} U_b) U_a \delta \psi_a + (V_i B_{ib} m_b) m_a \delta \psi_a + a^\mu U_a \delta(\partial_\mu \psi_a), \quad (86)$$

where we omit the variation with respect to ϕ_i , because it is irrelevant to the condition for the implicit function theorem.

Now let us evaluate the variation of $a^\mu G_\mu$, which is given as

$$\begin{aligned}
\delta(a^\mu G_\mu) &= a^\mu n_i \delta(\partial_\mu F_i) = n_i \delta(\partial_\mu \bar{\phi}_i) \\
&= a^\mu n_i \delta \left(\frac{\partial \bar{\phi}_i}{\partial(\partial_\alpha \psi_a)} \partial_\mu \partial_\alpha \psi_a + \frac{\partial \bar{\phi}_i}{\partial \psi_a} \partial_\mu \psi_a + \frac{\partial \bar{\phi}_i}{\partial x^\mu} \right) \\
&= a^\mu n_i \left[\left(\frac{\partial^2 \bar{\phi}_i}{\partial(\partial_\alpha \psi_a) \partial(\partial_\beta \psi_b)} \partial_\mu \partial_\alpha \psi_a + \frac{\partial^2 \bar{\phi}_i}{\partial \psi_a \partial(\partial_\beta \psi_b)} \partial_\mu \psi_a + \frac{\partial^2 \bar{\phi}_i}{\partial x^\mu \partial(\partial_\beta \psi_b)} \right) \delta(\partial_\beta \psi_b) \right. \\
&\quad \left. + \left(\frac{\partial^2 \bar{\phi}_i}{\partial(\partial_\alpha \psi_a) \partial \psi_b} \partial_\mu \partial_\alpha \psi_a + \frac{\partial^2 \bar{\phi}_i}{\partial \psi_a \partial \psi_b} \partial_\mu \psi_a + \frac{\partial^2 \bar{\phi}_i}{\partial x^\mu \partial \psi_b} \right) \delta \psi_b + B_{ia} \delta(\partial_\mu \psi_a) \right] \\
&= a^\mu n_i (\partial_\mu B_{ib}) U_b U_a \delta \psi_a + a^\mu n_i (\partial_\mu B_{ib}) m_b m_a \delta \psi_a + n_i (a^\mu \partial_\mu V_i + B_{ib} U_b) a^\nu U_a \delta(\partial_\nu \psi_a), \tag{87}
\end{aligned}$$

where we used $\partial_\mu F_i = 0$ at the first equality, and some formulae in section 2.3.3 such as the degeneracy conditions $n_i \frac{\partial \bar{\phi}_i}{\partial(\partial_\alpha \psi_a)} = n_i A_{ia}^\alpha = 0$, $n_i B_{ia} m_a = 0$ at the fourth equality. The key point is that, in Eqs. (85), (86) and (87), $\delta(\partial_\mu \psi_a)$ appears only in a linear combination $a^\mu U_a \delta(\partial_\mu \psi_a)$. The last term proportional to $a^\mu U_a \delta(\partial_\mu \psi_a)$ in Eq. (87) corresponds to the last line of Eq. (77) for $M = 1$. This allows us to use the result of section 3.1 by regarding ψ_a , $\partial_\mu \psi_a$ and $\partial_\mu \partial_\nu \psi_a$ as independent variables and also identifying x_a , y_b and z_c as ψ_a , a component of $\partial_\mu \psi_a$ and the other components of $(\partial_\mu \psi_a, \partial_\mu \partial_\nu \psi_a)$, respectively. Then, the condition of invertibility is

$$\begin{aligned}
0 \neq \det & \begin{pmatrix} V_i \frac{\partial F_i}{\partial \psi_a} U_a & V_i \frac{\partial F_i}{\partial \psi_a} m_a & V_i \frac{\partial F_i}{\partial(\partial_\nu \psi_a)} U_a \\ n_i \frac{\partial F_i}{\partial \psi_a} U_a & n_i \frac{\partial F_i}{\partial \psi_a} m_a & n_i \frac{\partial F_i}{\partial(\partial_\nu \psi_a)} U_a \\ \frac{\partial(a^\mu G_\mu)}{\partial \psi_a} U_a & \frac{\partial(a^\mu G_\mu)}{\partial \psi_a} m_a & \frac{\partial(a^\mu G_\mu)}{\partial(\partial_\nu \psi_a)} U_a \end{pmatrix} \\
&= \det \begin{pmatrix} V_i B_{ia} U_a & V_i B_{ia} m_a & a^\nu \\ n_i B_{ia} U_a & 0 & 0 \\ a^\mu n_i (\partial_\mu B_{ia}) U_a & a^\mu n_i (\partial_\mu B_{ia}) m_a & n_i (a^\mu \partial_\mu V_i + B_{ia} U_a) a^\nu \end{pmatrix} \\
&= -a^\nu (n_i B_{ia} U_a)^2 (V_j B_{jb} - a^\mu \partial_\mu U_b) m_b, \tag{88}
\end{aligned}$$

where the components of the matrix are expressed with respect to the bases (V_i, n_i) , (U_a, m_a) , and also the degenerate row and column whose components are completely zero are removed. Equation (88) is satisfied if $n_i B_{ia} U_a \neq 0$ and $(V_j B_{jb} - a^\mu \partial_\mu U_b) m_b \neq 0$. Therefore, the necessary conditions for the invertibility, Eqs. (50), (68) and (73), are also the sufficient conditions.

4 Examples of invertible transformations

Invertibility conditions for transformations $\phi_i = \phi_i(\psi_a, \partial \psi_a)$ are established in the previous sections. In this section, we propose some non-trivial examples of field transformations that satisfy the invertibility conditions. In principle, the most general invertible transformation could be constructed by finding the general solution of the invertibility conditions. Unfortunately it is not an easy task,⁷ so we proceed with introducing the following ansatz of field transformations:

1. $\phi_i = b(\psi_a, Y_\mu) V_i(\psi_a) + \tilde{V}_i(\psi_a) \quad (Y_\mu \equiv U_a(\psi_a) \partial_\mu \psi_a)$
2. $\phi_i = b(\psi_a, Y_a) V_i(\psi_a) + \tilde{V}_i(\psi_a) \quad (Y_a \equiv \tilde{a}^\mu(\psi_a) \partial_\mu \psi_a)$
3. $\phi_i = \phi_i(\psi_a, Y) \quad (Y \equiv \tilde{a}^\mu(\psi_a) U_a(\psi_a) \partial_\mu \psi_a)$

These ansatze are chosen so that the first degeneracy condition (50) among the invertibility conditions is automatically satisfied. The second degeneracy condition (68) and the non-degeneracy conditions (73) are

⁷See Ref. [1] for a previous attempt of constructing examples of invertible transformations.

then used to constrain the form of the transformations. Although these transformations may not span the most general transformation satisfying the invertibility conditions, they would give a good starting point for investigating the most-general invertible field transformation.

In the derivation below, we use the fact that the vector U_a can be set to a constant vector $(0, 1)$ if it depends only on ψ_a . We explain this method in Appendix C.

4.1 Ansatz 1: $\phi_i = b(\psi_a, U_a(\psi_a)\partial_\mu\psi_a)V_i(\psi_a) + \tilde{V}_i(\psi_a)$

We first consider the following ansatz for a field transformation given by

$$\phi_i = b(\psi_a, Y_\mu)V_i(\psi_a) + \tilde{V}_i(\psi_a) \quad (Y_\mu \equiv U_a(\psi_a)\partial_\mu\psi_a) , \quad (89)$$

where V_i is normalized as $V_iV_i = 1$. In this transformation, U_a can be set to $U_a = (0, 1)$ without loss of generality. It then follows

$$U_a = (0, 1), \quad m_a = (1, 0), \quad n_i = \epsilon_{ij}V_j . \quad (90)$$

For (89), the first degeneracy condition (50) is given by

$$A_{ia}^\mu = \frac{\partial\phi_i}{\partial(\partial_\mu\psi_a)} = \frac{\partial b}{\partial Y_\mu}U_aV_i \equiv a^\mu U_aV_i . \quad (91)$$

The second degeneracy condition (68) is evaluated as follows.

$$B_{ia} = \frac{\partial\phi_i}{\partial\psi_a} = \frac{\partial b}{\partial\psi_a}V_i + b\frac{\partial V_i}{\partial\psi_a} + \frac{\partial\tilde{V}_i}{\partial\psi_a} , \quad (92)$$

$$n_iB_{ia}m_a = b(\psi_a, Y_\mu)n_i\frac{\partial V_i}{\partial\psi_1} + n_i\frac{\partial\tilde{V}_i}{\partial\psi_1} . \quad (93)$$

In (93), only b depends on Y_μ while the other terms depend only on ψ_a . Because Eq. (93) should be satisfied identically for any ψ_a and Y_μ , it implies

$$b n_i \frac{\partial V_i}{\partial\psi_1} = 0 , \quad (94)$$

$$n_i \frac{\partial\tilde{V}_i}{\partial\psi_1} = 0 . \quad (95)$$

Equation (94) implies either $n_i\frac{\partial V_i}{\partial\psi_1} = 0$ or $b = 0$. $n_i\frac{\partial V_i}{\partial\psi_1} = 0$ implies $V_i = V_i(\psi_2)$ thanks to the normalization $V_iV_i = 1$, while $b = 0$ gives a transformation without derivatives $\phi_i = \phi_i(\psi_a)$. Below we focus on the case $n_i\frac{\partial V_i}{\partial\psi_1} = 0$, in which the transformation depends on derivatives $\phi_i = \phi_i(\psi_a, \partial\psi_a)$. In this case, Eq. (95) implies that $\partial\tilde{V}_i/\partial\psi_1$ is parallel to V_i , that is,

$$\frac{\partial\tilde{V}_i}{\partial\psi_1} = \tilde{c}(\psi_a)V_i(\psi_2) \quad \Rightarrow \quad \tilde{V}_i = c(\psi_a)V_i(\psi_2) + \hat{V}_i(\psi_2) \quad (96)$$

for some functions $\tilde{c}(\psi_a)$, $c(\psi_1) = \int \tilde{c}(\psi_a)d\psi_1$ and $\hat{V}_i(\psi_2)$. The cV_i term can be absorbed into the bV_i term in the ansatz (89), and the remainder $\hat{V}_i(\psi_2)$ depends only on ψ_2 . It implies that, once Eq. (90) is imposed, we may set $\tilde{V}_i = \hat{V}_i(\psi_2)$ in the ansatz (89) without loss of generality by absorbing the cV_i term. Below we continue the derivation under this assumption.

The non-degeneracy conditions (73) are given by

$$0 \neq n_iB_{ia}U_a = b n_i \frac{\partial V_i}{\partial\psi_2} + n_i \frac{\partial\tilde{V}_i}{\partial\psi_2} , \quad (97)$$

$$0 \neq (V_iB_{ia} - a^\mu\partial_\mu U_a)m_a = \left(\frac{\partial b}{\partial\psi_a} + \frac{\partial b}{\partial Y_\mu}\partial_\mu\psi_b\frac{\partial U_b}{\partial\psi_a} + V_i\frac{\partial\tilde{V}_i}{\partial\psi_a} - a^\mu\partial_\mu U_a \right) m_a = \frac{\partial b}{\partial\psi_1} , \quad (98)$$

which can be regarded as constraints on b , V_i and \tilde{V}_i .

To summarize, a transformation given by

$$\phi_i = b(\psi_a, Y_\mu) V_i(\psi_2) + \tilde{V}_i(\psi_2) \quad (Y_\mu \equiv \partial_\mu \psi_2) . \quad (99)$$

is invertible if Eqs. (97) and (98) are satisfied. One may further apply an invertible transformation $\psi_a = \psi_a(\tilde{\psi}_b)$ such that $\det(\partial\psi_a/\partial\tilde{\psi}_b) \neq 0$ on Eq. (99) to re-introduce nontrivial $U_a(\tilde{\psi}_a)$ and construct a transformation of the form of Eq. (89).

4.2 Ansatz 2: $\phi_i = b(\psi_a, \tilde{a}^\mu(\psi_a) \partial_\mu \psi_a) V_i(\psi_a) + \tilde{V}_i(\psi_a)$

The next ansatz is given by

$$\phi_i = b(\psi_a, Y_a) V_i(\psi_a) + \tilde{V}_i(\psi_a) \quad (Y_a \equiv \tilde{a}^\mu(\psi_a) \partial_\mu \psi_a) . \quad (100)$$

The first degeneracy condition (50) is automatically satisfied for this ansatz:

$$A_{ia}^\mu = \tilde{a}^\mu \frac{\partial b}{\partial Y_a} V_i \equiv a^\mu(\psi_a, Y_a) U_a(\psi_a, Y_a) V_i(\psi_a) . \quad (101)$$

The second degeneracy (68) condition is given as

$$B_{ia} = \frac{\partial b}{\partial \psi_a} V_i + b \frac{\partial V_i}{\partial \psi_a} + \frac{\partial \tilde{V}_i}{\partial \psi_a} , \quad (102)$$

$$\therefore n_i B_{ia} m_a = n_i \left(b \frac{\partial V_i}{\partial \psi_a} + \frac{\partial \tilde{V}_i}{\partial \psi_a} \right) m_a \propto n_i \left(b \frac{\partial V_i}{\partial \psi_a} + \frac{\partial \tilde{V}_i}{\partial \psi_a} \right) \epsilon_{ab} \frac{\partial b}{\partial Y_b} = 0 . \quad (103)$$

Equation (103) is equivalent to

$$\frac{n_i \left(b \frac{\partial V_i}{\partial \psi_1} + \frac{\partial \tilde{V}_i}{\partial \psi_1} \right)}{n_i \left(b \frac{\partial V_i}{\partial \psi_2} + \frac{\partial \tilde{V}_i}{\partial \psi_2} \right)} = \frac{\partial b}{\partial Y_1} \equiv -c(\psi_a, Y_a) \quad (104)$$

$$\Leftrightarrow n_i \left(b \frac{\partial V_i}{\partial \psi_1} + \frac{\partial \tilde{V}_i}{\partial \psi_1} \right) + c(\psi_a, Y_a) n_i \left(b \frac{\partial V_i}{\partial \psi_2} + \frac{\partial \tilde{V}_i}{\partial \psi_2} \right) = 0 , \quad \frac{\partial b}{\partial Y_1} + c(\psi_a, Y_a) \frac{\partial b}{\partial Y_2} = 0 . \quad (105)$$

Equation (105) may be solved for $\tilde{V}_i(\psi_a)$ and $b(\psi_a, Y_a)$ once $V_i(\psi_a)$ and $c(\psi_a, Y_a)$ are freely specified, and each solution gives an invertible transformation as long as it satisfies the non-degeneracy conditions (73).

In a special case where the function $c(\psi_a, Y_a)$ depends only on ψ_a but not on Y_a , one can construct invertible transformations more explicitly as follows. When $c = c(\psi_a)$, we may apply an invertible transformation $\psi_a = \psi_a(\tilde{\psi}_a)$ on Eq. (105) to eliminate $c(\psi_a)$, that is, Eq. (105) transforms under $\psi_a = \psi_a(\tilde{\psi}_a)$ as

$$\begin{aligned} 0 &= \frac{\partial \tilde{\psi}_a}{\partial \psi_1} n_i \left(b \frac{\partial V_i}{\partial \tilde{\psi}_a} + \frac{\partial \tilde{V}_i}{\partial \tilde{\psi}_a} \right) + c(\psi_a) n_i \frac{\partial \tilde{\psi}_a}{\partial \psi_2} (\psi_a, Y_a) \left(b \frac{\partial V_i}{\partial \tilde{\psi}_a} + \frac{\partial \tilde{V}_i}{\partial \tilde{\psi}_a} \right) \\ &= \left(\frac{\partial \tilde{\psi}_1}{\partial \psi_1} + c(\psi_a) \frac{\partial \tilde{\psi}_1}{\partial \psi_2} \right) n_i \left(b \frac{\partial V_i}{\partial \tilde{\psi}_1} + \frac{\partial \tilde{V}_i}{\partial \tilde{\psi}_1} \right) + \left(\frac{\partial \tilde{\psi}_2}{\partial \psi_1} + c(\psi_a) \frac{\partial \tilde{\psi}_2}{\partial \psi_2} \right) n_i \left(b \frac{\partial V_i}{\partial \tilde{\psi}_2} + \frac{\partial \tilde{V}_i}{\partial \tilde{\psi}_2} \right) , \end{aligned} \quad (106)$$

and then the function $\tilde{\psi}_a = \tilde{\psi}_a(\psi_a)$ may be chosen (at least locally) so that

$$\frac{\partial \tilde{\psi}_2}{\partial \psi_1} + c(\psi_a) \frac{\partial \tilde{\psi}_2}{\partial \psi_2} = 0 , \quad (107)$$

with which Eq. (106) reduces to

$$n_i \left(b \frac{\partial V_i}{\partial \tilde{\psi}_1} + \frac{\partial \tilde{V}_i}{\partial \tilde{\psi}_1} \right) = 0 . \quad (108)$$

This is equivalent to setting $c(\psi_a) = 0$ in Eq. (105) by means of a transformation in ψ_a space.

When $c(\psi_a) = 0$, Eq. (104) implies that

$$n_i(\psi_a) \left(b(\psi_a, Y_a) \frac{\partial V_i(\psi_a)}{\partial \psi_1} + \frac{\partial \tilde{V}_i(\psi_a)}{\partial \psi_1} \right) = 0, \quad \frac{\partial b}{\partial Y_1} = 0. \quad (109)$$

The second equation implies $b = b(\psi_a, Y_2)$, which gives $U_a = (0, 1)$ and $m_a = (1, 0)$. In the first equation, the first term involving $b(\psi_a, Y_2)$ depends on Y_2 while the second term is independent of Y_a , then it follows that

$$b(\psi_a, Y_2) n_i(\psi_a) \frac{\partial V_i(\psi_a)}{\partial \psi_1} = 0, \quad (110)$$

$$n_i(\psi_a) \frac{\partial \tilde{V}_i(\psi_a)}{\partial \psi_1} = 0. \quad (111)$$

Equation (110) implies either $V_i = V_i(\psi_2)$ or $b = 0$, for the latter of which the transformation (100) does not involve $\partial\psi_a$. Below we focus on the former case, which gives a transformation with derivatives. In this case, we can show that $\tilde{V}_i = f(\psi_a)V_i(\psi_2) + \hat{V}_i(\psi_2)$ for some function $f(\psi_a)$ and $\hat{V}_i(\psi_2)$ using Eq. (111) as we did around Eq. (96) for the previous ansatz. Then, once setting $U_a = (0, 1)$, by absorbing the $c V_i$ term into the $b V_i$ term we may set $\tilde{V}_i = \tilde{V}_i(\psi_2)$ without loss of generality.

To summarize the results above, a transformation

$$\phi_i = b(\psi_a, Y_2)V_i(\psi_2) + \tilde{V}_i(\psi_2) \quad (Y_a \equiv \tilde{a}^\mu(\psi_a)\partial_\mu\psi_a) \quad (112)$$

may be invertible if Eq. (111) is satisfied. Adding to that, the non-degeneracy conditions (73) must be satisfied for the invertibility, and also an invertible transformation $\psi_a = \psi_a(\tilde{\psi}_b)$ may be applied to construct a transformation with nontrivial $U_a(\psi_a)$. Actually this transformation is a special case of the transformation (99) examined in the previous section, although we have started from a different ansatz. This result follows from the assumption $c = c(\psi_a)$ imposed at Eq. (106), and more general transformations are obtained if we solve Eq. (105) for $c = c(\psi_a, Y_a)$.

4.3 Ansatz 3: $\phi_i = \phi_i(\psi_a, \tilde{a}^\mu(\psi_a)U_a(\psi_a)\partial_\mu\psi_a)$

The third ansatz we consider is

$$\phi_i = \phi_i(\psi_a, Y) \quad (Y \equiv \tilde{a}^\mu(\psi_a)U_a(\psi_a)\partial_\mu\psi_a) . \quad (113)$$

for which the first degeneracy condition (50) is satisfied automatically as

$$A_{ia}^\mu = \tilde{a}^\mu(\psi_a)U_a(\psi_a)\frac{\partial\phi_i}{\partial Y}(\psi_a, Y) \equiv a^\mu(\psi_a, Y)U_a(\psi_a)V_i(\psi_a, Y) . \quad (114)$$

In this expression V_i is normalized as $V_i V_i = 1$, that is,

$$V_i \equiv \frac{1}{v(\psi_a, Y)} \frac{\partial\phi_i}{\partial Y}(\psi_a, Y) , \quad a^\mu(\psi_a, Y) \equiv v(\psi_a, Y)\tilde{a}^\mu(\psi_a) , \quad V_i V_i = 1 , \quad (115)$$

where $v(\psi_a, Y)$ is a normalization factor. For simplicity, we assume $v \neq 0$ in the following.

Following the procedure explained in Appendix C, we set $U_a = (0, 1)$, $m_a = (1, 0)$ without loss of generality. Then, using $n_i = v^{-1}\epsilon_{ij}\partial\phi_j/\partial Y$, the second degeneracy condition (68) is evaluated as

$$B_{ia} = \frac{\partial\phi_i}{\partial\psi_a} + \frac{\partial\phi_i}{\partial Y} \frac{\partial\tilde{a}^\mu}{\partial\psi_a} \partial_\mu\psi_2 = \frac{\partial\phi_i}{\partial\psi_a} + v V_i \frac{\partial\tilde{a}^\mu}{\partial\psi_a} \partial_\mu\psi_2 , \quad (116)$$

$$n_i B_{ia} m_a = v^{-1}\epsilon_{ij} \frac{\partial\phi_i}{\partial\psi_1} \frac{\partial\phi_j}{\partial Y} = 0 . \quad (117)$$

Equation (117) implies that, provided $v \neq 0$,

$$\frac{\partial\phi_1}{\partial\psi_1} \Big/ \frac{\partial\phi_1}{\partial Y} = \frac{\partial\phi_2}{\partial\psi_1} \Big/ \frac{\partial\phi_2}{\partial Y} , \quad (118)$$

that is, gradient vectors $(\frac{\partial\phi_1}{\partial\psi_1}, \frac{\partial\phi_1}{\partial Y})$ and $(\frac{\partial\phi_2}{\partial\psi_1}, \frac{\partial\phi_2}{\partial Y})$ are parallel to each other in the (ψ_1, Y) space. Then it follows that

$$\phi_2 = F(\psi_2, \phi_1(\psi_a, Y)) , \quad (119)$$

where $F(\psi_2, \phi_1)$ is an arbitrary function of ϕ_1 and ψ_2 .⁸

Let us examine the non-degeneracy conditions (73). One of them is given by

$$\begin{aligned} 0 \neq n_i B_{ia} U_a &= n_i B_{i2} = v^{-1} \epsilon_{ij} \frac{\partial\phi_i}{\partial\psi_2} \frac{\partial\phi_j}{\partial Y} = v^{-1} \left(\frac{\partial\phi_1}{\partial\psi_2} \frac{\partial\phi_2}{\partial Y} - \frac{\partial\phi_2}{\partial\psi_2} \frac{\partial\phi_1}{\partial Y} \right) \\ &= v^{-1} \left[\frac{\partial\phi_1}{\partial\psi_2} \frac{\partial F}{\partial\phi_1} \frac{\partial\phi_1}{\partial Y} - \left(\frac{\partial F}{\partial\psi_2} + \frac{\partial F}{\partial\phi_1} \frac{\partial\phi_1}{\partial\phi_2} \right) \frac{\partial\phi_1}{\partial Y} \right] = -v^{-1} \frac{\partial F}{\partial\phi_1} \frac{\partial\phi_1}{\partial Y} . \end{aligned} \quad (120)$$

This equation implies that $v, \partial F/\partial\phi_1, \partial\phi_1/\partial Y$ must not vanish. The other non-degeneracy condition is given by

$$0 \neq (V_i B_{ia} - a^\mu \partial_\mu U_a) m_a = V_i \left(\frac{\partial\phi_i}{\partial\psi_1} + v V_i \frac{\partial\tilde{a}^\mu}{\partial\psi_1} \partial_\mu \psi_2 \right) = V_i \frac{\partial\phi_i}{\partial\psi_1} + v \frac{\partial\tilde{a}^\mu}{\partial\psi_1} \partial_\mu \psi_2 . \quad (121)$$

To summarize, a transformation (113) is invertible if ϕ_2 is given by Eq. (119) and Eqs. (120), (121) are satisfied.

5 Applications to disformal transformations with higher derivatives

5.1 No-go for disformal transformation with $\nabla\nabla\chi$ term

Here we apply our approach to disformal metric transformation. The disformal transformations involving only one derivative of the scalar field χ ,

$$\tilde{g}_{\mu\nu} = C(\chi, X) g_{\mu\nu} + D(\chi, X) \nabla_\mu \chi \nabla_\nu \chi , \quad (122)$$

where $X = \partial_\mu \chi \partial^\mu \chi$, are invertible, provided that $C (C - \frac{dC}{dX} X - \frac{dD}{dX} X^2) \neq 0$, see e.g. [7, 8]. This follows from the fact that from the above expression one can straightforwardly express the metric $g_{\mu\nu}$ in terms of $\tilde{g}_{\mu\nu}$, χ and $\nabla_\mu \chi$. Thus the transformation (122) is one-to-one change of variables $(\chi, g_{\mu\nu}) \leftrightarrow (\chi, \tilde{g}_{\mu\nu})$ as long as the above condition is satisfied. It is useful to check that the transformation (122) is invertible by applying our method. Although the method we developed above does not apply metric transformations in general because a metric is a tensor and has more than 2 components except for one dimensional spacetime, we can use it when applied to particular ansatze that contain scalar functions only. Indeed, let us restrict ourselves to the case of the homogeneous cosmology,

$$g_{\mu\nu} dx^\mu dx^\nu = -\mathbf{n}(t) dt^2 + \mathbf{a}(t) d\mathbf{x}^2, \quad \chi = \chi(t), \quad (123)$$

where $\mathbf{a}(t)$ is the squared of the scale factor and $\mathbf{n}(t)$ is the squared of the lapse function. Note that we keep $\mathbf{n}(t)$ in the ansatz (123), since it changes under the transformation (122). For (123) we obtain under the disformal transformation (122),

$$\tilde{\mathbf{n}} = C\mathbf{n} - D\dot{\chi}^2, \quad \tilde{\mathbf{a}} = C\mathbf{a}, \quad (124)$$

where a dot denotes a t derivative, and

$$C = C(\chi, X) = C\left(\chi(t), -\frac{\dot{\chi}^2(t)}{\mathbf{n}(t)}\right), \quad D = D\left(\chi(t), -\frac{\dot{\chi}^2(t)}{\mathbf{n}(t)}\right). \quad (125)$$

⁸When the (ψ_1, Y) space is separated into connected sets by borders on which the gradient vector $(\frac{\partial\phi_1}{\partial\psi_1}, \frac{\partial\phi_1}{\partial Y})$ vanishes, the function form of $F(\psi_2, \phi_1(\psi_a, Y))$ may be different on each connected sets on the (ψ_1, Y) space. The non-degeneracy conditions however imply $\partial\phi_1/\partial Y \neq 0$, and it guarantees that $F(\psi_2, \phi_1)$ is given uniquely on the entire (ψ_1, Y) space.

Note that the above transformation can be considered as the change of variables $\{\mathbf{n}, \mathbf{a} \rightarrow \tilde{\mathbf{n}}, \tilde{\mathbf{a}}\}$ (with the time-dependent external function $\chi(t)$), that does not involve derivatives. Therefore, by the virtue of the standard theorem on invertibility, the transformation (124) is invertible if $\det B_{ia} \neq 0$, where B_{ia} is given by,

$$B_{ia} = \begin{pmatrix} C - XC_X - X^2 D_X & 0 \\ -\frac{\mathbf{a}}{\mathbf{n}} XC_X & C \end{pmatrix}, \quad (126)$$

so that we obtain $C \left(C - \frac{dC}{dX} X - \frac{dD}{dX} X^2 \right) \neq 0$, which reproduces the result we cited above.

On the other hand, one can consider a more general disformal transformation by including two derivatives of the scalar as follows,

$$\tilde{g}_{\mu\nu} = C(\chi, X)g_{\mu\nu} + D(\chi, X)\nabla_\mu\chi\nabla_\nu\chi + E(\chi, X)\nabla_\mu\nabla_\nu\chi. \quad (127)$$

For the above transformation one cannot directly express $g_{\mu\nu}$ in terms of $\tilde{g}_{\mu\nu}$, since the last term of the r.h.s. of (127) also contains the metric $g_{\mu\nu}$. Therefore, it has been conjectured that the inverse transformation of (127) does not exist, see Ref. [9]. However, to the best of our knowledge, this has not yet been proven. Indeed, although the simple inverse of (127) does not exist, it does not necessarily mean that there is no more complicated inverse transformation. Using our method, however, we are able to demonstrate that the transformation (127) with non-zero E is not invertible indeed.

To do this, let us assume that the transformation (127) is invertible, and we will see that this assumption leads to a contradiction. The FRW homogeneous ansatz (132) expresses a subspace of functional space described with $g_{\mu\nu}$ and χ . The invertibility limited to this subspace gives the necessary condition of invertibility for full functional space. Here, we will show the violation of this necessary condition, which leads to a contradiction. From (127) and (123) we have,

$$\tilde{\mathbf{n}} = C\mathbf{n} - D\dot{\chi}^2 - E \left(\ddot{\chi} - \frac{\dot{\chi}\dot{\mathbf{n}}}{2\mathbf{n}} \right), \quad \tilde{\mathbf{a}} = C\mathbf{a} - E\frac{\dot{\chi}\dot{\mathbf{a}}}{2\mathbf{n}}, \quad (128)$$

which is a generalization of (124) for the case of non-zero E .

As in the case considered above, we treat (128) as a transformation relating a set of two variables $(\mathbf{n}(t), \mathbf{a}(t))$ with $(\tilde{\mathbf{n}}(t), \tilde{\mathbf{a}}(t))$ with the time-dependent external function $\chi(t)$. Contrary to standard disformal transformation, in this case the transformation involves first derivatives of the variables $\mathbf{n}(t)$ and $\mathbf{a}(t)$, therefore we should apply our method to check whether the conditions for this transformation to be invertible are satisfied. We have,

$$A_{ia} = \frac{E\dot{\chi}}{2\mathbf{n}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (129)$$

According to our discussion above, the leading-order condition for the transformation given by (127) to be invertible is the vanishing of the determinant of (129). As we can see, this condition is clearly violated for non-zero E , therefore we arrive to the conclusion that the transformation (127) is not invertible unless $E = 0$.

5.2 Invertibility of generalized disformal transformation with second derivatives

Once we generalize the structure of the higher-derivative term as [10, 11]

$$\tilde{g}_{\mu\nu} = \mathcal{F}_0 g_{\mu\nu} + \mathcal{F}_1 \chi_{,\mu} \chi_{,\nu} + \mathcal{F}_2 \chi_{\mu\nu} + \mathcal{F}_3 \chi_{(\mu} X_{\nu)} + \mathcal{F}_4 X_\mu X_\nu + \mathcal{F}_5 \chi_{,\mu}^\alpha \chi_{,\nu\alpha}, \quad (130)$$

where $\mathcal{F}_i = \mathcal{F}_i(\phi, X, \mathcal{B}, \mathcal{Y}, \mathcal{Z}, \mathcal{W})$ and

$$\mathcal{B} = \nabla_\mu \nabla^\mu \chi, \quad \mathcal{Y} = \nabla_\mu \chi \nabla^\mu X, \quad \mathcal{Z} = \nabla_\mu X \nabla^\mu X, \quad \mathcal{W} = \nabla^\mu \nabla^\nu \chi \nabla_\mu \nabla_\nu \chi, \quad (131)$$

the structure of A_{ia} is changed and the no-go result obtained above for the transformation (127) could be evaded. We examine such a possibility in this section, and find that there may be an invertible disformal transformation with higher derivatives if the metric is limited to the FRW type. Note that the investigation of the FRW type gives a necessary condition. While the violation of invertibility in the FRW subspace shows that in the full space of metric, the establishment of it in the subspace does not result in that in the full space.

5.2.1 Generalized disformal transformation with second derivatives

Let us take the FRW homogeneous ansatz for the metric $g_{\mu\nu}$,

$$g_{\mu\nu}dx^\mu dx^\nu = -\mathbf{n}(t)dt^2 + \mathbf{a}(t)d\mathbf{x}^2, \quad \chi = \chi(t). \quad (132)$$

For this ansatz we have

$$\begin{aligned} X &= -\frac{\dot{\chi}^2}{\mathbf{n}}, \\ \mathcal{B} &= \frac{1}{2\mathbf{n}} \left(\dot{\chi} \frac{\dot{\mathbf{n}}}{\mathbf{n}} - 3\dot{\chi} \frac{\dot{\mathbf{a}}}{\mathbf{a}} - 2\ddot{\chi} \right) = \frac{\mathcal{Y}}{2X} - \frac{3\dot{\chi} \dot{\mathbf{a}}}{2\mathbf{n} \mathbf{a}}, \\ \mathcal{Y} &= -\frac{\dot{\chi}^2}{\mathbf{n}^2} \left(\dot{\chi} \frac{\dot{\mathbf{n}}}{\mathbf{n}} - 2\ddot{\chi} \right), \\ \mathcal{Z} &= \frac{\dot{\chi}^2}{\mathbf{n}^3} \left(-\dot{\chi}^2 \frac{\dot{\mathbf{n}}^2}{\mathbf{n}^2} + 4\dot{\chi} \ddot{\chi} \frac{\dot{\mathbf{n}}}{\mathbf{n}} - 4\ddot{\chi}^2 \right) = -\frac{\dot{\chi}^2}{\mathbf{n}^3} \left(\dot{\chi} \frac{\dot{\mathbf{n}}}{\mathbf{n}} - 2\ddot{\chi} \right)^2 = \frac{\mathcal{Y}^2}{X}, \\ \mathcal{W} &= \frac{1}{\mathbf{n}^2} \left(\frac{\dot{\chi}^2 \dot{\mathbf{n}}^2}{4} + \frac{3\dot{\chi}^2 \dot{\mathbf{a}}^2}{4 \mathbf{a}^2} - \dot{\chi} \ddot{\chi} \frac{\dot{\mathbf{n}}}{\mathbf{n}} + \ddot{\chi}^2 \right) = \left(\frac{\mathcal{Y}}{2X} \right)^2 + \frac{1}{3} \left(\frac{\mathcal{Y}}{2X} - \mathcal{B} \right)^2. \end{aligned} \quad (133)$$

Hence, for the FRW ansatz, \mathcal{Z} and \mathcal{W} are expressed by X , \mathcal{B} and \mathcal{Y} and not independent. Evaluating also the terms appearing in Eq. (130), we find that the metric after the transformation \tilde{g} is expressed by the FRW ansatz

$$\tilde{g}_{\mu\nu} = \tilde{\mathbf{n}}(t)dt^2 + \tilde{\mathbf{a}}(t)d\mathbf{x}^2, \quad (134)$$

and $\tilde{\mathbf{n}}, \tilde{\mathbf{a}}$ are given by

$$\begin{aligned} \tilde{\mathbf{n}} &= \mathbf{n} \left(\mathcal{F}_0 + \mathcal{F}_1 X + \mathcal{F}_2 \frac{\mathcal{Y}}{2X} + \mathcal{F}_3 \mathcal{Y} + \mathcal{F}_4 \frac{\mathcal{Y}^2}{X} + \mathcal{F}_5 \left(\frac{\mathcal{Y}}{2X} \right)^2 \right) := \mathbf{n} F_{\mathbf{n}}(X, \mathcal{B}, \mathcal{Y}) \\ \tilde{\mathbf{a}} &= \mathbf{a} \left(\mathcal{F}_0 - \mathcal{F}_2 \frac{1}{3} \left(\frac{\mathcal{Y}}{2X} - \mathcal{B} \right) + \mathcal{F}_5 \frac{1}{9} \left(\frac{\mathcal{Y}}{2X} - \mathcal{B} \right)^2 \right) := \mathbf{a} F_{\mathbf{a}}(X, \mathcal{B}, \mathcal{Y}). \end{aligned} \quad (135)$$

This result implies that the generalized disformal transformation (130) is completely governed by the functions $F_{\mathbf{n}}(X, \mathcal{B}, \mathcal{Y})$ and $F_{\mathbf{a}}(X, \mathcal{B}, \mathcal{Y})$ for the FRW ansatz.

5.2.2 Invertibility of generalized disformal transformation on FRW spacetime

The transformation (135) involves only \mathbf{n}, \mathbf{a} and their first derivatives, hence its invertibility can be analyzed within the framework explained in section 2. Based on it, let us derive the invertibility conditions for the transformation (135) below.

Since the transformation (135) involves $\dot{\mathbf{n}}, \dot{\mathbf{a}}$, A_{ia} must be degenerate for the invertibility:

$$A_{ia} = \begin{pmatrix} \partial \tilde{\mathbf{n}} / \partial \dot{\mathbf{n}} & \partial \tilde{\mathbf{n}} / \partial \dot{\mathbf{a}} \\ \partial \tilde{\mathbf{a}} / \partial \dot{\mathbf{n}} & \partial \tilde{\mathbf{a}} / \partial \dot{\mathbf{a}} \end{pmatrix} = \frac{1}{2} \sqrt{\frac{-X}{\mathbf{n}}} \begin{pmatrix} F_{\mathbf{n}, \mathcal{B}} + 2X F_{\mathbf{n}, \mathcal{Y}} & -\frac{3\mathbf{n}}{\mathbf{a}} F_{\mathbf{n}, \mathcal{B}} \\ \frac{\mathbf{a}}{\mathbf{n}} (F_{\mathbf{a}, \mathcal{B}} + 2X F_{\mathbf{a}, \mathcal{Y}}) & -3F_{\mathbf{a}, \mathcal{B}} \end{pmatrix} \quad (136)$$

$$\det A = \frac{3X^2}{2\mathbf{n}} (F_{\mathbf{a}, \mathcal{B}} F_{\mathbf{n}, \mathcal{Y}} - F_{\mathbf{n}, \mathcal{B}} F_{\mathbf{a}, \mathcal{Y}}) = 0. \quad (137)$$

Below we assume that the rank of the matrix A_{ia} is 1. When the rank of A_{ia} is zero, all the components of A_{ia} vanishes and then the transformation (135) does not involve $\dot{\mathbf{n}}$ and $\dot{\mathbf{a}}$. In this case, the invertibility condition is simply given by $\det B_{ia} \neq 0$.

When the degeneracy condition (137) is satisfied, there exists eigenvectors for zero eigenvalue n_i, m_a and their dual vectors V_i, U_a satisfying

$$n_i A_{ia} = 0 = A_{ia} m_a, \quad n_i n_i = 1 = m_a m_a, \quad n_i = \epsilon_{ij} V_j, \quad m_a = \epsilon_{ab} U_b. \quad (138)$$

They are explicitly given as

$$\begin{aligned} n_i &= N^{-1} \left(F_{\mathbf{a},\mathcal{B}}, -\frac{\mathbf{n}}{\mathbf{a}} F_{\mathbf{n},\mathcal{B}} \right), & m_a &= M^{-1} \left(\frac{3\mathbf{n}}{\mathbf{a}} F_{\mathbf{n},\mathcal{B}}, 2X F_{\mathbf{n},\mathcal{Y}} + F_{\mathbf{n},\mathcal{B}} \right), \\ V_i &= N^{-1} \left(-\frac{\mathbf{n}}{\mathbf{a}} F_{\mathbf{n},\mathcal{B}}, -F_{\mathbf{a},\mathcal{B}} \right), & U_a &= M^{-1} \left(2X F_{\mathbf{n},\mathcal{Y}} + F_{\mathbf{n},\mathcal{B}}, -\frac{3\mathbf{n}}{\mathbf{a}} F_{\mathbf{n},\mathcal{B}} \right), \end{aligned} \quad (139)$$

where M and N are normalization coefficients given by

$$M = \sqrt{(F_{\mathbf{n},\mathcal{B}} + 2X F_{\mathbf{n},\mathcal{Y}})^2 + (3\mathbf{n} F_{\mathbf{n},\mathcal{B}}/\mathbf{a})^2}, \quad N = \sqrt{F_{\mathbf{a},\mathcal{B}}^2 + (\mathbf{n} F_{\mathbf{n},\mathcal{B}}/\mathbf{a})^2}. \quad (140)$$

We assume $M, N \neq 0$ below, and discuss the case where either M or N vanishes separately.

Using these expressions, the second degeneracy condition (68) in the invertibility conditions is evaluated as

$$0 = n_i B_{ia} m_a = \frac{1}{MN} \frac{\mathbf{n} F_{\mathbf{n},\mathcal{B}}}{\mathbf{a}} [3F_{\mathbf{n}} F_{\mathbf{a},\mathcal{B}} - F_{\mathbf{a}} (F_{\mathbf{n},\mathcal{B}} + 2X F_{\mathbf{n},\mathcal{Y}}) + 3X (F_{\mathbf{a},\mathcal{X}} F_{\mathbf{n},\mathcal{B}} - F_{\mathbf{a},\mathcal{B}} F_{\mathbf{n},\mathcal{X}})] , \quad (141)$$

where we eliminated $F_{\mathbf{a},\mathcal{Y}}$ using the first degeneracy condition (137).

The non-degeneracy conditions (73) in the invertibility conditions are given as follows.

$$0 \neq n_i B_{ia} U_a = \frac{M F_{\mathbf{a}}}{3N} , \quad (142)$$

where we used the first and second degeneracy conditions (137), (141) to simplify the expression. The other non-degeneracy condition can be derived after some calculations as

$$0 \neq \left(V_i B_{ia} - a \dot{U}_a \right) m_a = \frac{3N}{M} \left[-(-X)^{3/2} F_{\mathbf{n},\mathcal{B}} \left(\frac{F_{\mathbf{n},\mathcal{Y}}}{F_{\mathbf{n},\mathcal{B}}} \right)' - F_{\mathbf{n}} + X F_{\mathbf{n},\mathcal{X}} + \mathcal{B} F_{\mathbf{n},\mathcal{B}} + \mathcal{Y} F_{\mathbf{n},\mathcal{Y}} \right], \quad (143)$$

where a prime ($'$) denotes a derivative with respect to the proper time, that is, $f' \equiv \mathbf{n}^{-1/2} \dot{f}$. To simplify this equation, we used the degeneracy conditions (137), (141) and their t derivatives, that is, we assumed the degeneracy conditions (137), (141) are satisfied identically at any t .

To summarize, the transformation (135) becomes invertible when the conditions (137), (141), (142) and (143) are satisfied. These conditions should be regarded as necessary conditions for the invertibility of the generalized disformal transformation (130) for a general metric, because the above results are derived only for the FRW ansatz (132), in which the degrees of freedom of the metric is reduced to two functions $\mathbf{n}(t), \mathbf{a}(t)$ depending only on t . The invertibility conditions for a general metric should encompass the conditions (137), (141), (142) and (143), while they may contain more stringent conditions in general. An obvious next step is to derive the invertibility conditions for a more general metric, and also it would be interesting to construct examples of invertible transformation based on the invertibility conditions obtained above. We reserve those issues for future works.

5.2.3 $M, N = 0$ case

Let us briefly mention the case that either M or N given by Eq. (140) vanishes. In this case, it turns out that the invertibility conditions imply that $F_{\mathbf{n},\mathcal{B}} = F_{\mathbf{n},\mathcal{Y}} = F_{\mathbf{a},\mathcal{B}} = 0$, that is, $F_{\mathbf{n}} = F_{\mathbf{n}}(X)$ and $F_{\mathbf{a}} = F_{\mathbf{a}}(X, \mathcal{Y})$. For example, if we assume that $M = 0$ while keeping N arbitrary, Eq. (140) implies that $F_{\mathbf{n},\mathcal{B}}$ obey

$$F_{\mathbf{n},\mathcal{B}} + 2X F_{\mathbf{n},\mathcal{Y}} = 3\mathbf{n} F_{\mathbf{n},\mathcal{B}}/\mathbf{a} = 0 \quad \Leftrightarrow \quad F_{\mathbf{n},\mathcal{B}} = F_{\mathbf{n},\mathcal{Y}} = 0 . \quad (144)$$

Then it follows that

$$A_{ia} = \frac{1}{2} \sqrt{\frac{-X}{\mathbf{n}}} \begin{pmatrix} 0 & 0 \\ \frac{\mathbf{a}}{\mathbf{n}} (F_{\mathbf{a},\mathcal{B}} + 2X F_{\mathbf{a},\mathcal{Y}}) & -3F_{\mathbf{a},\mathcal{B}} \end{pmatrix} . \quad (145)$$

This A_{ia} satisfies the first degeneracy condition (137) automatically, and the eigenvectors for vanishing eigenvalue of A_{ia} are given by

$$m_a \propto M'^{-1} \left(3F_{\mathbf{a},\mathcal{B}}, \frac{\mathbf{a}}{\mathbf{n}} (F_{\mathbf{a},\mathcal{B}} + 2X F_{\mathbf{a},\mathcal{Y}}) \right), \quad n_i = (1, 0) , \quad (146)$$

where $M' \equiv \sqrt{(3F_{a,B})^2 + \frac{a^2}{n^2} (F_{a,B} + 2XF_{a,Y})^2}$. Using these vectors, the second degeneracy condition (141) is evaluated as

$$0 = n_i B_{ia} m_a = \frac{3}{M'} F_{a,B} (F_n - XF_{n,X}) . \quad (147)$$

By direct calculations, it can be shown that the non-degeneracy conditions (142), (143) are reduced to

$$F_n - XF_{n,X} \neq 0 , \quad F_a \neq 0 . \quad (148)$$

To make these conditions compatible with Eq. (147) it is necessary to set $F_{a,B} = 0$, hence $F_{n,B} = F_{n,Y} = F_{a,B} = 0$ is shown. The same conclusion follows even when we assume that $N = 0$ while keeping M arbitrary.

When $F_{n,B} = F_{n,Y} = F_{a,B} = 0$, which corresponds to the case $M = N = 0$, it follows that

$$A_{ia} = - \left(\frac{-X}{n} \right)^{3/2} \begin{pmatrix} 0 & 0 \\ \mathbf{a}F_{a,Y} & 0 \end{pmatrix} \quad (149)$$

and $V_i = (0, 1)$, $U_a = (1, 0)$, $n_i = (1, 0)$, $m_a = (0, 1)$. By direct calculations, it can be shown that the first and second degeneracy conditions (137), (141) are automatically satisfied, and the non-degeneracy conditions (142), (143) are given by Eq. (148).

5.2.4 Application to invertible generalized disformal transformation

Let us apply this result to recently constructed example of invertible disformal transformation with second derivative terms, which is given by [12]

$$\tilde{g}_{\mu\nu} = \mathcal{F}_0 g_{\mu\nu} + \mathcal{F}_1 \chi_\mu \chi_\nu + \mathcal{F}_3 \chi_{(\mu} X_{\nu)} + \mathcal{F}_4 X_\mu X_\nu , \quad (150)$$

where $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_4$ are functions of χ, X, \mathcal{Y} and \mathcal{Z} . This transformation corresponds to (130) with $\mathcal{F}_2 = \mathcal{F}_5 = 0$ and the dependence of the functions \mathcal{F}_i on \mathcal{B} is dropped. The sufficient conditions to make this transformation are given by

$$\mathcal{F}_0 \neq 0 , \quad \mathcal{F} \neq 0 , \quad \tilde{X}_{,X} \neq 0 , \quad \tilde{X}_{,Y} = \tilde{X}_{,Z} = 0 , \quad \left| \frac{\partial(\tilde{\mathcal{Y}}, \tilde{\mathcal{Z}})}{\partial(\mathcal{Y}, \mathcal{Z})} \right| \neq 0 , \quad (151)$$

where $\tilde{X}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Z}}, \mathcal{F}$ are defined as

$$\begin{aligned} \tilde{X} &\equiv \tilde{g}^{\mu\nu} \chi_\mu \chi_\nu = \frac{X\mathcal{F}_0 - \mathcal{F}_4(\mathcal{Y}^2 - X\mathcal{Z})}{\mathcal{F}} , \\ \tilde{\mathcal{Y}} &\equiv \tilde{g}^{\mu\nu} \chi_\mu \tilde{X}_\nu = \tilde{X}_{,X} \frac{\mathcal{Y}\mathcal{F}_0 + \frac{1}{2}\mathcal{F}_2(\mathcal{Y}^2 - X\mathcal{Z})}{\mathcal{F}} + \tilde{X}_{,X} \tilde{X} , \\ \tilde{\mathcal{Z}} &\equiv \tilde{g}^{\mu\nu} \tilde{X}_\mu \tilde{X}_\nu = \tilde{X}_{,X}^2 \frac{\mathcal{Z}\mathcal{F}_0 - \mathcal{F}_1(\mathcal{Y}^2 - X\mathcal{Z})}{\mathcal{F}} + 2\tilde{X}_{,X} \tilde{\mathcal{Y}} - \tilde{X}_{,X}^2 \tilde{X} , \\ \mathcal{F} &\equiv \mathcal{F}_0^2 + \mathcal{F}_0(X\mathcal{F}_1 + \mathcal{Y}\mathcal{F}_2 + \mathcal{Z}\mathcal{F}_3) + (\mathcal{F}_2^2 - \mathcal{F}_1\mathcal{F}_3)(\mathcal{Y}^2 - X\mathcal{Z}) . \end{aligned} \quad (152)$$

More precisely, to construct an invertible transformation of the form of Eq. (150), Ref. [12] assumed $\tilde{X}_{,Y} = \tilde{X}_{,Z} = 0$ (or equivalently $\tilde{X} = \tilde{X}(\chi, X)$), then the conditions other than $\tilde{X}_{,Y} = \tilde{X}_{,Z} = 0$ in Eq. (151) were imposed to make (150) invertible.

Below, we show that the invertibility conditions (151) for the transformation (150) proposed in Ref. [12] can be reproduced from the conditions derived in this work, which are Eqs. (137), (141), (142) and (143), when they are evaluated on the FRW spacetime. For this purpose we follow the strategy of section 5.2.1. The transformations of the FRW metric functions are given by

$$\begin{aligned} \tilde{\mathbf{n}} &= \mathbf{n} \left(\mathcal{F}_0 + \mathcal{F}_1 X + \mathcal{F}_3 \mathcal{Y} + \mathcal{F}_4 \frac{\mathcal{Y}^2}{X} \right) := \mathbf{n} F_n(X, \mathcal{Y}) , \\ \tilde{\mathbf{a}} &= \mathbf{a} \mathcal{F}_0 := \mathbf{a} F_a(X, \mathcal{Y}) , \end{aligned} \quad (153)$$

where we used $\mathcal{F}_2 = \mathcal{F}_5 = 0$ and also Eq. (133) to express \mathcal{Z} in terms of X and \mathcal{Y} . Hereafter, χ in arguments of various functions are not shown just to simplify the notation. Using Eq. (133), quantities defined in Eq. (133) are evaluated on the FRW background as

$$\tilde{X} = \frac{X}{F_n}, \quad \tilde{\mathcal{Y}} = \tilde{X}_{,X} \frac{\mathcal{Y}}{F_n} + \tilde{X}_{,X} \tilde{X}, \quad \tilde{\mathcal{Z}} = \tilde{X}_{,X}^2 \frac{\mathcal{Z}}{F_n} + 2\tilde{X}_{,X} \tilde{\mathcal{Y}} - \tilde{X}_{,X}^2 \tilde{X}, \quad \mathcal{F} = \mathcal{F}_0 F_n. \quad (154)$$

Since $\tilde{X} = \tilde{X}(X)$ is assumed, the first equation in Eq. (154) implies F_n is a function only of X (and χ , which is not shown in the argument explicitly). Then, the quantities appearing in the invertibility conditions (151) are evaluated as

$$\tilde{X}_{,X} = \frac{1}{F_n^2} (F_n - X F_{n,X}), \quad \left| \frac{\partial(\tilde{\mathcal{Y}}, \tilde{\mathcal{Z}})}{\partial(\mathcal{Y}, \mathcal{Z})} \right| = \frac{\tilde{X}_{,X}^3}{F_n^2}, \quad (155)$$

where we used $F_n = F_n(X)$. Using these expressions, it can be shown that the invertibility conditions (151) are equivalent to

$$\mathcal{F}_0 \neq 0, \quad F_n \neq 0, \quad \tilde{X} = \tilde{X}(X), \quad F_n - X F_{n,X} \neq 0. \quad (156)$$

Using these expressions now let us show that the conditions (156) are equivalent to the invertibility conditions (137), (141), (142) and (143) derived in this work, once $\tilde{X} = \tilde{X}(X)$ is assumed. Equation (153) and $F_n = F_n(X)$, which follows from $\tilde{X} = \tilde{X}(X)$, imply $F_{n,\mathcal{B}} = F_{n,\mathcal{Y}} = F_{\mathcal{a},\mathcal{B}} = 0$, hence the transformation given by (153) falls into the case discussed in section 5.2.3. In this case, the degeneracy conditions (137), (141) among the invertibility conditions are automatically satisfied, and the non-degeneracy conditions (142), (143) are given by Eq. (148). Since $F_{\mathcal{a}} = \mathcal{F}_0$ as shown by Eq. (153), the first and the last equation in Eq. (156) coincides with the invertibility condition (148) derived in section 5.2.3. The remaining condition in Eq. (156), $F_n \neq 0$, is necessary to keep $\tilde{n} \neq 0$ so that the metric after the transformation regular (see Eq. (153)).

In this section we showed that the generalized disformal transformation of the form of Eq. (150) falls into the special case discussed in section 5.2.3 when it is restricted on the FRW spacetime. On the other hand, the results of section 5.2.2 suggest that there may be more general transformations of the type of Eq. (130) that are invertible, though this result should be understood with caution because more stringent conditions may be obtained if we take more general spacetime and scalar field configuration as the background. It would be interesting to clarify if the transformation (150) is the unique invertible transformation among transformations of the form of Eq. (130), or there are more general invertible transformations.

6 Discussions

In this work we focused on field transformations that involve up to first-order derivative of fields, $\phi_i = \bar{\phi}_i(\psi_a, \partial_\alpha \psi_a, x^\mu)$, between two fields ψ_a and other two fields ϕ_i , and showed conditions for this transformation to be invertible. A field transformation of this type changes the number of derivatives acting on the fields, hence in general it changes number of degree of freedom. When the transformation function satisfies certain conditions, however, appearance of additional degrees of freedom is hindered and then the transformation can be invertible.

We emphasize that the degeneracy conditions and the procedure to derive the complete set of the invertibility conditions given in section 2.1 apply to field transformations that involve arbitrary number of fields and arbitrary-order derivatives, though we presented expressions only for transformations with first derivatives just for simplicity. In the following part of this work, for simplicity, we limited our scope to the simplest case that the transformation maps two fields ϕ_i ($i = 1, 2$) to other two fields ψ_a ($a = 1, 2$), and then derived the conditions for this transformation to be invertible. This is because, as we have shown, there is no invertible transformation with derivatives for one field case and hence two field case is the simplest case. To derive the necessary conditions for the invertibility, we employed the method of characteristics for a partial differential equations in section 2. If a transformation is invertible, the number of characteristic surfaces, which corresponds to the number of physical degrees of freedom, must

be invariant. The derivatives contained in the transformation generates extra characteristic surfaces in general, and then the necessary conditions are obtained by demanding that extra characteristic surfaces are removed so that the total number of characteristic surfaces is invariant. After deriving the necessary conditions, in section 3 we confirmed that they are actually sufficient. It turned out that the invertibility conditions are composed of two degeneracy conditions (50), (68) and two non-degeneracy conditions given in Eq. (73).

As an application of the thus derived invertibility conditions, in section 4 we showed some examples of invertible transformations satisfying the invertibility conditions. The invertibility conditions can be regarded as equations for the function of the field transformation, and if we could construct their general solution we could obtain the most-general invertible transformation. Instead of finding the general solution, we proposed some ansatz for the transformation for which a part of the invertibility conditions are automatically satisfied, and as a result we obtained three kinds of invertible transformations as non-trivial examples. Although they may not be most general, they span broad class of invertible transformations and would provide a basis for construction and classification of various invertible transformations.

As another application, in section 5 we considered a higher-derivative extension of the disformal transformation in gravity and examined its invertibility. Using our invertibility conditions, just by simple calculations we have explicitly shown that a disformal-type transformation associated with second derivative of the scalar field cannot be invertible, which is the first rigorous proof as far as we are aware of. We also derived the necessary conditions to make the generalized disformal transformation with higher derivatives of the form of Eq. (130) invertible. We studied the example of invertible disformal transformation with higher derivatives in Ref. [12], and our invertibility conditions reproduced those proposed in this literature when the transformation is reduced on the FRW spacetime background. Also our result imply that there might be more general invertible transformation of the form of Eq. (130), and it would be interesting to examine this possibility further.

Several directions of the future research are in order. In this work we considered the simplest case that the transformation involves only two scalar fields up to their first derivatives. Provided that our method can be generalized to transformations involving both a scalar field and a metric, we will be able to apply our results to studies on scalar-tensor theories and various modified gravity theories. For example, invertible disformal transformations were utilized to generate and classify the so-called DHOST theories from a simpler theory, the Horndeski theory. This scheme may be generalized to incorporate higher-order derivatives if we could generalize the disformal transformation by introducing higher derivatives. Such an applications to modified gravity theory will be an ultimate goal of this study. As a first step for such a goal, it would be useful to consider a generalization to involve more than two scalar fields and more than first-order derivatives. Such generalizations within transformations of scalar fields, and also further generalizations including more fields such as metric will be discussed in our future work.

Acknowledgments

N. T. is supported in part by JSPS Grant-in-Aid for Scientific Research No. JP18K03623 and 21H05189.

A Derivation of Eq. (6)

In this section we sketch the derivation of the formula (6), based on which the necessary conditions for the invertibility are derived. Acting ∂_{μ_n} on $\phi_i = \bar{\phi}_i(\psi_a, \partial_\alpha \psi_a, x^\mu)$ gives

$$\partial_{\mu_n} \phi_i(\psi_a, \partial_\alpha \psi_a, x^\mu) = \frac{\partial \bar{\phi}_i}{\partial \psi_a} \partial_{\mu_n} \psi_a + \frac{\partial \bar{\phi}_i}{\partial (\partial_\alpha \psi_a)} \partial_{\mu_n} \partial_\alpha \psi_a + \frac{\partial \bar{\phi}_i}{\partial x^{\mu_n}}, \quad (157)$$

where the third term is the partial derivative with respect to the explicit x^μ dependence of $\bar{\phi}_i$. Then, the highest derivative term of ψ_a contained in $\partial_{\mu_1 \dots \mu_n} \phi_i$ is generated when all the derivatives other than ∂_{μ_n}

act on $\partial_{\mu_n} \partial_\alpha \psi_a$ in Eq. (157), that is,

$$\partial_{\mu_1 \dots \mu_n} \phi_i \ni \frac{\partial \bar{\phi}_i}{\partial (\partial_\alpha \psi_a)} \partial_{\mu_1 \dots \mu_n \alpha} \psi_a = A_{ia}^\alpha \partial_{\mu_1 \dots \mu_n \alpha} \psi_a . \quad (158)$$

This term is the origin of the leading $\partial^{n+1} \psi_a$ term of Eq. (6). The subleading $\partial^n \psi_a$ term is composed of the following two contributions. The first one is generated when $\partial_{\mu_1 \dots \mu_{n-1}}$ acts on $\partial_{\mu_n} \psi_a$ in Eq. (157), that is,

$$\partial_{\mu_1 \dots \mu_n} \phi_i \ni \frac{\partial \bar{\phi}_i}{\partial \psi_a} \partial_{\mu_1 \dots \mu_n} \psi_a = B_{ia} \partial_{\mu_1 \dots \mu_n} \psi_a . \quad (159)$$

The second one is generated when $n-1$ derivatives among $\partial_{\mu_1 \dots \mu_n}$ are consumed to generate $A_{ia}^{\mu_n} \partial^n \psi_a$ and the other one derivative acts directly on ϕ_i , that is,

$$\partial_{\mu_1 \dots \mu_n} \phi_i \ni \sum_{k=1}^n \frac{\partial^2 \bar{\phi}_i}{\partial x^{\mu_k} \partial (\partial_\alpha \psi_a)} \partial_{\mu_1 \dots \mu_{k-1} \mu_{k+1} \dots \mu_n \alpha} \psi_a = \sum_{k=1}^n \partial_{\mu_k} A_{ia}^\alpha \partial_{\mu_1 \dots \mu_{k-1} \mu_{k+1} \dots \mu_n \alpha} \psi_a . \quad (160)$$

Contracting with a totally-symmetric coefficient $K_{bi}^{(\mu_1 \dots \mu_n)}$, the expression appearing in Eq. (6) is obtained as

$$K_{bi}^{(\mu_1 \dots \mu_n)} \partial_{\mu_1 \dots \mu_n} \phi_i \ni K_{bi}^{(\mu_1 \dots \mu_n)} \sum_{k=1}^n \partial_{\mu_k} A_{ia}^\alpha \partial_{\mu_1 \dots \mu_{k-1} \mu_{k+1} \dots \mu_n \alpha} \psi_a = n K_{bi}^{(\alpha \mu_1 \dots \mu_{n-1})} \partial_\alpha A_{ia}^{\mu_n} \partial_{\mu_1 \dots \mu_n} \psi_a . \quad (161)$$

The $\partial^{n-k} \psi_a$ term of Eq. (6) is obtained in a similar manner. The $\partial^k B_{ia}$ term is obtained by using $n-k$ derivatives to generate $\partial^{n-k} \psi_a$ and by acting the other derivatives on ϕ_i directly. The $\partial^{k+1} A_{ia}^\mu$ term is obtained by using $n-k-1$ derivatives to generate $\partial^{n-k} \psi_a$ and by acting the other derivatives on ϕ_i directly. The coefficients of each term, $\binom{n}{k}$ and $\binom{n}{k+1}$, correspond to the number of combinations of the derivatives among ∂^n used to generate $\partial^k \psi_a$.

B Derivation of Eq. (40)

We give the derivation of Eq. (40) based on Eq. (6) and using the degeneracy conditions in the two-field case, Eqs. (32) and (37). We also use results given in section 2.2.

We start the derivation from

$$\begin{aligned} & \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \mu_n + 2\mu_{n+3}} \partial_{\mu_1 \dots \mu_{n+3}} \phi_i \\ & - \left[\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2, bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} - \left(n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \mu_{n+1}} + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \mu_{n+1} \alpha} \right) \partial_\alpha \bar{A}_{bi}^{\mu_{n+2}} \right] \partial_{\mu_1 \dots \mu_{n+2}} \phi_i, \end{aligned} \quad (162)$$

which is given by $\partial^{n+2} \psi$ terms and lower-order terms once rewritten in terms of ψ using Eq. (6), as explained in section 2.2.

Using Eq. (6), the first term of Eq. (162) is expressed in terms of ψ as

$$\begin{aligned} & \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \mu_n + 2\mu_{n+3}} \partial_{\mu_1 \dots \mu_{n+3}} \phi_i \\ & = [\partial^{n+3} \psi \text{ term}] \\ & + \left\{ \left[\frac{n(n-1)}{2} \tilde{K}^{(\alpha_1 \alpha_2 \mu_1 \dots \mu_{n-2})} \bar{A}_{bi}^{\mu_{n-1}} \bar{A}^{2, \mu_n \mu_{n+1}} + n \tilde{K}^{(\alpha_1 \mu_1 \dots \mu_{n-1})} \bar{A}_{bi}^{\alpha_2} \bar{A}^{2, \mu_n \mu_{n+1}} \right. \right. \\ & \quad \left. \left. + 2n \tilde{K}^{(\alpha_1 \mu_1 \dots \mu_{n-1})} \bar{A}_{bi}^{\mu_n} \bar{A}^{2, \mu_{n+1} \alpha_2} + \tilde{K}^{(\mu_1 \dots \mu_n)} (2 \bar{A}_{bi}^{\alpha_1} \bar{A}^{2, \mu_{n+1} \alpha_2} + \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \alpha_1 \alpha_2}) \right] \partial_{\alpha_1 \alpha_2} A_{ia}^{\mu_{n+2}} \right. \\ & \quad \left. + \left[n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}_{bi}^{\mu_n} \bar{A}^{2, \mu_{n+1} \mu_{n+2}} + \tilde{K}^{(\mu_1 \dots \mu_n)} (\bar{A}_{bi}^\alpha \bar{A}^{2, \mu_{n+1} \mu_{n+2}} + 2 \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \mu_n + 2\alpha}) \right] \partial_\alpha B_{ia} \right\} \partial_{\mu_1 \dots \mu_{n+2}} \psi_a \\ & + \mathcal{O}(\partial^{n+1} \psi_a), \end{aligned} \quad (163)$$

where the terms appearing on the right-hand side are classified according to the positions of the indices $\alpha, \alpha_1, \alpha_2$. Likewise, the second term of Eq. (162) is expressed as

$$\begin{aligned}
& \left[\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} - \left(n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \mu_{n+1}} + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \mu_{n+1} \alpha} \right) \partial_\alpha \bar{A}_{bi}^{\mu_{n+2}} \right] \partial_{\mu_1 \dots \mu_{n+2}} \phi_i \\
&= [\partial^{n+3} \psi \text{ term}] \\
&+ \left\{ \left[\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} - \left(n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \mu_{n+1}} + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \mu_{n+1} \alpha} \right) \partial_\alpha \bar{A}_{bi}^{\mu_{n+2}} \right] B_{ia} \right. \\
&+ \left\{ n \tilde{K}^{(\alpha_2 \mu_1 \dots \mu_{n-1})} \mathcal{A}_{2,bc}^{\mu_n} \tilde{A}_{ci}^{\mu_{n+1}} + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,bc}^{(\alpha_2)} \tilde{A}_{ci}^{\mu_{n+1}} \right. \\
&- n \left[(n-1) \tilde{K}^{(\alpha_1 \alpha_2 \mu_1 \dots \mu_{n-2})} \bar{A}^{2, \mu_{n-1} \mu_n} \partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+1}} + \tilde{K}^{(\alpha_1 \mu_1 \dots \mu_{n-1})} (2 \bar{A}^{2, \alpha_2 \mu_n} \partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+1}} + \bar{A}^{2, \mu_n \mu_{n+1}} \partial_{\alpha_1} \bar{A}_{bi}^{\alpha_2}) \right] \\
&- \left. \left. 2 \left(n \tilde{K}^{(\alpha_2 \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \alpha_1} \partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+1}} + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \alpha_1 (\alpha_2)} \partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+1}} \right) \right\} \partial_{\alpha_2} A_{ia}^{\mu_{n+2}} \right\} \partial_{\mu_1 \dots \mu_{n+2}} \psi_a \\
&+ \mathcal{O}(\partial^{n+1} \psi_a), \tag{164}
\end{aligned}$$

where the $\partial^{n+3} \psi$ term on the right-hand side is the same as that in Eq. (163). Then, in Eq. (162) the $\partial^{n+3} \psi$ terms cancel out, and the remaining terms are given by the difference between Eqs. (163) and (164). Reorganizing the terms according to their tensor structures, it can be expressed as

$$\begin{aligned}
& \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}_{bi}^{\mu_{n+1}} \bar{A}^{2, \mu_{n+2} \mu_{n+3}} \partial_{\mu_1 \dots \mu_{n+3}} \phi_i \\
&- \left[\tilde{K}^{(\mu_1 \dots \mu_n)} \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} - \left(n \tilde{K}^{(\alpha \mu_1 \dots \mu_{n-1})} \bar{A}^{2, \mu_n \mu_{n+1}} + 2 \tilde{K}^{(\mu_1 \dots \mu_n)} \bar{A}^{2, \mu_{n+1} \alpha} \right) \partial_\alpha \bar{A}_{bi}^{\mu_{n+2}} \right] \partial_{\mu_1 \dots \mu_{n+2}} \phi_i \\
&= \left\{ \frac{n(n-1)}{2} \tilde{K}^{(\alpha_1 \alpha_2 \mu_1 \dots \mu_{n-2})} \bar{A}^{2, \mu_{n-1} \mu_n} \left[\bar{A}_{bi}^{\mu_{n+1}} \partial_{\alpha_1 \alpha_2} A_{ia}^{\mu_{n+2}} + 2 (\partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+1}}) \partial_{\alpha_2} A_{ia}^{\mu_{n+2}} \right] \right. \\
&+ n \tilde{K}^{(\alpha_1 \mu_1 \dots \mu_{n-1})} \left\{ - \mathcal{A}_{2,bc}^{\mu_n} \tilde{A}_{ci}^{\mu_{n+1}} \partial_{\alpha_1} A_{ia}^{\mu_{n+2}} \right. \\
&+ 2 \bar{A}^{2, \mu_n \alpha_2} \left[\bar{A}_{bi}^{\mu_{n+1}} \partial_{\alpha_1 \alpha_2} A_{ia}^{\mu_{n+2}} + (\partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+1}}) \partial_{\alpha_2} A_{ia}^{\mu_{n+2}} + (\partial_{\alpha_2} \bar{A}_{bi}^{\mu_{n+1}}) \partial_{\alpha_1} A_{ia}^{\mu_{n+2}} \right] \\
&+ \bar{A}^{2, \mu_n \mu_{n+1}} \left[\bar{A}_{bi}^{\mu_{n+2}} \partial_{\alpha_1} B_{ia} + (\partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+2}}) B_{ia} + \bar{A}_{bi}^{\alpha_2} \partial_{\alpha_1 \alpha_2} A_{ia}^{\mu_{n+2}} + (\partial_{\alpha_1} \bar{A}_{bi}^{\alpha_2}) \partial_{\alpha_2} A_{ia}^{\mu_{n+2}} \right] \left. \right\} \\
&+ \tilde{K}^{(\mu_1 \dots \mu_n)} \left[\bar{A}_{bi}^{\alpha_1} \bar{A}^{2, \mu_{n+1} \mu_{n+2}} \partial_{\alpha_1} B_{ia} - 2 \mathcal{A}_{2,bc}^{(\alpha_2)} \tilde{A}_{ci}^{\mu_{n+1}} \partial_{\alpha_2} A_{ia}^{\mu_{n+2}} - \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} B_{ia} \right. \\
&+ 2 \bar{A}^{2, \mu_{n+1} \alpha_1} \left(\partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+2}} B_{ia} + \bar{A}_{bi}^{\mu_{n+2}} \partial_{\alpha_1} B_{ia} + \bar{A}_{bi}^{\alpha_2} \partial_{\alpha_1 \alpha_2} A_{ia}^{\mu_{n+2}} + \partial_{\alpha_1} \bar{A}_{bi}^{\alpha_2} \partial_{\alpha_2} A_{ia}^{\mu_{n+2}} \right) \\
&+ \left. \bar{A}^{2, \alpha_1 \alpha_2} \left(\bar{A}_{bi}^{\mu_{n+1}} \partial_{\alpha_1 \alpha_2} A_{ia}^{\mu_{n+2}} + 2 \partial_{\alpha_1} \bar{A}_{bi}^{\mu_{n+1}} \partial_{\alpha_2} A_{ia}^{\mu_{n+2}} \right) \right] \left. \right\} \partial_{\mu_1 \dots \mu_{n+2}} \psi_a + \mathcal{O}(\partial^{n+1} \psi_a) \\
&= \left\{ - \frac{n(n-1)}{2} \tilde{K}^{(\alpha_1 \alpha_2 \mu_1 \dots \mu_{n-2})} \bar{A}^{2, \mu_{n-1} \mu_n} (\partial_{\alpha_1 \alpha_2} \bar{A}_{bi}^{\mu_{n+1}}) A_{ia}^{\mu_{n+2}} \right. \\
&+ n \tilde{K}^{(\alpha_1 \mu_1 \dots \mu_{n-1})} \left[- \mathcal{A}_{2,bc}^{\mu_n} \tilde{A}_{ci}^{\mu_{n+1}} \partial_{\alpha_1} A_{ia}^{\mu_{n+2}} - 2 \bar{A}^{2, \mu_n \alpha_2} (\partial_{\alpha_1 \alpha_2} \bar{A}_{bi}^{\mu_{n+1}}) A_{ia}^{\mu_{n+2}} + \bar{A}^{2, \mu_n \mu_{n+1}} \partial_{\alpha_1} \mathcal{A}_{ba}^{\mu_{n+2}} \right] \\
&+ \tilde{K}^{(\mu_1 \dots \mu_n)} \left[\bar{A}^{2, \mu_{n+1} \mu_{n+2}} \bar{A}_{bi}^{\alpha} \partial_\alpha B_{ia} - 2 \mathcal{A}_{2,bc}^{(\alpha)} \tilde{A}_{ci}^{\mu_{n+1}} \partial_\alpha A_{ia}^{\mu_{n+2}} - \mathcal{A}_{2,bc}^{\mu_{n+1}} \tilde{A}_{ci}^{\mu_{n+2}} B_{ia} + 2 \bar{A}^{2, \mu_{n+1} \alpha} \partial_\alpha \mathcal{A}_{2,ba}^{\mu_{n+2}} \right. \\
&- \left. \bar{A}^{2, \alpha_1 \alpha_2} (\partial_{\alpha_1 \alpha_2} \bar{A}_{bi}^{\mu_{n+1}}) A_{ia}^{\mu_{n+2}} \right] \left. \right\} \partial_{\mu_1 \dots \mu_{n+2}} \psi_a + \mathcal{O}(\partial^{n+1} \psi_a). \tag{165}
\end{aligned}$$

At the final equality, we used Eq. (28) and the identity that follows from the second derivative of Eq. (12) in the two-field case, which is given by

$$\partial_\mu \partial_\nu \left(\bar{A}_{bi}^{(\alpha} A_{ia}^{\beta)} \right) = \left(\partial_\mu \partial_\nu \bar{A}_{bi}^{(\alpha} \right) A_{ia}^{\beta)} + \left(\partial_\mu \bar{A}_{bi}^{(\alpha} \right) \partial_\nu A_{ia}^{\beta)} + \left(\partial_\nu \bar{A}_{bi}^{(\alpha} \right) \partial_\mu A_{ia}^{\beta)} + \bar{A}_{bi}^{(\alpha} \left(\partial_\mu \partial_\nu A_{ia}^{\beta)} \right) = 0. \tag{166}$$

As argued in section 2.2, to find the last condition for the invertibility, we should focus on the coefficient of the highest derivative term in the ψ^\perp space, and this coefficient is obtained by replacing $\partial_{\mu_1 \dots \mu_{n+2}} \psi_a$ by $\bar{A}_{aj}^{\mu_3}$ and symmetrizing over μ_1, μ_2, μ_3 in Eq. (165). Below, let us evaluate each term in Eq. (165) after

this replacement. First, the $\tilde{K}^{(\alpha_1\alpha_2\mu_1\dots\mu_{n-2})}$ term vanishes because it contains $A_{ia}^{(\mu_{n+2})}\bar{A}_{aj}^{(\mu_3)} = 0$, which is enforced by Eq. (28). Next, the $\tilde{K}^{(\alpha_1\mu_1\dots\mu_{n-1})}$ term also vanishes as follows.

$$\begin{aligned}
& \left[-\mathcal{A}_{2,bc}^{(\mu_n)}\tilde{A}_{ci}^{\mu_{n+1}}\partial_{\alpha_1}A_{ia}^{\mu_{n+2}} - 2\bar{A}^{2,\alpha_2(\mu_n)}(\partial_{\alpha_1\alpha_2}\tilde{A}_{bi}^{\mu_{n+1}})A_{ia}^{\mu_{n+2}} + \bar{A}^{2,(\mu_n\mu_{n+1})}\partial_{\alpha_1}\mathcal{A}_{ba}^{\mu_{n+2}} \right] \bar{A}_{aj}^{(\mu_3)} \\
&= \left(-\mathcal{A}_{2,bc}^{(\mu_n)}\tilde{A}_{ci}^{\mu_{n+1}}\partial_{\alpha_1}A_{ia}^{\mu_{n+2}} + \bar{A}^{2,(\mu_n\mu_{n+1})}\partial_{\alpha_1}\mathcal{A}_{ba}^{\mu_{n+2}} \right) \bar{A}_{aj}^{(\mu_3)} \\
&= \left(\mathcal{A}_{2,bc}^{(\mu_n)}\tilde{A}_{ci}^{\mu_{n+1}}A_{ia}^{\mu_{n+2}} - \bar{A}^{2,(\mu_n\mu_{n+1})}\mathcal{A}_{ba}^{\mu_{n+2}} \right) \partial_{\alpha_1}\bar{A}_{aj}^{(\mu_3)} \\
&= \left[\mathcal{A}_{2,bc}^{(\mu_n)}\tilde{A}_{ci}^{\mu_{n+1}}A_{ia}^{\mu_{n+2}} - \mathcal{A}_{bc}^{(\mu_n)} \left(\bar{A}_{ci}^{\mu_{n+1}}\bar{A}_{ai}^{\mu_{n+2}} + \tilde{A}_{ci}^{\mu_{n+1}}A_{ia}^{\mu_{n+2}} \right) \right] \partial_{\alpha_1}\bar{A}_{aj}^{(\mu_3)} = 0. \tag{167}
\end{aligned}$$

The first equality follows from Eq. (28). The second equality follows from Eq. (14) and the first derivative of Eq. (37), that is,

$$\partial_{\alpha_1}\mathcal{A}_{ba}^{(\mu_{n+2})}\bar{A}_{aj}^{(\mu_3)} = -\mathcal{A}_{ba}^{(\mu_{n+2})}\partial_{\alpha_1}\bar{A}_{aj}^{(\mu_3)}. \tag{168}$$

At the third equality we replaced $\bar{A}^{2,\mu_n\mu_{n+1}}$ using Eq. (30). Then this equation can be shown to vanish using Eq. (37). Lastly, in the $\tilde{K}^{(\mu_1\dots\mu_n)}$ term of Eq. (165), the last part having $A_{ia}^{\mu_{n+2}}$ vanishes thanks to Eq. (12). The remaining term gives Eq. (41), once ∂B is rewritten into \mathcal{B} using Eq. (28).

C Setting $U_a = (0, 1)$ in $\phi_i = \phi_i(\psi_a, U_a(\psi_a)\partial_\mu\psi_a)$

In this appendix, we consider a transformation in which derivative of ψ_a appears only in a combination $U_a(\psi_a)\partial_\mu\psi_a$, that is,

$$\phi_i = \phi_i(\psi_a, U_a(\psi_a)\partial_\mu\psi_a), \tag{169}$$

and show that $U_a(\psi_a)$ can be set to a constant vector $U_a = (0, 1)$ without loss of generality by a field transformation $\psi_a = \psi_a(\tilde{\Psi}_a)$. This technique is used in section 4 to simplify examples of invertible transformations.

C.1 Field transformation to set $U_a = (0, 1)$

As a first step to set $U_a(\psi_a)$ to a constant vector, we rewrite this transformation as

$$\phi_i = \phi_i(\psi_a, \tilde{U}_a(\psi_a)\partial_\mu\psi_a), \tag{170}$$

where

$$\tilde{U}_a \equiv c(\psi_a)U_a(\psi_a) \tag{171}$$

is a local rescaling of U_a such that \tilde{U}_a is an irrotational vector in the ψ_a space, that is,

$$\frac{\partial\tilde{U}_1}{\partial\psi_2} - \frac{\partial\tilde{U}_2}{\partial\psi_1} = 0. \tag{172}$$

This \tilde{U}_a can be obtained by choosing the rescaling factor $c(\psi_a)$ appropriately. We show the construction method of $c(\psi_a)$ and $\tilde{U}_a(\psi_a)$ in the next section.

When (172) is satisfied, due to the Poincaré lemma there exists a scalar function $\Psi_1(\psi_k)$ satisfying

$$\tilde{U}_a = \frac{\partial\Psi_2}{\partial\psi_a}. \tag{173}$$

Let us also introduce another scalar function $\Psi_1(\psi_a)$ that is functionally independent from $\Psi_1(\psi_a)$ (*i.e.* $\det(\partial\Psi_a/\partial\psi_b) \neq 0$). Then there exists a one-to-one mapping between ψ_a and Ψ_a , and the transformation (170) may be expressed in terms of Ψ_a as

$$\phi_i = \phi_i(\Psi_a, \partial_\mu\Psi_2). \tag{174}$$

This is equivalent to setting $U_a = (0, 1)$ in the transformation (169). Hence, we may impose $U_i = (0, 1)$ without loss of generality in the transformation (169), as long as there exists a rescaling $c(\psi_i)$ satisfying the condition (172).

C.2 Finding irrotational \tilde{U}_i

In the argument above, it is crucial that there exists a rescaling (171) that makes \tilde{U}_a an irrotational vector satisfying (172). We show how to find such a rescaling in this section.

Using Eq. (171), the condition (172) can be rewritten as

$$U_1 \frac{\partial c}{\partial \psi_2} - U_2 \frac{\partial c}{\partial \psi_1} + c \left(\frac{\partial U_1}{\partial \psi_2} - \frac{\partial U_2}{\partial \psi_1} \right) = 0 . \quad (175)$$

This equation is a first-order partial differential equation for $\log c(\psi_1, \psi_2)$, and it can be solved at least locally once an appropriate boundary condition for c is given. For example, when $U_1 \neq 0$ we may solve Eq. (175) as an evolution equation in the “time” direction ψ_2 for an initial condition given by $c = 1$ on a $\psi_2 = \text{constant}$ line on the ψ_a space, regarding ψ_1 as the “spatial” coordinate.

References

- [1] E. Babichev, K. Izumi, N. Tanahashi and M. Yamaguchi, [arXiv:1907.12333 [hep-th]].
- [2] H. Motohashi, T. Suyama and K. Takahashi, Phys. Rev. D **94**, no.12, 124021 (2016) [arXiv:1608.00071 [gr-qc]].
- [3] J. Ben Achour, D. Langlois and K. Noui, Phys. Rev. D **93**, no.12, 124005 (2016) [arXiv:1602.08398 [gr-qc]].
- [4] G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space,” Int. J. Theor. Phys. **10**, 363 (1974).
- [5] J. D. Bekenstein, Phys. Rev. D **48**, 3641 (1993) [gr-qc/9211017].
- [6] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 2, Wiley-Interscience, Inc., New York (1962).
- [7] M. Zumalacárregui and J. García-Bellido, Phys. Rev. D **89** (2014), 064046 [arXiv:1308.4685 [gr-qc]].
- [8] G. Domènech, S. Mukohyama, R. Namba, A. Naruko, R. Saitou and Y. Watanabe, Phys. Rev. D **92** (2015) no.8, 084027 [arXiv:1507.05390 [hep-th]].
- [9] J. M. Ezquiaga, J. García-Bellido and M. Zumalacárregui, Phys. Rev. D **95** (2017) no.8, 084039 [arXiv:1701.05476 [hep-th]].
- [10] M. Minamitsuji, Phys. Lett. B **816**, 136240 (2021) [arXiv:2104.03662 [gr-qc]].
- [11] A. L. Alinea and T. Kubota, Int. J. Mod. Phys. D **30**, no.08, 2150057 (2021) [arXiv:2005.12747 [gr-qc]].
- [12] K. Takahashi, H. Motohashi and M. Minamitsuji, Phys. Rev. D **105**, no.2, 024015 (2022) doi:10.1103/PhysRevD.105.024015 [arXiv:2111.11634 [gr-qc]].