学籍番号	氏名

常微分方程式 演習 [2019年度後期 月曜1限]　第12回 (1/6(月))

(1) 以下の公式を導出せよ。なるべく自力で導出してみること。

(a) 微分のラプラス変換の式　$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$

(b) 積分のラプラス変換の式　$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}[f(t)]$

(c) ラプラス変換の微分の式　$\mathcal{L}[tf(t)] = -F'(s)$

(d) ラプラス変換の積分の式　$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(\sigma)d\sigma$

(a) $\mathcal{L}[f''(x)] = \int_0^\infty f''(x)e^{-sx}dx = \underbrace{[f'(x)e^{-sx}]_0^\infty}_{=-f'(0)} - \int_0^\infty f'(x)(-s)e^{-sx}dx$

$= -f'(0) + s\{\underbrace{[f(x)e^{-sx}]_0^\infty}_{=-f(0)} - \int_0^\infty f(x)e^{-sx}dx\} = -f'(0) - sf(0) + s^2\mathcal{L}[f(x)].$

(b) $\mathcal{L}[\int_0^x f(\tau)d\tau] = \int_0^\infty (\int_0^x f(\tau)d\tau)e^{-sx}dx$

$= \underbrace{[(\int_0^x f(\tau)d\tau)(-\frac{1}{s})e^{-sx}]_0^\infty}_{=0} - \int_0^\infty \frac{d}{dx}(\int_0^x f(\tau)d\tau)(-\frac{1}{s})e^{-sx}dx = \frac{1}{s}\int_0^\infty f(x)e^{-sx}dx$

$= \frac{1}{s}\mathcal{L}[f(x)]$

(c) $\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(x)e^{-sx}dx = \int_0^\infty f(x)\frac{d}{ds}e^{-sx}dx = \int_0^\infty f(x)(-x)e^{-sx}dx = -\int_0^\infty xf(x)e^{-sx}dx$

$= -\mathcal{L}[xf(x)].$

(d) $\int_s^\infty F(\sigma)d\sigma = \int_s^\infty(\int_0^\infty f(x)e^{-\sigma x}dx)d\sigma = \int_0^\infty f(x)\underbrace{(\int_s^\infty e^{-\sigma x}d\sigma)}_{=[-\frac{1}{x}e^{-\sigma x}]_{\sigma=s}^{\sigma=\infty} = +\frac{1}{x}e^{-sx}}ds$

$= \int_0^\infty \frac{1}{x}f(x)e^{-sx}dx = \mathcal{L}[\frac{1}{x}f(x)].$

(2) ラプラス変換を使って $y(t)$ についての初期値問題 $y' - 4y = 2,\ y(0) = -3$ を解け。
$\mathcal{L}[C] = C/s,\ \mathcal{L}[e^{\alpha t}] = 1/(s-\alpha)$ などを用いてよい。

$\mathcal{L}[y' - 4y] = sY(s) - \underbrace{y(0)}_{=-3} - 4Y(s) = (s-4)Y(s) + 3,\quad \mathcal{L}[2] = \frac{2}{s}$

$\therefore y' - 4y = 2,\ y(0) = -3 \iff (s-4)Y + 3 = \frac{2}{s} \quad \therefore Y(s) = \frac{1}{s-4}(\frac{2}{s} - 3) = \frac{-3s+2}{s(s-4)}.$

このY(s)を部分分数分解すると

$$Y(s) = \frac{-3s+2}{s(s-4)} = \frac{c_1}{s} + \frac{c_2}{s-4} = \frac{(c_1+c_2)s - 4c_1}{s(s-4)} \quad (c_1, c_2 : \text{定数})$$

両辺を比較して $c_1 = -\frac{1}{2},\ c_2 = -\frac{5}{2}$ $\therefore Y(s) = -\frac{1}{2}(\frac{1}{s} + \frac{5}{s-4})$

$\mathcal{L}^{-1}[\frac{1}{s}] = 1,\ \mathcal{L}^{-1}[\frac{1}{s-4}] = e^{4x}$ より

$y(x) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[-\frac{1}{2}(\frac{1}{s} + \frac{5}{s-4})] = -\frac{1}{2}(\mathcal{L}^{-1}[\frac{1}{s}] + 5\mathcal{L}^{-1}[\frac{1}{s-4}]) = -\frac{1}{2}(1 + 5e^{4x}).$